Calculus for Computer Scientists

Lecture Notes

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Chapter 1

Calculus – FAQ

Calculus is probably not the most popular course for computer scientists. After all, if someone has a particularly great desire to study calculus, he or she probably studies mathematics, not computer science. And strangely enough computer science freshmen, eager to write their first lines of professional software code are forced to study questions like : "does this infinite sum converge?". It turns out, that among the mandatory courses on the first year you find calculus!

I would like to address some frequently arising questions or doubts, and convince you that this course did not make it to the schedule by a mistake. Actually, it is one of the most important courses of the first few years, and its worthy to devote it your attention.

One frequently encounters the following question: why does the computer scientist need mathematics. Well if a need ever arises to apply a particular mathematical concept or result, you can always read up the necessary stuff, or you can consult the specialist. Actually, this line of reasoning is a misunderstanding. The basic course of calculus is not in any way a specialized knowledge. You should not expect that notions and theorems we will study in this course are going to apply to any of your particular projects. All that calculus stuff is simply a language that we use when we want to formulate or understand a problem. It is a universal language throughout engineering sciences, also in computer science. In today's world, if one wants to be a true, creative professional, practically in any field one has to command English. The professional literature, Internet (simplifying a little bit) are all in English, and any professional foreign stay will not be a success, if you do not know English. It is similar with calculus. You have to become accustomed to notions like convergence, continuity, approximation, integral, power series and the likes. Notions of this type appear everywhere, and will accompany you throughout your future career. Many of you will leave to gain experience in foreign countries, for example to the Microsoft headquarters in Redmond on the shores of lake Washington. Remember that every graduate of a university in the engineering field (and that includes computer science) in the United States has at least 3 semesters of calculus. Those people will form your environment, with them you will do your projects. Without the knowledge of the basic language of the technical trades you will be, so to speak, professionally illiterate. Let me stress that: the basic course in calculus is not a specialized knowledge, which might become useful or it might as well not. It is the basic notions and the relations among them that will appear perpetually, throughout your studies, and then in your professional everyday life. During your further studies you will be offered various other mathematical or borderline mathematical courses. A lot of them will be optional – you can take them or opt out. But calculus, as well as, for example, logic, plays a different role – it is basic, and it is mandatory.

Another problem arises frequently. Students say: "All right, if you insist that badly we will study calculus. But why do you justify everything in such a detail, and why do you give us proofs of theorems. Some of your proofs pour over an entire page! We trust you, if you say that the theorems are true. Instead of proofs, cover more material." Well, this is still the same misunderstanding. In this course our aim is to learn notions, dependencies between them, the way in which they influence one another. The way of arguing is just as important as the facts themselves. In this course the question "what?" is just as important as "why?". Observe, that most proofs are really short and clear. If the proof is not immediate I always try to stress the idea. First, we try intuitively to grasp, why the theorem should hold, and once we get the general idea, we try to make it precise, and we "dress it up" with the right words. If we sense from the start what the proof is trying to accomplish the whole thing is neither hard nor complicated.

Many students make the following comment: "This course is merely a repetition of what we had in high school. Most problems on the mid-terms and the final are shamefully easy. We want, and we can, do more!" It is true, that a lot of the material of this course is in the high school program. But please remember that this course is not aimed at breaking scientific world records. We want to systematically develop the basic knowledge, which is the calculus. There is not much new material, but everything gets laid out with details, without hiding the troublesome odds and ends. In the problem sessions we will do a lot of exercises. As the Americans say: "What is the basis of thorough knowledge? Repetition, repetition, repetition!" But do not worry, if you are looking for in-depth, quality knowledge you have found yourself in the right place. Besides calculus a lot of other courses await you, and you will not get bored. If you are interested in calculus, or other mathematical subjects, then in the building next door you will find courses in virtually any mathematical field and on virtually any level. Many computer science students attend courses in the Mathematical Institute, and many math students come to classes in the Computer Science Institute. It is not by accident, that these two buildings are adjacent to one another, and you can go from one to the other "with the dry foot". Even the library is common. You are always welcome at the office hours, where you can talk to your lecturer who, so to speak, has eaten his mathematical oats.

Another question arises: "The lecture notes have 15 chapters, roughly the same as the number of weeks for the course. Thus we have the work plan, and additionally the notes. Can we then skip the classes? Why should we drag ourselves out of bed for a class at noon, just to watch you copy your notes to the blackboard? Why should we go to the problem sessions to watch someone solve a simple exercise?" Well, the answer is no, you should definitely attend both the lecture and the problem sessions. Listening to the lecture is something completely different than reading the notes. It is not just the matter of questions or ideas appearing. From experience we know, that each lecture is different. Sometimes same topic is covered in 15 minutes some other time the same topic takes an hour. Most certainly a lecture does not mean simply copying notes to the blackboard. The same goes for the problem sessions. You cannot master the material without doing exercises on you own. I think one could use an analogy with studying a foreign language. You have to practice, you have to try, and of course you have to go to the blackboard, and solve the problem in public. Also you have to try not to "fall behind". In a course like calculus it is easy to get lost and lose contact at some point. Notions and ideas once introduced are used later repeatedly. Your attendence is not formally checked, but please remember that not coming to a lecture or to a problem session you can get yourself into trouble. It is not easy to master the material by simply reading the notes. Besides the final exam during the semester we will have 3 mid-term exams, roughly one a month. The mid-terms should give you a "real time" clear image of how you are doing.

If you have other questions – please ask. My address is

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Chapter 2

Real and complex numbers

Real numbers

We are not going to go into details of construction of the set of real numbers. Constructing real numbers, all the arithmetic operations, establishing all their properties is an interesting subject, and is certainly worthy of interest. But in this course we will only recall the more important facts, and we will basically assume that everybody knows the real numbers. The set of real numbers is denoted by $\mathbf{R}$, and we understand the real number as a decimal expansion (sequence of decimal digits), for example $123,357290\ldots$. Decimal expansion contains a decimal point (in this part of the world it is the comma), it is finite to the left, and finite or infinite to the right. The expansion can have a sign $-$, and then it is called a negative number. We all know how to add, subtract, multiply and divide numbers like that, and we know the properties of such arithmetic such as connectivity. Let us recall important facts:

1. If certain sequence of digits repeats itself right of the decimal point, we say that the expansion is periodic, and we enclose the repeated sequence between the parenthesis: $0,03212512512\cdots = 0,032(125)$.

2. If from certain place right of the decimal point the expansion consists entirely of zeros, we call such expansion finite, and we omit he trailing zeros: $3,234000000\cdots = 3,234(0) = 3,234$.

3. In principle different decimal expansions mean different real numbers. There are, however, exceptions and it may happen, that 2 different expansions denote the same number. Such an exception happens if the expansion, from certain point (right of the decimal), consists of only 9. Such an expansion represents the same number as the expansion,

in which all repeating 9 are dropped and the last non-9 is increased by 1. For example $0,09999\cdots = 0,0(9) = 0,1$. This can be easily proved, using the properties of arithmetic (for example, using the fact that multiplying a number by 10 means shifting the decimal point of its expansion right by one place). Let $x = 0,0(9)$. We then have

$$10 \cdot x = 0,(9) = 0,9 + 0,0(9) = 0,9 + x \Rightarrow 9 \cdot x = 0,9 \Rightarrow x = 0,1.$$

Real numbers, whose expansions have only zeros to the right of the decimal point are called integers, and the set of integers is denoted by **Z**. Positive integers $1, 2, \ldots$ (without zero) are called natural numbers, or naturals, and the set of naturals is denoted by **N**.

Rational numbers

Numbers whose decimal expansions are finite or periodic are called rational numbers. We denote the set of rational numbers by **Q**. Rational numbers can be written as fractions $\frac{m}{n}$, where m, n are integers, and $n \neq 0$. If n is a natural number, and m and n have no common divisor, then the expression of x as the fraction $\frac{m}{n}$ is unique, and we call such fraction irreducible. Each rational number can be expressed as an irreducible fraction.

Examples: (a) $\frac{1}{7} = 0,1428571428\cdots = 0,(142857)$. The decimal expansion can be obtained by applying the "long division" procedure. Dividing, at certain point we observe, that the remainder repeats a past value. At that point the entire expansion starts repeating a period. It is not hard to observe, that the period is no longer than the value of the denominator minus 1.

(b) $0,123 = \frac{123}{1000}$. This is an irreducible fraction, since, as can be easily checked the numerator and the denominator have no common divisors, and the denominator is positive.

(c) $0,(a_1 a_2 \cdots a_k) = \frac{a_1 a_2 \cdots a_k}{99\cdots 9}$ (k – nines in the denominator). It is easy to prove, writing out and solving an appropriate equation for $x = 0,(a_1 \cdots a_k)$.

(d) Let us convert the following decimal expansion into a fraction

$$\begin{aligned} 0,123(45) = 0,123 + 0,000(45) &= \frac{123}{1000} + \frac{0,(45)}{1000} \\ &= \frac{123}{1000} + \frac{1}{1000}\frac{45}{99} = \frac{99 \cdot 123 + 45}{99000} = \frac{12222}{99000}. \end{aligned}$$

Irrational numbers

Real numbers which are not rational, that is those with decimal expansions neither finite nor periodic, are called irrational numbers.

Examples: (a) Let us write out an expansion which contains consecutively longer sequences of zeros, separated by single ones:

$$x = 0,101001000100001 \cdots 10 \cdots 010 \cdots .$$

The series of zeros are progressively longer, and so the expansion is not periodic. It is not finite either, since ones keep appearing, although more and more scarcely. x is thus an irrational real number.

(b) Another example of an irrational number is $\sqrt[3]{15}$. We will show, that $\sqrt[3]{15}$ is not rational. This will be a typical reasoning, and it can be adapted to many examples. First of all recall, that a root of arbitrary order can be extracted from any non-negative real. This is a property of reals, and we assume it is known. Thus $\sqrt[3]{15}$ by definition is the unique positive real number, such that raised to the third power recovers 15. Let us assume it is rational. This is an example of indirect reasoning. We assume something, and show that such assumption leads to contradiction. By the rules of logic this shows, that the initial assumption was false. So, again let us reason indirectly, and let us assume that $\sqrt[3]{15}$ is rational. Let us then express it as an irreducible fraction

$$\sqrt[3]{15} = \frac{m}{n} \quad \Rightarrow \quad 15 = \frac{m^3}{n^3} \quad \Rightarrow \quad n^3 \cdot 15 = m^3.$$

3 divides the left hand side of the last equality, and so it must divide the right hand side. 3 is a prime number, so if it divides a product of numbers it must divide one of the factors (it is a property of primes). So 3 must divide m, and so the right hand side, as a cube, has to be divisible by 27. In that case on the left hand side n^3 has to be divisible by 3 (since 15 can only be divided by 3), and thus again, since 3 divides n^3 it must divide n. The fraction $\frac{m}{n}$ is thus not irreducible, which contradicts our assumption. The assumption that $\sqrt[3]{15}$ is a rational number has to be false.

Remarks: (i) A prime number is a natural number, greater than 1, which has no other divisors than 1 and itself. Prime numbers have the following property: if p is a prime, and $p|m \cdot n$ (p divides $m \cdot n$), then $p|m$ or $p|n$.

(ii) The above reasoning in (i) is an application of the decomposition of a natural number as a product of factors, each of which is prime. Such a decomposition is called a decomposition into prime factors, and such decomposition is unique. In the equality

$$n^3 \cdot 15 = m^3$$

the prime factors of n^3 and m^3 come in triples, and the prime factors of 15, that is 3 and 5 are lone. We have used this to arrive at a contradiction. The existence and the uniqueness of the prime factor decomposition is a property of the set $\mathbf{N}$, which we will not prove, but which is worth remembering. As an exercise in which prime factor decomposition can be useful let us mention the following question: how many trailing zeros does the number (1000)! (1000 factorial) have?

(iii) The root appearing in the above example, sa the logarithm and powers appearing below are examples of elementary functions. We assume that we know elementary functions, and we will not provide detailed definitions. In the next chapter we will briefly recall the most important facts about elementary functions.

(c) $\log_2 3$. We will reason in the same way as in the previous example, that is indirectly. Let us assume that $\log_2 3$ is a rational number and let $\log_2 3 = \frac{m}{n}$ be an irreducible fraction.

$$\log_2 3 = \frac{m}{n} \quad \Rightarrow \quad 2^{\frac{m}{n}} = 3 \quad \Rightarrow \quad 2^m = 3^n.$$

We have arrived at a contradiction, since the left hand side of the above equality only contains twos as its prime factors, while the right hand side only contains threes. The assumption that $\log_2 3$ is rational must thus be false.

(d) The sum, difference, product and fraction of rational numbers are all rational (of course one cannot divide by zero). The sum, difference, product and fraction of an irrational number by a rational number are irrational (unless, in the case of product and fraction the rational number is zero). The result of an arithmetic operation on two irrational numbers depends, can be rational or irrational, depending on the particular values.

Geometric interpretation

We can think of real numbers as points on a line. On that line we mark places for zero and one, and we mark with the arrow the positive direction, which is determined by the relative position of zero and one. Traditionally the positive direction points to the right. Each real number can then be assigned a unique point on such line. The line with the real numbers assigned to its points is called the real axis.

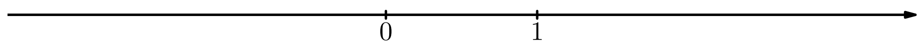

Figure 2.1: The real axis.

The ordering of the set R

If $x-y$ is a positive number we write $x > y$ ("x is greater than y"), if it is a non negative number we write $x \geq y$ ("x is greater or equal than y"). Similarly, if $x-y$ is negative we write $x < y$, and if non-positive we write $x \leq y$. Therefore, we see that for for any two real numbers x, y we have either $x = y$ or $x < y$ or $x > y$. We say that the set **R** is ordered. O the real axis $x > y$ if x is more to the right than y – this is symbolized by the arrow – to the right the numbers grow.

Symbols

$\forall$ we read "for all", $\exists$ we read "exists", $\Leftrightarrow$ we read 'if and only if", $(\cdots) \Rightarrow (\cdots)$ we read "$(\cdots)$ implies $(\cdots)$", $\in$ we read "belongs to", $\subset$ we read "is a subset of". The symbol $\wedge$ we read 'and", while the symbol $\vee$ we read "or".

Let us recall two properties of the set of real numbers: the Archimedean axiom and the continuity axiom.

Archimedean axiom

The real numbers have the following property, which intuitively is clear: for any $x, y > 0$ there exists a natural number n such that

$$nx > y.$$

Using the above introduced notation we can write the axiom as follows

$$\forall\ x, y > 0 \quad \exists\ n \in \mathbf{N} \quad nx > y.$$

It follows from the Archimedean axiom that, for example, there are natural numbers arbitrarily large (larger than any fixed real number). Since multiplying by -1 reverses the inequalities, then it also follows from the axiom that there exist integers arbitrarily small (smaller than any fixed real number). Let us observe, that it also follows from the axiom that there are positive numbers arbitrarily small (positive, but smaller than arbitrary other fixed positive number). We will be using all these facts, without directly referring to the Archimedean axiom.

The extrema

We say that the set $A \subset \mathbf{R}$ is:

- bounded from above, if

$$\exists\, c \quad \forall\, x \in A \quad x \leq c,$$

- bounded from below, if

$$\exists\, d \quad \forall\, x \in A \quad x \geq d,$$

- bounded, if it is bounded from above and from below.

The constants c and d in the above conditions are respectively called the bound of the set A from above (or the upper bound), and the bound of the set A from below (the lower bound). The set of natural numbers is bounded from below (the bound from below is, for example, the number 1), but is not bounded from above (it follows from the Archimedean axiom that one cannot find a c which is a bound of $\mathbf{N}$ from above). If the set $A \subset \mathbf{R}$ is bounded from above, then the smallest upper bound of A is called its supremum, and is denoted

$$\sup A \quad \text{(supremum of } A\text{)}.$$

If $A \subset \mathbf{R}$ is bounded from below, then the largest lower bound of A is called its infimum, and is denoted

$$\inf A \quad \text{(infimum of } A\text{)}.$$

Thus, $s = \sup A$ if

- $\forall\, x \in A \quad x \leq s$,
- $\forall\, u < s \quad \exists\, x \in A \quad x > u$.

The first condition says that A is bounded from above, and s is its upper bound, while the second condition says that no number smaller than s is an upper bound of A. Thus, both conditions together say that s is the smallest upper bound of A. We can similarly summarize the definition of the infimum: $k = \inf A$ if the following two conditions are satisfied simultaneously:

- $\forall\, x \in A \quad x \geq k$,
- $\forall\, l > k \quad \exists\, x \in A \quad x < l$.

The notions of supremum and infimum have been introduced in the case of a set bounded from above and from below respectively. In addition to that, if A is not bounded from above we will write

$$\sup A = +\infty,$$

and if A is not bounded from below we will write

$$\inf A = -\infty.$$

For example

$$\inf \mathbf{N} = 1 \quad \text{and} \quad \sup \mathbf{N} = +\infty.$$

Continuity axiom

This axiom states that every set $A \subset \mathbf{R}$, bounded from above has a supremum. This is a property of the set of real numbers: from all upper limits of A, bounded from above, one can choose the smallest one. Thinking geometrically, this property says that the real numbers fill out the entire real axis, with no holes left. This property, the continuity axiom, can be equivalently formulated in terms of lower bounds: every set bounded from below has an infimum.

Remark: A set can contain its supremum or infimum or not. Consider, for example

$$\sup\{x : \; x < 1\} = \sup\{x : \; x \leq 1\} = 1.$$

The first set does not contain 1, while the second one does.

Example: Let us consider the following set

$$A = \left\{ \frac{m^2 + n^2}{2mn} : \; m, n \in \mathbf{N}, m < n \right\}.$$

Let us observe, that A is not bounded from above. Indeed, that set A contains all numbers of the form $\frac{m^2+1}{2m}$, $m \in \mathbf{N}$, $m > 1$. Each such number is larger than $\frac{m}{2}$, and numbers of that form, with arbitrary $m \in \mathbf{N}$ include all natural numbers. Thus A contains elements larger than arbitrary fixed natural number. It is therefore not bounded from above. On the other hand, let us observe, that A is bounded from below, and that 1 is a lower bound. To this end, let us recall a well known inequality:

$$2ab \leq a^2 + b^2 \quad \Rightarrow \quad \frac{m^2 + n^2}{2mn} \geq 1 \quad \text{for } m, n > 0.$$

We will now prove that 1 is the largest lower bound of A. Let $c > 1$. Then $\frac{1}{c-1}$ is a positive number, and from the Archimedean axiom it follows, that there exists a natural number m larger that $\frac{1}{c-1}$. Additionally let $m \geq 2$, which we can always assume, increasing m if necessary. Then

$$\frac{1}{c-1} < m < 2m(m-1) \quad \Rightarrow \quad 1 + \frac{1}{2m(m-1)} < c.$$

We thus have

$$\frac{m^2 + (m-1)^2}{2m(m-1)} = \frac{m^2 + m^2 - 2m + 1}{2m(m-1)} = \\ = \frac{2m(m-1) + 1}{2m(m-1)} = 1 + \frac{1}{2m(m-1)} < c.$$

Assuming that $c > 1$ we found in A an element $\frac{m^2+(m-1)^2}{2m(m-1)}$, smaller that c. Thus, no $c > 1$ can be a lower bound of A, and so 1 is the largest lower bound of A, that is $\inf A = 1$. In addition, let us observe that $1 \notin A$: if $1 \in A$ then there would be $m, n \in \mathbf{N}$, $n \neq m$, such that $m^2 + n^2 = 2mn$. But we know that such equality is equivalent to $(m-n)^2 = 0$, so $m = n$, which is a contradiction.

Intervals

We denote intervals in the following way:

$$\begin{aligned} (a, b) &= \{x : \ a < x < b\}, && \text{(open interval)}, \\ [a, b] &= \{x : \ a \leq x \leq b\}, && \text{(closed interval)}, \\ (a, b] &= \{x : \ a < x \leq b\}, && \text{(left-hand open interval)}, \\ [a, b) &= \{x : \ a \leq x < b\}, && \text{(right-hand open interval)}. \end{aligned}$$

In the case of intervals (a, b) and $(a, b]$ we allow $a = -\infty$, and in the case of intervals (a, b) and $[a, b)$ we allow $b = \infty$. Such intervals then denote the appropriate half-axes. We assume by default that $a < b$, and in case of the closed interval $[a, b]$ we allow $a = b$.

Absolute value

We define the absolute value of a real number in a following way

$$|x| = \begin{cases} x & : \quad x \geq 0, \\ -x & : \quad x < 0. \end{cases}$$

The absolute value has the following properties:

1. $|-x| = |x|$,
2. $-|x| \leq x \leq |x|$,
3. $|x + y| \leq |x| + |y|$ (triangle inequality),
4. $\big||x| - |y|\big| \leq |x - y|$,
5. $|x - y|$ represents the distance from x to y on the real line,
6. $|x \cdot y| = |x| \cdot |y|$,
7. $|x| = \sqrt{x^2}$,
8. $|x| \geq 0$ and $|x| = 0 \Leftrightarrow x = 0$,
9. $x \leq y \ \wedge \ -x \leq y \ \Rightarrow \ |x| \leq y$.

As an example let us prove the triangle inequality 3. We consider separately two cases
(a) x and y have the same sign $\pm$. Their sum again has the same sign, so

$$|x + y| = \pm(x + y) = \pm x + \pm y = |x| + |y|.$$

In this case we see, that the triangle inequality is actually an equality.
(b) x and y have opposite signs. We can assume $x \leq 0 \leq y$, if not we simply rename x and y. If $x + y \geq 0$ then

$$|x + y| = x + y \leq -x + y = |x| + |y|,$$

while if $x + y < 0$ then

$$|x + y| = -(x + y) = -x - y \leq -x + y = |x| + |y|.$$

In this case, if none of x and y is zero, the inequality is actually sharp.

Integral part and fractional part

The integral part of x is the largest integer not greater that x (clearly, such greatest integer exists). The integral part of x is denoted by $[x]$. The fractional part of x is $\{x\} = x - [x]$. The integral part has the following properties

- $[x] \in \mathbf{Z}$,
- $[x] \leq x < x + 1$ and $x - 1 < [x] \leq x$,
- $[x] = x \Leftrightarrow x \in \mathbf{Z}$.

Examples: $[1,5] = 1,\ [-1,5] = -2,\ \{-1,5\} = 0,5$.

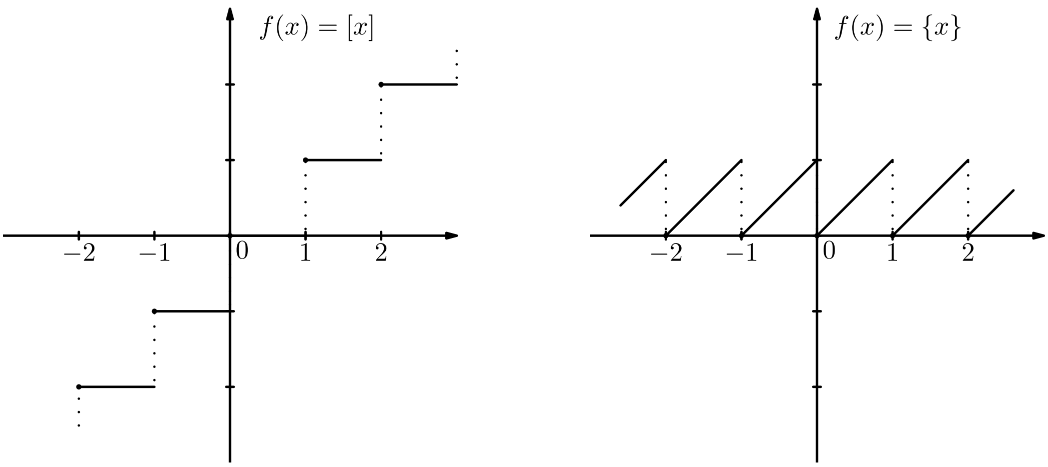

Figure 2.2: The integral part and the fractional part.

The density of the rational and the irrational numbers

Both a rational and an irrational numbers exist in every interval (a, b). Let (a, b) ba an arbitrary interval (remember that $a < b$, so the interval is not an empty set). We will show that there is a rational number in (a, b). The irrational number is let as an exercise. We have $\frac{1}{b-a} > 0$, so it follows from the Archimedean axiom that there exists a number $n \in \mathbf{N}$ such that $n > \frac{1}{b-a}$, that is $\frac{1}{n} < (b - a)$. Let us consider the set of numbers of the form

$$\left\{ \frac{k}{n} :\ k \in \mathbf{Z} \right\}.$$

We will show now, that one of the numbers from the above set must fall into our interval (a, b). The idea is clear: numbers from the above set are uniformly spaced on the real line, with the distance $\frac{1}{n}$ between adjacent. This spread this smaller than the size of our interval. Let us now make this idea precise. Let k_0 be the largest if the integers k such that

$$k \leq na.$$

The set of integers k satisfying the above condition is clearly bounded from above, and clearly its supremum is an integer, so such largest k_0 exists. Observe, that $\frac{k_0+1}{n} > a$ and since $\frac{k_0}{n} \leq a$ and $\frac{1}{n} < (b - a)$, then $\frac{k_0+1}{n} < a + (b - a) = b$. Therefore $\frac{k_0+1}{n} \in (a, b)$, and is of course rational.

Mathematical induction

The set of natural numbers has the following property: each of its non-empty subsets has a smallest element. From this property we obtain the following principle of mathematical induction. Let $T(n)$, $n \geq n_0$ be some sequence of theorems. In applications often these are equalities or inequalities, with the natural parameter n. Let:

1. $T(n_0)$ be true (the starting point for induction),
2. $\forall\, n \geq n_0$ the following implication is true $(T(n) - \text{true}) \Rightarrow (T(n+1) - \text{true})$ (the induction step).

Then all theorems $T(n)$, $n \geq n_0$ are true. The principle of mathematical induction is intuitively obvious, and it can be easily proved: If not all of the theorems $T(n)$, $n \geq n_0$ are true, then the set $A \subset \mathbf{N}$ of those $n \geq n_0$, for which $T(n)$ is false is non-empty. A has the smallest element, which we denote $\tilde{n}$.Observe, that it follows from 1. that we must have $\tilde{n} > n_0$. So we have $T(\tilde{n})$ false (since $\tilde{n} \in A$), but $T(\tilde{n} - 1)$ true, since $\tilde{n} - 1 \notin A$. This contradicts 2., since from the fact that $T(\tilde{n} - 1)$ is true it should follow that $T(\tilde{n})$ is also true.

Example: We will show, that for every $n \in \mathbf{N}$ the following inequality is true: $10n < 2^n + 25$. This inequality is our theorem $T(n)$. We first try to prove the induction step, that is, we prove 2. Let us thus assume

$$10n < 2^n + 25,$$

and let us try, assuming the above, to prove

$$10(n + 1) < 2^{n+1} + 25. \tag{2.1}$$

We thus have

$$10(n+1) = 10n + 10 < 2^n + 25 + 10. \tag{2.2}$$

To conclude the proof, and arrive at the right hand side of (2.1) we need the inequality $10 \leq 2^n$, which, unfortunately is only true for $n \geq 4$. We thus restrict ourselves to $n_0 = 4$, and conclude (2.2):

$$2^n + 25 + 10 < 2^n + 2^n + 25 = 2^{n+1} + 25,$$

that is we have the induction step proved for $n \geq 4$. The induction principle can be only used with the starting point $n_0 = 4$. We still have to check $T(4)$, and additionally $T(1)$, $T(2)$ and $T(3)$, which could not be "reached" by induction. We easily check these particular cases by hand.
$n = 1:$ $10 < 2 + 25$ true,
$n = 2:$ $20 < 2^2 + 25$ true,
$n = 3:$ $30 < 2^3 + 25$ true, and finally
$n = 4:$ $40 < 2^4 + 25 = 41$ also true.

We have used the principle of induction to conduct the proof for $n \geq 4$, and we did the remaining cases directly. This is the typical approach: attempting to make the induction step we identify the conditions (lower bound) on n under which the induction step can be proved. To this lower bound we adjust the induction starting point, and we verify the eventual leftover cases "by hand".

Complex numbers

The set of complex numbers $\mathbf{C}$ is the set of symbols $a + b\,i$, where $a, b \in \mathbf{R}$. Such symbols are added, subtracted and multiplied according to the formulas

$$(a + b\,i) \pm (c + d\,i) = (a \pm c) + (b \pm d)\,i,$$
$$(a + b\,i) \cdot (c + d\,i) = (ac - bd) + (ad + cb)\,i.$$

We can also divide by non zero complex numbers:

$$\frac{a + b\,i}{c + d\,i} = \frac{(ac + bd) + (-ad + bc)\,i}{c^2 + d^2}, \quad c^2 + d^2 > 0.$$

We treat real numbers as a subset of the complex numbers $\mathbf{R} \subset \mathbf{C}$ by identifying $x \sim x + 0\,i$. Observe, that such identification preserves the arithmetic operations: for example $(a + 0\,i) + (b + 0\,i) = (a + b) + 0\,i$. Also, observe that $(i)^2 = (0 + 1\,i)^2 = -1 + 0\,i = -1$. With the above identification we have $i^2 = -1$, and we treat complex numbers as an expansion of the set of real numbers. The set **C** has an advantage: each polynomial with complex coefficients factors into a product of linear terms. Thanks to this the complex numbers are an important tool for both mathematicians and engineers (also for computer scientists:-)). Let us introduce the following notions:

- $\Re(a + b\,i)$ = the real part of $(a + b\,i) = a$,
- $\Im(a + b\,i)$ = the imaginary part of $(a + b\,i) = b$,
- $\overline{a + b\,i}$ = the conjugate of $(a + b\,i) = a - b\,i$.

We have the following properties

1. $\overline{(\overline{z})} = z, \quad \overline{z + w} = \overline{z} + \overline{w}, \quad \overline{z \cdot w} = \overline{z} \cdot \overline{w}$,
2. $\Re(z) = \frac{z+\overline{z}}{2}, \quad \Im(z) = \frac{z-\overline{z}}{2\,i}$,
3. $z = \overline{z} \Leftrightarrow z \in \mathbf{R}$,
4. $z \cdot \overline{z} = \Re(z)^2 + \Im(z)^2$ – a nonnegative real number.

The modulus

The modulus of a complex number is defined as

$$|z| = \sqrt{\Re(z)^2 + \Im(z)^2}.$$

Examples: $| - 1 + 2\,i| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$, $|i| = |0 + 1\,i| = 1$.
The modulus of a complex number correspondes to the absolute value of a real number. If z happens to be real ($\Im z = 0$), then $|z|$ is the same number, regardless of whether we think of it as the absolute value of a real number, or as the modulus of a complex numbers. Both names "the modulus" and 'the absolute value" are often used interchangeably. We have the following properties of the modulus

- $|z| \geq 0$ i $|z| = 0 \Leftrightarrow z = 0$,
- $|z| = | - z| = |\overline{z}|$, $|\alpha z| = |\alpha| \cdot |z|$ for $\alpha \in \mathbf{R}$,
- $|z \cdot w| = |z| \cdot |w|$,
- $|z + w| \leq |z| + |w|$ (the triangle inequality),
- $|z - w| \geq \big||z| - |w|\big|$.

The geometric interpretation

The complex numbers, that is the expressions of the form $a + b\,i$ can be identified with points in the plane $\mathbf{R}^2 = \{(x, y) : \ x, y \in \mathbf{R}\}$. With this geo-

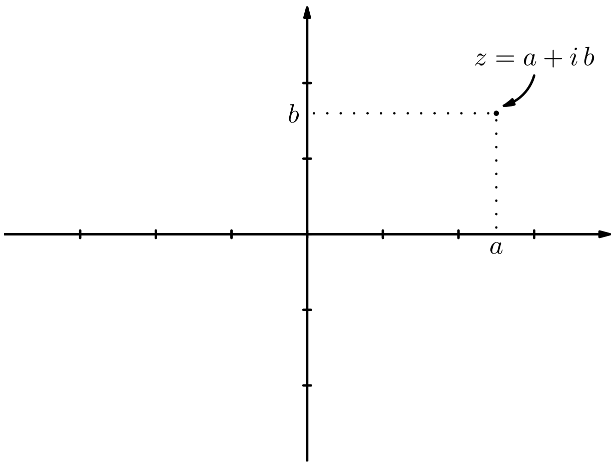

Figure 2.3: Complex plane.

metric interpretation the addition corresponds to vector addition (according to the parallelogram rule), and multiplication by a real number corresponds to multiplication by a scalar. The operation of conjugation is a reflection with respect to the horizontal axis, and the modulus represents the Euclidean distance from the origin of the coordinate system.

