

CALCULUS
PROBLEMS LIST 4
22.10.2010

- (1) Prove the inequality: $2^k < (k+1)!$ for each natural $k \geq 2$.
(2) Prove the Bernoulli's inequality: for $x > -1$ and any $n \in \mathbf{N}$

$$(1+x)^n \geq 1+nx.$$

- (3) Show that for $x > 0$ and any $n \in \mathbf{N}$ we have

$$(1+x)^n > 1 + \frac{n(n-1)}{2}x^2.$$

- (4) Prove, that for any $n \in \mathbf{N}$ the following inequalities hold

$$(a) \quad \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n,$$

$$(b) \quad \sum_{\substack{k=1 \\ k\text{-odd}}}^n \binom{n}{k} = \sum_{\substack{k=0 \\ k\text{-even}}}^n \binom{n}{k}.$$

- (5) Show, that for any natural number n we have the inequality $\binom{2n}{n} < 4^n$.

- (6) Prove, that for any number $a \in \mathbf{R}$ or $a \in \mathbf{C}$ satisfying the condition $|a| < 1$ we have $\lim_{n \rightarrow \infty} a^n = 0$.

- (7) Find the limits:

$$(a) \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^2}\right)^n, \quad (b) \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n.$$

- (8) Find the limits of sequences:

$$(a) a_n = \sqrt[n]{2^n + 3^n}, \quad (b) a_n = \sqrt[n]{2^n + 3^n + 5^n}.$$

- (9) For which real α does the limit

$$\lim_{n \rightarrow \infty} \sqrt[3]{n + n^\alpha} - \sqrt[3]{n}$$

exist? Find this limit for those α for which it exists.

- (10) Compute the limits:

$$(a) \lim_{n \rightarrow \infty} \frac{1+2+3+\cdots+n}{n^2}, \quad (b) \lim_{n \rightarrow \infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}.$$

- (11) Compute the limits of sequences:

$$(a) a_n = \frac{\sin^2 n}{n}, \quad (b) a_n = \sqrt[n]{\log n},$$

$$(c) a_n = \frac{1}{n^2} \log \left(1 + \frac{(-1)^n}{n}\right).$$

- (12) Prove, that if $a_n \xrightarrow{n \rightarrow \infty} g$ then the sequence of absolute values $\{|a_n|\}$ is also convergent, and

$$\lim_{n \rightarrow \infty} |a_n| = |g|.$$

Show that the above theorem does not hold the other way around, that is find a sequence $\{a_n\}$ which is not convergent, even though $\{|a_n|\}$ does converge.

- (13) Prove, that if $|a_n| \xrightarrow{n \rightarrow \infty} 0$ then $\{a_n\}$ also converges to 0.
 (14) Prove, that if sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_n \leq b_n$ and are convergent, then

$$\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n.$$

- (15) The sequence a_n is given in the following way: $a_1 = 0$, $a_2 = 1$, and

$$a_{n+2} = \frac{a_n + a_{n+1}}{2}, \quad \text{for } n = 1, 2, \dots$$

Show that

$$\lim_{n \rightarrow \infty} a_n = \frac{2}{3}.$$

- (16) Show that if $a_n \xrightarrow{n \rightarrow \infty} 0$ and the sequence $\{b_n\}$ is bounded, then

$$\lim_{n \rightarrow \infty} (a_n \cdot b_n) = 0.$$

- (17) Show that if $a_n > 0$ for all $n \in \mathbf{N}$ and $a_n \xrightarrow{n \rightarrow \infty} 0$ then

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} = \infty$$

(improper limit).

- (18) Given is a sequence $\{b_n\}$, about which it is known, that

$$\forall \epsilon > 0 \quad \forall n \geq 10/\epsilon \quad |b_n + 2| < \epsilon.$$

Find M such that

$$\forall n \in \mathbf{N} \quad |b_n| < M,$$

n_1 such that

$$\forall n \geq n_1 \quad b_n < 0,$$

n_2 such that

$$\forall n \geq n_2 \quad b_n > -3,$$

and n_3 such that

$$\forall n \geq n_3 \quad |b_n - 2| > \frac{1}{10}.$$

- (19) Let $a_n = \frac{\sqrt{n^2 + n}}{n}$ and $\epsilon = \frac{1}{100}$. Find $n_0 \in \mathbf{N}$ such, that for $n \geq n_0$ we have $|a_n - 1| < \epsilon$.