The trigonometric form

A complex number $a+b\,i$ can be written in the so-called trigonometric form. In this form numbers can be easily multiplied, raised to the power, and roots can be easily extracted. Let $z = a + b\,i \neq 0$

$$z = a + b\,i = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}\,i\right).$$

We can find a number $\varphi \in [0, 2\pi)$, such that

$$\cos\varphi = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\varphi = \frac{b}{\sqrt{a^2+b^2}}.$$

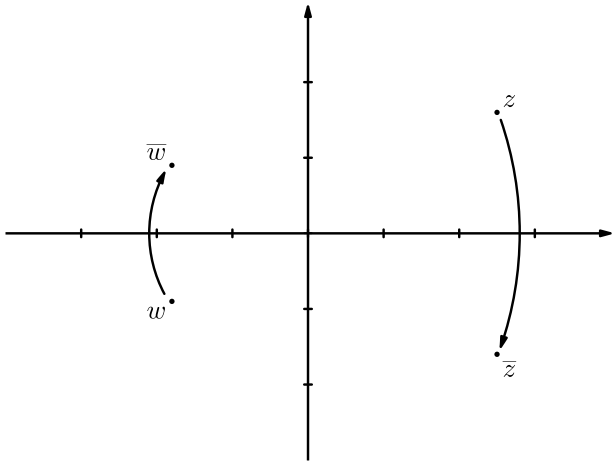

Figure 2.4: The conjugation of the complex number.

This can be plugged into the formula for z, and we obtain the so called trigonometric form of z

$$z = |z|\,(\cos\varphi + i\,\sin\varphi).$$

Using the geometric interpretation, writing a complex number $a + b\,i$ in the trigonometric form $r(\cos\varphi + i\,\sin\varphi)$ corresponds to presenting a point (a, b) on the plane in polar coordinates (r, φ). The number φ is called the argument of z. Since functions sin and cos are periodic with period 2π, so each complex number z has infinitely many arguments, which differ precisely by an integer multiple of 2π. This one of them, which falls into the interval $[0, 2\pi)$ (there is precisely one such) is called the principal argument of z.

Example: $z = 1 - i = \sqrt{2}(\frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\,i)$. We are looking for $\varphi \in [0, 2\pi)$, such that

$$\cos\varphi = \frac{1}{\sqrt{2}}, \quad \sin\varphi = -\frac{1}{\sqrt{2}}.$$

It is easy to observe, that $\varphi = \frac{7}{4}\pi$.

Remarks: (i) Two complex numbers are equal, if both their real and imaginary parts are equal. In the case these numbers are written in the trigonometric form we have

$$r\,(\cos\varphi + i\,\sin\varphi) = s\,(\cos\psi + i\,\sin\psi)$$

if $r = s$ and $\varphi - \psi$ is an integer multiple of 2π.

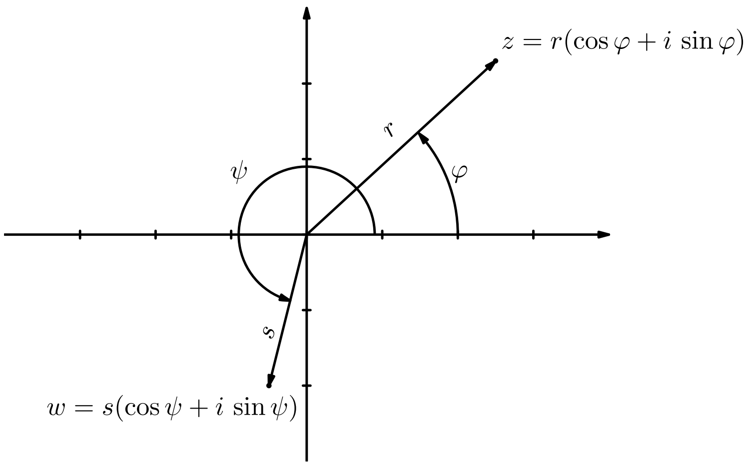

Figure 2.5: The trigonometric form of a complex number.

(ii) The product can be easily expressed in the trigonometric form

$$r\,(\cos\varphi + i\,\sin\varphi)\cdot s\,(\cos\psi + i\,\sin\psi) = rs\,(\cos(\varphi+\psi) + i\,\sin(\varphi+\psi)).$$

In other words, we multiply the moduli and we add the arguments.

(iii) As an immediate consequence of (ii) we obtain

$$z = r(\cos\varphi + i\,\sin\varphi) \Rightarrow z^n = r^n(\cos(n\varphi) + i\,\sin(n\varphi)).$$

Observe, that the above holds for all $n \in \mathbf{Z}$, bot positive and negative.

(iv) A root of a complex number z of order $n \in \mathbf{N}$ is a number w such, that $w^n = z$. Using the trigonometric form we will show, that every complex number $z \neq 0$ has exactly n distinct roots of order n. Let

$$z = r\,(\cos\varphi + i\,\sin\varphi),$$

and $n \in \mathbf{N}$. Let us introduce the following numbers

$$w_k = \sqrt[n]{r}\,(\cos\psi_k + i\,\sin\psi_k),$$

where

$$\psi_k = \frac{\varphi + 2k\pi}{n} \quad k = 0, 1, \ldots, n-1.$$

Observe, that each of the numbers w_k is indeed the root of order n of z (this is immediate from (iii)), and that they are all distinct. By definition we have

$$\psi_k - \psi_l = \frac{k-l}{n}\,2\pi,$$

and $-1 < \frac{k-l}{n} < 1$. The only integer satisfying these inequalities is zero, so if $w_k = w_l$ we have $k = l$. We thus have n distinct roots. There can be no more, since each root of order n of number z is also a root of the polynomial of order n

$$P(w) = w^n - z.$$

We know, that polynomials of order n have at most n distinct roots.

Example: We will compute all roots of order 4 of $1 - i$. By the above procedure,

$$\sqrt[4]{1-i} = \sqrt[8]{2}\,(\cos\psi_k + i\,\sin\psi_k),$$

where

$$\psi_k = \frac{\frac{7}{4}\pi + 2k\pi}{4} = \frac{7}{16} + \frac{k\,\pi}{2}, \qquad k = 0, 1, 2, 3.$$

Chapter 3

Functions

Let us recall some basics about functions, that we will use. Let $A \subset \mathbf{R}$ be a subset of the real numbers. A real valued function $f(x)$ defined on A is a way of assigning some real number to every element of A. A complex valued function is, similarly, a way of assigning a complex number to each element of A. We write

$$f : M \to \mathbf{R} \quad \text{or} \quad f : M \to \mathbf{C}.$$

The set A is called the domain of the function $f(x)$ and is often denoted by D_f. The set

$$\{y : \ \exists\, x \in D_f \quad f(x) = y\}$$

is called the range of $f(x)$, or the set of values of $f(x)$. Defining a function (that is the way of assigning values to elements of the domain) most often takes the form of a formula. The formula is often split between parts of the domain. A function so defined is called a function "spliced" from its parts. Often we do not specify the domain D_f. In such case we assume that the function is defined on the largest set on which the formula (or formulas) defining it makes sense. Such maximal set is called the natural domain of the function $f(x)$. When we refer to a function, we write it together with its typical argument: $f(x)$. We will be careful not to confuse this with the value of the function at a particular point x.

The monotonicity

For real valued $f(x)$ we say that $f(x)$ is increasing (or strictly increasing), if

$$x < y \quad \Rightarrow \quad f(x) < f(y).$$

We say that it is weakly increasing (or non-decreasing), if

$$x < y \quad \Rightarrow \quad f(x) \le f(y).$$

Similarly, $f(x)$ is decreasing (strictly decreasing), if

$$x < y \quad \Rightarrow \quad f(x) > f(y)$$

and weakly decreasing (non-increasing), if

$$x < y \quad \Rightarrow \quad f(x) \geq f(y).$$

In other words and increasing function can be applied to the sides of an inequality, and the inequality is preserved, while if the function is decreasing the inequality is reversed. We say, that $f(x)$ is monotonic, if it is either increasing or decreasing, and the same with adjectives "strictly" and "weakly". Functions can be piecewise monotonic. For example, $f(x) = x^3$ is strictly increasing, and thus inequalities can be raised to the third power. On the other hand $f(x) = x^2$ is only piecewise monotonic – decreasing for $x \leq 0$, and increasing for $x \geq 0$. Inequalities can be therefore raised to the second power (squared), provided they relate nonnegative numbers.

The graph

If $f(x)$ is real valued, then its graph is the following subset of the plane

$$\{(x, y) : \; x \in D_f, \; y = f(x)\} \subset \mathbf{R}^2.$$

When analyzing a function it is always a good idea to try to sketch its graph. A graph visualizes properties, which are usually not so easy to deduce from the formulas. Of course, the sketch of the graph is not a replacement for the proper definition.

Elementary functions

Functions that are encountered most often are the so-called elementary functions. Let us recall the some of the elementary functions

(a) Polynomials are functions of the form $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. n is called the degree of the polynomial $f(x)$ – provided $a_n \neq 0$. The coefficients can be real (then the polynomial is real valued) or complex (then the polynomial is complex valued). $D_f = \mathbf{R}$. A polynomial of degree n has at most n roots (points where it is zero). A polynomial with real coefficients with odd degree has at least 1 root, while that with even degree might have no roots at all. For values $|x|$ large the polynomial behaves like its leading term $a_n x^n$.

(b) Rational functions are functions of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ i

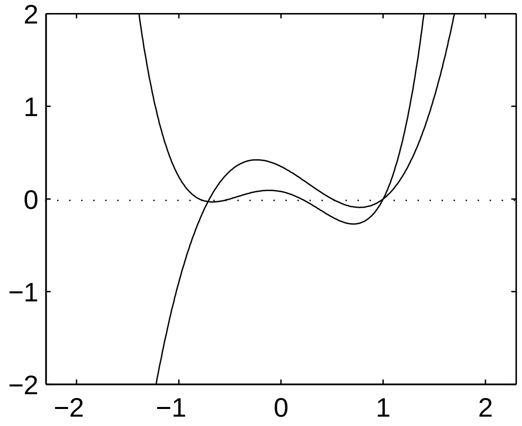

Figure 3.1: Polynomials of degree 3 and 4

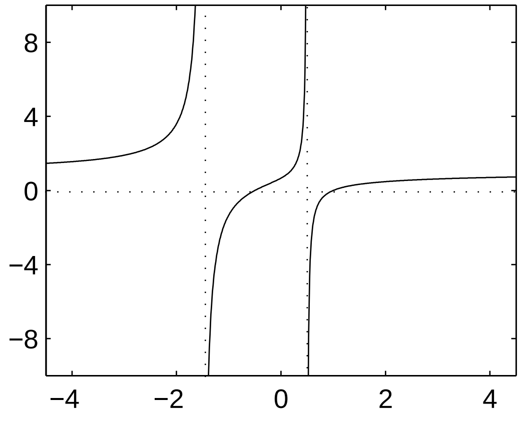

Figure 3.2: An example of a rational function

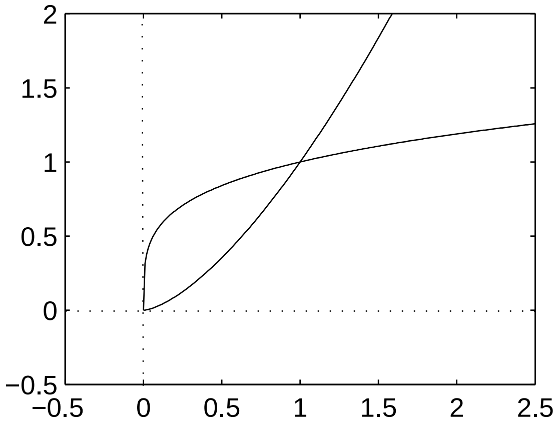

Figure 3.3: Two power functions, with exponents $0,25$ i $1,5$.

$Q(x)$ are polynomials. $D_f = \{x : Q(x) \neq 0\}$.

(c) The power function $f(x) = x^\alpha$. D_f depends on α If $\alpha = \frac{m}{n}$ and $m, n \in \mathbf{N}$, then $x^\alpha = \sqrt[n]{x^m}$. $x^0 = 1$ for every x, and for $m < 0$ we let $x^m = \frac{1}{x^{-m}}$. If α is irrational, and $x > 0$ we define

$$\begin{aligned} x^\alpha &= \sup\{x^q : \ q \in \mathbf{Q}, \ q < \alpha\}, \qquad x \geq 1, \\ x^\alpha &= \inf\{x^q : \ q \in \mathbf{Q}, \ q < \alpha\}, \qquad x < 1. \end{aligned}$$

Apart from particular cases of α (for example, $\alpha \in \mathbf{N}$) we have $D_f = \mathbf{R}^+ = \{x \in \mathbf{R} : \ x > 0\}$. If $\alpha > 0$ then the power function is increasing, while if $\alpha < 0$ the function is decreasing. Of course, if $\alpha = 0$, the the power function is constant equal to one.

(d) The exponential function $f(x) = a^x, \ a > 0$. $D_f = \mathbf{R}$. The arithmetic operation is the same as in the case of the power function, but now it is the exponent that is variable, while the base is fixed. If $a > 1$ then the function is increasing, while if $a < 1$ it is decreasing. Of course, if the base $a = 1$ then the function is constant, equal to 1.

(e) The logarithm $f(x) = \log_a x, \ a > 0, a \neq 1$. $D_f = \mathbf{R}R^+$. The logarithm is a function inverse to the exponential, that is $y = \log_a x \Leftrightarrow a^y = x$. If the base $a > 1$ then the logarithm is increasing, while if $a < 1$ the logarithm is decreasing. The case $a = 1$ is excluded.

The power function, the exponential function and the logarithm are all related to raising numbers to some powers. We have thus the following properties (in each case we must remember possible limitations on the range of

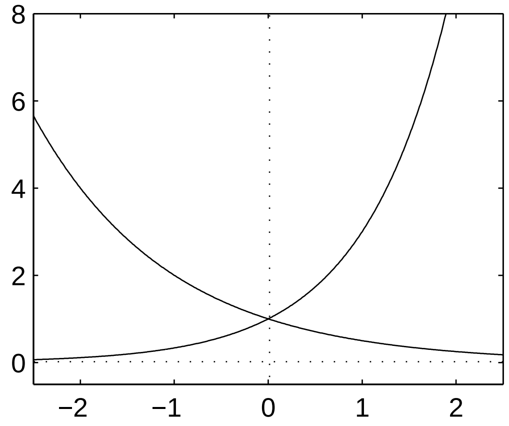

Figure 3.4: Exponential functions with bases greater and smaller than 1.

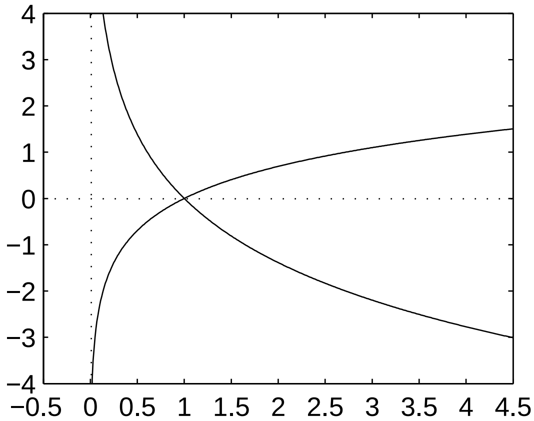

Figure 3.5: Logarithms with bases larger and smaller than 1.

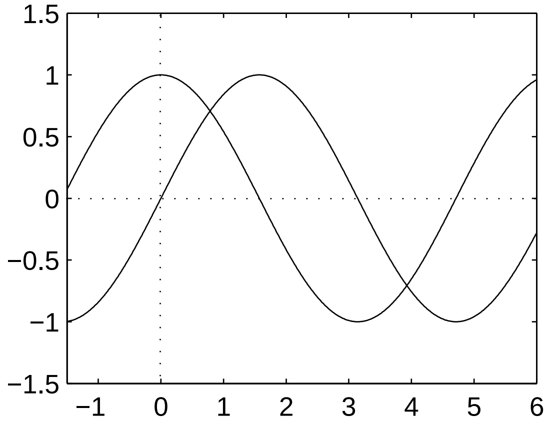

Figure 3.6: Functions $\sin(x)$ and $\cos(x)$.

variables): $(x^\alpha)^\beta = x^{\alpha\cdot\beta}$, $(x \cdot y)^\alpha = x^\alpha y^\alpha$, $x^\alpha x^\beta = x^{\alpha+\beta}$, $a^{x+y} = a^x \cdot a^y$, $\log_a(x \cdot y) = \log_a x + \log_a y$, $\log_a(x^\alpha) = \alpha \log_a x$, $\log_b x = \frac{\log_a x}{\log_a b}$.

(f) Trigonometric functions. On the unit circle on the plane we measure, from the point $(1, 0)$ the distance φ (on the circumference), counterclockwise if $\varphi > 0$ and clockwise if $\varphi < 0$. We arrive at some point (x, y) on the unit circle, depending on φ. The coordinates of this point are called $\cos(\varphi)$ (cosine) and $\sin(\varphi)$ (sine) respectively:

$$x = \cos\varphi, \quad y = \sin\varphi.$$

Functions $\cos(x)$ and $\sin(x)$ are periodic with period 2π, that is both satisfy $f(x + 2\pi) = f(x)$ (since the length of the complete circumference is 2π). We also have $\sin^2 x + \cos^2 x = 1$ (since the radius of the defining circle is 1, and the equalities

$$\cos(\varphi + \psi) = \cos\varphi\cos\psi - \sin\varphi\sin\psi,$$
$$\sin(\varphi + \psi) = \cos\varphi\sin\psi + \sin\varphi\cos\psi.$$

Operations on functions

At each point of the domain the value of the function is a number, so it can be added, subtracted, multiplied and divided. In that case the same arithmetic operations can be carried over to the functions. If we have two functions, $f(x)$ and $g(x)$, with domains D_f and D_g, then we can define functions

$$(f \pm g)(x), \quad \text{where} \quad (f \pm g)(x) = f(x) \pm g(x),$$

$$(f \cdot g)(x), \quad \text{where} \quad (f \cdot g)(x) = f(x) \cdot g(x),$$
$$\left(\frac{f}{g}\right)(x), \quad \text{where} \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

The domain of these functions is the common part of the domains d_f and D_g, with, in the case of division the points where the denominator is zero removed (we cannot divide by zero).

Example: The function $\tan x$ is a fraction of the sine by the cosine:

$$\tan x = \frac{\sin x}{\cos x}, \quad x \neq \frac{\pi}{2} + k\,\pi, \quad k = 0, \pm 1, \pm 2, \ldots .$$

Composition of functions

If we have functions $f(x)$ and $g(x)$, and the range of $f(x)$ falls into the domain of $g(x)$ we can define the composition of $g(x)$ with $f(x)$:

$$(g \circ f)(x) = g(f(x)).$$

Chapter 4

Sequences

Definition 4.1. *A real valued sequence is a function* $a : \mathbf{N} \to \mathbf{R}$*, and a complex valued sequence is a function* $a : \mathbf{N} \to \mathbf{C}$*.*

In the case of sequences the value of a at n is called the n-th term of the sequence, and instead $a(n)$ we write a_n. The sequence with terms a_n is denoted $\{a_n\}_{n=1}^{\infty}$ or, more compactly $\{a_n\}$. We will be mostly concerned with real sequences, and will occasionally remark about complex sequences.

Examples: (a) A geometric sequence (traditionally called a geometric progression): $a, aq, aq^2, \dots, a_n = aq^{n-1}$.

(b) A constant sequence $a_n = c$.

(c) The harmonic sequence $a_n = \frac{1}{n}$.

(d) $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$.

(e) Fibonacci sequence $a_1 = a_2 = 1$, $a_{n+2} = a_n + a_{n+1}$.

To define a sequence we must describe the way in which all terms a_n are to be computed. We can do this with one formula like in examples (a)–(c) or recursively like in examples (d) and (e). The recursive definition (also called an inductive definition) describes way in which a consecutive term in the sequence is computed when all previous terms are already known. Also, one has to define sufficiently many initial terms. For example, in the definition of the Fibonacci sequence consecutive terms are computed using two previous terms, so as a staring point we have to know initial two terms. The rest of the sequence is then uniquely defined by these two initial terms, in the case of Fibonacci sequence, they are $a_1 = a_2 = 1$.

We say that the sequence is:

- strictly increasing, if $a_n < a_{n+1}$, and strictly decreasing, if $a_n > a_{n+1}$,

- weakly increasing if $a_n \leq a_{n+1}$, and weakly decreasing, if $a_n \geq a_{n+1}$,
- strictly monotonic if it is either strictly increasing or strictly decreasing, and weakly monotonic if it is either weakly increasing or weakly decreasing

Sometimes we will just say, that a sequence is increasing or decreasing, if it is not important whether strictly or weakly. Also, sometimes we refer to weakly increasing sequences as non-decreasing, and similarly for weakly decreasing. The above monotonicity notions, clearly, correspond to the monotonicity of the sequence as a function.

The harmonic sequence from example (c) is strictly decreasing, while sequences from examples (d) and (e) are strictly increasing. The example (c) follows directly from the formula: $a_n > a_{n+1}$ is nothing else than $n + 1 > n$. Examples (d) and (e) can be dealt with using induction. In (d) we first prove that all terms a_n are smaller than 2, and than, using that, we prove that the sequence is strictly increasing. Both proofs can be carried out by induction. Similarly in example (e), the Fibonacci's sequence, we first prove, using induction, that all terms are strictly positive $a_n > 0$. Then directly from the recursive formula we show that the sequence is increasing: $a_{n+2} = a_n + a_{n+1} > a_{n+1}$. This is a typical situation – if the sequence is defined inductively, then its properties can be usually established using induction.

Operations on sequences

We add, subtract, multiply and divide sequences as we do functions: $(a \pm b)_n = a_n + b_n$, $(a \cdot b)_n = a_n \cdot b_n$, $\left(\frac{a}{b}\right)_n = \frac{a_n}{b_n}$, $b_n \neq 0$.

Bounded sequences

We say that a sequence is bounded, if

$$\exists\, M \quad \forall\, n \in \mathbf{N} \quad |a_n| \leq M,$$

we say that it is bounded from above, if

$$\exists\, M \quad \forall\, n \in \mathbf{N} \quad a_n \leq M,$$

and we say that it is bounded from below, if

$$\exists\, M \quad \forall\, n \in \mathbf{N} \quad a_n \geq M.$$

Examples: (a) The harmonic sequence $a_n = \frac{1}{n}$ is bounded, from below by 0, and from above by $a_1 = 1$. More generally, a decreasing sequence is always bounded from above by its first term, and similarly an increasing sequence is always bounded from below by its first term.

(b) The Fibonacci's sequence is not bounded from above. We have already mentioned, that its terms are positive. Similarly, inductively we can show that its elements satisfy $a_n \geq n$ for $n \geq 6$. From this we can deduce immediately, that the sequence is not bounded from above.

(c) The sequence $a_n = \sqrt{n+1} - \sqrt{n}$ is bounded. We can see immediately, that its terms are positive (square root is an increasing function), that is the sequence is bounded from below by 0. We will show, that it is also bounded from above.

$$\begin{aligned} a_n = \sqrt{n+1} - \sqrt{n} &= (\sqrt{n+1} - \sqrt{n})\,\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{1+1}+1} \leq \frac{1}{2}. \end{aligned}$$

(d) The geometric progression $a_n = aq^{n-1}$ is bounded if $|q| \leq 1$, and unbounded if $|q| > 1$ and $a \neq 0$. The first statement is immediate: $|a_n| = |aq^{n-1}| = |a||q|^{n-1} \leq |a|$. The second statement requires a proof. We can, for example use the following important inequality, which can be proved, for example, inductively: for $\epsilon > 0$

$$(1+\epsilon)^n > 1 + n\epsilon. \tag{4.1}$$

If $|q| > 1$ then $|q| = (1+\epsilon)$ for some $\epsilon > 0$. We thus have

$$|a_n| = |a| \cdot |q|^{n-1} = \frac{|a|}{|q|}\,(1+\epsilon)^n > \frac{|a|}{|q|}\,(1+n\epsilon).$$

If $|a_n| \leq M$, then

$$\frac{|a|}{|q|}\,(1+n\epsilon) \leq M \quad \Rightarrow \quad n \leq \frac{1}{\epsilon}\left(M\,\frac{|q|}{|a|} - 1\right).$$

It is clear from the above estimate, that the sequence a_n cannot be bounded.

Convergent sequences

We now pass to the most important for us notion concerning sequences

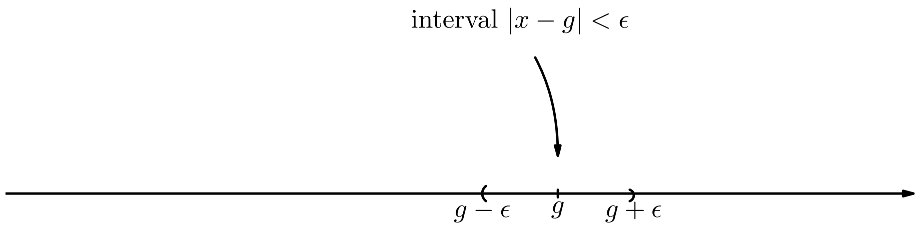

Figure 4.1: The limit of a sequence.

Definition 4.2. *We say that the sequence $\{a_n\}$ is convergent to a number g, if*

$$\forall\ \epsilon > 0 \quad \exists\ n_0 \in \mathbf{N} \quad \forall\ n \geq n_0 \quad |a_n - g| < \epsilon.$$

We write this

$$\lim_{n\to\infty} a_n = g \quad or \quad a_n \xrightarrow{n\to\infty} g.$$

The definition can be applied both to the real and complex valued sequences, in the latter case the limit might also be a complex number, and $|\ \cdots\ |$ would then denote the modulus of a complex number.

Examples: (a) $a_n = \frac{1}{n}$. We can easily prove, that $\lim_{n\to\infty} a_n = 0$. To do so, le us see, that

$$n \geq n_0 = \left[\frac{1}{\epsilon}\right] + 1 \ \Rightarrow\ n > \frac{1}{\epsilon} \ \Rightarrow\ \frac{1}{n} < \epsilon.$$

The absolute value in the last inequality can be dropped, since the terms of the sequence are all positive.

(b) $a_n = \sqrt{n+1} - \sqrt{n} \xrightarrow{n\to\infty} 0$. Let us prove that.

$$|a_n - 0| = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{2\ \sqrt{n}}.$$

It is thus enough to solve the inequality $\frac{1}{2\ \sqrt{n}} < \epsilon$

$$\frac{1}{2\ \sqrt{n}} < \epsilon \Leftrightarrow 2\sqrt{n} > \frac{1}{\epsilon} \Leftrightarrow n > \frac{1}{4\ \epsilon^2}.$$

For given $\epsilon > 0$ there thus exists $n_0 = \left[\frac{1}{4\epsilon^2}\right] + 1$ satisfying the condition in the definition.

(c) $a_n = \frac{n^2+2}{2\,n^2-1} \xrightarrow{n\to\infty} \frac{1}{2}$. Similarly as in the previous example we will solve an appropriate inequality. As in the previous example, we will aid ourselves in the calculations by using estimates, instead finding an exact solution.

$$\left|\frac{n^2+2}{2\,n^2-1} - \frac{1}{2}\right| = \frac{5}{2\;(2n^2-1)} \leq \frac{5}{2\,n}.$$

The last estimate, that is $2(2n^2-1) \geq 2n$, is true for all $n \in \mathbf{N}$, and it can be proved by solving a quadratic inequality. Finally, it is enough to solve a simple inequality $\frac{5}{2n} < \epsilon$, which gives us $n > \frac{5}{2\epsilon}$. Let then, for a given $\epsilon > 0$ be $n_0 = [\frac{5}{2\epsilon}] + 1$. This n_0 satisfies the condition in the definition.

(d) The constant sequence $a_n = c$ has limit $\lim_{n\to\infty} a_n = c$.

To prove the convergence of sequences to the given limits in the above examples we used the definition directly. In practice we usually establish the convergence using various properties of limits. For example, we have the following basic theorem

Theorem 4.3. *If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then the sequences $\{(a \pm b)_n\}$ i $\{(a \cdot b)_n\}$ converge, and*

$$\lim_{n\to\infty} (a \pm b)_n = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = a \pm b,$$
$$\lim_{n\to\infty} (a \cdot b)_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = a \cdot b.$$

If additionally $b_n \neq 0$ for all $n \in \mathbf{N}$ and $b \neq 0$ then the sequence of fractions $\{(\frac{a}{b})_n\}$ converges, and

$$\lim_{n\to\infty} \left(\frac{a}{b}\right)_n = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{a}{b}.$$

In the proof we will use the following observations

Fact 4.4. *(i) A convergent sequence is bounded. To see that let sequence $\{a_n\}$ converge to a and let us take arbitrary $\epsilon > 0$, for example $\epsilon = 1$. Then, by the definition, there exists $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ we have $|a_n| - |a| \leq |a_n - a| < 1$, from which we get $|a_n| < |a| + 1$. Let*

$$M = \max\{|a_1|, |a_2|, \ldots, |a_{n_0-1}|, |a| + 1\}.$$

Then the sequence $\{a_n\}$ is bounded by M: $\forall\, n \in \mathbf{N} \quad |a_n| \leq M$.
(ii) A sequence $\{b_n\}$ of numbers different than zero, convergent to a limit b different than zero is "bounded away from zero":

$$\exists\, \delta > 0 \quad \forall\, n \in \mathbf{N} \quad |b_n| \geq \delta.$$

To see that, let $\epsilon = \frac{|b|}{2}$*. Then, from the definition of convergence, there exists* $n_0 \in \mathbf{N}$ *such, that* $|b| - |b_n| \leq |b - b_n| < \frac{|b|}{2}$*, thus* $|b_n| > |b| - \frac{|b|}{2} = \frac{|b|}{2}$*. Let*

$$\delta = \min\left\{|b_1|, |b_2|, \ldots, |b_{n_0-1}|, \frac{|b|}{2}\right\} > 0.$$

Then $\forall\ n \in \mathbf{N}$ *we have* $|b_n| \geq \delta$*.*

Proof of the theorem. We will carry out the proof for the product, and we leave the other cases as exercises. For the product, the inequality we will be trying to solve for n will be

$$|a_n \cdot b_n - a \cdot b| < \epsilon.$$

Let au do the following

$$\begin{aligned}|a.b_n - a \cdot b| &= |a.b_n - a \cdot b_n + a \cdot b_n - a \cdot b| \\ &\leq |a_n \cdot b_n - a \cdot b_n| + |a \cdot b_n - a \cdot b| \\ &= |a_n - a| \cdot |b_n| + |a| \cdot |b_n - b|.\end{aligned}$$

The expression on the left hand side can be estimated using the fact, that we can estimate the expressions on the right hand side. We know, that the sequence $\{b_n\}$ is bounded (since it is convergent), so let M be the bound $|b_n| \leq M$. Let $\tilde{M} = \max\{M, |a|, 1\}$, and $\epsilon > 0$ be arbitrary. Let us fix $\tilde{\epsilon} = \frac{\epsilon}{2\tilde{M}} > 0$ (we can divide, since we know, that $\tilde{M} > 0$). Then there exists $n_1 \in \mathbf{N}$ such that $|a_n - a| < \tilde{\epsilon}$ for $n \geq n_1$ and there exists $n_2 \in \mathbf{N}$ such that $|b_n - b| < \tilde{\epsilon}$ for $n \geq n_2$. Finally, let $n_0 = \max\{n_1, n_2\}$. Then $|a_n - a| < \tilde{\epsilon}$ and $|b_n - b| < \tilde{\epsilon}$ for $n \geq n_0$. We thus have, for $n \geq n_0$

$$\begin{aligned}|a_n \cdot b_n - a \cdot b| &\leq |a_n - a| \cdot |b_n| + |a| \cdot |b_n - b| \\ &\leq |a_n - a|\ \tilde{M} + |b_n - b|\ \tilde{M} \\ &< \tilde{\epsilon}\ \tilde{M} + \tilde{\epsilon}\ \tilde{M} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon,\end{aligned}$$

which finishes the proof □

Example: Let

$$a_n = \frac{n^2 + 2}{2\,n^2 - 1} = \frac{1 + \frac{2}{n^2}}{2 - \frac{1}{n^2}}.$$

We have $\frac{1}{n} \to 0 \Rightarrow \frac{1}{n^2} = \frac{1}{n} \cdot \frac{1}{n} \to 0 \Rightarrow \frac{2}{n^2} = 2 \cdot \frac{1}{n^2} \to 0$ so the numerator converges to 1, and the denominator to 2, so

$$a_n = \frac{1 + \frac{2}{n^2}}{2 - \frac{1}{n^2}} \xrightarrow{n\to\infty} \frac{1}{2}.$$

We have found this limit earlier, using just the definition, but now we did it much easier.

Improper limits

Definition 4.5. *A real valued sequence* $\{a_n\}$ *has an improper limit* $+\infty$ *(we say that it diverges to* $+\infty$*) if*

$$\forall\ M \quad \exists\ n_0 \in \mathbf{N} \quad \forall\ n \geq n_0 \quad a_n > M.$$

The real valued sequence $\{a_n\}$ *has an improper limit* $-\infty$ *(diverges to* $-\infty$*) if*

$$\forall\ M \quad \exists\ n_0 \in \mathbf{N} \quad \forall\ n \geq n_0 \quad a_n < M.$$

A complex valued sequence $\{a_n\}$ *has improper limit* ∞ *(diverges to* ∞*) if*

$$\forall\ M \quad \exists\ n_0 \in \mathbf{N} \quad \forall\ n \geq n_0 \quad |a_n| > M,$$

(In the case of complex valued sequences we do not distinguish infinities).

Example: The sequence $a_n = \frac{n^2-3}{n+1}$ diverges to $+\infty$: for $n \geq 3$ we have

$$\frac{n^2 - 3}{n + 1} \geq \frac{\frac{1}{2}\,n^2}{2\,n} = \frac{n}{4},$$

while $\frac{n}{4} > M \Leftrightarrow n \geq [4M] + 1$. Let then $n_0 = \max\{3, [4M] + 1\}$, then for $n \geq n_0$ we have $|a_n| > M$.

The theorem about arithmetic operations on limits extends to <u>some</u> cases of improper limits. For example, let $a_n \to a$, $b_n \to b$ (real valued sequences). Then

$$a = +\infty,\ b > 0 \Rightarrow a_n \cdot b_n \to +\infty,$$
$$a = +\infty,\ b < 0 \Rightarrow a_n \cdot b_n \to -\infty.$$

The Cauchy's condition

Theorem 4.6. *The sequence* $\{a_n\}$ *is convergent* $\Leftrightarrow$ *(if and only if) it satisfies the so-called Cauchy's condition:*

$$\forall\ \epsilon > 0 \quad \exists\ n_0 \in \mathbf{N} \quad \forall\ m, n \geq n_0 \quad |a_m - a_n| < \epsilon.$$

Proof. We will prove the theorem for real valued sequences. Extension to the case of complex valued sequences is then an easy exercise. The proof has two parts: the Cauchy's condition from convergence(the "$\Rightarrow$" part), and from the Cauchy's condition the convergence(the "$\Leftarrow$" part).

$\Rightarrow$ We assume that $\{a_n\}$ is convergent to a. Let $\epsilon > 0$ be arbitrary. Them from the definition of convergence there exists an $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ we have $|a_n - a| < \epsilon/2$. Let both $m, n \geq n_0$, then $|a_m - a| < \epsilon/2$ and $|a_n - a| < \epsilon/2$, and so

$$|a_m - a_n| = |a_m - a + a - a_n| \leq |a_m - a| + |a_n - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

The Cauchy's condition is thus satisfied.

$\Leftarrow$ Let us assume that the sequence $\{a_n\}$ satisfies the Cauchy's condition. Observe, that in that case the sequence $\{a_n\}$ has to be bounded: let $\epsilon = 1$, so

$$\exists\ n_0 \in \mathbf{N} \quad \forall\ m, n \geq n_0 \quad |a_m - a_n| < 1.$$

So, taking $n = n_0$ we obtain, for every $m \geq n_0$ $|a_m - a_{n_0}| < 1 \Rightarrow |a_m| < |a_{n_0}| + 1$. The terms of the sequence $\{a_n\}$, from the n_0-th on, all sit in the interval $(-|a_{n_0}| - 1, |a_{n_0} + 1)$. Finally, let

$$M = \max\{a_1, |a_2|, \ldots, |a_{n_0-1}|, |a_{n_0}| + 1\}.$$

Then, for every $n \in \mathbf{N}$ we have $|a_n| \leq M$.

Let us define the following two auxillary sequences

$$\begin{aligned} \alpha_k &= \inf\{a_n : \ n \geq k\} \qquad \text{a non-decreasing sequence,} \\ \beta_k &= \sup\{a_n : \ n \geq k\} \qquad \text{a non-increasing sequence,} \end{aligned}$$

and let

$$\begin{aligned} A &= \sup\{\alpha_k : \ k \in \mathbf{N}\}, \\ B &= \inf\{\beta_k : \ k \in \mathbf{N}\}. \end{aligned} \tag{4.2}$$

The observation that $\{\alpha_k\}$ is non-decreasing and $\{\beta_k\}$ is non-increasing is obvious, the terms are the infimum and supremum of a set, that is getting smaller with increasing k. The original sequence $\{a_n\}$ is bounded, so all the extrema exist. In the first step we will prove, that $A \leq B$. This inequality is true for all bounded sequences, not only those satisfying the Cauchy's condition. It simply reflects the definition of auxillary sequences: by definition we always have $\alpha_k \leq \beta_k$. Let us carry out this proof in detail. We will argue indirectly, so let us assume that $A > B$, and then show that this leads to a contradiction. If $A > B$ then let $0 < \epsilon < \frac{A-B}{2}$. From the definition of the extrema we can find $k_1 \in \mathbf{N}$ such that

$$\alpha_{k_1} > A - \epsilon.$$

Since the sequence $\{\alpha_k\}$ is non-decreasing, then the above inequality holds for all $k \geq k_1$. Similarly there has to exist $k_2 \in \mathbf{N}$ such that

$$\beta_{k_2} < B + \epsilon, \quad \Rightarrow \quad \forall\ k \geq k_2 \quad \beta_k < B + \epsilon.$$

Now let $k_0 = \max\{k_1, k_2\}$, so we have

$$A - \epsilon < \alpha_{k_0} \leq \beta_{k_0} < B + \epsilon \quad \Rightarrow \quad \frac{A - B}{2} < \epsilon,$$

that is the contradiction. We have arrived at a contradiction, indeed, we must have

$$A \leq B.$$

As we have already mentioned, the above inequality is a consequence of the definition of A and B, and is true for all bounded sequences $\{a_n\}$, not just those satisfying the Cauchy's condition. We will now show, that for sequences satisfying the Cauchy's condition actually the equality holds: $A = B$. We will again argue indirectly. Let $A < B$, and let $0 < \epsilon < \frac{B-A}{2}$. There exists $n_0 \in \mathbf{N}$ such that for all $m, n \geq n_0$ we have $|a_m - a_n| < \epsilon$, in particular

$$\forall\ n \geq n_0 \quad |a_{n_0} - a_n| < \epsilon \Rightarrow a_{n_0} - \epsilon < a_n < a_{n_0} + \epsilon.$$

From this it follows that

$$\begin{aligned} \alpha_{n_0} = \inf\{a_n :\ n \geq n_0\} \geq a_{n_0} - \epsilon \quad &\Rightarrow \quad A \geq a_{n_0} - \epsilon \\ \beta_{n_0} = \sup\{a_n :\ n \geq n_0\} \leq a_{n_0} + \epsilon \quad &\Rightarrow \quad B \leq a_{n_0} + \epsilon. \end{aligned}$$

We thus have

$$B - A \leq a_{n_0} + \epsilon - a_{n_0} + \epsilon = 2\epsilon \quad \Rightarrow \quad \epsilon \geq \frac{B - A}{2},$$

that is, the contradiction. We therefore must have $A = B$. Finally, let $g = A = B$. From the definition of the extrema

$$\forall\, \epsilon > 0 \quad \exists\, n_0 \in \mathbf{N} \quad \forall\, n \geq n_0 \quad g - \epsilon < \alpha_n \quad \text{and} \quad \beta_n < g + \epsilon.$$

In the above we have used the fact that $g = A = B$ is both the infimum of all $\beta_k' s$ and the supremum of all α_k's, and that both sequences are appropriately monotonic ($\{\alpha_k\}$ non-decreasing and $\{\beta_k\}$ non-increasing). Since for all n we obviously have $\alpha_n \leq a_n \leq \beta_n$, thus, for $n \geq n_0$

$$g - \epsilon < \alpha_n \leq a_n \leq \beta_n < g + \epsilon \;\Rightarrow\; |a_n - g| < \epsilon.$$

This clearly finishes the proof □

Remark: The constants A and B defined in the above proof have sense for any bounded sequence $\{a_n\}$. They are called the lower limit and the upper limit of the sequence $\{a_n\}$. Soon we will discuss these issues in more detail.

Examples: (a) The sequence $a_n = (-1)^n$ does not satisfy the Cauchy's condition. Let $\epsilon = 1$, then $|a_n - a_{n+1}| = 2 > \epsilon$ for all n. Clearly an n_0 from the Cauchy's condition cannot exist.

(b) The sequence $a_n = \frac{n-1}{n}$ does satisfy the Cauchy's condition. Let us verify that: let $m > n$, then

$$|a_m - a_n| = \frac{m-1}{m} - \frac{n-1}{n} = \frac{(m-1)n - (n-1)m}{m \cdot n} = \frac{m-n}{m \cdot n} < \frac{m}{m \cdot n} = \frac{1}{n}.$$

Clearly then, it is enough to take $n_0 = [\frac{1}{\epsilon}] + 1$, then for $m, n \geq n_0$ we have $\frac{1}{m}, \frac{1}{n} < \epsilon$ and the Cauchy's condition is satisfied.

Theorem 4.7. *(i) A monotonic bounded sequence has a limit (proper).*
(ii) A monotonic unbounded sequence has an improper limit.

Remark: Weak monotonicity and only from a certain point on is sufficient. Also, in (ii) we mean unbounded from above, if it is increasing, and unbounded from below, if it is decreasing. Clearly, a monotonic sequence is automatically bounded on one end: increasing bounded from below, and decreasing bounded from above.

Proof. (i) Suppose that $\{a_n\}$ is weakly increasing, and bounded, that is

$$a_n \leq a_{n+1} \quad \text{and} \quad |a_n| \leq M \quad \text{for} \quad n = 1, 2, \ldots.$$

It thus has a supremum

$$g = \sup\{a_n : \; n = 1, 2, \ldots\}.$$

From the definition of supremum we have

$$\forall\, n \in \mathbf{N} \quad a_n \leq g \quad \text{and} \quad \forall\, \epsilon > 0 \exists\, n_0 \in \mathbf{N} \quad a_{n_0} > g - \epsilon.$$

Since $\{a_n\}$ is weakly increasing, then for all $n \geq n_0$ we have $a_n \geq a_{n_0} > g - \epsilon$, that is $g - \epsilon < a_n \leq g \Rightarrow |a_n - g| < \epsilon$.
(ii) Let us suppose that the sequence $\{a_n\}$ is wekly increasing, and not bounded from above. Let a number m be given. Since the sequence $\{a_n\}$ is not bounded from above, then there exists $n_0 \in \mathbf{N}$ such, that $a_{n_0} > M$. Since the sequence is weakly increasing, we have

$$\forall\, n \geq n_0 \quad a_n \geq a_{n_0} > M.$$

The condition in the definition of an improper limit $+\infty$ is therefore satisfied.
The case of decreasing sequences can be proved in the same way. □

Remark: Observe, that at the same time we have proved, that if the sequence $\{a_n\}$ is increasing and bounded, then

$$\lim_{n\to\infty} a_n = \sup\{a_n :\ n \geq 1\},$$

and if it is decreasing and bounded, then

$$\lim_{n\to\infty} a_n = \inf\{a_n :\ n \geq 1\}.$$

Binomial expansion

Let us recall the following formula, the so-called binomial expansion. For $n \in \mathbf{N}$ the factorial of n is the product of all natural numbers $k \leq n$: $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$. We also write $0! = 1$. For $0 \leq k \leq n$ let us introduce the so-called binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}, \quad k, n \in \mathbf{Z}, \quad 0 \leq k \leq n.$$

The following formula is called the binomial expansion. It can be proved, for example, by induction. It is one of the formulas that we will constantly use, so it is worthy to learn it well. Let $a, b \in \mathbf{R},\ n \in \mathbf{N}$, then

$$\begin{aligned}(a+b)^n &= \binom{n}{0} a^0\, b^n + \binom{n}{1} a^1\, b^{n-1} + \binom{n}{2} a^2\, b^{n-2} + \cdots + \binom{n}{n} a^n\, b^0 \\ &= \sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k}.\end{aligned}$$

The last equality is simply an expansion of the summation symbol Σ. We will often use this symbol. It simply means the sum of the expression for all values of the parameter k in the given range, in this case $k = 0, 1, \ldots, n$.

The number e

Let us consider the following sequence: $a_n = (1 + \frac{1}{n})^n$. We will show, that this sequence is increasing nd bounded, and thus convergent. Let us observe right away, that neither the fact that $\{a_n\}$ is increasing nor that it is bounded is obvious: even though the power increases, but the base decreases to 1. For example,

$$\begin{gathered} a_1 = 2, \quad a_2 = \left(\frac{3}{2}\right)^2 = 2{,}25, \quad a_3 = \left(\frac{4}{3}\right)^3 = 2{,}370\ldots, \\ a_4 = \left(\frac{5}{4}\right)^4 = 2{,}441\ldots, \quad a_5 = \left(\frac{6}{5}\right)^5 = 2{,}488\ldots\ . \end{gathered}$$

We will now show, that the sequence $\{a_n\}$ is increasing. Let us observe the following equality for $k = 0, 1, \ldots, n$

$$\begin{aligned} \binom{n}{k}\left(\frac{1}{n}\right)^k &= \frac{n!}{k!\,(n-k)!}\left(\frac{1}{n}\right)^k \\ &= \frac{(n-k+1)\cdot(n-k+2)\cdot\ldots\cdot(n-1)\cdot(n)}{k!\,n\cdot n\cdot\ldots\cdot n\cdot n} \\ &= \frac{1}{k!}\,\frac{n-1}{n}\cdot\frac{n-2}{n}\cdot\ldots\cdot\frac{n-(k-1)}{n} \\ &= \frac{1}{k!}\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right). \end{aligned}$$

We will now expand the terms of $\{a_n\}$ using the binomial expansion, and then use the above formula.

$$\begin{aligned} a_n &= \binom{n}{0}\left(\frac{1}{n}\right)^0 + \binom{n}{1}\left(\frac{1}{n}\right)^1 + \binom{n}{2}\left(\frac{1}{n}\right)^2 + \ldots \\ &\qquad \ldots + \binom{n}{k}\left(\frac{1}{n}\right)^k + \ldots + \binom{n}{n}\left(\frac{1}{n}\right)^n \\ &= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots \qquad (4.3) \\ &\qquad \ldots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right) + \ldots \\ &\qquad \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{n}\right). \end{aligned}$$

Observe, that written in the above form the term of a_n, with increase of n, contains more components, and each of the components is becoming greater (with the exception of the first two terms, $1+1$, which are left unchanged). Therefore, having written the terms of $\{a_n\}$ in the above form we more easily see, that this sequence is increasing. In addition observe that we can estimate a_n from above

$$a_n \leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}. \tag{4.4}$$

The first inequality is obtained from (4.3) by omitting factors which are less than 1, while the second inequality is obtained by replacing the factors larger that 2 in the denominators by 2. The denominators therefore become smaller, and the fractions larger. What remains to do, is to use the formula for the sum of the geometric progression: for $q \neq 1$ and $l \in \mathbf{N}$ we have

$$1 + q + q^2 + \cdots + q^{l-1} = \frac{1 - q^l}{1 - q}. \tag{4.5}$$

The above equality can be proved, for example, by induction. It is one of those formulas, that we have to constantly remember, and which will constantly keep appearing. The sum on the right hand side of the estimate (4.4) is precisely the sum of a geometric progression, with $q = \frac{1}{2}$, and with an extra 1 at front. We thus have

$$a_n < 1 + \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} < 1 + \frac{1}{\frac{1}{2}} = 3.$$

We have shown that the sequence $\{a_n\}$ is increasing and bounded, and thus convergent. The limit of this sequence is called e.

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

We also know from our estimates, that $2 < e \leq 3$. e is an important number and it will be constantly present throughout our lecture, mostly as the base for logarithms and exponential functions.

Theorem 4.8 (The 3 sequence theorem)**.** *Suppose we have 3 sequences satisfying inequalities*

$$a_n \leq b_n \leq c_n, \qquad n = 1, 2, 3, \ldots, \tag{4.6}$$

and the "outside" sequences $\{a_n\}$ and $\{c_n\}$ converge to a common limit

$$a = \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n.$$

Then the sequence $\{b_n\}$ also converges to the same, common limit

$$a = \lim_{n \to \infty} b_n.$$

Remark: It is sufficient that the sequences satisfy the inequalities (4.6) from certain $n_0 \in \mathbf{N}$ onwards.

Proof of the theorem. Let $\epsilon > 0$ and let $n_1 \in \mathbf{N}$ be such, that for $n \geq n_1$

$$|a_n - a| < \epsilon \quad \Rightarrow \quad a_n > a - \epsilon,$$

and let $n_2 \in \mathbf{N}$ be such, that for $n \geq n_2$

$$|c_n - a| < \epsilon \quad \Rightarrow \quad c_n < a + \epsilon.$$

The existence of such n_1 and n_2 is a consequence of the convergence of the sequences $\{a_n\}$ and $\{c_n\}$ to the common limit a. Then, for $n \geq n_0 = \max\{n_1, n_2\}$ we have

$$a - \epsilon < a_n \leq b_n \leq c_n < a + \epsilon \quad \Rightarrow \quad |b_n - a| < \epsilon. \tag{4.7}$$

Let us also observe, that if the inequalities hold only from certain point, say for $n \geq k$,then it is enough to modify the definition of n_0: let $n_0 = \max\{n_1, n_2, k\}$, and the inequality (4.7) holds. This way we have justified the note below the statement of the theorem. □

Examples: (a) Let $a_n = \sqrt{n}\,(\sqrt{n+1} - \sqrt{n})$. We will employ the 3 sequence theorem, and to do so we need to do some computations. We have seen earlier how to transform a difference of two square roots

$$\sqrt{n}\,(\sqrt{n+1} - \sqrt{n}) = \sqrt{n}\;\frac{1}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{\frac{n+1}{n}} + 1} = \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}.$$

Then

$$1 \leq \sqrt{1 + \frac{1}{n}} \leq 1 + \frac{1}{n} \quad \Rightarrow \quad \frac{1}{1 + \frac{1}{n} + 1} \leq a_n \leq \frac{1}{2}.$$

The two outside sequences share a common limit $\frac{1}{2}$, and so $a_n \to \frac{1}{2}$.

(b) Let $a > 1$ and $a_n = \sqrt[n]{a}$. The terms of the sequence are roots of increasing degree of a number greater that 1. Let us observe at once, that such sequence must converge, since it is decreasing, and bounded from below by 1. We do not actually need to use this observation, since we will use the 3 sequence theorem. First of all, since $a > 1$ we must have $a_n > 1$ for all n. Let $\epsilon_n = a_n - 1 > 0$. We will use the inequality (4.1), and obtain

$$a = (1 + \epsilon_n)^n \geq 1 + n\epsilon_n \quad \Rightarrow \quad 0 < \epsilon_n \leq \frac{a-1}{n}.$$

The outside sequences both converge to 0, so also $\epsilon_n \to 0$ that is

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{a} = 1.$$

(c) Let $a_n = \sqrt[n]{n}$. Similarly as in the previous example let us write $a_n = 1+\epsilon_n$, and thus $\epsilon_n > 0$. We will now use another simple inequality, which is true for $x > 0$ and $n \geq 2$

$$(1+x)^n > \binom{n}{2} x^2 = \frac{n\,(n-1)}{2}\, x^2.$$

The above inequality can be proved using the binomial expansion (this is also one way of proving (4.1)). Using the above inequality for $x = \epsilon_n$ we obtain, for $n \geq 2$

$$n = (1+\epsilon_n)^n > \frac{n\,(n-1)}{2}\,\epsilon_n^2 \quad \Rightarrow \quad \epsilon_n^2 < \frac{2\,n}{n\,(n-1)} \quad \Rightarrow \quad 0 < \epsilon_n < \sqrt{\frac{2}{n-1}}.$$

The rightmost sequence converges to 0, which can be easily shown using the definition, or applying the theorem about limits of roots which we will present next. We can thus apply the 3 sequence theorem and conclude that $\epsilon_n \to 0$, so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Theorem 4.9. *Let $a_n \to a$, $a_n \geq 0$ and $m \in \mathbf{N}$. Then*

$$\lim_{n\to\infty} \sqrt[m]{a_n} = \sqrt[m]{a}.$$

Proof. We will consider 2 cases: $a = 0$ and $a > 0$. If $a = 0$ (this is a slightly simpler case) let $\epsilon > 0$ be arbitrary, and let $\tilde{\epsilon} = \epsilon^m$. From the definition of the limit

$$\exists\, n_0 \in \mathbf{N} \quad \forall\, n \geq n_0 \quad 0 \leq a_n < \tilde{\epsilon} \quad \Rightarrow \quad 0 \leq \sqrt[m]{a_n} < \epsilon.$$

In the case $a = 0$ the theorem is thus proved. Let us consider the remaining case of $a > 0$. We will use the following formula, for $\alpha, \beta \geq 0$, $m \in \mathbf{N}$

$$(\alpha - \beta)\,(\alpha^{m-1} + \alpha^{m-2}\,\beta + \cdots + \alpha\,\beta^{m-2} + \beta^{m-1}) = \alpha^m - \beta^m.$$

This equality can be proved directly (for example inductively), or it can be deduced from the formula for the sum of the geometric progression (4.5). We thus have

$$|\sqrt[m]{a_n} - \sqrt[m]{a}| =$$

$$= \frac{|a_n - a|}{\left((\sqrt[m]{a_n})^{m-1} + (\sqrt[m]{a_n})^{m-2}\sqrt[m]{a} + \cdots + (\sqrt[m]{a})^{m-1}\right)} \leq \frac{|a_n - a|}{\left(\sqrt[m]{a}\right)^{m-1}}.$$

It is sufficient now, as in the precious case, to take $\tilde{\epsilon} = (\sqrt[m]{a})^{m-1}\epsilon$ and then we have

$$|a_n - a| < \tilde{\epsilon} \quad \Rightarrow \quad |\sqrt[m]{a_n} - \sqrt[m]{a}| < \epsilon.$$

□

Observe, that the above theorem allows us to "enter with the limit under" an arbitrary rational power, provided a_n and a are such, that the power can be applied.

$$\lim_{n\to\infty} (a_n)^{\frac{p}{q}} = \big(\lim_{n\to\infty} a_n\big)^{\frac{p}{q}}, \quad p \in \mathbf{Z},\ q \in \mathbf{N}.$$

Example: Let $a_1 = \sqrt{2}$ and let $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$. We have already considered this sequence, and we have shown that $\{a_n\}$ is increasing and bounded, and thus convergent. We will now use this to find the limit.

$$g = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n} = \sqrt{2 + \lim_{n\to\infty} a_n} = \sqrt{2 + g}.$$

The limit g (which we know from previous considerations exists) must therefore satisfy the quadratic equation $g^2 - g - 2 = 0$. This equation has two roots $g = -1$ and $g = 2$. The limit cannot be negative, since the sequence has positive terms, so the only possibility is $g = 2$.

Remark: We have used the following fact: if $a_n \to a$ and $a_n \geq 0$ then $a \geq 0$. This fact can be formulated more generally: if $a_n \to a$ and $b_n \to b$ and $a_n \leq b_n$ (at least from some point on), then $a \leq b$. We leave this fact as an exercise.

Subsequences

Definition 4.10. *A subsequence of the sequence $\{a_n\}$ is the sequence of the form $\{a_{n_k}\}_{k=1}^{\infty}$, where $\{n_k\}$ is strictly increasing sequence of natural numbers.*

Remark: In the definition it is important that the sequence of indices $\{n_k\}$ be *strictly* increasing. In other words, $a_1, a_5, a_6, a_{17}, \ldots$ could be a subsequence of the sequence $\{a_n\}$, but $a_1, a_2, a_2, \ldots$ or $a_1, a_5, a_2, \ldots$ are not subsequences. Let us observe, that according to the definition the entire sequence $\{a_n\}$ is its own subsequence, it is enough to take $n_k = k$. The definition of the subsequence reduces to extracting from the original sequence only certain terms, but respecting their order.

Example: The sequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots, \frac{1}{n^2}, \dots$ is a subsequence of the harmonic sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$. Here $a_n = \frac{1}{n}$ and $n_k = k^2$, so $a_{n_k} = \frac{1}{k^2}$.

Theorem 4.11. *Every subsequence of a convergent sequence is also convergent, to the same limit.*

Proof. Let $\{a_n\}$ be a sequence, converging to a limit g. Suppose we have a subsequence of this sequence, with indices $\{n_k\}$. Let $\epsilon > 0$, and let $n_0 \in \mathbf{N}$ be such, that for $n \geq n_0$ we have $|a_n - g| < \epsilon$. That we have from the the fact that $a_n \to g$. Next, let

$$k_0 = \min\{k \in \mathbf{N} : \; n_k \geq n_0\}.$$

So, if $k \geq k_0$ then $n_k \geq n_{k_0} \geq n_0$ (the sequence n_k increases), and so $|a_{n_k} - g| < \epsilon$. This finishes the proof. □

Example: Let $l \in \mathbf{N}$ and

$$a_n = \left(1 + \frac{\frac{1}{l}}{n}\right)^n.$$

Let $n_l = lk$. It is a strictly increasing sequence, and

$$a_n = \left(1 + \frac{1}{l\,n}\right)^n = \left(\left(1 + \frac{1}{l\,n}\right)^{l\,n}\right)^{\frac{1}{l}},$$

so, if $b_n = (1 + \frac{1}{n})^n$ (the sequence defining the number e), then $a_k = \sqrt[l]{b_{n_k}}$. We know, that $b_n \to e$, and thus

$$b_{n_k} \xrightarrow{k \to \infty} e \quad \Rightarrow \quad \sqrt[l]{b_{n_k}} \xrightarrow{k \to \infty} \sqrt[l]{e},$$

and so we have

$$\lim_{n \to \infty} a_n = e^{\frac{1}{l}} = \sqrt[l]{e}.$$

Theorem 4.12 (Bolzano-Weierstrass)**.** *Each bounded sequence contains a convergent subsequence.*

Proof. Let the sequence $\{a_n\}$ be bounded. Let us recall the construction from Theorem 4.6, which was about the Cauchy's condition.

$$\alpha_k = \inf\{a_n : \; n \geq k\}, \quad A = \sup\{\alpha_k : \; k \geq 1\} = \lim_{k \to \infty} \alpha_k.$$

We know, that the infima and supremum exist, since the sequence $\{a_n\}$ is bounded by assumption. This was already remarked in the proof of Theorem 4.6. We also know, that A is the supremum of the set of values of sequence $\{\alpha_k\}$, and also its limit, since this sequence is increasing, possibly weakly. We will now show, that there exists a subsequence $\{a_{n_k}\}$ converging to A. The construction of this subsequence is as follows. Let a_{n_1} be the element of the sequence $\{a_n\}$ which is closer to α_1 than $\frac{1}{2}$. We know such element exists, from the definition of the infimum. We thus have

$$\alpha_1 \leq a_{n_1} < \alpha_1 + \frac{1}{2}.$$

From now on the construction of the subsequence will be inductive. We will choose the next element in the subsequence from among a_n's with indices larger than n_1, so let a_{n_2} be the element of the sequence $\{a_n\}$, $n \geq n_1 + 1$, which lies closer to α_{n_1+1} than $\frac{1}{4}$. We thus have $n_2 > n_1$ and

$$\alpha_{n_1+1} \leq a_{n_2} < \alpha_{n_1+1} + \frac{1}{2^2}.$$

Let us now describe the step of the inductive definition. Suppose we have already constructed the a piece of the subsequence $a_{n_1}, a_{n_2}, \ldots, a_{n_m}$ such, that $n_1 < n_2 < \cdots < n_m$, and

$$\alpha_{n_l+1} \leq a_{n_{l+1}} < \alpha_{n_l+1} + \frac{1}{2^{l+1}}, \quad l = 1, 2, \ldots, m-1.$$

This is precisely what we have done for $m = 2$. Let the next index n_{m+1} be such that, firstly, $n_{m+1} \geq n_m + 1$ (the indices have to increase strictly)) and, secondly,

$$\alpha_{n_m+1} \leq a_{n_{m+1}} < \alpha_{n_m+1} + \frac{1}{2^{m+1}}.$$

Let us notice, that such choice is always possible, by the definition of the sequence $\{\alpha_k\}$ as the sequence of infima. In this way, we have inductively defined a subsequence $\{a_{n_k}\}$ satisfying

$$\alpha_{n_{k-1}+1} \leq a_{n_k} < \alpha_{n_{k-1}+1} + \frac{1}{2^k}, \quad k = 2, 3, \ldots.$$

On the outside of the above chain of inequalities we have sequences converging to A ($\{\alpha_{n_{k-1}+1}\}$ is a subsequence of the sequence $\{\alpha_n\}$ and $\frac{1}{2^k} \to 0$), so applying the 3 sequence theorem we get

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} \alpha_k = A.$$

□

Remark: The theorem is intuitively clear. If the sequence is bounded, then its terms (of which there are infinitely many) have to accumulate somewhere. The above proof makes this intuitive statement precise.

Definition 4.13. *A number α is called an accumulation point of the sequence $\{a_n\}$ if there exists a subsequence $\{a_{n_k}\}$ converging to α.*

Theorem 4.14. *α is an accumulation point of the sequence $\{a_n\}$ if, and only if*

$$\forall\ \epsilon > 0 \quad \forall\ n_0 \in \mathbf{N} \quad \exists\ n \geq n_0 \quad |a_n - g| < \epsilon. \tag{4.8}$$

In other words, any neighborhood of the point α contains terms of the sequence $\{a_n\}$ with arbitrarily far indices (in particular any neighborhood of the point α contains infinitely many terms of the sequence $\{a_n\}$).

Proof. If α is an accumulation point of the sequence $\{a_n\}$ then from the definition there exists a subsequence $\{a_{n_k}\}$ converging to α. Let then $\epsilon > 0$, and k_0 be such, that for any $k \geq k_0$ we have $|a_{n_k} - \alpha| < \epsilon$. If there is given $n_0 \in \mathbf{N}$, then let some $k \geq k_0$ satisfy $n_k \geq n_0$. Such k must exist, since the sequence of indices $\{n_k\}$ diverges to $+\infty$. The index n_k is the required index in (4.8). In the opposite direction, let the condition (4.8) be satisfied. We will construct inductively a subsequence $\{a_{n_k}\}$ converging to α. To start our construction let n_1 be the index of such element of the sequence, which satisfies

$$|a_{n_1} - \alpha| < \frac{1}{2}.$$

The existence of such element follows from (4.8). Further, assume we have already constructed an increasing sequence of indices $n_1 < n_2 < \cdots < n_k$ satisfying

$$|a_{n_l} - \alpha| < \frac{1}{2^l}, \quad l = 1, 2, \dots, k.$$

Let n_{k+1} be the index of the element of the sequence $\{a_n\}$ which satisfies

$$|a_{n_{k+1}} - \alpha| < \frac{1}{2^{k+1}},$$

and $n_{k+1} > n_k$. By (4.8) such element must exist. This way, we obtain a subsequence $\{a_{n_k}\}$ satisfying

$$0 \leq |a_{n_k} - \alpha| < \frac{1}{2^k}, \quad k = 1, 2, \dots.$$

We now see, from the definition, that $a_{n_k} \to \alpha$, so α is indeed an accumulation point of the sequence $\{a_n\}$. □

If the sequence $\{a_n\}$ is bounded, then the set of its accumulation points (which by the Bolzano-Weierstrass theorem is not empty) is also bounded (we leave this as an exercise). This set therefore has its extrema

Definition 4.15. *The infimum and the supremum of the set of accumulation points of a sequence $\{a_n\}$, are called the lower and the upper limits of this sequence respectively, and denoted by*

$$\liminf_{n\to\infty} a_n \quad \textit{lower limit} \qquad \limsup_{n\to\infty} \quad \textit{upper limit.}$$

Remarks: (i) The lower limit is less then or equal to the upper limit.

(ii) The set of accumulation points of a bounded sequence contains its extrema. The lower limit is thus the smallest accumulation point, and the upper limit is the largest accumulation point.

(iii) The bounded sequence is convergent if and only if its lower and upper limits are equal. In other words, a bounded sequence is convergent if and only if it has exactly one accumulation point.

(iv) The constants A and B which appeared in (4.2) in the proof of Theorem 4.6 are the lower and upper limits respectively of the squence $\{a_n\}$.

Example: Let $m \in \mathbf{N}$ be fixed, and let $a_n = (1 + \frac{m}{n})^n$. We will show that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{m}{n}\right)^n = e^m.$$

First, let $0 \leq x < 1$, and let $b_n = (1 + \frac{1}{n+x})^n$. Observe, that we have the following estimate

$$\left(1 + \frac{1}{n+1}\right)^n < \left(1 + \frac{1}{n+x}\right)^n \leq \left(1 + \frac{1}{n}\right)^n,$$

The right hand side converges to e, and the left hand side, as can be easily seen, also:

$$\left(1 + \frac{1}{n+1}\right)^n = \left(1 + \frac{1}{n+1}\right)^{n+1} \left(\frac{n+1}{n+2}\right) \xrightarrow{n\to\infty} e \cdot 1 = e. \qquad (4.9)$$

Applying the 3 sequence theorem we thus see, that $b_n \to e$, regardless of x. Let us now fix $l = 0, \dots, m-1$, and let $n_k = mk + l$. Observe, that that the corresponding subsequence of the sequence $\{a_n\}$ zbiega do e^m, regardless of l:

$$a_{n_k} = \left(1 + \frac{m}{mk+l}\right)^{mk+l} = \left(1 + \frac{1}{k + \frac{l}{m}}\right)^{mk+l}$$

$$= \left(\left(1+\frac{1}{k+\frac{l}{m}}\right)^k\right)^m \cdot \left(1+\frac{1}{k+\frac{l}{m}}\right)^l \xrightarrow{k\to\infty} e^m,$$

by (4.9). All such subsequences, for different l share the same limit e^m. Each element of the sequence $\{a_n\}$ belongs to one of these subsequences, of which there is finitely many. It follows, that $\{a_n\}$ is convergent, and its limit is e^m. Let us prove this last statement. Let $\{n_k^l\}$ be the sequence $n_k^l = mk + l$ for $l = 0, 1, \ldots, m-1$. We know that each subsequence $\{a_{n_k^l}\}$ converges to e^m, so for every $\epsilon > 0$ there exist $k_0^l \in \mathbf{N}$ such that for $k \geq k_0^l$ we have

$$|a_{n_k^l} - e^m| < \epsilon, \quad l = 0, \ldots, m-1$$

Let $n_0 = \max\{mk_0^0, mk_0^1 + 1, \ldots, mk_0^{m-1} + m - 1\}$. If $n \geq n_0$, then n has to be in one of the subsequences n_k^l and additionally $k \geq k_0^l$. a_n thus satisfies $|a_n - e^m| < \epsilon$. Let $m, k \in \mathbf{N}$. As a direct consequence of the above computations we have

$$\left(1+\frac{\frac{m}{k}}{n}\right)^n = \left(\left(1+\frac{m}{k\cdot n}\right)^{k\cdot n}\right)^{\frac{1}{k}} = (a_{kn})^{\frac{1}{k}} \xrightarrow{n\to\infty} (e^m)^{\frac{1}{k}} = e^{\frac{m}{k}}.$$

For $p = \frac{m}{k}$, $m, k \in \mathbf{N}$ we thus have

$$\lim_{n\to\infty}\left(1+\frac{p}{n}\right)^n = e^p.$$

The above equality can be first extended to arbitrary $p \in \mathbf{R}$, $p > 0$, and then on an arbitrary $p \in \mathbf{R}$. We leave this as an exercise.

Chapter 5

Series

Series are infinite sums. To define them we use the notion of convergence, introduced in the previous chapter. The infinite sums are nothing exotic, they frequently appear in practice, for example if we want to compute the area of various planar shapes. Suppose we have a given sequence $\{a_n\}$, and we form a sequence of consecutive sums

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \quad s_n = a_1 + a_2 + \cdots + a_n, \dots$$

Definition 5.1. *If the sequence $\{s_n\}$ has a limit s we say that the series (infinite sum) $\sum_{n=1}^{\infty} a_n$ converges, and that its sum is s. We write $s = \sum_{n=1}^{\infty} a_n$. We call s_k the k-th partial sum of the series $\sum a_n$, and the sequence $\{s_n\}$ the sequence of partial sums. If the sequence $\{s_n\}$ does not converge, we say that the series $\sum a_n$ diverges, and the expression $\sum a_n$ is only a symbol and has no numerical interpretation.*

Examples: (a) Let $a_n = (\frac{2}{3})^n$. Then

$$s_n = \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^n = \frac{2}{3}\,\frac{1 - (\frac{2}{3})^n}{1 - \frac{2}{3}} = 2\left(1 - \left(\frac{2}{3}\right)^n\right) \xrightarrow{n\to\infty} 2.$$

The series $\sum_{n=1}^{\infty} (\frac{2}{3})^n$ is thus convergent, and $\sum_{n=1}^{\infty} (\frac{2}{3})^n = 2$ (it is an example of a geometric series). We have used the formula for the sum of the geometric progression (4.5)

(b) Let $a_n = \frac{1}{n\,(n+1)}$. Let us observe, that $a_n = \frac{1}{n} - \frac{1}{n+1}$. We thus have

$$\begin{aligned} s_n &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \xrightarrow{n\to\infty} 1. \end{aligned}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)}$ is therefore convergent, and its sum is 1.

(c) The series $\sum_{n=1}^{\infty}(-1)^n$ diverges, since $s_n = -1$ or 0, depending on the parity of n.

Operations on series

The theorem about arithmetic operations on limits carries over to series:

$$\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$
$$\sum_{n=1}^{\infty}(c \cdot a_n) = c \cdot \sum_{n=1}^{\infty} a_n,$$

with the assumption that the series on the right hand side are convergent. The theorem about the limit of a product or a fraction does not apply here.

Theorem 5.2. *If the series* $\sum a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof. The series being convergent means that the sequence of partial sums $s_n = a_1 + \cdots + a_n$ is convergent. For $n \geq 2$ $a_n = s_n - s_{n-1}$, so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = 0.$$

□

Remark: The above theorem provides the so-called necessary condition for convergence. $\lim a_n = 0$ does not guarantee the convergence of the series $\sum_{n=1}^{\infty} a_n$. The theorem is mostly useful to show divergence.

Example: Let $a_n = \frac{1}{n}$. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. It is the so-called harmonic series. We will show, that the sequence of partial sums is not bounded, and thus cannot be convergent. It is enough to produce a subsequence of the sequence $\{s_n\}$, which diverges to $+\infty$.

$$\begin{aligned} s_{2^n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{2^n} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots \\ &\qquad \cdots + \left(\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \cdots + \frac{1}{2^n}\right). \end{aligned}$$

Between each consecutive pair of parentheses we have $2^k - 2^{k-1} = 2^{k-1}$ terms, and each term is $\geq \frac{1}{2^k}$. The sum within each pair of parentheses is therefore larger than $2^{k-1} \cdot \frac{1}{2^k} = \frac{1}{2}$

$$\begin{aligned} s_{2^n} &\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n}\right) \\ &= 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{n \text{ times}} = 1 + n \cdot \frac{1}{2} = 1 + \frac{n}{2}. \end{aligned}$$

So, we have $s_{2^n} \geq 1 + \frac{n}{2}$, so the sequence of partial sums $\{s_n\}$ is not bounded, and thus not convergent.

Theorem 5.3. *The series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the sequence of partial sums $\{s_n\}$ satisfies the Cauchy's condition:*

$$\forall\ \epsilon > 0 \quad \exists\ n_0 \in \mathbf{N} \quad \forall\ m, n \geq n_0 \quad |s_m - s_n| < \epsilon.$$

This condition can be reformulated:

$$\forall\ \epsilon > 0 \quad \exists\ n_0 \in \mathbf{N} \quad \forall\ m \geq n \geq n_0 \quad |a_n + a_{n+1} + \cdots + a_m| < \epsilon.$$

Proof. The theorem is an immediate consequence of Theorem 4.6 for sequences. □

Example: If $|r| < 1$ then the series $\sum_{n=0}^{\infty} r^n$ is convergent. Let $m \geq n \geq n_0$.

$$\begin{aligned} |a_n + a_{n+1} + \cdots + a_m| &\leq |a_n| + |a_{n+1}| + \cdots + |a_m| = |r^n| + |r^{n+1}| + \cdots + |r^m| \\ &= |r|^n + |r|^{n+1} + \cdots + |r|^m = |r|^n(1 + |r| + \cdots + |r|^{m-n}) \\ &= |r|^n \cdot \frac{1 - |r|^{m-n+1}}{1 - |r|} < \frac{|r|^n}{1 - |r|} \leq \frac{|r|^{n_0}}{1 - |r|}. \end{aligned}$$

The sequence $\frac{|r|^n}{1-|r|}$ converges to 0, so it is enough to find appropriate n_0 for given ϵ.

Convergence criteria

Establishing the convergence of series in most cases can be reduced to the application of one of the following convergence criteria.

Theorem 5.4 (Comparative criterion)**.**

(i) If $|a_n| \leq b_n$ and the series $\sum_{n=1}^{\infty} b_n$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is also convergent.

(ii) If $0 \le a_n \le b_n$ *and the series* $\sum_{n=1}^{\infty} a_n$ *is divergent, then the series* $\sum_{n=1}^{\infty} b_n$ *is also divergent.*

Proof. (i) Since $\sum b_n$ converges, then its sequence of partial sums satisfies the Cauchy's condition. On the other hand we have

$$\begin{aligned}|a_n + a_{n+1} + \cdots + a_{n+k}| &\le |a_n| + |a_{n+1}| + \cdots + |a_{n+k}| \\ &\le b_n + b_{n+1} + \cdots + b_{n+k} = |b_n + b_{n+1} + \cdots + b_{n+k}|.\end{aligned}$$

Thus the sequence of partial sums of the series $\sum a_n$ also satisfies the Cauchy's condition.

(ii) The series $\sum a_n$ has non-negative terms, and is divergent, so its sequence of partial sums is increasing (possibly weakly), and not convergent. It must be therefore unbounded. The sequence of partial sums of the series $\sum b_n$ has terms which are not smaller, so it is also unbounded, and thus cannot be convergent. □

Remark: It is enough, that the relevant inequalities are satisfied only from a certain point onwards.

Examples: (a) The series $\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$ is convergent, since

$$\frac{1}{n^2+2n} \le \frac{1}{n^2+n} = \frac{1}{n(n+1)}.$$

(b) The series $\sum_{n=1}^{\infty} \frac{1}{n+1}$ is divergent, since

$$\frac{1}{n+1} \ge \frac{1}{n+n} = \frac{1}{2n},$$

and the series $\sum \frac{1}{2n}$ diverges. Observe, that in this case the estimate $\frac{1}{n+1} < \frac{1}{n}$ is not useful.

Theorem 5.5. *Let the sequence* $\{a_n\}$ *be non-negative, and weakly decreasing,* $a_1 \ge a_2 \ge \cdots \ge 0$*. Then the series* $\sum_{n=1}^{\infty} a_n$ *converges if and only if, the series* $\sum_{n=1}^{\infty} 2^n a_{2^n}$ *converges.*

The above criterion does not directly decide whether the given series converges or not, but it reduces the study of the convergence of one series to the study of convergence of another series.

Proof. Let us denote by $\{s_n\}$ the sequence of partial sums of the series $\sum a_n$, and by $\{s'_n\}$ the sequence of partial sums of the series $\sum 2^n a_{2^n}$. Since the terms of both series are non-negative, both sequences of partial sums are non-decreasing. Using the appropriate estimates we will show, that these sequences are simultaneously either bounded or unbounded. We have

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \cdots + a_n, \\ s'_n &= 2 \cdot a_2 + 4 \cdot a_4 + 8 \cdot a_8 + \ldots 2^n \cdot a_{2^n} \\ &= 2\,(a_2 + 2 \cdot a_4 + 4 \cdot a_8 + \cdots + 2^{n-1} \cdot a_{2^n}). \end{aligned}$$

Let us observe then, that

$$\begin{aligned} \frac{1}{2}\, s'_n &= a_2 + 2 \cdot a_4 + 4 \cdot a_8 + \cdots + 2^{n-1} \cdot a_{2^n} \\ &\le a_1 + a_2 + a_3 + a_4 + \cdots + a_{2^n-1} + a_{2^n} \\ &= s_{2^n} \end{aligned}$$

To the sum on the left hand side we have added $a_1 \ge 0$, and each component of the sum $2^{k-1} \cdot a_{2^k}$ we have replaced by a greater or equal expression $a_{2^{k-1}+1} + \cdots + a_{2^k}$, $k = 1, \ldots, n$. If the sequence $\{s_n\}$ is bounded then the sequence $\{s'_n\}$ is also bounded.

On the other hand observe, that

$$\begin{aligned} s_{2^{n+1}-1} &= a_1 + a_2 + a_3 + a_4 + \cdots + a_{2^{n+1}-1} \\ &\le a_1 + 2 \cdot a_2 + 4 \cdot a_4 + \cdots + 2^n \cdot a_{2^n} \\ &= a_1 + s'_n. \end{aligned}$$

We obtained the inequality by replacing te sums $a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1}$ (2^k terms in the sum) by non-smaller expression $2^k \cdot a_{2^k}$, $k = 1, \ldots, n$. If the sequence $\{s'_n\}$ is bounded, then it follows from the above inequality that the subsequence $\{s_{2^{n+1}-1}\}$ of the sequence $\{s_n\}$ is also bounded. The sequence $\{s_n\}$ is non-decreasing, and contains a bounded subsequence, thus it must be bounded (exercise). □

Example: Let us consider the series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. If $p \le 0$ then the sequence $\{\frac{1}{n^p}\}$ is not convergent to 0, so the series cannot converge. If $p > 0$ then the sequence $\{\frac{1}{n^p}\}$ is positive and decreasing, so it satisfies the assumptions of the above criterion 5.5. Instead of the series $\sum \frac{1}{n^p}$ let us consider the series with terms

$$2^n\, \frac{1}{(2^n)^p} = 2^n\, \frac{1}{2^{n \cdot p}} = \frac{1}{2^{n \cdot (p-1)}} = \left(\frac{1}{2^{p-1}}\right)^n.$$

The series $\sum(\frac{1}{2^{p-1}})^n$ is a geometric series. If $p-1>0$ then the ratio of the series $\frac{1}{2^{p-1}}<1$ and the series converges, while, if $p-1\leq 1$, then the ratio $\frac{1}{2^{p-1}}\geq 1$, and the series does not converge. We thus have

$$\sum_{n=1}^{\infty}\frac{1}{n^p} \qquad \leftarrow \qquad \begin{cases} \text{convergent if } p>1, \\ \text{divergent if } p\leq 1. \end{cases} \tag{5.1}$$

Let us observe, that the case $p=1$ was already considered before (we called it the harmonic series). The series of this form often prove useful. If the terms of some series could be somehow estimated by a power function, then such estimate can be compared with series (5.1), for which the convergence or divergence has been established, depending on p.

Theorem 5.6 (d'Alembert's criterion)**.** *Let $\{a_n\}$ be a sequence with non-zero terms. Then*

(i) If $\limsup_{n\to\infty}|\frac{a_{n+1}}{a_n}|<1$ *then the series* $\sum a_n$ *converges,*

(ii) If $\liminf_{n\to\infty}|\frac{a_{n+1}}{a_n}|>1$ *then the series* $\sum a_n$ *diverges (this includes the case of the improper limit* $\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|=+\infty$*).*

Proof. (i) Let us observe, that since the upper limit of the sequence $|\frac{a_{n+1}}{a_n}|$ is less than 1, then there exist $0<c<1$ and $n_0\in\mathbf{N}$ such that for $n\geq n_0$

$$\left|\frac{a_{n+1}}{a_n}\right|\leq c,$$

and in particular, for $k\geq 0$

$$\begin{aligned} |a_{n_0+k}| &= \left|\frac{a_{n_0+k}}{a_{n_0+k-1}}\right|\cdot\left|\frac{a_{n_0+k-1}}{a_{n_0+k-2}}\right|\cdot\ldots\cdot\left|\frac{a_{n_0+1}}{a_{n_0}}\right|\cdot|a_{n_0}| \\ &\leq |a_{n_0}|\cdot c^k=\frac{|a_{n_0}|}{c^{n_0}}\cdot c^{n_0+k}. \end{aligned} \tag{5.2}$$

The product appearing in (5.2) is sometimes called a telescopic product, since we use it by extending or contracting the required number of terms. The sequence $\{a_n\}$ thus satisfies (for $n\geq n_0$) the inequality

$$|a_n|\leq\frac{|a_{n_0}|}{c^{n_0}}\cdot c^n,\quad 0<c<1,$$

so it converges by the comparative criterion 5.4.

(ii) Let us observe, that since the lower limit of the sequence $|\frac{a_{n+1}}{a_n}|$ is greater than 1 (similarly if this sequence has an improper limit $+\infty$), then there exist $c > 1$ and $n_0 \in \mathbf{N}$ such that for $n \geq n_0$

$$\left|\frac{a_{n+1}}{a_n}\right| \geq c.$$

As in case (i), for $k \geq 0$ we have

$$|a_{n_0+k}| = \left|\frac{a_{n_0+k}}{a_{n_0+k-1}}\right| \cdot \left|\frac{a_{n_0+k-1}}{a_{n_0+k-2}}\right| \cdot \ldots \cdot \left|\frac{a_{n_0+1}}{a_{n_0}}\right| \cdot |a_{n_0}| \geq |a_{n_0}| \cdot c^k \geq |a_{n_0}|,$$

so the sequence $\{a_n\}$ does not converge to 0. The series $\sum a_n$ must then diverge. □

The d'Alembert's criterion leaves certain cases unsettled. For example for series of the form $\sum \frac{1}{n^p}$ we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n^p}{(n+1)^p} = \left(\lim_{n\to\infty}\frac{n}{n+1}\right)^p = 1.$$

This series are not covered by the either case the d'Alembert's criterion. Indeed, we know that these series can be convergent or divergent, depending on the parameter p.

Theorem 5.7 (Cauchy's criterion)**.** *Let the sequence $\{a_n\}$ be given, and let*

$$g = \limsup_{n\to\infty} \sqrt[n]{|a_n|},$$

(proper or improper limit). Then

(i) If $g < 1$ then the series $\sum a_n$ converges,

(ii) If $g > 1$ then the series $\sum a_n$ diverges (this covers also the case of an improper upper limit $g = +\infty$).

Proof. (i) Just like in the case of the d'Alemert's criterion, there exist $0 < c < 1$ and $n_0 \in \mathbf{N}$ such that for $n \geq n_0$

$$\sqrt[n]{|a_n|} \leq c \quad \Rightarrow \quad |a_n| \leq c^n,$$

so from the comparative criterion the series $\sum a_n$ converges.

(ii) If $g > 1$, then there exists a subsequence $\{a_{n_k}\}$ such that $|a_{n_k}| \geq 1$. The sequence $\{a_n\}$ thus cannot converge to 0, so the series $\sum a_n$ does not converge, □

Remarks: (i) Similarly as in the case of the d'Alembert's criterion the Cauchy's criterion leaves the case $g = 1$ unresolved. In this cases certain series can converge and others not. The Cauchy's criterion does not help in this case.

(ii) Both above criteria apply to the series with complex terms. The absolute value is then the complex modulus. and both the statements and the proofs basically remain valid.

Example: The series $\sum_{n=0}^{\infty} \frac{1}{n!}$. We have $a_n = \frac{1}{n!}$, and so

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \xrightarrow{n\to\infty} 0.$$

Using d'Alembert's criterion we get, that the series $\sum \frac{1}{n!}$ converges. Computing the actual sum of the series is much harder, than proving its existence. We will now prove, that

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e. \tag{5.3}$$

Let us recall that e is the limit

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Proving the existence of this limit we have shown in (4.4), that

$$\left(1 + \frac{1}{n}\right)^n \leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = s_n,$$

where $\{s_n\}$ is the sequence of the partial sums of our series $\sum_{n=0}^{\infty} \frac{1}{n!}$. Passing to the limit $n \to \infty$ on both sides of the inequality, we obtain

$$e \leq \sum_{n=0}^{\infty} \frac{1}{n!}. \tag{5.4}$$

On the other hand, let us fix $k \in \mathbf{N}$ and let $n \geq k$. From the expansion (4.3) (truncating the expansion after the k-th term) we have

$$\left(1 + \frac{1}{n}\right)^n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \dots$$
$$\dots + \frac{1}{k!}\left(1 - \frac{1}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right).$$

Passing to the limit $n \to \infty$ on both sides of the inequality (leaving k fixed) we obtain

$$e \geq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} = s_k,$$

for each $k \in \mathbf{N}$. Now passing to the limit as $k \to \infty$ (the left hand side is constant) we obtain

$$e \geq \sum_{n=0}^{\infty} \frac{1}{n!},$$

which, together with (5.4) gives us (5.3). Equation (5.3) is sometimes used as the definition of the number e.

Absolutely convergent series

Definition 5.8. *If the series $\sum |a_n|$ converges, we say that the series $\sum a_n$ converges absolutely. If the series $\sum a_n$ converges, but does not converge absolutely (that is the series $\sum |a_n|$ diverges), we say that the series $\sum a_n$ converges conditionally.*

Remarks: (i) If the series converges absolutely, then it also converges in the normal sense. This is the consequence of the Cauchy's condition:

$$|a_{n+1} + a_{n+2} + \cdots + a_m| \leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_m|.$$

If the series $\sum |a_n|$ converges, then it satisfies the Cauchy's condition, so the series $\sum a_n$ also satisfies the Cauchy's condition, so it also converges. The absolute convergence is therefore a special type of convergence.

(ii) If the terms of the series $\sum a_n$ all have the same sign, then the absolute convergence follows from the ordinary convergence, and both convergence types are equivalent. The absolute convergence is thus an issue for series whose terms change signs.

(iii) Let us observe, that all convergence criteria we have so far considered deal with absolute convergence. None of these criterions allow to establish conditional convergence.

(iv) Absolute convergence is important – it is only for series converging absolutely that the convergence and the sum are independent of the order of summation, and eventual placement of the parentheses.

Alternating series

We say that the series $\sum a_n$ is alternating if its terms alternate the sign, that is they are alternately positive and negative: $a_n = (-1)^n \cdot b_n$ and $b_n \geq 0$ or $b_n \leq 0$ for all n.

Theorem 5.9 (The Leibniz's criterion)**.** *If the series $\{a_n\}$ is decreasing (possibly weakly) and $\lim_{n\to\infty} a_n = 0$, then the alternating series*

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

is convergent.

Proof. Let $s_n = a_1 - a_2 + a_3 - a_4 + \cdots + \pm a_n$ be the sequence of partial sums. Let us observe, that the subsequence of terms with even indices s_{2n} is increasing:

$$s_{2(n+1)} = s_{2n} + a_{2n+1} - a_{2n+2} \geq s_{2n},$$

and the subsequence of terms with odd indices s_{2n+1} is decreasing:

$$s_{2(n+1)+1} = s_{2n+1} - a_{2n+2} + a_{2n+3} = s_{2n+1} - (a_{2n+2} - a_{2n+3}) \leq s_{2n+1}.$$

Let us observe, that the subsequence s_{2n} (which is increasing) is bounded from above:

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + \cdots - a_{2n} \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-1} - a_{2n}) \leq a_1, \end{aligned}$$

while the subsequence s_{2n+1} (which is decreasing) is bounded from below

$$\begin{aligned} s_{2n+1} &= a_1 - a_2 + a_3 - a_4 + \cdots - a_{2n} + a_{2n+1} \\ &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) + a_{2n+1} \geq 0. \end{aligned}$$

Bot subsequences are thus convergent. Let $s = \lim_{n\to\infty} s_{2n}$. Then

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty}(s_{2n} + a_{2n+1}) = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} a_{2n+1} = s + 0 = s.$$

Both subsequences thus have a common limit. The sequence $\{s_n\}$ therefore splits into two subsequences, The terms with even indices and the terms with odd indices. Each element of the sequence $\{s_n\}$ belongs to one of the two subsequences, and both subsequences share the common limit s. It follows, that the entire sequence $\{s_n\}$ converges to s. Let us write down the above sketch of the reasoning. Let $\epsilon > 0$. From the fact that $\lim_{n\to\infty} s_{2n} = s$ it follows that

$$\exists\ k_1 \in \mathbf{N} \quad \forall\ k \geq k_1 \quad |s_{2k} - s| < \epsilon,$$

and from the fact that $\lim_{n\to\infty} s_{2n+1} = s$ we have

$$\exists\ k_2 \in \mathbf{N} \quad \forall\ k \geq k_2 \quad |s_{2k+1} - s| < \epsilon.$$

Let $n_0 = \max\{2k_1, 2k_2 + 1\}$. Then, for $n \geq n_0$ there is $n = 2k$, $k \geq k_1$ or $n = 2k + 1$, $k \geq k_2$, depending on the parity of n. In both cases

$$|s_n - s| < \epsilon.$$

$\square$

Remark: Let us observe, that from the proof follows the following estimate for the sum s. For any $k, l \in \mathbf{N}$

$$s_{2l} \leq \sum_{n=1}^{\infty} (-1)^{n+1} a_n \leq s_{2k+1}.$$

The sum is greater or equal that any even partial sum, and less or equal than any even partial sum. This applies to the alternating series whose even terms are ≤ 0 and odd are ≥ 0.

Example: The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges, but not absolutely. In the nearest future we will see, that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2,$$

but, again, the computing the actual sum is much harder than simply proving the convergence. In this case the convergence (conditional) follows directly from the Leibniz's criterion.

The power series

Definition 5.10. *A series of the form* $\sum_{n=0}^{\infty} a_n x^n$*, where the sequence of coefficients* $\{a_n\}$ *and the number* x *can be real or complex, is called a power series.*

Remarks: (i) For a fixed sequence of coefficients $\{a_n\}$ the power series can converge or not, depending on the number x. It always converges for $x = 0$.

(ii) At those points x, in which the power series converges it defines a function:

$$f(x) = \sum_{n=1}^{\infty} a_n\, x^n.$$

The functions which are sums of convergent power series are very important. These are the so-called analytic functions, and we will see that practically any function, in particular all elementary functions can be written this way (sometimes we say that a function "can be expanded into power series").

(iii) Of course, any series can be written in the form of a power series. Therefore the term "power series" concerns the way of presenting of the series.

(iv) In the sequel we will concentrate on series with real valued terms.

Theorem 5.11. *The power series $\sum_{n=0}^{\infty} a_n x^n$ is either convergent absolutely for every $x \in \mathbf{R}$, or there is a number $R \geq 0$ such that*

(i) For $x \in (-R, R)$ the series converges absolutely.

(ii) For $x \notin [-R, R]$ the series diverges.

The set of those x for which the powers series $\sum_{n=0}^{\infty} a_n x^n$ is convergent has therefore the form of the interval, containing one or both ends, or perhaps without ends (it can be the entire real line $\mathbf{R}$). This set is called the "interval of convergence" of the power series. The number R is called the "radius of convergence" (in the case when the interval of convergence is $(-\infty, \infty)$, we say that the radius of convergence is infinite).

Remark: On the endpoints of the interval of convergence we can have convergence or not. For example, the series $\sum x^n$ has the interval of convergence $(-1, 1)$, the series $\sum \frac{1}{n} x^n$ has the interval of convergence $[-1, 1)$, while the series $\sum \frac{1}{n^2} x^n$ has the interval of convergence $[-1, 1]$.

The proof of the theorem. If for $x_0 \in \mathbf{R}$ the series $\sum a_n x_0^n$ converges, then the sequence $\{a_n x_0^n\}$ converges to 0, and in particular is bounded:

$$\exists\, M \quad \forall\, n \in \mathbf{N} \quad |a_n x_0^n| \leq M.$$

If $|x| < |x_0|$ then let $q = \frac{|x|}{|x_0|} < 1$. We then have

$$|a_n x^n| = \left| a_n\, x_0^n\, \frac{x^n}{x_0^n} \right| = |a_n\, x_0^n| \cdot \left(\frac{|x|}{|x_0|} \right)^n \leq M \cdot q^n.$$

The geometric series with terms q^n converges, since $0 \le q < 1$. From the comparative criterion the series $\sum a_n x^n$ thus converges absolutely. Let

$$A = \left\{ |x| : \sum_{n=0}^{\infty} a_n\, x^n \text{ converges, } x \in \mathbf{R} \right\}.$$

If A is not bounded, then the series converges absolutely for all $x \in \mathbf{R}$. This is because for every $x \in \mathbf{R}$ we can find x_0 such that $|x_0| > |x|$, and the series $\sum a_n x_0^n$ converges. If A is bounded, then let

$$R = \sup A.$$

The R so defined satisfies the properties stated in the theorem. If $|x| < R$, then we can find x_0 such that $|x_0| > |x|$, and the series $\sum a_n x_0^n$ converges. In this case the series $\sum a_n x^n$ converges absolutely. On the other hand, if $|x| > R$ then the series $\sum a_n x^n$ cannot converge: if it did, we would have $|x| \in A$, so $|x| \le R$. □

Examples: (a) The series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ has the radius of convergence $R = 1$, which can be checked using d'Alembert's criterion. At $x = 1$ it diverges (it is tere the harmonic series), and at the point $x = -1$ it converges, which follows from the Leibniz's criterion.

(b) The series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has an infinite radius of convergence, which can be verified using the d'Alembert's criterion:

$$\left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = |x| \cdot \frac{1}{n+1} \xrightarrow{n\to\infty} 0,$$

independently of x.

(c) The series $\sum_{n=0}^{\infty} n^n x^n$ has the radius of convergence $R = 0$:

$$\left| \frac{(n+1)^{n+1}\, x^{n+1}}{n^n\, x^n} \right| = |x| \cdot (n+1) \left(1 + \frac{1}{n}\right)^n \xrightarrow{n\to\infty} +\infty,$$

for each $x \ne 0$.

Using the known criteria of convergence of the series we obtain various formulas for the radius of convergence.

Theorem 5.12. *Let us consider the power series $\sum a_n x^n$ and let*

$$g = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

If $g = 0$ then the radius of convergence of the series is infinite, if $g = +\infty$ then $R = 0$, and if $0 < g < \infty$ then

$$R = \frac{1}{g}.$$

Proof. We will apply the Cauchy's criterion to the series $\sum a_n x^n$.

$$\limsup_{n\to\infty} \sqrt[n]{|a_n\, x^n|} = \limsup_{n\to\infty} |x| \sqrt[n]{|a_n|} = |x| \cdot \limsup_{n\to\infty} \sqrt[n]{|a_n|} = |x| \cdot g.$$

If $g = 0$ then the series converges (absolutely) for every $x \in \mathbf{R}$, that is the radius of convergence is infinite. If $g = +\infty$ diverges, independently of $x \neq 0$, so $R = 0$. Finally, if $0 < g < \infty$ then the series converges (absolutely) for $|x| < \frac{1}{g}$ and diverges for $|x| > \frac{1}{g}$, so $R = \frac{1}{g}$. □

Remark: Using the d'Alembert's criterion in a similar way we obtain the following theorem: if

$$g = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists (proper or improper), then $R = \frac{1}{g}$ (we understand that $R = 0$ for $g = +\infty$ and R infinite for $g = 0$).

Example: Let us consider the series $\sum_{n=0}^{\infty} \frac{n^n}{n!} x^n$. Using the above observation we compute:

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)^n}{n!} \cdot \frac{n!}{n^n} = \left(1 + \frac{1}{n}\right)^n \xrightarrow{n\to\infty} e.$$

Thus $R = \frac{1}{e}$. We note, comparing this to Theorem 5.12 that we can conclude, that

$$\limsup_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}} = \limsup_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e.$$

Let us show now, that the sequence $\{\sqrt[n]{\frac{n^n}{n!}}\}$ is increasing, and if so, then its upper limit is its limit (an exercise), and we have the following corollary, which is worth remembering

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e.$$

Let us check that this sequence is indeed increasing. Let us introduce the notation $c_n = (1 + \frac{1}{n})^n$. We know, that $2 = c_1 < c_2 < c_3 < \cdots < e$, and so

$$\frac{(n+1)^{n^2}}{n^{n^2}} = \left(\frac{n+1}{n} \right)^{n^2} = \left(1 + \frac{1}{n} \right)^{n^2} = c_n^n > c_1 \cdot c_2 \cdot \ldots \cdot c_{n-1} = \frac{n^n}{n!}. \quad (5.5)$$

The last equality can be proved inductively: for $n = 2$ we have $c_1 = \frac{2^2}{2} = 2$, so the equality holds. Next we perform the induction step:

$$
\begin{aligned}
c_1 \cdot \dots \cdot c_{n-1} \cdot c_n &= \frac{n^n}{n!} \cdot c_n = \frac{n^n}{n!}\left(1 + \frac{1}{n}\right)^n \\
&= \frac{n^n}{n!}\left(\frac{n+1}{n}\right)^n = \frac{(n+1)^n}{n!} = \frac{(n+1)^{n+1}}{(n+1)!}.
\end{aligned}
$$

We thus have proved the inequality (5.5). It follows immediately that:

$$
\frac{(n+1)^{n^2}}{n^{n(n+1)}} > \frac{1}{n!} \quad \Rightarrow \quad \frac{(n+1)^{n^2}}{(n!)^n} > \frac{n^{n(n+1)}}{(n!)^{n+1}}.
$$

Now it is enough to extract roots of order $n(n+1)$ from both sides, and we obtain

$$
\sqrt[n+1]{\frac{(n+1)^{n+1}}{(n+1)!}} > \sqrt[n]{\frac{n^n}{n!}},
$$

which is what we wanted.

Chapter 6

Limit of a function at a point

Let $f(x)$ be a function of real variable with real values, that is $f : D_f \to \mathbf{R}$, where $D_f \subset \mathbf{R}$ is the domain of $f(x)$. Let $\overline{D_f}$ be the "completion" of D_f, that is the set of all those points x, for which there is a sequence $\{x_n\} \subset D_f$, $x_n \neq x$, converging to x. For example, the natural domain of the function $f(x) = \frac{1}{x}$ is the set $D_f = \{x : x \neq 0\}$. Then $\overline{D_f} = \mathbf{R}$. The notion of a limit of a function at a point will be introduced for points in $\overline{D_f}$, that is those points which belong to the domain of $f(x)$ (but are not isolated), or those, which do not belong to the domain, but are "at the very edge" of the domain.

Definition 6.1. *We say that the function $f(x)$ has at the point $x_0 \in \overline{D_f}$ a limit g, if*

$$\forall\ \epsilon > 0 \quad \exists\ \delta > 0 \quad \forall\ x \in D_f \qquad 0 < |x - x_0| < \delta \Rightarrow |f(x) - g| < \epsilon.$$

In this case we write

$$\lim_{x \to x_0} f(x) = g.$$

We say, that the function $f(x)$ has at the point $x_0 \in \overline{D_f}$ an improper limit ∞ (or $-\infty$) if

$$\forall\ M \in \mathbf{R} \quad \exists\ \delta > 0 \quad \forall\ x \in D_f \qquad 0 < |x - x_0| < \delta \Rightarrow f(x) > M,$$

(or $f(x) < M$). In such case we write

$$\lim_{x \to x_0} f(x) = \pm\infty.$$

The definition of the limit of the function at a point can be readily interpreted in the language of convergence of sequences:

Theorem 6.2. *Let* $x_0 \in \overline{D_f}$. *Then* $\lim_{x\to x_0} f(x) = g$ *if and only if, for each sequence* $\{x_n\} \subset D_f,\ x_n \neq x_0$ *such that* $\lim_{n\to\infty} x_n = x_0$ *there is*

$$\lim_{n\to\infty} f(x_n) = g.$$

Similarly, $\lim_{x\to x_0} f(x) = \pm\infty$ *if and only if for each sequence* $\{x_n\} \subset D_f,\ x_n \neq x_0$ *such that* $\lim_{n\to\infty} x_n = x_0$ *there is*

$$\lim_{n\to\infty} f(x_n) = \pm\infty.$$

Proof. Let $f(x)$ have at a point x_0 the limit g

$$g = \lim_{x\to x_0} f(x).$$

Let $\{x_n\}$ be an arbitrary sequence from D_f, converging to x_0, $x_n \neq x_0$. We will show, that the sequence $\{f(x_n)\}$ converges to g. Let $\epsilon > 0$. From the definition of the limit we know that there exists $\delta > 0$ such that if $x \in D_f$ and $x \neq x_0$ then

$$|x - x_0| < \delta \Rightarrow |f(x) - g| < \epsilon. \tag{6.1}$$

Since $x_n \to x_0$ thus (δ plays the role of ϵ from the definition of the sequence limit) there exists $n_0 \in \mathbf{N}$ such that $\forall\ n \geq n_0$ we have $|x_n - x_0| < \delta$, so, using (6.1)

$$|f(x_n) - g| < \epsilon.$$

This way we have proved that $\lim_{n\to\infty} f(x_n) = g$.

Now the proof in the opposite direction. Let $f(x_n) \to g$ for every sequence $x_n \to x_0$, satisfying $x_n \neq x_0$ and $\{x_n\} \subset D_f$. We will show, that $f(x)$ ha at the point x_0 the limit g. We will proceed with the proof indirectly. Suppose that $f(x)$ does not have the limit g at x_0, that is the condition from the definition of the limit does not hold:

$$\exists\ \epsilon_0 > 0 \quad \forall\ \delta > 0 \quad \exists\ x \in D_f \qquad 0 < |x - x_0| < \delta\ \wedge\ |f(x) - g| \geq \epsilon_0.$$

Using the above we define a sequence $\{x_n\}$ which will give us the contradiction. The sequence $\{x_n\}$ is defined in the following way. For $n \in \mathbf{N}$ let $\delta = \frac{1}{n}$, and x_n be this element of D_f, which satisfies $0 < |x_n - x_0| < \frac{1}{n} \wedge |f(x_n) - g| \geq \epsilon_0$. Let us observe, that such sequence $\{x_n\}$ satisfies $\{x_n\} \subset D_f$, $x_n \neq x_0$ and $x_n \to x_0$, but $f(x_n) \not\to g$. We have thus obtained the contradiction.

We leave the case of improper limits as an exercise for our dear reader. :-) □

Using the above theorem, and theorems about convergence and limits of sequences we have the following corollary.

Corollary 6.3. *(i) If* $a = \lim_{x\to x_0} f(x)$ *and* $b = \lim_{x\to x_0} g(x)$ *then*

$$\lim_{x\to x_0}(f \pm g)(x) = a \pm b, \qquad \lim_{x\to x_0}(f \cdot g)(x) = a \cdot b,$$

and if additionally $b \neq 0$ *then*

$$\lim_{x\to x_0}\left(\frac{f}{g}\right)(x) = \frac{a}{b}.$$

(ii) If in some neighborhood of x_0 *we have*

$$g(x) \leq f(x) \leq h(x),$$

and

$$\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = a,$$

then also

$$\lim_{x\to x_0} f(x) = a.$$

(iii) We can "take the limit under" roots, that is

$$\lim_{x\to x_0}\sqrt[k]{f(x)} = \sqrt[k]{\lim_{x\to x_0} f(x)},$$

provided given roots are defined ($f \geq 0$ *for even* k)*.*

Examples: (a) Let us compute the limit

$$\lim_{x\to 2}\frac{3x-5}{x^3-1}.$$

The denominator $x^3 - 1$ is non-zero in some neighborhood of $x_0 = 2$, so 2, together with some neighborhood belongs to the natural domain of the function, which is the set of all $x \in \mathbf{R}$ for which $x^3 \neq 1$. Let $x_n \to 2$, $x_n \neq 2$, and $x_n^3 \neq 1$. Then

$$\frac{3x_n - 5}{x_n^3 - 1} \to \frac{3 \cdot 2 - 5}{2^3 - 1} = \frac{1}{7},$$

that is $\lim_{x\to 2}\frac{3x-5}{x^3-1} = \frac{1}{7}$.

(b) $\lim_{x\to 0}\sin x$. We will use the following estimate: $0 \leq x \leq \frac{\pi}{2}$

$$0 \leq \sin x \leq x, \quad 0 \leq x \leq \frac{\pi}{2}. \tag{6.2}$$

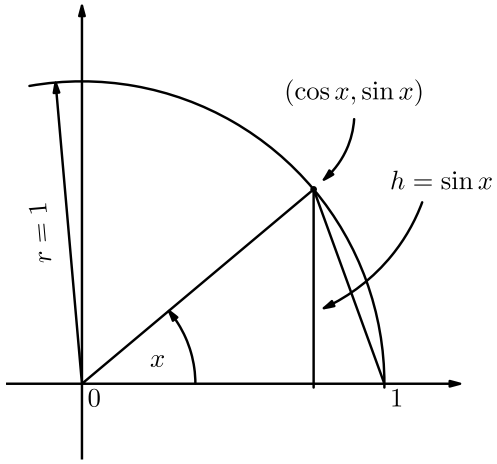

Figure 6.1: The estimate for $\sin x$.

It follows from Fig. 6.1. $\frac{x}{2}$ is the area of the disc sector cut out by the central angle x while $\frac{\sin x}{2}$ is the area of the triangle inside this sector (the triangle has the base 1, and height $\sin x$). The estimate (6.2) is thus clear. Further we have that $\sin(-x) = -\sin(x)$ (that is $\sin(x)$ is an odd function), so for $-\frac{\pi}{2} \leq x \leq 0$ we obtain from the above

$$x \leq \sin x \leq 0.$$

Thus, for $|x| \leq \frac{\pi}{2}$ we have

$$0 \leq |\sin x| \leq |x|,$$

that is $\lim_{x\to 0} \sin x = 0$.

(c) In the case of $\cos x$ we can use the facts we have already shown for $\sin x$. In the neighborhood of zero $\cos x$ is positive, and so

$$\lim_{x\to 0} \cos x = \lim_{x\to 0} \sqrt{1 - \sin^2 x} = \sqrt{1 - \big(\lim_{x\to 0} \sin x\big)^2} = 1.$$

(d) Using the trigonometric identities we find limits in other points

$$\begin{aligned} \lim_{x\to x_0} \sin x &= \lim_{x\to 0} \sin(x + x_0) \\ &= \lim_{x\to 0}(\sin x \cos x_0 + \cos x \sin x_0) \\ &= \cos x_0 \lim_{x\to 0} \sin x + \sin x_0 \lim_{x\to 0} \cos x \end{aligned}$$

$$= \sin x_0,$$

and

$$\begin{aligned}
\lim_{x\to x_0} \cos x &= \lim_{x\to 0} \cos(x + x_0) \\
&= \lim_{x\to 0} (\cos x \cos x_0 - \sin x \sin x_0) \\
&= \cos x_0 \lim_{x\to 0} \cos x - \sin x_0 \lim_{x\to 0} \sin x \\
&= \cos x_0.
\end{aligned}$$

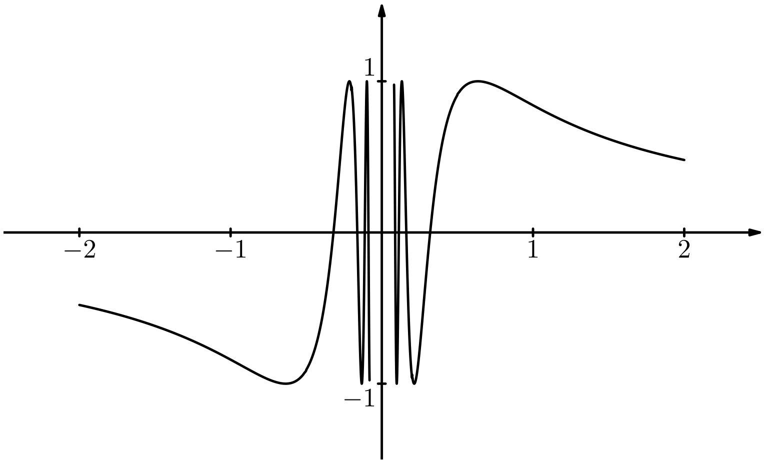

Figure 6.2: The function $\sin\frac{1}{x}$ in the neighborhood of 0.

(e) Let us observe, that the limit $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist. Let us consider two sequences, $x_n = \frac{1}{\pi/2+2n\pi}$ and $y_n = \frac{1}{3\pi/2+2n\pi}$. Observe that

$$\begin{aligned}
\sin\frac{1}{x_n} &= \sin\Big(\frac{\pi}{2} + 2n\pi\Big) = \sin\Big(\frac{\pi}{2}\Big) = 1, \\
\sin\frac{1}{y_n} &= \sin\Big(\frac{3\,\pi}{2} + 2n\pi\Big) = \sin\Big(\frac{3\,\pi}{2}\Big) = -1,
\end{aligned}$$

so $\lim_{n\to\infty} f(x_n) = 1$, and $\lim_{n\to\infty} f(y_n) = -1$. The situation is clarified by Fig. 6.2.

(f) Let $a > 1$. We will show, that $\lim_{x\to 0} a^x = 1$. Let $\epsilon > 0$ and $x > 0$. We thus have $a^x > 1$. Let $n_0 \in \mathbf{N}$ be such, that $\sqrt[n]{a} - 1 < \epsilon$ for $n \geq n_0$. We use the fact that $\sqrt[n]{a} \to 1$. Let $\delta_0 = \frac{1}{n_0}$. Then, if

$$0 < x < \delta_0 \;\Rightarrow\; 1 < a^x < a^{\frac{1}{n_0}} \;\Rightarrow\; 0 < a^x - 1 < \sqrt[n_0]{a} - 1 < \epsilon.$$

Let now $x < 0$. We know that

$$\sqrt[n]{\frac{1}{a}} = \frac{1}{\sqrt[n]{a}} \to 1,$$

and thus let $n_1 \in \mathbf{N}$ be such, that for $n \geq n_1$ we have $0 < 1 - \sqrt[n]{1/a} < \epsilon$. Let $\delta_1 = \frac{1}{n_1}$, then for $-\delta_1 < x < 0$ it follows that

$$a^{-\frac{1}{n_1}} < a^x < 1 \;\Rightarrow\; \sqrt[n]{\frac{1}{a}} < a^x < 1 \;\Rightarrow\; 0 < 1 - a^x < 1 - \sqrt[n_1]{\frac{1}{a}} < \epsilon.$$

Finally let $\delta = \min\{\delta_0, \delta_1\}$, then $0 < |x| < \delta$ implies $|1 - a^x| < \epsilon$.

(g) Let $a > 1$, then $\lim_{x \to x_0} a^x = a^{x_0}$. This is because we have

$$\lim_{x \to x_0} a^x = \lim_{x \to 0} a^{x + x_0} = \lim_{x \to 0} a^x \cdot a^{x_0} = a^{x_0} \cdot \lim_{x \to 0} a^x = a^{x_0}.$$

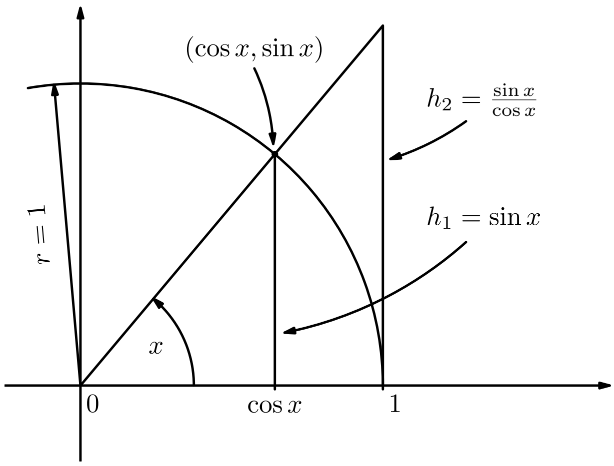

Figure 6.3: Further estimates for $\sin x$.

(h) Let us again recall the definition of the function $\sin x$, and let us compare the area of sector of the unit circle, cut out by the central angle x, and the area of the large triangle (Fig. 6.3). The area of the sector is equal to $\frac{x}{2}$, while the large triangle has height $h_2 = \frac{\sin x}{\cos x}$ and base 1, so the area equal to $\frac{1}{2} \cdot \frac{\sin x}{\cos x}$. The formula for h_2 follows from the comparison with the small triangle, which is built on the same angle x, which has the height $h_1 = \sin x$ and the base $\cos x$. For $0 \leq x \leq \frac{\pi}{2}$ we thus have

$$\frac{x}{2} \leq \frac{\sin x}{2 \cos x},$$

so, combining this with (6.2) we obtain the double estimate

$$\cos x \leq \frac{\sin x}{x} \leq 1. \tag{6.3}$$

Considering the fact that all functions above are even, we get (6.3) also for $|x| \leq \frac{\pi}{2}$. The functions on the sides have limit 1 at zero, so

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

It is one of the important limits, which we will use repeatedly in the future.

One sided limits

If we restrict the definition only to $x > x_0$ (or $x < x_0$) and the condition holds, we say that the function has at x_0 a right-hand limit (or left-hand limit). For example, for proper (finite) limits the condition for the existence of the right-hand limit is the following

$$\forall\, \epsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x \in D_f \qquad 0 < x - x_0 < \delta \Rightarrow |f(x) - g| < \epsilon.$$

For the left-hand limit it is the following

$$\forall\, \epsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x \in D_f \qquad 0 < x_0 - x < \delta \Rightarrow |f(x) - g| < \epsilon.$$

The one sided limits are denoted respectively

$$\lim_{x\to x_0^+} f(x), \qquad \text{and} \qquad \lim_{x\to x_0^-} f(x).$$

For improper limits these conditions have to be modified in the usual way.

Corollary 6.4. *(i)* $g = \lim_{x\to x_0^\pm} f(x)$ *if for arbitrary sequence* $\{x_n\} \subset D_f$, $x_n > x_0$ *(or respectively* $x_n < x_0$*) and* $x_n \to x_0$ *we have* $f(x_n) \to g$*. The situation is in complete analogy to Theorem 6.2.*

(ii) The function $f(x)$ *has at a point* x_0 *the limit* g *(proper or improper) if and only if it has at* x_0 *both one-sided limits, and they are both equal. This follows directly from the definition.*

(iii) Theorems concerning arithmetic operations on limits apply to one-sided limits. For example

$$\lim_{x\to x_0^+} (f+g)(x) = \lim_{x\to x_0^+} f(x) + \lim_{x\to x_0^+} g(x).$$

Examples: (a) $f(x) = [x]$. If $x_0 \in \mathbf{Z}$ then, as can be checked easily

$$\lim_{x \to x_0^+} f(x) = x_0, \qquad \text{and} \qquad \lim_{x \to x_0^-} f(x) = x_0 - 1.$$

In points $x_0 \in \mathbf{Z}$ the function $f(x)$ thus has different one-sided limits, and therefore does not have a normal (both-sided) limit. At other points the function $f(x)$ does have the normal limit.

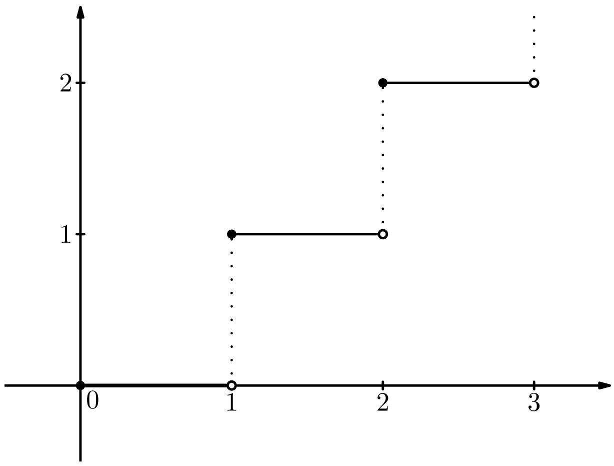

Figure 6.4: One sided limits of the function $[x]$.

(b) $f(x) = 2^{\frac{1}{x}}$. The domain $D_f = \{x : x \neq 0\}$, and so $0 \in \overline{D_f}$. We have

$$\lim_{x \to 0^+} f(x) = +\infty \qquad \text{and} \qquad \lim_{x \to 0^-} f(x) = 0.$$

The first limit follows from the fact that the function 2^y is increasing and unbounded.

Limits at infinities

If the domain of the function allows this, we can consider the limits of the function at $+\infty$ and $-\infty$. These limits can be proper (finite), or improper (infinite).

Definition 6.5. *We say that a function $f(x)$ has at $+\infty$ (or $-\infty$) a limit g, if*

$$\forall\, \epsilon > 0 \quad \exists\, M \quad \forall\, x \in D_f \quad x > M \Rightarrow |f(x) - g| < \epsilon \quad (x < M \Rightarrow |f(x) - g| < \epsilon).$$

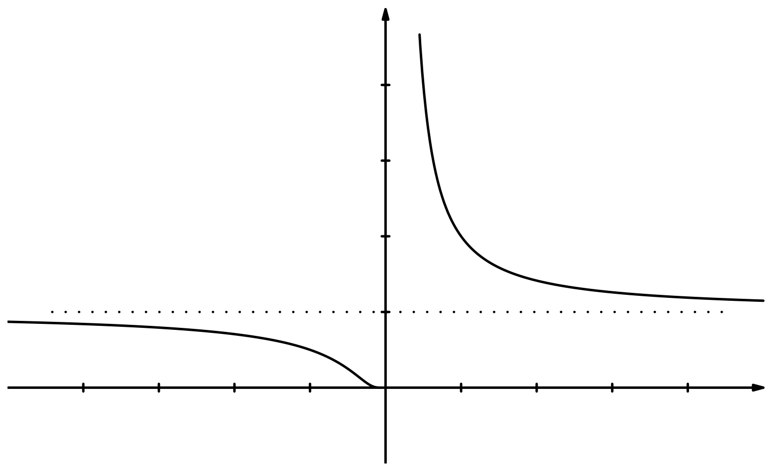

Figure 6.5: The limits at 0 of function $2^{\frac{1}{x}}$.

In such case we write

$$g = \lim_{x\to\pm\infty} f(x).$$

Similarly we define the improper limits. For example, $\lim_{x\to+\infty} f(x) = +\infty$ *if*

$$\forall\ M \quad \exists\ K \quad \forall\ x \in D_f \qquad x > K \Rightarrow f(x) > M.$$

Corollary 6.6. *The above definition can be also expressed in the language of sequences. For example,* $\lim_{x\to+\infty} f(x) = +\infty$ *if and only if for every sequence* $\{x_n\}$ *from the domain of* $f(x)$*, diverging to* $+\infty$ *the sequence* $\{f(x_n)\}$ *also diverges to* $+\infty$*.*

Examples: (a) We will find the limit at $+\infty$ of the function $f(x) = \frac{e^x}{x}$. Of course, this function has limit 0 at $-\infty$. On the other hand, if $x \to +\infty$ both the numerator and the denominator tend to $+\infty$. First we consider the sequence

$$\frac{e^n}{n} = \left(\frac{e}{\sqrt[n]{n}}\right)^n.$$

Since $\sqrt[n]{n} \to 1$, Thus

$$\frac{e}{\sqrt[n]{n}} \to e.$$

Therefore,

$$\exists\ n_0 \in \mathbf{N} \quad \forall\ n \geq n_0 \qquad \frac{e}{\sqrt[n]{n}} > 2\ \Rightarrow\ \frac{e^n}{n} > 2^n.$$

The sequence 2^n diverges to $+\infty$, we thus have the improper limit

$$\lim_{n\to\infty} \frac{e^n}{n} = +\infty.$$

We also have the following estimates. Let us momentarily denote $\epsilon = x - [x]$, so $0 \leq \epsilon < 1$, and so

$$\frac{e^x}{x} = \frac{e^{[x]+\epsilon}}{[x]+\epsilon} \geq \frac{e^{[x]}}{[x]+1} = \frac{1}{e}\,\frac{e^{[x]+1}}{[x]+1}.$$

Let $x_n \to +\infty$ and let $M > 0$. Then

$$\exists\; n_0 \in \mathbf{N} \quad \forall\; n \geq n_0 \qquad \frac{e^n}{n} \geq e\,\cdot\, M,$$

and

$$\exists\; n_1 \in \mathbf{N} \quad \forall\; n \geq n_1 \qquad x_n \geq n_0 \;\Rightarrow\; [x_n] \geq n_0.$$

So, for $n \geq n_1$ we have

$$\frac{e^{x_n}}{x_n} \geq \frac{1}{e}\,\frac{e^{[x_n]+1}}{[x_n]+1} \geq \frac{1}{e}\,\cdot\, e\,\cdot\, M = M.$$

We have thus proved, that

$$\lim_{x\to+\infty} \frac{e^x}{x} = +\infty.$$

This can be understood in the following way. When x increases to ∞ then the exponential function e^x grows faster than x. Let us observe, that the above reasoning can be easily modified to show that the exponential function grows faster than any polynomial.

(b) Let us consider the limit

$$\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x.$$

We know the limit of the respective sequence (when $x = n$), it is, by definition, the number e. Now we would like to adapt the reasoning from example (a), and estimate the values of the function at points x by its values in some natural points n. We will use various estimates, but the idea is simple. Let $\epsilon > 0$, the sequence $x_n \to \infty$, and let us denote $k_n = [x_n]$. Let us observe, that $k_n \to \infty$ and that they satisfy

$$k_n \leq x_n < k_n + 1$$

$$\frac{1}{k_n+1} < \frac{1}{x_n} \leq \frac{1}{k_n},$$

(enough that $x_n \geq 1$), so further

$$1+\frac{1}{k_n+1} < 1+\frac{1}{x_n} \leq 1+\frac{1}{k_n}$$

$$\left(1+\frac{1}{k_n+1}\right)^{k_n} < \left(1+\frac{1}{x_n}\right)^{x_n} < \left(1+\frac{1}{k_n}\right)^{k_n+1}$$

$$\left(1+\frac{1}{k_n+1}\right)^{k_n+1}\frac{1}{1+\frac{1}{k_n+1}} < \left(1+\frac{1}{x_n}\right)^{x_n} < \left(1+\frac{1}{k_n}\right)^{k_n}\left(1+\frac{1}{k_n}\right)$$

We know that the sequences

$$\left(1+\frac{1}{n}\right)^{n}\cdot\left(1+\frac{1}{n}\right) \qquad \text{and} \qquad \left(1+\frac{1}{n+1}\right)^{n+1}\cdot\frac{n+1}{n+2}$$

converge to e, so there exists $n_1 \in \mathbf{N}$ such that for $n \geq n_1$

$$\left(1+\frac{1}{n}\right)^{n}\cdot\left(1+\frac{1}{n}\right) < e+\epsilon \qquad \text{and} \qquad \left(1+\frac{1}{n+1}\right)^{n+1}\cdot\frac{n+1}{n+2} > e-\epsilon.$$

Let $n_0 \in \mathbf{N}$ be such, that for $n \geq n_0$ there is $x_n \geq n_1$ so $k_n = [x_n] \geq n_1$. Then

$$e-\epsilon < \left(1+\frac{1}{k_n+1}\right)^{k_n+1}\frac{1}{1+\frac{1}{k_n+1}} <$$
$$< \left(1+\frac{1}{x_n}\right)^{x_n} < \left(1+\frac{1}{k_n}\right)^{k_n}\left(1+\frac{1}{k_n}\right) < e+\epsilon,$$

which implies

$$\left|\left(1+\frac{1}{x_n}\right)^{x_n} - e\right| < \epsilon.$$

Similarly we can show, that the limit of this function at $-\infty$ is also equal to e. The proof is analogous, and uses the limit

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} = \frac{1}{e},$$

which was on the exercise list.

Chapter 7

Continuous functions

Definition 7.1. *We say that the function $f(x)$ is continuous at a point x_0 from its domain, if*

$$f(x_0) = \lim_{x \to x_0} f(x).$$

We say, that a function is continuous on a set $A \subset D_f$ if it is continuous at every point $x_0 \in A$. If a function is continuous at every point of its domain, we simply say that it is continuous.

Speaking informally we may say, that a function is continuous, if one can "enter" with a limit "under" such function. Intuitively, a continuous function is such whose graph is an unbroken curve.

Remarks: (i) Using the definition of the limit at a point we obtain the following condition for continuity of a function at a point x_0

$$\forall\ \epsilon > 0 \quad \exists\ \delta > 0 \quad \forall\ x \in D_f \qquad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

Using the language of sequences, that is Theorem 6.2 we obtain the condition

$$\forall\ \{x_n\} \in D_f, \quad x_n \to x_0 \ \Rightarrow\ f(x_n) \to f(x_0).$$

(ii) If $x_0 \in D_f$ is an isolated point of the domain, then the limit of the function at such point is not defined. Let us settle, that at each isolated point of the domain the function is, by definition, continuous. Of course, isolated points of the domain are rare, but they actually can appear. For example the function $f(x) = \sqrt{x^3 - x^2}$ has the natural domain $D_f = \{0\} \cup [1, \infty)$.

Corollary 7.2. *All elementary functions, that is polynomials, rational functions, the exponential, power, trigonometric functions, and the logarithm are continuous.*

Theorem 7.3. *The sum, difference, product, fraction and the composition of continuous functions are continuous at each point at which the given operation can be performed.*

Proof. We will prove the case of composition. All other operations on continuous functions are an immediate consequence of the theorem about arithmetic operations on limits of functions. Thus, let for each $x \in D_f$ $f(x) \in D_g$, and let both functions $f(x)$ and $g(x)$ be continuous. The composition $(g \circ f)(x) = g(f(x))$ is then defined for every $x \in D_f$. Let $x_n \to x_0$, $x_n, x_0 \in D_f$. Then $f(x_n) \to f(x_0)$ (this follows from the continuity of $f(x)$ at x_0) and $g(f(x_n)) \to g(f(x_0)$ (this follows from the continuity of $g(x)$ at $f(x_0)$). The composition is thus continuous. □

Example: Let $f(x) = x^x$ for $x > 0$ and let $f(0) = 1$. We will show, that $f(x)$ is continuous at 1, that is

$$\lim_{x \to 0^+} x^x = 1.$$

We will use the limits we already know

$$\lim_{x \to \infty} \frac{e^x}{x} = \infty \ \Rightarrow \ \lim_{x \to \infty} \frac{x}{\log x} = \infty.$$

The above follows directly from the definition of a limit, and from the fact that $x_n \to \infty \Leftrightarrow \log x_n \to \infty$. Passing to inverses we obtain

$$\lim_{x \to \infty} \frac{\log x}{x} = 0 \ \Rightarrow \ \lim_{x \to 0^+} x \log x = 0.$$

The last limit follows from the previous one, since $x_n \to \infty \Leftrightarrow \frac{1}{x} \to 0^+$. At last we get

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \log x} = e^{\lim_{x \to 0^+} x \log x} = e^0 = 1.$$

As can be easily observed, the continuity of the function $f(x)$ at other points of its domain can be easily obtained, once we prove the continuity of the function $\log x$. This, in turn, will follow from the theorem about the continuity of the inverse function, which we will soon prove (recall, that $\log x$ is an inverse to e^x).

Remark: A function may be discontinuous from various reasons. For example, the limit of a function at a point may exists

$$g = \lim_{x \to x_0} f(x),$$

but $g \neq f(x_0)$. This is the situation we encounter in the case of $f(x) = [-|x|]$. If $0 < |x| < 1$ then $-1 < -|x| < 0$ that is $f(x) = -1$ and thus

$$\lim_{x \to 0} f(x) = -1.$$

On the other hand $f(0) = 0$. This type of discontinuity is called removable. It is enough to redefine the value of the function at a single point to obtain continuity at that point.

A different type of discontinuity is the so-called jump discontinuity. If the one-sided limits of a function exist at a point, but are not equal, we say that the function has a jump discontinuity. An example can be seen with $f(x) = [x]$, which has jump discontinuities at integers

$$\lim_{x \to k^-} f(x) = k - 1, \qquad \lim_{x \to k^+} f(x) = k, \qquad k \in \mathbf{Z}.$$

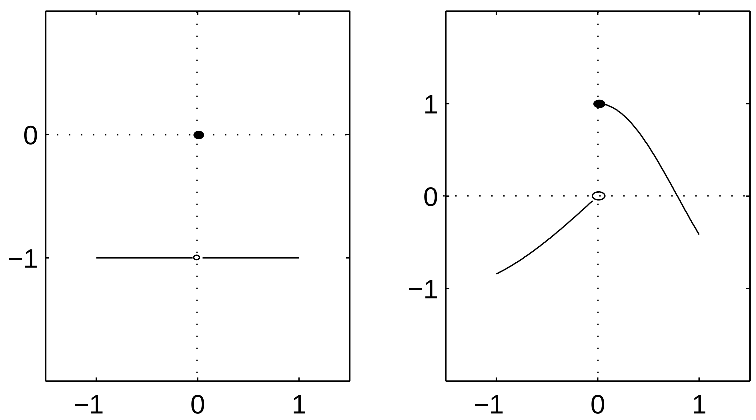

Figure 7.1: Removable discontinuity and jump discontinuity.

The function

$$f(x) = \begin{cases} \sin \frac{1}{x} & : x \neq 0 \\ 0 & : x = 0, \end{cases}$$

has a discontinuity of another type, the limit of $f(x)$ at zero does not exist, not even one-sided limits.

Properties of continuous functions

Continuous functions have a number of important properties, some of which we will now prove. Namely, we will prove, that a continuous function on a finite closed interval $[a, b]$ is bounded, and reaches its highest and lowest values, and also assumes all intermediate values. We will use our knowledge of sequences, and the main tool, as we will see, will be the Bolzano-Weierstrass theorem (theorem 4.12). The properties that we will prove lead to numerous numerical algorithms, for example in root-finding. The continuity of the function is the basic required condition.

Theorem 7.4. *A function $f(x)$, continuous on the interval $[a, b]$ (finite and closed), is bounded.*

Proof. We will prove that $f(x)$ is bounded from above. The proof that it is also bounded from below we leave as an exercise. Observe, that this can be done either by appropriately adapting the proof of bound from above, or by observing that the function $-f(x)$ is bounded from above if and only if $f(x)$ is bounded from below. We now proceed with the proof that $f(x)$ is bounded from above, and we will do that indirectly. Suppose then, that $f(x)$ is not bounded from above. Then there exists a sequence of points $\{x_n\} \subset [a, b]$, for which

$$f(x_n) > n, \qquad n = 1, 2, \dots.$$

This sequence is constructed by using the fact, that $f(x)$ is not bounded from above by 1 - this gives us x_1 such, that $f(x_1) > 1$, then we use that the function is not bounded by 2, 3, and so on. The sequence lies within $[a, b]$, so it is bounded, and thus we can extract a subsequence $\{x_{n_k}\}$ which converges to some $c \in [a, b]$ (Theorem 4.12):

$$x_{n_k} \to c.$$

From the definition of continuity we have $f(x_{n_k}) \to f(c)$, which is a contradiction, because the sequence $\{f(x_{n_k})\}$ is not bounded. □

Remark: It is crucial, that the interval $[a, b]$ is finite, and contains the endpoints. Without these assumptions the function might be unbounded. For example, $f(x) = x$ is continuous on $[0, \infty)$, while $f(x) = \frac{1}{x}$ is continuous on $(0, 1]$, and neither is bounded. The same remark applies to the following theorem.

Theorem 7.5. *The function $f(x)$ which is continuous on the interval $[a, b]$ (finite and closed) assumes its maximal and minimal values.*

Proof. Again, we will only show that $f(x)$ assumes its maximal value, and leave the rest as an exercise. Let

$$M = \sup\{y : y = f(x),\ x \in [a, b]\}.$$

We know that the set of values is bounded, so the above supremum is finite. Form the definition of supremum there exists a sequence $\{x_n\} \subset [a, b]$ such that $f(x_n) \to M$. The sequence $\{x_n\}$ is bounded, so we can extract a subsequence $\{x_{n_k}\}$ convergent to some $c \in [a, b]$ (again, theorem 4.12). We thus have $f(x_{n_k}) \to f(c)$, so $f(c) = M$. □

Theorem 7.6 (The Darboux property)**.** *A function $f(x)$ continuous on an interval $[a, b]$ assumes all intermediate values between its minimal value m, and its maximal value M. In other words, the set of values of the function continuous on an interval $[a, b]$ is a closed and finite interval $[m, M]$.*

Proof. We know, that the function $f(x)$ assumes its extremal values, that is there exist numbers $c, d \in [a, b]$ such that $f(c) = m$ and $f(d) = M$. Let us assume $c < d$. Observe, that we do not actually lose any generality, since if $c = d$ the function is clearly constant, and in the case $c > d$ we can consider $-f(x)$ in the place of $f(x)$, or, alternately, we may modify the present argument. Also, assume $m < M$, since otherwise the function is constant, and there is nothing to prove. So, suppose $c < d$. Let $y_0 \in (m, M)$, that is y_0 is an intermediate value, between the minimal and the maximal. Let

$$x_0 = \sup\{t \in [c, d] : f(x) < y_0 \text{ for } x \in [c, t]\}.$$

We know that the above set is non-empty because it contains at least c (recall, that $f(c) = m < y_0$), and it is bounded. The supremum thus exists. We will now show, that the following must hold

$$f(x_0) = y_0, \tag{7.1}$$

so, indeed, y_0 is a value of the function $f(x)$. Let us assume, that $f(x_0) < y_0$. Then, since $f(x)$ is continuous, there exists $\delta > 0$ such, that $f(x) < y_0$ for $x \in [x_0-\delta, x_0+\delta]$. We thus see, that $f(x) < y_0$ on a larger interval $[c, x_0+\delta]$, which contradicts the definition of x_0. Therefore, it cannot happen that $f(x_0) < y_0$. Let us then assume that $f(x_0) > y_0$. This time, from the continuity of $f(x)$ at x_0 we have, that $f(x) > y_0$ on some interval $[x_0-\delta, x_0+\delta]$, for some $\delta > 0$. But it follows from the definition of x_0 that $f(x) < y_0$ for $x < x_0$, which is a contradiction. Thus the only possibility for the value of $f(x_0)$ is (7.1). □

Remark: The above theorem can be used to find approximate roots of equations. If we know, that a function $f(x)$ is continuous, and that $f(a) \cdot$

$f(b) < 0$ ($f(a)$ and $f(b)$ have opposite signs), then $f(x)$ has a root in the interval (a, b):

$$f(x) = 0 \qquad \text{for some } x \in (a, b).$$

The algorithm of approximating this root, the so-called "halving" method is recursive. Let $c = \frac{a+b}{2}$. Either $f(c) = 0$, and then the root is found, or $f(c) \neq 0$ and then we must have $f(a) \cdot f(c) < 0$ or $f(c) \cdot f(b) < 0$. In other words, the root must be either in the left half of the interval $[a, b]$, or in the right half. We are thus in the starting point again (that is we have a root localized in some interval), but with the interval half shorter. For example, to numerically compute $\sqrt{2}$ we can seek the root of the equation

$$f(x) = x^2 - 2 = 0.$$

We have $f(1) \cdot f(2) = -2 < 0$, and the function $f(x)$ is continuous, so there exists a root in the interval $(1, 2)$ (big deal :-)). It is easy to see, that using the halving method we acquire 3 additional decimal digits of accuracy of the approximation for every 10 iterations. Each iteration (in the current example) reduces to 1 multiplication, so the algorithm is rapidly convergent – 3 digits of accuracy per 10 multiplications.

Theorem 7.7. *If a function $f(x)$ is continuous on the interval $[a, b]$ and bijective (each value assumed only once), then the function $g(y)$, inverse to $f(x)$, is continuous on the set of values of $f(x)$, that is on the interval $[m, M]$, where the constants m and M denote, as in the previous theorem, the minimal and maximal values of the function $f(x)$ on $[a, b]$.*

Proof. The range – the set of values of the function $f(x)$, according to theorem 7.6, is an interval $[m, M]$, so it is the domain of the inverse function $g(y)$. If $m \leq y_0 \leq M$ then $g(y)$ is defined at the point y_0, by the formula $g(y_0) = x_0$, where x_0 is the unique point for which $f(x_0) = y_0$ (recall that $f(x)$ is bijective). Let $y_n \to y_0$ and $y_n \in [m, M]$ for $n = 1, 2, \ldots$. Since y_n and y_0 are in the set of values of $f(x)$, then there are $x_0, x_n \in [a, b]$ such, that $f(x_n) = y_n$ and $f(x_0) = y_0$. The sequence $\{x_n\}$ is bounded. Denote its upper limit by x_0', and the lower limit by x_0''. Suppose the subsequences $\{x_{n_k'}\}$ and $\{x_{n_k''}\}$ converge to x_0' and x_0'' respectively. From the continuity of $f(x)$ it follows that

$$f(x_0') = \lim_{k\to\infty} f(x_{n_k'}) = \lim_{k\to\infty} y_{n_k'} = y_0,$$

and similarly

$$f(x_0'') = \lim_{k\to\infty} f(x_{n_k''}) = \lim_{k\to\infty} y_{n_k''} = y_0.$$

We thus have $f(x_0') = f(x_0'') = y_0 = f(x_0)$. But since $f(x)$ was bijective, we must have $x_0 = x_0' = x_0''$. The upper and the lower limits of the sequence $\{x_n\}$ are both equal to x_0, and so the sequence actually converges to x_0. We thus have

$$g(y_n) = x_n \xrightarrow{n\to\infty} x_0 = g(y_0),$$

so $g(y)$ is continuous at y_0. □

Corollary 7.8. *The function* $\log_a x$ *is continuous on* $(0, \infty)$*, since it is the inverse function of the continuous function* a^x *(*$a > 0$*,* $a \neq 1$*).*

Remark: A function continuous, bijective on the interval $[a, b]$ must be strictly monotonic. We leave the proof of this simple fact as an exercise.

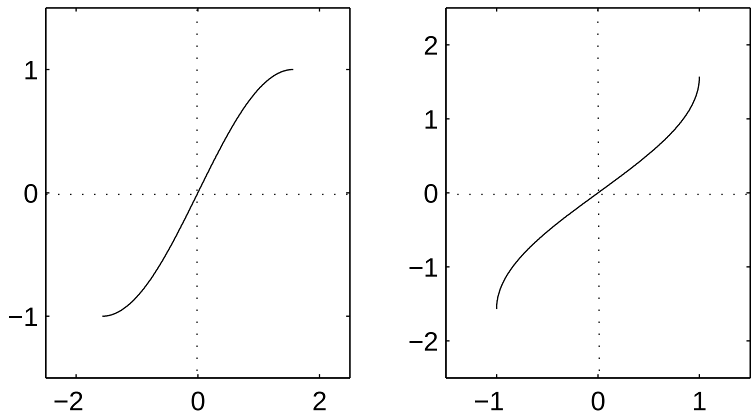

Figure 7.2: Functions $\sin x$ and $\arcsin x$.

Inverse trigonometric functions

Functions $\sin x$ i $\cos x$ are not bijective, and thus are not invertible. We can, however consider them on a restricted domain, on which they are bijective. The function $\sin x$ with its domain restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is strictly increasing -1 do 1, and thus bijective and invertible.

The inverse function, defined on $[-1, 1]$ is called $\arcsin x$, and according to the above theorem is continuous. Similarly, $\cos x$, with its domain restricted to the interval $[0, \pi]$ is strictly decreasing from 1 to -1, and thus invertible. The inverse function, defined on the interval $[-1, 1]$ is called $\arccos x$, and is also continuous.

The function $\tan x$ is periodic, with period π, and consists of separated "branches". With its domain restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$ it is strictly increasing

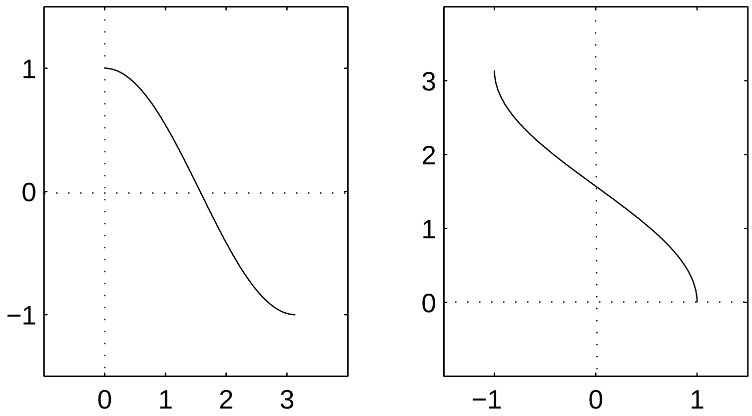

Figure 7.3: Functions $\cos x$ and $\arccos x$.

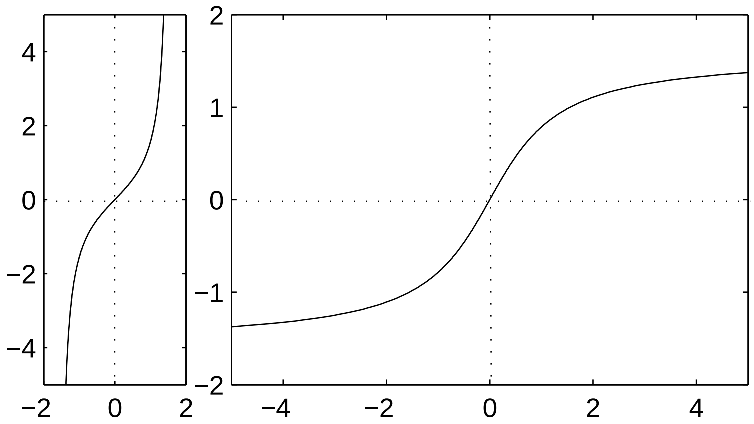

Figure 7.4: A branch of $\tan x$ and its inverse $\arctan x$.

and invertible. The inverse function, defined on the entire real line $\mathbf{R}$ is called $\arctan x$ and is continuous.

Chapter 8

The derivative

The derivative of a function is the momentary speed with which it changes .

Definition 8.1. *The derivative of a function $f(x)$ at a point x_0 is the limit*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, \tag{8.1}$$

provided this limit exists. If it does, we say that $f(x)$ is differentiable at the point x_0 (or that it has a derivative at the point x_0). The derivative of the function $f(x)$ at a point x_0 is denoted by

$$f'(x_0) \quad (\text{“}f \text{ prime”}) \quad \text{or} \quad \frac{df}{dx}(x_0) \quad (\text{“}df \text{ with respect to } dx \text{ at } x_0\text{”}).$$

Remarks: (i) The derivative of the function $f(x)$ is also a function, with the domain consisting of these points, at which $f(x)$ is differentiable. Computing the derivative is called “differentiating” a function.

(ii) The quotient appearing in the limit (8.1) is called the “differential quotient”. The differential quotient, that is the increase of the function divided by the increase of the argument determines the average rate of increase of the function $f(x)$ over the interval $[x, x + h]$ (if $h > 0$, otherwise this has to be reformulated). Thus the interpretation of the derivative as the momentary rate of change of the given function. The derivative also has geometric interpretation. The differential quotient in (8.1) is the tan of the angle of inclination φ of the secant to the graph of $f(x)$. This secant intersects the graph in two points $(x, f(x))$ and $(x + h, f(x + h))$. As $h \to 0$ the secant turns into the tangent, and thus geometrically the derivative is the tan of the angle of inclination of the tangent to the graph at a given point. The existence of the derivative means simply the existence of the tangent to the graph.

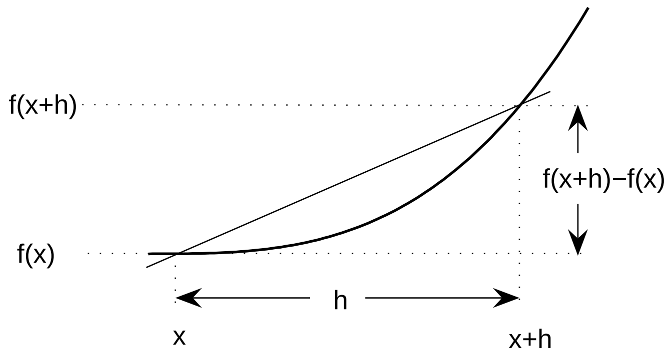

Figure 8.1: The differential quotient and the secant.

(iii) The derivative may fail to exist. For example, for the function $f(x) = |x|$ we have

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1, \quad \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

The differential quotient has different one-sided limits at zero, and thus $f(x)$ is not differentiable at 0. The geometric interpretation of this lack of differentiability at 0 is particularly suggestive: the graph of $f(x)$ at the point $(0,0)$ has an "angle", and thus has no tangent.

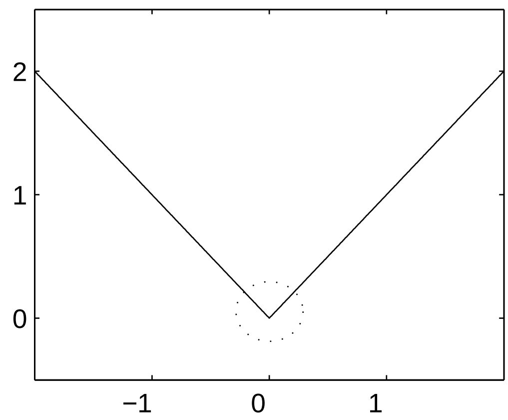

Figure 8.2: The graph of $f(x) = |x|$ and the non-differentiable "angle".

(iv) The derivative ot the function $f(x)$ is defined at the "internal" points of the domain, that is these points from $x \in D_f$, for which some interval $(x-\delta, x+\delta) \subset D_f$.

Theorem 8.2. *If the function $f(x)$ is differentiable at a point x_0 then it must be continuous at x_0.*

Proof. Let us observe that

$$\begin{aligned}
\lim_{x\to x_0} f(x) - f(x_0) &= \lim_{x\to x_0} (f(x) - f(x_0)) \\
&= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \cdot (f(x) - f(x_0)) \\
&= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \cdot \lim_{x\to x_0} (f(x) - f(x_0)) \\
&= f'(x_0) \cdot 0 = 0.
\end{aligned}$$

□

Theorem 8.3. *If functions $f(x)$ and $g(x)$ are both differentiable at a point x_0, then the following functions are also differentiable at x_0, and we have the formulas*

- $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$,
- $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$ *(the Leibniz rule),*
- $(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}$, *provided $g(x_0) \neq 0$.*

Proof. We will show the product and the fraction, while the sum and the difference are left to the enthusiastic reader. Let us start with the product.

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h} \\
&\quad = \lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0+h) + f(x_0)g(x_0+h) - f(x_0)g(x_0)}{h} \\
&\quad = \lim_{h\to 0} \left(\frac{(f(x_0+h) - f(x_0))g(x_0+h)}{h} + \frac{(g(x_0+h) - g(x_0))f(x_0)}{h} \right) \\
&\quad = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \cdot g(x_0+h) + \lim_{h\to 0} f(x_0) \cdot \frac{g(x_0+h) - g(x_0)}{h} \\
&\quad = f'(x_0)g(x_0) + f(x_0)g'(x_0),
\end{aligned}$$

provided the derivatives on the right both exist. Let us recall, that if $g(x)$ is differentiable at a point x_0 then it is also continuous at that point, and thus, a wic $g(x_0+h) \to g(x_0)$ as $h \to 0$. Now the quotient. If $g(x_0) \neq 0$ and $g(x)$ is continuous at x_0, then $g(x_0+h) \neq 0$ for sufficiently small h. Computing the limits when $h \to 0$ we can restrict ourselves to h from an arbitrarily small interval around 0.

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{f(x_0+h)}{g(x_0+h)} - \frac{f(x_0)}{g(x_0)}}{h} &= \lim_{h\to 0} \frac{f(x_0+h)g(x_0) - f(x_0)g(x_0+h)}{h\, g(x_0+h)\, g(x_0)} \\
&= \lim_{h\to 0} \frac{\frac{f(x_0+h)g(x_0) - f(x_0)g(x_0) - (f(x_0)g(x_0+h) - f(x_0)g(x_0))}{h}}{g(x_0+h)\, g(x_0)} \\
&= \frac{\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \cdot g(x_0) - f(x_0) \cdot \lim_{h\to 0} \frac{g(x_0+h)-g(x_0)}{h}}{g^2(x_0)} \\
&= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}.
\end{aligned}$$

□

Examples: (a) The constant function $f(x) = c$. $f'(x) = \lim_{h\to 0} \frac{c-c}{h} = 0$. The derivative of a constant function is a constant 0.

(b) $f(x) = x$. We have $f'(x) = \lim_{h\to 0} \frac{x+h-x}{h} = 1$.

(c) $f(x) = x^n$, for $n \in \mathbf{N}$. We have $f'(x) = nx^{n-1}$, (the Leibniz rule plus the induction, or alternately the binomial expansion).

(d) A polynomial of degree n: $f(x) = a_n x^n + \cdots + a_1 x + a_0$. $f'(x) = na_n^{n-1} + \cdots + a_1$, a polynomial of degree $n-1$.

(e) $f(x) = \sin x$. We have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{2\, \sin(\frac{1}{2}h)\, \cos(x + \frac{1}{2}\, h)}{h} \\
&= \lim_{h\to 0} \frac{\sin(\frac{1}{2}h)}{\frac{1}{2}h} \cdot \lim_{h\to 0} \cos(x + \frac{1}{2}\, h) \\
&= \cos x.
\end{aligned}$$

We have used a trigonometric identity

$$\sin(a+b) - \sin(a-b) = 2\, \sin b\, \cos a,$$

with $a = x + \frac{1}{2}h$ and $b = \frac{1}{2}h$. Similarly, in the next example we will use the identity

$$\cos(a+b) - \cos(a-b) = -2\, \sin a\, \sin b.$$

(f) $f(x) = \cos x$. Similarly as in (e)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\ &= \lim_{h\to 0} \frac{-2\,\sin(\frac{1}{2}h)\,\sin(x+\frac{1}{2}\,h)}{h} \\ &= -\lim_{h\to 0} \frac{\sin(\frac{1}{2}h)}{\frac{1}{2}h} \cdot \lim_{h\to 0} \sin(x+\frac{1}{2}\,h) \\ &= -\sin x. \end{aligned}$$

(g) $f(x) = \log x$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\log(x+h) - \log(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\,\log\left(\frac{x+h}{x}\right) \\ &= \lim_{h\to 0} \log\left(\frac{x+h}{x}\right)^{\frac{1}{h}}. \end{aligned}$$

As we know, the logarithm is a continuous function (Corollary 7.8), so we can "take the limit" under the logarithm. Let us see then, what is the limit under the logarithm.

$$\lim_{h\to 0}\left(\frac{x+h}{x}\right)^{\frac{1}{h}} = \lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h}} = \lim_{h\to 0}\left(\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}}\right)^{\frac{1}{x}}.$$

We also know, that we can "take the limit" under an arbitrary power. Let us observe, that as $h \to 0^+$ then $\frac{x}{h} \to +\infty$, and while $h \to 0^-$ then $\frac{x}{h} \to -\infty$ (we only consider $x \in D_{\log}$, that is $x > 0$). We remember that

$$\lim_{t\to\pm\infty}\left(1+\frac{1}{t}\right)^{t} = e, \quad \text{and thus} \quad \lim_{h\to 0}\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}} = e,$$

(both 1-sided limits are equal to e). Putting together the pieces we obtain

$$f'(x) = \log e^{\frac{1}{x}} = \frac{1}{x}\log e = \frac{1}{x}.$$

Theorem 8.4 (Differentiating the inverse function)**.** *Let $f : [a,b] \to \mathbf{R}$ be continuous and 1-1, and also differentiable at the point $x_0 \in (a,b)$, with $f'(x_0) \neq 0$. Let $g(y)$ be the function inverse to $f(x)$ $f(x)$, that is defined on* *the interval $[m, M]$, where m and M are the extremes of the set of values of $f(x)$. Let $y_0 = f(x_0)$. Then $g(y)$ is differentiable at the point y_0 and*

$$g'(y_0) = \frac{1}{f'(x_0)}.$$

Proof. Let us denote $k = f(x_0+h) - f(x_0)$ (k depends on h). So $g(y_0+k) = x_0 + h$. For $k \to 0$ we thus have $h \to 0$, since $g(y)$ is continuous. Therefore

$$\begin{aligned} g'(y_0) &= \lim_{k \to 0} \frac{g(y_0 + k) - g(y_0)}{k} \\ &= \lim_{k \to 0} \frac{h}{f(x_0 + h) - f(x_0)} \\ &= \frac{1}{\lim\limits_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}} \\ &= \frac{1}{f'(x_0)}. \end{aligned}$$

□

Corollary 8.5. *For $f(x) = e^x$ and $y_0 = e^{x_0}$ we have*

$$f'(x_0) = \frac{1}{\log'(y_0)} = \frac{1}{\frac{1}{y_0}} = y_0 = e^{x_0}.$$

The derivative of the function e^x is the same function e^x.

Function extrema

We say that a function $f(x)$ has at a point x_0 a maximum (or local maximum), if

$$f(x) \leq f(x_0),$$

for $x \in D_f$, in some neighborhood of x_0. Similarly, we say that it has at x_0 a minimum (local minimum), if

$$f(x) \geq f(x_0),$$

for $x \in D_f$, in some neighborhood of x_0. In general, we say that $f(x)$ has at x_0 an extremum, if it has at this point a maximum or a minimum.

Theorem 8.6. *If a function $f(x)$ has at the point x_0 an extremum, and is differentiable at that point, then*

$$f'(x_0) = 0.$$

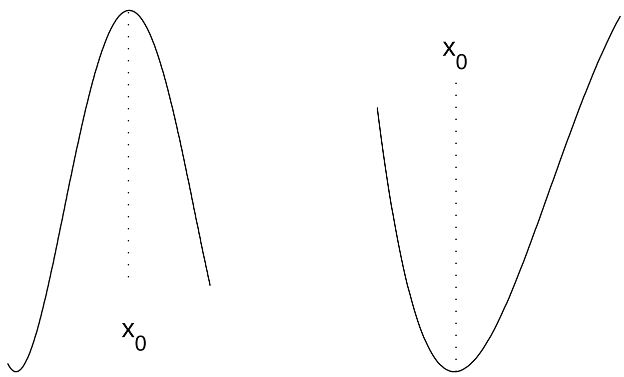

Figure 8.3: Local maximum and minimum.

Proof. Let us consider the case when $f(x)$ has at x_0 a maximum. The case of a minimum is similar. For h from some neighborhood of 0 we thus have

$$f(x_0 + h) \leq f(x_0),$$

that is for $h > 0$ we have $\frac{f(x_0+h)-f(x_0)}{h} \leq 0$, and for $h < 0$ we have $\frac{f(x_0+h)-f(x_0)}{h} \geq 0$. The right-hand limit of the differential quotient at zero thus cannot be positive, while the left-hand limit cannot be negative. Since they are equal, their common value is necessarily zero. □

Remarks: (i) If $f(x)$ has an extremum at the point x_0, then $f'(x_0) = 0$, but not the other way around. For example, the function $f(x) = x^3$ satisfies $f'(0) = 0$, but does not have an extremum at 0.

(ii) The above theorem is useful in finding the maximum and minimum values of a function. The maximal and minimal values of a function can be attained at a point where it is not differentiable (for example at an end-point of the interval where the function is defined), or otherwise at a point where the derivative is 0. This narrows down the search.

(iii) The theorem 8.6 is obvious geometrically. If a function has at a given point an extremum, then the tangent to the graph of this function (if it exists) is necessarily horizontal.

Theorem 8.7 (Rolle's theorem)**.** *Let $f(x)$ be continuous over the interval $[a, b]$ (including the endpoints), and differentiable in (a, b). Assume that $f(a) = f(b)$. Then there exists a $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. The function $f(x)$ attains its maximal and minimal values. If both are attained at the end-points a, b, then the function is a constant, and its derivative is a constant $f'(x) \equiv 0$ over the entire interval (a, b). Otherwise, one of the extrema, the minimum or the maximum, has to be attained at an inside point $c \in (a, b)$. By theorem [?] we must have $f'(c) = 0$. □

The following theorem is important from the point of view of both, the theory and the applications.

Theorem 8.8 (the mean value theorem)**.** *If the function $f(x)$ is continuous on $[a, b]$, and differentiable in (a, b), then there exists a point $c \in (a, b)$ such, that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof. Let us observe, that an auxiliary function

$$g(x) = f(x) - \left(f(a) + (x - a)\,\frac{f(b) - f(a)}{b - a} \right)$$

satisfies the assumptions of the Rolle's theorem: $g(a) = g(b) = 0$. Simply speaking, we have subtracted from $f(x)$ a linear function with the same value as $f(x)$ at a, and with the same increase over the interval $[a, b]$. It thus follows from the Rolle's theorem, that there exists a point $c \in (a, b)$ such, that $g'(c) = 0$. But $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, which finishes the proof. □

From the mean value theorem we immediately obtain the following corollary.

Corollary 8.9. *If, over an interval (a, b) we have*

- $f'(x) \geq 0$ *then the function $f(x)$ is increasing on (a, b),*
- $f'(x) \leq 0$ *then the function $f(x)$ is decreasing on (a, b),*
- $f'(x) = 0$ *then the function $f(x)$ is constant on (a, b),*

If the inequalities are sharp, then the monotonicity is strict.

Remarks: (i) Let us observe, that in the above corollary we assume an inequality on the entire interval. This is important since, for example the function $\frac{1}{x}$ has a derivative, which is strictly negative on its entire domain, while the function itself is not decreasing. It is decreasing on each of the intervals $(-\infty, 0)$ and $(0, \infty)$ which together make up the domain, but not decreasing on the entire domain.

(ii) It follows directly from the definition of the derivative, that if in some neighborhood of x_0 the function $f(x)$ is increasing, then $f'(x_0) \geq 0$ if it exists). Similarly for a decreasing function. We thus see, that monotonicity is closely related to the sign of the derivative.

We state the following theorem without proof.

Theorem 8.10 (Chain rule)**.** *Let functions $f(x)$ and $g(x)$ be differentiable, and let their composite $g \circ f$ be defined, that is the values of $f(x)$ fall into the domain of $g(x)$. Then the composite $(g \circ f)(x)$ is differentiable, and*

$$(g \circ f)'(x) = g'(f(x))\, f'(x).$$

It is not hard to see how to proceed with the proof:

$$\frac{g(f(x+h)) - g(f(x))}{h} = \frac{g(f(x+h)) - g(f(x))}{f(x+h) - f(x)} \cdot \frac{f(x+h) - f(x)}{h}.$$

We omit the technical details.

Corollary 8.11. *Let $f(x) = x^a$, $x > 0$, where a is an arbitrary real power. We then have*

$$x^a = e^{a \log x} \;\Rightarrow\; (x^a)' = e^{a \log x}(a \log x)' = x^a \cdot \frac{a}{x} = a\, x^{a-1}.$$

We have proved this formula earlier in the case of a natural power.

The following theorem is the so called de l'Hôspital's rule. It is a very simple theorem, which is surprisingly useful. We will use it repeatedly. It allows to find limits (if they exist) of expressions of the form

$$\lim_{x \to x_0} \frac{f(x)}{g(x)},$$

where both functions have limits 0. A limit of this kind is referred to, for obvious reasons, as an indeterminate expression of type $\frac{0}{0}$.

Theorem 8.12 (de l'Hôspital's rule)**.** *If functions $f(x)$ and $g(x)$ are differentiable in an interval (a, b), and satisfy*

$$\lim_{x \to a+} f(x) = \lim_{x \to a^+} g(x) = 0,$$

then if the limit

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

exists (proper or improper), then the limit

$$\lim_{x\to a^+} \frac{f(x)}{g(x)}$$

also exists, and both limits are equal:

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Proof. If we let $f(a) = g(a) = 0$, then both functions by assumption are continuous on $[a, b)$. Analogously as in the proof of the mean value theorem we will introduce an auxiliary function, and apply the Rolle's theorem. Let $h > 0$ such that $a + h < b$ be fixed, and for $x \in [a, a + h]$ let

$$\Phi(x) = f(x) - g(x)\,\frac{f(a+h)}{g(a+h)}.$$

We have $\Phi(a) = \Phi(a + h) = 0$, so there exists $c \in (a, a + h)$ such, that $\Phi'(c) = 0$. This means that

$$f'(c) - g'(c)\,\frac{f(a+h)}{g(a+h)} = 0 \;\Rightarrow\; \frac{f'(c)}{g'(c)} = \frac{f(a+h)}{g(a+h)}.$$

Observe also that when $h \to 0$ then also $c \to 0$. If the limit

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)}, \tag{8.2}$$

exists, then also the limit

$$\lim_{h\to 0^+} \frac{f'(c)}{g'(c)},$$

necessarily exists, and is equal to the limit (8.2). We thus have

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{h\to 0^+} \frac{f(a+h)}{g(a+h)} = \lim_{h\to 0+} \frac{f'(c)}{g'(c)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists. □

Remarks: (i) The theorem remains true if instead of right-hand limits at a we consider left-hand limits at b. The same thus follows for both-sided limits.

(ii) The de l'Hôspital's rule can be iterated. For example, let us consider the limit

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}.$$

Let us observe right away, that this is, indeed, and indeterminate expression of the type $\frac{0}{0}$. Differentiating the numerator and the denominator we again arrive at an indeterminate expression of the type $\frac{0}{0}$: $\frac{\cos x - 1}{3x^2}$. Differentiating the numerator and the denominator again we obtain $\frac{-\sin x}{6x}$, which is still $\frac{0}{0}$, but this time we happen to know this limit, it is equal to $-\frac{1}{6}$. We then go back, applying the de l'Hôspital's rule twice.

$$\lim_{x\to 0}\frac{-\sin x}{6x} = -\frac{1}{6} \ \Rightarrow\ \lim_{x\to 0}\frac{\cos x - 1}{3x^2} = -\frac{1}{6} \ \Rightarrow\ \lim_{x\to 0}\frac{\sin x - x}{x^3} = -\frac{1}{6}.$$

(iii) We can apply the de l'Hôspital's rule to limits of other "types", transforming them appropriately. For example, let us consider the limit

$$\lim_{x\to +\infty}\frac{f(x)}{g(x)},$$

where $f(x) \to 0$ and $g(x) \to 0$ as $x \to +\infty$. Let us introduce the notation

$$\varphi(t) = f\left(\frac{1}{t}\right) \quad \text{and} \quad \psi(t) = g\left(\frac{1}{t}\right),$$

then $\varphi(t) \to 0$ and $\psi(t) \to 0$ as $t \to 0^+$, and additionally

$$\varphi'(t) = f'\left(\frac{1}{t}\right)\cdot\left(\frac{-1}{t^2}\right), \quad \text{and} \quad \psi'(t) = g'\left(\frac{1}{t}\right)\cdot\left(\frac{-1}{t^2}\right).$$

We thus obtain

$$\frac{\varphi'(t)}{\psi'(t)} = \frac{f'(\frac{1}{t})(-\frac{1}{t^2})}{g'(\frac{1}{t})(-\frac{1}{t^2})} = \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})}.$$

It follows from the above, that

$$\lim_{x\to +\infty}\frac{f'(x)}{g'(x)}, \quad \text{and} \quad \lim_{t\to 0^+}\frac{\varphi'(t)}{\psi'(t)}$$

are the same limits, the existence of one is equivalent to the existence of the other, and they are both equal. We thus have

$$A = \lim_{x\to +\infty}\frac{f'(x)}{g'(x)} \ \Rightarrow\ A = \lim_{t\to 0^+}\frac{\varphi'(t)}{\psi'(t)}$$
$$\Rightarrow\ A = \lim_{t\to 0^+}\frac{\varphi(t)}{\psi(t)} \ \Rightarrow\ A = \lim_{x\to +\infty}\frac{f(x)}{g(x)}$$

(iv) One can also prove (in a roughly similar fashion) a version of the de l'Hôspital's rule for indeterminate expressions of the type $\frac{\infty}{\infty}$: if $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = \infty$ and $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = A$ (proper or improper) then also $\lim_{x\to a^+} \frac{f(x)}{g(x)} = A$. Similarly for $a = \infty$ (an indeterminate expression of the type $\frac{\infty}{\infty}$ at ∞).

Examples: (a) $\lim_{x\to 0} \frac{\log(1+x)}{x}$. It is clearly an indeterminate expression of the type $\frac{0}{0}$, so we have

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1.$$

(b) $\lim_{x\to 0^+} x \log x$. It is an expression of the "type" $0 \cdot \infty$, but moving x to the denominator we obtain an expression of the type $\frac{\infty}{\infty}$. We thus have

$$\lim_{x\to 0^+} x \log x = \lim_{x\to 0^+} \frac{\log x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = -\lim_{x\to 0^+} x = 0.$$

(c) $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x}}$. It is an expression of the "type" 1^∞. We transform it in the usual way

$$(\cos x)^{\frac{1}{x}} = e^{\frac{1}{x}\log\cos x} = e^{\frac{\log\cos x}{x}}.$$

In the exponent we have an expression of the type $\frac{0}{0}$, so we compute the limit in the exponent

$$\lim_{x\to 0^+} \frac{\log\cos x}{x} = \lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{1} = 0 \;\Rightarrow\; \lim_{x\to 0^+} (\cos x)^{\frac{1}{x}} = e^0 = 1.$$

(d) $\lim_{x\to +\infty} \frac{\log x}{\sqrt{x}}$. It is an expression of the type $\frac{\infty}{\infty}$ at ∞, so we have

$$\lim_{x\to +\infty} \frac{\log x}{\sqrt{x}} = \lim_{x\to +\infty} \frac{\frac{1}{x}}{\frac{1}{2}\frac{1}{\sqrt{x}}} = \lim_{x\to +\infty} \frac{2}{\sqrt{x}} = 0.$$

The logarithm increases to ∞ less rapidly than the square root.

Derivatives of the inverse trigonometric functions

(a) $f(x) = \arcsin x$. $f(x)$ is defined on the interval $[-1, 1]$, and is a function inverse to the function $\sin x$ restricted to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Let $x_0 \in (-1, 1)$

and $x_0 = \sin(t_0)$ for some $t_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. From Theorem 8.4 we know, that $f(x)$ is differentiable at x_0 and

$$f'(x_0) = \frac{1}{\sin' t_0} = \frac{1}{\cos t_0} = \frac{1}{\cos \arcsin x_0}.$$

The last expression can be simplified. For $t_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we have $\cos t_0 > 0$, and so $\cos t_0 = \sqrt{1 - \sin^2 t_0}$. We thus have

$$f'(x_0) = \frac{1}{\cos \arcsin x_0} = \frac{1}{\sqrt{1 - x_0^2}}.$$

(b) $f(x) = \arccos x$. It is a function defined on $[-1, 1]$, inverse to the function $\cos x$ restricted to the interval $[0, \pi]$. Let $x_0 \in (-1, 1)$, and $x_0 = \cos t_0$ for some $t_0 \in (0, \pi)$.

$$f'(x_0) = \frac{1}{\cos' t_0} = \frac{1}{-\sin t_0} = \frac{-1}{\sin \arccos x_0}.$$

Similarly as above, $\sin x$ is positive on $(0, \pi)$, so $\sin t_0 = \sqrt{1 - \cos^2 t_0}$, thus

$$f'(x_0) = \frac{-1}{\sin \arccos x_0} = \frac{-1}{\sqrt{1 - x_0^2}}.$$

(c) $f(x) = \arctan x$. Function $f(x)$ is defined on the entire real line $\mathbf{R}$, and is the inverse to the function $\tan x$ restricted to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Let $x_0 = \tan t_0$ for some $t_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. We have

$$f'(x_0) = \frac{1}{\tan' t_0} = \frac{1}{\frac{1}{\cos^2 t_0}} = \cos^2 t_0.$$

On the other hand

$$\cos^2 t_0 = \frac{\cos^2 t_0}{\cos^2 t_0 + \sin^2 t_0} = \frac{1}{1 + (\frac{\sin t_0}{\cos t_0})^2} = \frac{1}{1 + \tan^2 t_0} = \frac{1}{1 + x_0^2}.$$

We finally get

$$f'(x_0) = \frac{1}{1 + x_0^2}.$$

Derivatives of higher orders

If the derivative $f'(x)$ is itself differentiable, then we can compute its derivative, the so called second derivative, or derivative of the second order of the function $f(x)$

$$(f')'(x) = f''(x) = f^{(2)}(x).$$

Similarly, we can compute derivatives of arbitrary orders $f^{(n)}(x)$ (provided the function $f(x)$ is differentiable sufficiently many times). We write $f^{(0)}(x) = f(x)$.

Examples: (a)

$$\sin^{(n)}(x) = \begin{cases} (-1)^{\frac{n-1}{2}} \cos x & n - \text{odd}, \\ (-1)^{\frac{n}{2}} \sin x & n - \text{even}. \end{cases}$$

(b)

$$f(x) = \begin{cases} x^3 & x \geq 0, \\ 0 & x < 0. \end{cases}$$

The function $f(x)$ is differentiable at each point $x \neq 0$ and $f'(x) = 3x^2$ for $x > 0$ and $f'(x) = 0$ for $x < 0$. The function is also differentiable at zero, and $f'(0) = 0$:

$$\lim_{x \to 0^+} \frac{f(x) - 0}{x} = \lim_{x \to 0^+} \frac{x^3}{x} = 0, \quad \lim_{x \to 0^-} \frac{f(x) - 0}{x} = \lim_{x \to 0^-} \frac{0}{x} = 0.$$

$f(x)$ is thus differentiable everywhere, and

$$f'(x) = \begin{cases} 3x^2 & x \geq 0, \\ 0 & x \leq 0. \end{cases}$$

We now compute the derivative of $f'(x)$. For $x > 0$ $f''(x) = 6x$, and for $x < 0$ $f''(x) = 0$. At zero $f'(x)$ is again differentiable, and $f''(0) = 0$:

$$\lim_{x \to 0^+} \frac{f'(x) - 0}{x} = \lim_{x \to 0^+} \frac{3x^2}{x} = 0, \quad \lim_{x \to 0^-} \frac{f'(x) - 0}{x} = \lim_{x \to 0^-} \frac{0}{x} = 0.$$

$f(x)$ is thus differentiable everywhere twice, and

$$f''(x) = \begin{cases} 6x & x \geq 0, \\ 0 & x \leq 0. \end{cases}$$

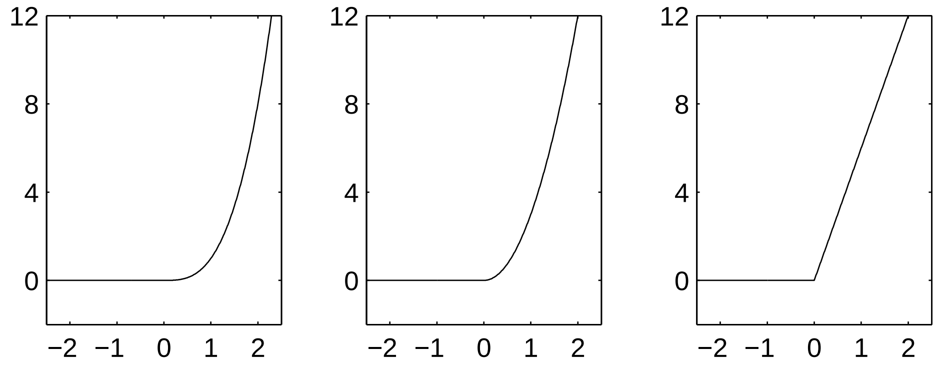

Figure 8.4: Functions $f(x)$, $f'(x)$ and $f''(x)$ from Example (b).

Observe, that $f''(x)$ is not differentiable at zero:

$$\lim_{x\to 0^+} \frac{f''(x) - 0}{x} = \lim_{x\to 0^+} \frac{6x}{x} = 6, \quad \lim_{x\to 0^-} \frac{f''(x) - 0}{x} = \lim_{x\to 0^-} \frac{0}{x} = 0.$$

$f(x)$ therefore has derivatives of order 2 everywhere, but not of order 3 at zero.

Remark: All elementary functions have derivatives of all orders at each point of their respective domains.

Examining functions' behavior

We will now discuss a procedure to examine a function's behavior. Such examination is a typical subject in applications.

Remark: The procedure we will discuss now assumes the function is sufficiently regular. There are functions whose graphs could hardly be even sketched. For example

$$f(x) = \begin{cases} 1 & x \in \mathbf{Q}, \\ 0 & x \notin \mathbf{Q}. \end{cases}$$

The functions we will consider will usually be at least piecewise continuous.

Attempting to analyse a function we usually proceed as follows.

(1) We establish the domain of the function, if it is not given. We establish points of continuity, discontinuity and differentiability. Usually the function is piecewise continuous and piecewise differentiable, that is its domain an be split into adjacent intervals in which it is continuous and differentiable. We

thus establish these intervals.

(2) We check for oddness and parity. If $f(x)$ is even, that is $f(-x) = f(x)$ or odd, that is $f(-x) = -f(x)$, then it is enough to analyse it for $x \geq 0$ and then transfer the results appropriately to $x < 0$. If the function is periodic, that is there exists a T such, that $f(x+T) = f(x)$, then it is enough to analyse the function over its period, for example the interval $[0, T]$.

(3) We find the roots of $f(x)$, that is points x_0 at which

$$f(x_0) = 0,$$

and we establish intervals over which the function preserves its sign.

(4) We find intervals of monotonicity and the local extrema. We test the sign of the derivative. This can give hints about the extrema. Sometimes we find useful the following theorem.

Theorem 8.13. *If $f'(x_0) = 0$ and $f''(x_0) \neq 0$ then $f(x)$ has an extremum at x_0. If $f''(x_0) < 0$ then it is a maximum, and if $f''(x_0) > 0$ then it is a minimum.*

Proof. If $f''(x_0) > 0$ then there exists a neighborhood of x_0 where the differential quotients are positive, that is there exists a $\delta > 0$ such that for $h \in (-\delta, \delta)$ we have

$$0 < \frac{f'(x_0+h) - f'(x_0)}{h} = \frac{f'(x_0+h)}{h}.$$

From this we see, that on the interval $(x_0 - \delta, x_0)$ $f'(x)$ is negative ($f(x)$ decreases), and on the interval $(x_0, x_0 + \delta)$ $f'(x)$ is positive ($f(x)$ increases). At x_0 the function therefore has a minimum. A similar analysis applies to the case $f''(x_0) < 0$ (maximum). □

One has to remember, that extrema can be located at points where the function is not differentiable. Such points require special attention.

(5) If the function $f(x)$ is twice differentiable, and in some interval we have $f''(x) > 0$, then we say that the function is convex in this interval. If on some interval $f''(x) < 0$ then we say that the function is concave. If at some point the function changes from convex to concave or the other way around, then we say that there it has an inflection point. Such point is an extremum of the first derivative. We try to localize the inflection points, and determine the intervals of convexity/concavity. The convexity and concavity have a clear geometric interpretation. If a function is convex on some interval, then the

tangents to its graph all lie below the graph, while the secants lie above the graph. If the function is concave, the opposite holds, the tangents are above, and the secants are below.

(6) We find eventual asymptotes. The asymptotes can be of different kinds.
(a) If at some point x_0 we have $\lim_{x\to x_0^\pm} f(x) = \pm\infty$, then the vertical line with an equation $x = x_0$ is called a vertical asymptote.
(b) If there exists a proper (finite) limit $\lim_{x\to\pm\infty} f(x) = A$, then the horizontal line $y = A$ is called a horizontal asymptote of $f(x)$ at $+\infty$ (or at $-\infty$).
(c) If there exists a constant m such that a proper limit $\lim_{x\to\pm\infty}(f(x) - mx) = c$ exists, then the line $y = mx+c$ is called an asymptote at $+\infty$ (or at $-\infty$). The horizontal asymptote is thus a particular kind of an asymptote, corresponding to $m = 0$. The asymptotes at $+\infty$ and $-\infty$ are independent, we check for asymptotes at both infinities, and at each an asymptote can exist or not. If at $+\infty$ or $-\infty$ the function $f(x)$ does have an asymptote, then the constant m is necessarily equal to either of the limits (which then must exist)

$$\lim_{x\to\pm\infty} \frac{f(x)}{x}, \quad \lim_{x\to\pm\infty} (f(x+1) - f(x)), \quad \lim_{x\to\pm\infty} f'(x),$$

(the last limit might not exist, even if the asymptote exists). One has to remember, that the existence of any of the above limits does not guarantee the existence of an asymptote. For the asymptote to exist the following limit has to exist too

$$\lim_{x\to\pm\infty} (f(x) - mx) = c.$$

Examples: (a) $f(x) = \frac{1}{x}$ has a vertical asymptote $x = 0$ and horizontal asymptotes $y = 0$ at both infinities (Fig. 8.5).

(b) $f(x) = \frac{x^3-2x^2+3}{2x^2}$. The function clearly has a vertical asymptote $x = 0$. We will look for other asymptotes.

$$\begin{aligned}
\frac{f(x)}{x} &= \frac{x^3 - 2x^2 + 3}{2x^3} = \frac{1}{2} - \frac{1}{x} + \frac{3}{2x^3} \xrightarrow{x\to\pm\infty} \frac{1}{2}, \\
f(x) - \frac{1}{2}\,x &= \frac{x^3 - 2x^2 + 3}{2x^2} - \frac{x}{2} = \frac{x^3 - 2x^2 + 3 - x^3}{2x^2} \\
&= \frac{-2x^2 + 3}{2x^2} = -1 + \frac{3}{2x^2} \xrightarrow{x\to\pm\infty} -1.
\end{aligned}$$

The function $f(x)$ thus has an asymptote $y = \frac{1}{2}x - 1$ at both infinities (Fig.8.6).

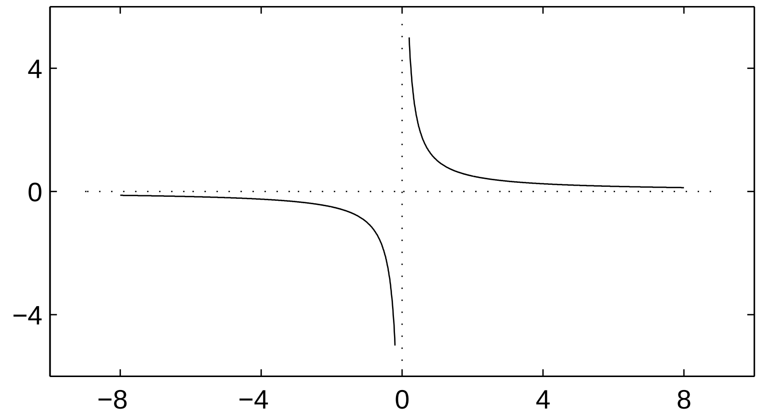

Figure 8.5: Asymptotes of the function $f(x) = \frac{1}{x}$.

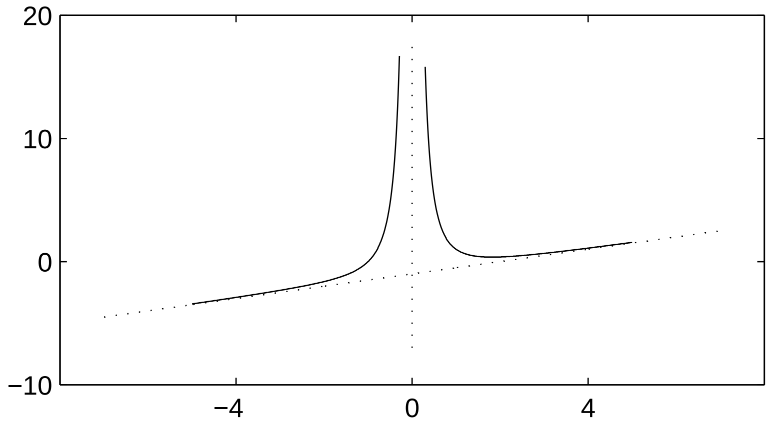

Figure 8.6: Asymptotes of the function from Example (b).

(c) Let us examine the function

$$f(x) = \frac{2\sqrt[3]{x^2}}{x+1}.$$

Its natural domain is $D_f = \mathbf{R} \setminus \{-1\}$, and the function is clearly continuous at each point of the domain, and differentiable at each point $x \neq 0$. The intervals of continuity are $(-\infty, -1)$ and $(-1, +\infty)$, while the intervals of differentiability are $(-\infty, -1)$, $(-1, 0)$ and $(0, +\infty)$. The function is neither odd nor even nor periodic. Its only root is the root of the numerator, that is $x_0 = 0$. $f(x)$ is positive for $x > -1,\ x \neq 0$ and negative for $x < -1$. Let us compute the derivative

$$
\begin{aligned}
f'(x) &= \frac{2\,\frac{2}{3}\,x^{-\frac{1}{3}}\,(1+x) - 2\,x^{\frac{2}{3}}}{(x+1)^2} \\
&= \frac{2\,x^{\frac{2}{3}}}{(x+1)^2} \cdot \left(\frac{2}{3}\,\frac{1+x}{x} - 1\right) \\
&= \frac{2\,x^{\frac{2}{3}}}{3\,(x+1)^2}\left(\frac{2}{x} - 1\right).
\end{aligned}
$$

The firs term is always positive, so the sign of the derivative is determined by the sign of $(\frac{2}{x}-1)$. After easy computations we conclude that the derivative is positive on the interval $(0,2)$ and negative on the intervals $(-\infty,-1)$, $(-1,0)$ and $(2,+\infty)$. The function $f(x)$ therefore increases on the interval $(0,2)$, and decreases on all of the other intervals. We thus see, that it has a minimum at zero (it is a point of non-differentiability), and a maximum at 2. We see, that it has a vertical asymptote $x=-1$, and a horizontal $y=0$ at $\pm\infty$. Let us now examine the convexity. Let us compute the second derivative.

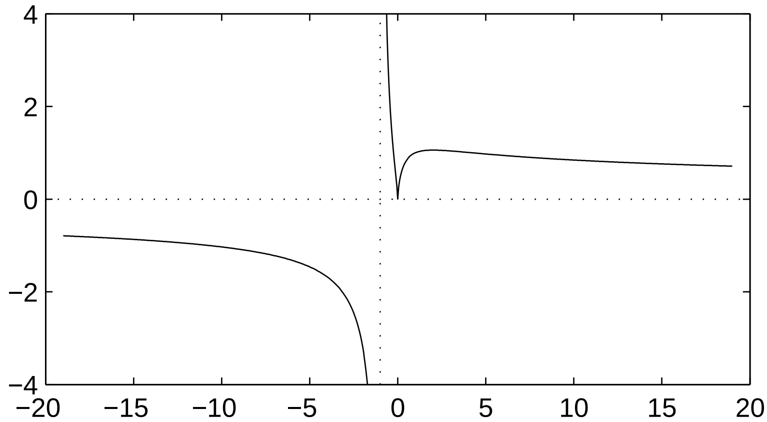

Figure 8.7: The graph of the function from Example (c).

$$
\begin{aligned}
f''(x) &= \left(\frac{\frac{4}{3}\,x^{-\frac{1}{3}}}{x+1} - \frac{2\,x^{\frac{2}{3}}}{(x+1)^2}\right)' \\
&= \frac{-\frac{4}{9}\,x^{-\frac{4}{3}}\,(x+1) - \frac{4}{3}\,x^{-\frac{1}{3}}}{(x+1)^2} - \frac{\frac{4}{3}\,x^{-\frac{1}{3}}\,(x+1)^2 - 2\,x^{\frac{2}{3}}\,2\,(x+1)}{(x+1)^4} \\
&= \frac{-\frac{4}{9}\,x^{-\frac{4}{3}}\,(x+1)^2 - \frac{8}{3}\,x^{-\frac{1}{3}}\,(x+1) + 4\,x^{\frac{2}{3}}}{(x+1)^3}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{4}{9}\, x^{-\frac{4}{3}}\, \frac{(x+1)^2 + 6\,x\,(x+1) - 9\,x^2}{(x+1)^3} \\
&= \frac{4}{9}\, x^{-\frac{4}{3}}\, \frac{2\,x^2 - 8\,x - 1}{(x+1)^3}.
\end{aligned}
$$

The expression $\frac{4}{9}x^{-\frac{4}{3}}$ is always positive, and the denominator is < 0 for $x < -1$ and > 0 for $x > -1$. On the other hand the numerator is positive for $x \notin (2 - \frac{3}{\sqrt{2}}, 2 + \frac{3}{\sqrt{2}})$ and negative for $x \in (2 - \frac{3}{\sqrt{2}}, 2 + \frac{3}{\sqrt{2}})$. Let us further observe, that $-1 < 2 - \frac{3}{\sqrt{2}} < 0$, and so $f(x)$ is:

- concave on $(-\infty, -1)$, $(2 - \frac{3}{\sqrt{2}}, 0)$ and $(0, 2 + \frac{3}{\sqrt{2}})$,
- convex on $(-1, 2 - \frac{3}{\sqrt{2}})$ and $(2 + \frac{3}{\sqrt{2}}, +\infty)$,
- has inflection points in $2 \pm \frac{3}{\sqrt{2}}$.

We know everything we wanted to know, and we may sketch the graph of the function (Fig. 8.7).

Proving inequalities

The methods of examining functions that we described above can be applied to proving inequalities.

Examples: (a) We will prove the inequality $(1+x)^p \geq 1 + px$ for $x > -1$. We have proved such an inequality earlier for a natural exponent p. Now we will prove it for an arbitrary $p \geq 1$. Let us consider the function

$$f(x) = (1+x)^p - 1 - px, \quad x \geq -1.$$

We have

$$f'(x) = p\,(1+x)^{p-1} - p.$$

For $x \geq 0$ $1 + x \geq 1$ and $p - 1 \geq 0$, so $(1+x)^{p-1} \geq 1$ that is $f'(x) \geq 0$. On the other hand for $x \leq 0$ we have $1 + x \leq 1$ so $(1+x)^{p-1} \leq 1$, and thus $f'(x) \leq 0$. The function $f(x)$ therefore decreases for $x < 0$ and increases for $x > 0$, and therefore attains its minimal value at zero

$$f(x) \geq f(0) = 0.$$

The function is thus always ≥ 0, that is

$$(1+x)^p \geq 1 + p\,x.$$

(b) For $x \geq 0$ we have $x - \frac{x^3}{6} \leq \sin x \leq x$. The right part of the inequalities is obvious, and was proved earlier. We will prove the left part. Let

$$f(x) = \sin x - x + \frac{x^3}{6}.$$

We have $f'(x) = \cos x - 1 + \frac{x^2}{2}$, $f'(0) = 0$, $f''(x) = -\sin x + x$. $f''(x) \geq 0$ for $x \geq 0$, so $f'(x)$ increases for $x \geq 0$, and since $f'(0) = 0$, then $f'(x) \geq 0$ for $x \geq 0$. The function $f(x)$ itself therefore increases for $x \geq 0$, and so

$$f(x) \geq f(0) = 0, \quad \text{for } x \geq 0.$$

Taylor's formula

The mean value theorem con be written in the form

$$f(b) = f(a) + (b-a)f'(c), \qquad \text{for some } c \in (a,b).$$

Introducing $h = b - a$, the formula becomes

$$f(a+h) = f(a) + h\, f'(a + \theta\, h), \qquad \text{for some } \theta \in (0,1).$$

The mean value theorem written in this form is a special case of the so called Taylor's formula.

Theorem 8.14 (Taylor's formula)**.** *Suppose the function $f(x)$ is n – times differentiable in an interval $(a-\delta, a+\delta)$ around a, for some $\delta > 0$. Then, for any h, $|h| < \delta$ there exists a $\theta \in (0,1)$ such, that*

$$\begin{aligned} f(a+h) &= f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n \\ &= \sum_{k=0}^{n-1} \frac{h^k}{k!} f^{(k)}(a) + R_n, \end{aligned}$$

where

$$R_n = \frac{h^n}{n!} f^{(n)}(a + \theta h).$$

Proof. Suppose h is given, and $|h| < \delta$. Let us denote $b = a + h$, and let us introduce auxiliary functions

$$\begin{aligned} \varphi(x) =& f(b) - f(x) - \frac{(b-x)}{1!}f'(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) \\ &= f(b) - \sum_{k=0}^{n-1} \frac{(b-x)^k}{k!} f^{(k)}(x), \end{aligned}$$

$$\Phi(x) = \varphi(x) - \frac{\varphi(a)}{(b-a)^n}\,(b-x)^n.$$

Notice, that $\Phi(x)$ satisfies the assumptions of the Rolle's theorem on the interval with endpoints a, b ($[a, b]$ or $[b, a]$ depending on the sign of h).

$$\Phi(a) = \varphi(a) - \frac{\varphi(a)}{(b-a)^n}\,(b-a)^n = 0, \quad \Phi(b) = \varphi(b) - 0 = 0.$$

We thus deduce that there exists a point c inside the interval with endpoints a, b, such that

$$\Phi'(c) = 0.$$

The point c can be written in the form $a + \theta\,(b-a) = a + \theta\,h$ for some $\theta \in (0, 1)$. We thus have

$$\Phi'(a+\theta\,h) = \varphi'(a+\theta\,h) - \frac{\varphi(a)}{(b-a)^n}\,n\,(h-\theta\,h)^{n-1}\,(-1) = 0. \tag{8.3}$$

We have to compute the derivative $\varphi'(x)$:

$$\begin{aligned}
\varphi'(x) &= -f'(x) - \left(\sum_{k=1}^{n-1}\frac{(b-x)^k}{k!}\,f^{(k)}(x)\right)' \\
&= -f'(x) - \sum_{k=1}^{n-1}\left(\frac{(b-x)^k}{k!}\,f^{(k)}(x)\right)' \\
&= -f'(x) - \sum_{k=1}^{n-1}\left(-\frac{(b-x)^{k-1}}{(k-1)!}\,f^{(k)}(x) + \frac{(b-x)^k}{k!}\,f^{(k+1)}(x)\right) \\
&= -f'(x) + \sum_{k=0}^{n-2}\frac{(b-x)^k}{k!}\,f^{(k+1)}(x) - \sum_{k=1}^{n-1}\frac{(b-x)^k}{k!}\,f^{(k+1)}(x) \\
&= -f'(x) + f'(x) - \frac{(b-x)^{(n-1)}}{(n-1)!}\,f^{(n)}(x) \\
&= -\frac{(b-x)^{(n-1)}}{(n-1)!}\,f^{(n)}(x).
\end{aligned}$$

Plugging this into (8.3) we obtain

$$-\frac{(h-\theta\,h)^{(n-1)}}{(n-1)!}\,f^{(n)}(a+\theta\,h) + \frac{\varphi(a)}{(b-a)^n}\,n\,(h-\theta\,h)^{n-1} = 0,$$

that is

$$\varphi(a) = \frac{(b-a)^n}{n!}\,f^{(n)}(a+\theta\,h) = \frac{h^n}{n!}\,f^{(n)}(a+\theta\,h).$$

It remains to observe that $\varphi(a)$ is precisely the remainder R_n:

$$
\begin{aligned}
f(b) &= \sum_{k=0}^{n-1} \frac{h^k}{k!}\, f^{(k)}(a) + \varphi(a) \\
&= f(a) + \frac{h}{1!}\, f'(a) + \frac{h^2}{2!}\, f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}\, f^{(n-1)}(a) + \frac{h^n}{n!}\, f^{(n)}(a + \theta\, h).
\end{aligned}
$$

□

Remarks: (i) R_n is the so-called remainder. The Taylor's formula can be viewed as an approximation of the function $f(x)$ by a polynomial, in the neighborhood of the point a (the so-called Taylor's polynomial at a), and then R_n is the error of this approximation.

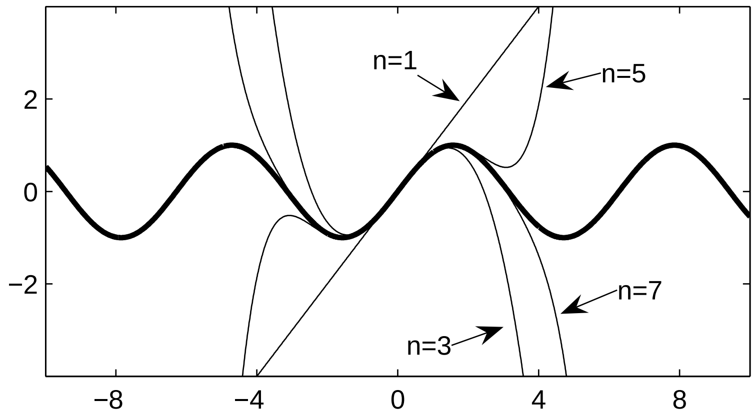

Figure 8.8: Taylor's polynomials of the function $\sin x$.

(ii) The accuracy of this approximation depends on the magnitude of R_n in the neighborhood of a (R_n of course depends on h). The more derivatives the function $f(x)$ has in the neighborhood of the point a (as we say, the more "smooth" $f(x)$ is near the point a) the more accurate the approximation is.

(iii) The reaminder R_n in the Taylor's formula can be expressed in various forms. The form given in the above theorem is the so-called Lagrange form. Various forms of the remainder come useful, when we want to estimate the error of the Taylor's approximation. Later we will see another simple proof of the above theorem, where the remainder will take the so-called integral form.

(iv) Let us observe, that if our function $f(x)$ has derivatives of all orders, and $\sup\{|f^{(n)}(x)|; x \in (a-\delta, a+\delta), n \in \mathbf{N}\}$ exists, then for any h, $|h| < \delta$ we have $R_n \to 0$ as $n \to \infty$, that is

$$f(a+h) = \sum_{k=0}^{\infty} \frac{h^k}{k!} f^{(k)}(a). \tag{8.4}$$

This is the so-called Taylor's series of the function $f(x)$ at a point a. Let us remember, that in general the Taylor's series need not converge, and even if it does converge, it might happen that

$$f(a+h) \neq \sum_{k=0}^{\infty} \frac{h^k}{k!} f^{(k)}(a).$$

On each occasion we need to verify the convergence of the remainderR_n to zero. It is only this convergence that gives the convergence of the Taylor's series, and the formula (8.4).

(v) If $a = 0$ we obtain the particular case of the Taylor's series, the so-called Maclaurin's series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0).$$

Examples: (a) $f(x) = \sin x$, $a = 0$. We know that

$$f^{(n)}(0) = \begin{cases} (-1)^{\frac{n-1}{2}} & n - \text{odd} \\ 0 & n - \text{even}. \end{cases}$$

We know also, that $|f^{(n)}(x)| \leq 1$ for all x, n. We thus have $R_n \to 0$ for any h, and we obtain the expansion of the function $\sin x$ into a Maclaurin's series

$$\begin{aligned} \sin x &= \sum_{\substack{n=0 \\ n\text{-odd.}}}^{\infty} (-1)^{\frac{n-1}{2}} \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots . \end{aligned}$$

(b) $f(x) = e^x$, $a = 0$. $f^{(n)}(x) = e^x$, therefore $f^{(n)}(0) = 1$. Let us observe, that if $|h| \leq M$ then $|f^{(n)}(\theta\, h)| \leq e^M$. The remainders converge to zero, and so

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

(c) $f(x) = \log(1+x)$, $a = 0$. Let us compute the derivatives

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = (-1)\,\frac{1}{(1+x)^2},$$
$$f'''(x) = 2\,\frac{1}{(1+x)^3}, \quad f^{(4)}(x) = (-1)\,2\cdot 3\,\frac{1}{(1+x)^4}.$$

We obtain

$$f^{(n)}(x) = (-1)^{n+1}\,\frac{(n-1)!}{(1+x)^n} \;\Rightarrow\; f^{(n)}(0) = (-1)^{n+1}\,(n-1)!.$$

We have to estimate the remainder

$$|R_n| \le \frac{|h|^n}{n!}\cdot\frac{(n-1)!}{(1-|h|)^n} = \frac{1}{n}\left(\frac{|h|}{1-|h|}\right)^n,$$

so for $|h| \le \frac{1}{2}$ we have

$$\frac{|h|}{1-|h|} \le 1 \Rightarrow |R_n| \to 0.$$

We thus have, for $|x| \le \frac{1}{2}$

$$\log(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots.$$

Remark: Using sharper estimates we may show, that the above formula remains true for $x \in (-1, 1]$.

Computing approximate values of functions

We will use the Taylor's formula for approximate computations
medskip
(a) We will compute the approximate value of e

$$e = \sum_{k=0}^{n-1}\frac{1}{k!} + \frac{e^{\theta}}{n!} \;\Rightarrow\; e \simeq \sum_{k=0}^{n-1}\frac{1}{k!} \quad \text{and the error } \le \frac{3}{n!}.$$

By the way: e is not rational. Let us suppose that it actually is rational, and $e = \frac{m}{n}$, for some $m, n \in \mathbf{N}$. Then

$$e = \frac{m}{n} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \frac{e^{\theta}}{(n+1)!}$$

$$\Rightarrow \left(\frac{m}{n} - 1 - 1 - \frac{1}{2!} - \cdots - \frac{1}{n!} \right) \cdot n! = \frac{e^\theta}{n+1}.$$

Let us observe, that the left hand side is an integer, that is $\frac{e^\theta}{n+1}$ also has to be an integer. But this is clearly impossible, since $1 < e^\theta < 3$, so we would have

$$\frac{1}{n+1} < \frac{e^\theta}{n+1} < \frac{3}{n+1}.$$

The only possibility is $n = 1$, so e would have to be a natural number, and it is easy to check that it is not (it is strictly between 2 and 3).

(b) We will compute an approximate value of $\sqrt[3]{9}$. Let $f(x) = x^{\frac{1}{3}}$, and observe that $f(9) = f(8+1)$, while $f(8) = 2$. We compute a few derivatives

$$f'(x) = \frac{1}{3}\, x^{-\frac{2}{3}}, \quad f''(x) = (-1)\, \frac{1}{3}\, \frac{2}{3}\, x^{-\frac{5}{3}},$$
$$f'''(x) = \frac{1}{3}\, \frac{2}{3}\, \frac{5}{3}\, x^{-\frac{8}{3}}, \quad f^{(4)}(x) = (-1) \frac{1}{3}\, \frac{2}{3}\, \frac{5}{3}\, \frac{8}{3}\, x^{-\frac{11}{3}}.$$

It is easy to notice that

$$f^{(n)}(x) = (-1)^{n+1}\, \frac{1}{3}\, \frac{2}{3}\, \frac{5}{3} \cdots \frac{3n-4}{3}\, x^{-\frac{3n-1}{3}}.$$

In that case

$$f^{(n)}(8) = (-1)^{n+1} \frac{2 \cdot 5 \ldots (3n-4)}{3^n}\, 8^{-\frac{3n-1}{3}} = (-1)^{n+1} \frac{2 \cdot 5 \ldots (3n-4)}{3^n\, 8^n}\, 2.$$

Plugging this into Taylor's formula, with $n = 3$ we obtain

$$\begin{aligned} \sqrt[3]{9} = f(8+1) &= f(8) + f'(8) + \frac{f''(8)}{2} + R_3 \\ &= 2 + \frac{1}{12} - \frac{2}{2 \cdot 3 \cdot 3 \cdot 32} + R_3 \\ &= 2 + \frac{1}{12} - \frac{1}{288} + R_3. \end{aligned}$$

We estimate the error of the approximation

$$\begin{aligned} |R_3| \leq \frac{2 \cdot 5}{3! 3 \cdot 3 \cdot 3}\, \frac{1}{(8+\theta)^{\frac{8}{3}}} &< \frac{10}{162}\, \frac{1}{8^{\frac{8}{3}}} = \frac{10}{162 \cdot 256} \\ &= \frac{10}{41472} < \frac{10}{40000} = \frac{1}{4000} = 0,00025. \end{aligned}$$

Chapter 9

Integrals

The antiderivative

Definition 9.1. *A function $F(x)$ is called an antiderivative of the function $f(x)$ (or a primitive of $f(x)$), if $F(x)$ is differentiable and $F'(x) = f(x)$ for each $x \in D_f$.*

Remark: (i) A function $f(x)$ might not have an antiderivative. If it does, then it has infinitely many of them:

$$F'(x) = f(x) \;\Rightarrow\; (F(x) + c)' = F'(x) = f(x).$$

In other words, if $F(x)$ is an antiderivative of the function $f(x)$, then for any constant c the function $F(x) + c$ is also an antiderivative of $f(x)$.

(ii) If both $F(x)$ and $G(x)$ are antiderivatives of the same function $f(x)$, then $(F - G)'(x) = F'(x) - G'(x) = 0$, for all $x \in D_f$. Thus on each interval contained in D_f the antiderivatives $F(x)$ and $G(x)$ differ by a constant. This constant can, however, be different on different intervals. If the domain of the function $f(x)$ is comprised of only one interval (for example, the entire real line)m then any two antiderivatives of $f(x)$ differ by a constant.

Indefinite integral

Definition 9.2. *If a function $f(x)$ has an antiderivative, then we say that it is integrable. Any of the antiderivatives of an integrable function $f(x)$ is called its indefinite integral, and is denoted*

$$\int f(x)\, dx.$$

The term "indefinite integral of $f(x)$" refers to a whole family of functions, which on different (separated) intervals of D_f differ by a constant. We often stress that, by adding a constant c to any formula for the indefinite integral that we obtain. In the notation for the integral $\int \ldots dx$ is one whole inseparable symbol, which always goes together. The part dx of the symbol underscores the variable with respect to which the antiderivative has been computed. Sometimes, in the case when the formula for the "integrated" function contains a fraction the symbol dx is placed in the numerator, for example

$$\int \frac{1}{x}\,dx = \int \frac{dx}{x}.$$

Examples:

$$(a) \qquad \int 0\,dx = c,$$

$$(b) \qquad \int a\,dx = ax + c, \quad \text{for any constant } a,$$

$$(c) \qquad \int x^a\,dx = \frac{1}{a+1}\,x^{a+1} + c \quad a \neq -1,\ x > 0,$$

$$(d) \qquad \int \cos x\,dx = \sin x + c,$$

$$(e) \qquad \int \sin x\,dx = -\cos x + c,$$

$$(f) \qquad \int \frac{dx}{\cos^2 x} = \tan x + c,$$

(the constant c can be taken different on different intervals),

$$(g) \qquad \int \frac{dx}{x} = \log|x| + c, \quad \text{(similar comment about the constant)},$$

$$(h) \qquad \int e^x\,dx = e^x + c.$$

The proof of any of the above formulas comes down to computing the derivative of the right hand side, and comparing it to the integrated function. The constants c introduced on the right hand sides are not very important (we know, that adding a constant does not change the derivative), but it is good to remember about them. Also, it is good to remember, that if the domain D_f is comprised of more than one connected intervals (like, for example, for $\frac{1}{x}$ we have $D_{1/x} = (-\infty, 0) \cup (0, \infty)$) the notation $+c$ in the formula for the integral is understood as that the constant can be different of any of the separate intervals of the domain.

From the formulas for derivatives we obtain the following formulas for the indefinite integrals:

$$\text{(a)} \qquad \int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx,$$

$$\text{(b)} \qquad \int a\, f(x)\, dx = a \int f(x)\, dx, \quad a \text{ - arbitrary constant}$$

$$\text{(c)} \qquad \int f'(x) g(x)\, dx = f(x)g(x) - \int f(x) g'(x)\, dx,$$

(the so-called formula for integrating by parts)

$$\text{(d)} \qquad \int (g \circ f)(x) f'(x)\, dx = \int g(y)\, dy \quad \text{where} \quad y = f(x),$$

(the so-called formula for integration by substitution).

Examples: (a) The integral of a polynomial is also a polynomial, of degree higher by 1:

$$\int (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)\, dx =$$
$$= \frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \cdots + \frac{a_1}{2} x^2 + a_0 x + c.$$

(b) We will apply the formula for inegration by parts:

$$\begin{aligned} \int \log x\, dx &= \int (x)' \log x\, dx \\ &= x \log x - \int x\, (\log x)'\, dx \\ &= x \log x - \int x \cdot \frac{1}{x}\, dx \\ &= x \log x - \int 1 \cdot dx \\ &= x \log x - x + c. \end{aligned}$$

Let us verify: $(x \log x - x + c)' = \log x + x \cdot \frac{1}{x} - 1 = \log x$.

(c) We use the formula for integration by substitution:

$$\int \frac{x}{1+x^2}\, dx = \frac{1}{2} \int \frac{1}{1+x^2} \cdot 2x\, dx \quad \text{let } f(x) = 1 + x^2$$

$$
\begin{aligned}
&= \frac{1}{2} \int \frac{1}{f(x)}\, f'(x)\, dx \\
&= \frac{1}{2} \int \frac{1}{y}\, dy \qquad f(x) = y \\
&= \frac{1}{2} \log|y| + c \\
&= \frac{1}{2} \log|1 + x^2| + c \\
&= \log\sqrt{1 + x^2} + c.
\end{aligned}
$$

Let us observe, that $(\log|x|)' = \frac{1}{x}$. For $x > 0$ this is clear, and for $x < 0$ we have $|x| = -x$, so

$$
(\log|x|)' = (\log(-x))' = \frac{1}{(-x)} \cdot (-x)' = \frac{-1}{-x} = \frac{1}{x}.
$$

In our example $1 + x^2 > 0$, so the absolute value does not change anything. Let us verify: $(\log\sqrt{1 + x^2})' = \frac{1}{\sqrt{1+x^2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1+x^2}} \cdot 2\,x = \frac{x}{1+x^2}$, so everything agrees.

(d) Once more integration by substitution:

$$
\begin{aligned}
\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx \\
&= -\int \frac{-\sin x}{\cos x}\, dx \\
&= -\int \frac{1}{f(x)}\, f'(x)\, dx \qquad f(x) = \cos x \\
&= -\int \frac{1}{y}\, dy \qquad y = \cos x \\
&= -\log|y| + c \\
&= -\log|\cos x| + c.
\end{aligned}
$$

(e) The following integral is immediate, if we remember the derivatives of the inverse trigonometric functions:

$$
\int \frac{1}{\sqrt{1 - x^2}}\, dx = \arcsin x + c, \qquad |x| < 1.
$$

If we happen not to remember these important formulas, we can still compute the integral, using substitution:

$$
\int \frac{1}{\sqrt{1 - x^2}}\, dx = \int \frac{1}{\sqrt{1 - f(t)^2}}\, f'(t)\, dt, \quad x = f(t)
$$

Figure 9.1: The graph of the function $f(x) = -\log|\cos x|$.

$$\begin{aligned}
&= \int \frac{1}{\sqrt{1-\sin^2 t}} \cos t \, dt, \quad f(t) = \sin t, \quad t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\
&= \int \frac{1}{\cos t} \cos t \, dt \\
&= \int 1 \, dt \\
&= t + c
\end{aligned}$$

We have used the fact that for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we have $\cos t > 0$ and so $\sqrt{1-\sin^2 t} = \cos t$. Since $x = \sin t$, then $t = \arcsin x$, and we obtain the same formula as before.

(f) We will compute the integral of the function $\sin^2 x$ in two ways. We can apply the trigonometric identity

$$\cos 2x = 1 - 2\sin^2 x \;\Rightarrow\; \sin^2 x = \frac{1-\cos 2x}{2}.$$

Then we obtain

$$\begin{aligned}
\int \sin^2 x \, dx &= \frac{1}{2} \int (1 - \cos 2x) \, dx \\
&= \frac{1}{2} \left(x - \frac{\sin 2x}{2}\right) + c \\
&= \frac{x}{2} - \frac{\sin 2x}{4} + c.
\end{aligned}$$

Alternately, we can apply integration by parts. Integrating by parts we do not obtain an integral which is easier to compute, but rather an equation for the integral we want to compute, which can be then solved.

$$\begin{aligned}
\int \sin^2 x\,dx &= \int \sin x \cdot \sin x\,dx \\
&= \int \sin x \cdot (-\cos x)'\,dx \\
&= -\sin x \cos x + \int (\sin x)' \cos x\,dx \\
&= -\sin x \cos x + \int \cos x \cdot \cos x\,dx \\
&= -\sin x \cos x + \int \cos^2 x\,dx \\
&= -\sin x \cos x + \int (1 - \sin^2 x)\,dx \\
&= -\sin x \cos x + x - \int \sin^2 x\,dx.
\end{aligned}$$

What we have obtained is an equation for our integral. Shifting the integral from the right hand side to the left, and dividing by 2 we obtain

$$\int \sin^2 x\,dx = \frac{-\sin x \cos x + x}{2}.$$

Integrability of functions

Integrating rational functions

Recall, that rational functions are functions of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials. Simple fractions are a special kind of rational functions, having the form

$$\frac{A}{(x-a)^n}, \qquad \frac{Bx+C}{(x^2+px+q)^n}, \qquad n = 1, 2, \ldots, \tag{9.1}$$

where A, B, C, a, p and q are arbitrary constants, and the quadratic expression $x^2 + px + q$ has no real roots, that is $p^2 - 4q < 0$. It turns out that any rational function can be decomposed as a sum of simple fractions, plus, eventually, a polynomial. On the other hand there are formulas for indefinite integrals of simple fractions. These two facts together give us a procedure to compute indefinite integrals of arbitrary rational functions.

Theorem 9.3. *Any rational function can be decomposed as a sum of simple fractions and a polynomial.*

The decomposition procedure: Rather than the technical proof we will sketch a procedure to decompose a function. This sketch can be refined to become the rigorous proof, but we leave that to the reader. Given a particular function $f(x) = \frac{P(x)}{Q(x)}$ we first divide polynomials $P(x)$ by $Q(x)$ "with the remainder", that is we find polynomials $W(x)$ (the quotient) and $R(x)$ (the remainder) such that

$$P(x) = W(x) \cdot Q(x) + R(x), \Rightarrow \frac{P(x)}{Q(x)} = W(x) + \frac{R(x)}{Q(x)},$$

and the degree of the remainder is less then the degree of $Q(x)$. We perform the division using the usual procedure of "long division", exactly the same way as we divide natural numbers.

Examples: (a) $\frac{x^3-2x^2-1}{x^2-1} = (x-2) + \frac{x-3}{x^2-1}$

(b) $\frac{x^4-2x^3-35}{x^3-2x^2+3x-6} = x + \frac{-3x^2+6x-35}{x^3-2x^2+3x-6}$.

After dividing out the polynomial part (which we know how to integrate), we are left with a fraction $\frac{R(x)}{Q(x)}$, in which the numerator has the degree lower that the denominator. In the next step we factorize the denominator, that is we decompose it as a product of polynomials which cannot be decomposed any further. The indecomposable polynomials are linear terms $(x-a)$, and quadratic terms (x^2+px+q), which have no real roots, that is $p^2-4q<0$. Let us recall, that in the case of polynomials in which we allow complex coefficients the only indecomposable terms are linear. Each polynomial of degree higher than 1 can be further factorized. In the case of polynomials with only real roots there can happen indecomposable factors (by the Bezout's theorem these can have no real roots) but it turns out that such indecomposable terms may have degree at most 2. In our procedure we now express the denominator $Q(x)$ as a product of expressions of the form

$$(x-a)^n \quad \text{and} \quad (x^2+px+q)^n. \tag{9.2}$$

The factorization of the denominator into indecomposable terms is, in practice, the main problem in the proedure of integrating rational functions. We know that such factorization exists, but in general we have no one way to find it. In the we will deal with either the factorization will be more or less obvious, or it will be provided. In our examples the factorization is rather simple: $x^2-1 = (x-1)(x+1)$ and $x^3-2x^2+3x-6 = (x-2)(x^2+3)$. If the polynomial has integer coefficients, and the coefficient of the leading power is 1 then first of all we look for its roots among the integer divisors of the free term. If we happen to hit upon a root, we divide the polynomial by the appropriate linear term, and we are left with a polynomial of a lower degree to factorize. We then repeat our attempts on finding a root. If the polynomial has no roots we try something else. For example let us consider the polynomial $Q(x) = x^4 + 1$, which clearly has no roots, so there is no linear factor to look for. We know this polynomial decomposes into a product of quadratic terms. The only thing to do is to find the coefficients of these quadratic terms. Start with two arbitrary quadratic polynomials with leading coefficients equal to 1 (we start with polynomial to fatorize with leading coefficient 1). We write such decomposition, with so far unknown coefficients, and multiply everything out:

$$x^4+1 = (x^2+ax+b)(x^2+cx+d) = x^4+(a+c)x^3+(b+d+ac)x^2+(ad+bc)x+bd.$$

Comparing terms on both sides we obtain a system of equations with a number of unknowns, which we will try to solve. In our example the solution is simple:

$$x^4 + 1 = (x^2 - \sqrt{2}x + 1) \cdot (x^2 + \sqrt{2}x + 1).$$

Once we have factored the denominator into undecomposable factors of the form (9.2) we can write a prototype of the decomposition of our rational function into simple fractions. As a first step we write out all simple fractions of the form (9.1) that will need to appear. For each factor of the form $(x-a)^n$ in our factorization we write out n simple fractions

$$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n},$$

and for each factor of the form $(x^2 + px + q)^n$ in the factorization of the determinant we write n fractions

$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Let us observe, that in writing out the necessary simple fractions we have introduced exactly as many unknowns A_i, B_i and C_i (to be determined shortly) as the degree of the original denominator. Now let us illustrate the just described step on our examples:

(a):

$$\frac{x-3}{x^2-1} = \frac{x-3}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1},$$

(b):

$$\frac{-3x^2 + 6x - 35}{x^3 - 2x^2 + 3x - 6} = \frac{-3x^2 + 6x - 35}{(x-2)(x^2+3)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 + 3}.$$

As the second step we determine all the constants in the numerators of the introduced simple fractions. To do that, we bring the sum of simple fractions we have obtained to a common denominator. It is easy to observe, that this common denominator is exactly the original denominator we have factorized. After bringing all simple fractions to the common denominator and combining them we compare the resulting numerator with the original numerator of the rational function. Both are polynomials of degree at most one less than the degree of the original denominator $Q(x)$. Sine both numerators have to agree, the coefficients of like powers of x must be equal. This gives us a number of equations, which is exactly the same as the number of unknowns. It turns out this system of equations can be solved. We will not prove this, but let us see how it works on our examples.

(a):

$$\frac{x-3}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{(A+B)x + (A-B)}{(x-1)(x+1)},$$

thus we must have $A+B=1$ and $A-B=-3$, which gives $A=-1$ and $B=2$, and so finally

$$\frac{x-3}{x^2-1} = \frac{-1}{x-1} + \frac{2}{x+1}.$$

(b):

$$\begin{aligned}\frac{-3x^2+6x-35}{x^3-2x^2+3x-6} &= \frac{A}{x-2} + \frac{Bx+C}{x^2+3} \\ &= \frac{A(x^2+3) + (Bx+C)(x-2)}{(x-2)(x^2+3)} \\ &= \frac{(A+B)x^2 + (-2B+C)x + (3A-2C)}{(x-2)(x^2+3)},\end{aligned}$$

thus $A+B=-3$, $-2B+C=6$ and $3A-2C=-35$. Solving this we get $A=-5$, $B=2$ and $C=10$, so finally

$$\frac{-3x^2+6x-35}{x^3-2x^2+3x-6} = \frac{-5}{x-2} + \frac{2x+10}{x^2+3}.$$

After we have determined all unknowns our decomposition of a rational function into a sum of simple fractions is complete. The procedure we have described is almost a proof of Theorem [**?**]. The only point which has not been made precise is that the final system of equations with unknowns indeed must have a solution. We leave this justification to the interested reader.

Integrating simple fraction

So far we have described how a rational function can be split into a sum of a polynomial and simple fractions. We know how to integrate a polynomial, and now we will show how to integrate simple fractions. The first type can be integrated easily. We have the following formulas:

$$\int \frac{dx}{x-a} = \log|x-a| + c,$$
$$\int \frac{dx}{(x-a)^n} = \frac{-1}{n-1} \cdot \frac{1}{(x-a)^{n-1}} + c, \quad n > 1.$$

A simple fraction of the second type can be written as a sum:

$$\frac{Bx+C}{(x^2+px+q)^n} = \frac{B}{2} \cdot \frac{2x+p}{(x^2+px+q)^n} + \frac{D}{(x^2+px+q)^n}, \qquad D = C - \frac{1}{2}\, B\, p. \tag{9.3}$$

The first of the fractions on the right hand side can be integrated using the substitution $t = x^2 + px + q$,

$$\int \frac{2x+p}{(x^2+px+q)^n}\, dx = \int \frac{dt}{t^n} = \begin{cases} \log(x^2+px+q) + c & : n = 1, \\ \dfrac{-1}{(n-1)(x^2+px+q)^{n-1}} + c & : n > 1. \end{cases}$$

Let us observe that since the polynomial x^2+px+q has no real roots, thus it is always positive, so the absolute value under the logarithm is not necessary. We are left with one final fraction to integrate, that is the second fraction on the right hand side of (9.3). We perform asimple transformation and then substitution:

$$\int \frac{dx}{(x^2+px+q)^n} = \int \frac{dx}{\Big((x+\frac{p}{2})^2 + (q - \frac{p^2}{4})\Big)^n} = \frac{\sqrt{a}}{a^n} \int \frac{dt}{(t^2+1)^n},$$

where

$$t = \frac{x + \frac{p}{2}}{\sqrt{a}}, \quad a = q - \frac{p^2}{4} > 0.$$

If $n = 1$ we have

$$\int \frac{dt}{t^2+1} = \arctan t + c,$$

while if $n > 1$ we have a recurrence relation. Let $k > 0$, then integrating by parts we get

$$\int \frac{dt}{(t^2+1)^k} = (t^2+1)^{-k} \cdot t'\, dt$$

$$
\begin{aligned}
&= \int \frac{t}{(t^2+1)^k} - \int (-k)\,\frac{2t}{(t^2+1)^{k+1}} \cdot t\,dt \\
&= \frac{t}{(t^2+1)^k} + 2\,k \int \frac{t^2}{(t^2+1)^{k+1}}\,dt \\
&= \frac{t}{(t^2+1)^k} + 2\,k \int \left(\frac{t^2+1}{(t^2+1)^{k+1}} - \frac{1}{(t^2+1)^{k+1}} \right) dt \\
&= \frac{t}{(t^2+1)^k} + 2\,k \int \frac{dt}{(t^2+1)^k} - 2\,k \int \frac{dt}{(t^2+1)^{k+1}}.
\end{aligned}
$$

We thus have

$$
2\,k \int \frac{dt}{(t^2+1)^{k+1}} = \frac{t}{(t^2+1)^k} + (2\,k-1) \int \frac{dt}{(t^2+1)^k},
$$

so, for $n > 1$

$$
\int \frac{dt}{(t^2+1)^n} = \frac{t}{2\,(n-1)\,(t^2+1)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{dt}{(t^2+1)^{n-1}}.
$$

We are now ready to integrate functions from both of our examples.

Examples: (a):

$$
\begin{aligned}
\int \frac{x^3-2x^2-1}{x^2-1}\,dx &= \int \left((x-2) + \frac{2}{x+1} - \frac{1}{x-1} \right) dx \\
&= \frac{x^2}{2} - 2x + 2\log|x+1| - \log|x-1| + c.
\end{aligned}
$$

(b):

$$
\begin{aligned}
\int \frac{x^4-2x^3-35}{x^3-2x^2+3x-6}\,dx &= \int \left(x - \frac{5}{x-2} + \frac{2x+10}{x^2+3} \right) dx \\
&= \frac{x^2}{2} - 5\log|x-2| + \int \frac{2x}{x^2+3}\,dx + \frac{10}{3} \int \frac{dx}{(\frac{x}{\sqrt{3}})^2+1} \\
&= \frac{x^2}{2} - 5\log|x-2| + \log(x^2+3) + \frac{10}{\sqrt{3}} \int \frac{\frac{1}{\sqrt{3}}\,dx}{(\frac{x}{\sqrt{3}})^2+1} \\
&= \frac{x^2}{2} - 5\log|x-2| + \log(x^2+3) + \frac{10}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + c.
\end{aligned}
$$

Chapter 10

Definite integral

The definite integral, intuitively, measures the "size" of the function, in a sense similarly to the way in which the surface area measures the size of a region in the plane. Let a function $f(x)$ be given, non-negative on the interval $[a, b]$ and let us consider the region under the graph of $f(x)$. We will try to measure the surface area of this region. We will use properties of surface area, which are intuitively clear, for example the property that a larger region has a greater surface area. Let $f(x) = x$ and we consider the region bounded from above by the graph of $f(x)$, over the interval $[0, a]$. The region is a triangle with both height and base equal to a. The surface area of such triangle is equal to $P = \frac{1}{2} a^2$. Let us now consider the region under the graph of $f(x) = x^2$, over the same interval $[0, a]$. We will build two polygons, one contained inside our region, and the other containing the region.

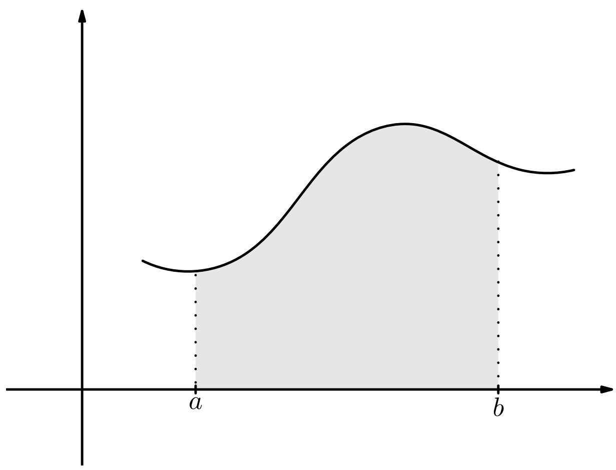

Figure 10.1: Region under a graph of a function.

The surface area of the region under the graph must be a number between the areas of the smaller and larger polygons. Thus to compute the surface area of the region under the graph we need to construct pairs of polygons which fit the region more and more tightly. We now describe this this construction. Let $n \in \mathbf{N}$ and let us divide the interval $[0, a]$ into n subintervals of equal length:

$$[0, a] = [0, \frac{a}{n}] \cup [\frac{a}{n}, 2\,\frac{a}{n}] \cup \cdots \cup [(n-2)\,\frac{a}{n}, (n-1)\,\frac{a}{n}] \cup [(n-1)\frac{a}{n}, a].$$

Over each of the subintervals of the partition $[k\,\frac{a}{n}, (k+1)\,\frac{a}{n}]$ we construct two rectangles, a smaller one with height $f(\frac{k\,a}{n})$ and a larger one with the height $f(\frac{(k+1)\,a}{n})$. Our function $f(x)$ is increasing, so indeed the second rectangle is larger. Let L_n be the combined surface area of all smaller rectangles, and U_n the combined surface area of all larger rectangles.

$$L_n = \sum_{k=0}^{n-1} f\left(\frac{k\,a}{n}\right)\frac{a}{n} = \sum_{k=0}^{n-1}\left(\frac{k\,a}{n}\right)^2\frac{a}{n} = \sum_{k=1}^{n-1}\left(\frac{k\,a}{n}\right)^2\frac{a}{n},$$
$$U_n = \sum_{k=0}^{n-1} f\left(\frac{(k+1)\,a}{n}\right)\frac{a}{n} = \sum_{k=0}^{n-1}\left(\frac{(k+1)\,a}{n}\right)^2\frac{a}{n} = \sum_{k=1}^{n}\left(\frac{k\,a}{n}\right)^2\frac{a}{n}.$$

Each smaller rectangle is completely included within the region under the graph, and thus the polygon, which is made up of all of them is included in the region under the graph. On the other hand the polygon made up of the larger rectangles clearly contains the region under the graph. The surface area of the smaller polygon is L_n, and the surface area of the larger polygon is U_n. If we denote the surface area of the region under the graph by A, we must then have

$$L_n \leq A \leq U_n,$$

for every $n \in \mathbf{N}$. Let us observe, that L_n and U_n have a common limit as $n \to \infty$. To show this we use the formula

$$1^2 + 2^2 + 3^2 + \cdots + m^2 = \frac{m(m+1)(2m+1)}{6},$$

which can be proved inductively.We thus have

$$\begin{aligned} L_n &= \sum_{k=1}^{n-1}\left(\frac{k\,a}{n}\right)^2\frac{a}{n} \\ &= \frac{a^3}{n^3}\sum_{k=1}^{n-1} k^2 \end{aligned}$$

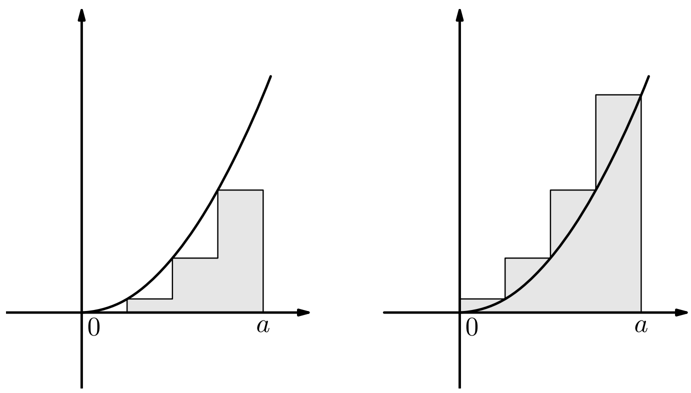

Figure 10.2: The two polygons with areas L_n and U_n.

$$= \frac{a^3}{n^3}\,\frac{(n-1)\cdot n\cdot(2n-1)}{6}$$
$$= a^3\,\frac{2n^3-3n^2+n}{6n^3} \xrightarrow{n\to\infty} a^3\,\frac{1}{3}.$$

Similarly,

$$U_n = L_n + a^2\cdot\frac{a}{n} \xrightarrow{n\to\infty} a^3\,\frac{1}{3}.$$

We see, that the surface area of the region under the graph must be equal to $A = \frac{a^3}{3}$. The construction of the definite integral is exactly analogous.

Lower and upper sums

Let the function $f(x)$ be bounded on the interval $[a, b]$, and let us denote by m and M the lower and upper bounds of $f(x)$ respectively. In other words we have

$$m \le f(x) \le M, \qquad x \in [a, b].$$

Let P be a partition of the interval $[a, b]$ into subintervals, that is let $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ (this is the set of partition points),

$$[a, b] = [a, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-2}, x_{n-1}] \cup [x_{n-1}, b].$$

On each of the subintervals $[x_i, x_{i+1}]$ we introduce the notation

$$m_i = \inf\{f(x);\ x \in [x_i, x_{i+1}]\}, \quad M_i = \sup\{f(x);\ x \in [x_i, x_{i+1}]\}, \quad i = 0, 1, \ldots, n-1.$$

We thus have $m \le m_i \le M_i \le M$.

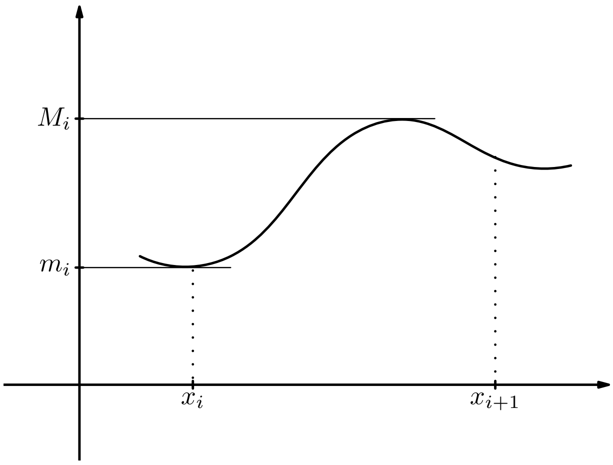

Figure 10.3: m_i and M_i.

Given the partition P we write the sums

$$L(P,f) = \sum_{i=0}^{n-1} m_i\,(x_{i+1} - x_i), \quad U(P,f) = \sum_{i=0}^{n-1} M_i\,(x_{i+1} - x_i).$$

$L(P,f)$ is called the lower sum and $U(P,f)$ is called the upper sum based on the partition P. Let us observe that these sums depend on the function $f(x)$, the interval $[a,b]$, and the partition P of that interval. Comparing this with the example above, in which we were computing the area of the region under the graph we see, that if $f(x)$ is non-negative, then the surface area of the region under the graph is a number between every lower sum, and every upper sum, regardless of the respective partitions. Observe that we obviously have

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a). \tag{10.1}$$

The lower integral of the function $f(x)$ over the interval $[a,b]$ is defined as

$$\underline{\int_a^b f(x)\,dx} = \sup\{L(P,f);\ P \text{ - a partition of } [a,b]\},$$

and the upper integral as

$$\overline{\int_a^b f(x)\,dx} = \inf\{U(P,f);\ P \text{ - a partition } [a,b]\}.$$

Definition 10.1. *Because of* (10.1) *the* sup *and* inf *make sense as numbers. If the lower integral of* $f(x)$ *is equal to its upper integral we say that the function* $f(x)$ *is integrable in the sense of Riemann, and the common value of the upper and lower integrals is called the Riemann integral of the function* $f(x)$ *over the interval* $[a, b]$*, and denoted*

$$\int_a^b f(x)\,dx.$$

Remarks: (i) Let us observe, as in (10.1), that for any partition P we have

$$L(P, f) = \sum_{i=0}^{n-1} m_i\,(x_{i+1} - x_i) \geq m \sum_{i=0}^{n-1} (x_{i+1} - x_i) = m(b - a),$$
$$U(P, f) = \sum_{i=0}^{n-1} M_i\,(x_{i+1} - x_i) \leq M \sum_{i=0}^{n-1} (x_{i+1} - x_i) = M(b - a).$$

Thus the integral, if it exists, satisfies

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a).$$

(ii) Let us recall, that the definition we gave requires that the function $f(x)$ be bounded, and that $a < b$. We will later introduce additional notation, which will allow the integration limits a and b to be arbitrary numbers, and we will describe how, sometimes one can integrate unbounded functions (these will be the so-called improper integrals).

(iii) The integral can fail to exist. As an example let $f(x)$ be given by the formula

$$f(x) = \begin{cases} 1 & : x \in \mathbf{Q}, \\ 0 & : x \notin \mathbf{Q}. \end{cases}$$

Then, for each partition P and for each i we have $m_i = 0$ and $M_i = 1$, so always $L(P, f) = 0$ and $U(P, f) = (b - a)$, and so

$$\underline{\int_a^b f(x)\,dx} = 0, \quad \text{and} \quad \overline{\int_a^b f(x)\,dx} = b - a.$$

(iv) The Riemann integral is closely related with the notion of surface area. If the function $f(x)$ is non-negative, then the integral is the surface area of the region under the graph. If the function $f(x)$ is non-positive, then the integral is the surface area of the region over the graph, under the axis OX, taken with the minus sign.

(v) We call the Riemann integral the definite integral. In the literature one can find other constructions of definite integral, but we restrict ourselves to the above Riemann construction. It will be our goal now to prove that continuous functions are integrable in the sense of Riemann. To do this we will now prove a number of simple theorems.

Theorem 10.2. *The lower integral is less than or equal to the upper integral:*

$$\underline{\int_a^b f(x)\,dx} \le \overline{\int_a^b f(x)\,dx}.$$

Proof. We have to show, that any lower sum is less than or equal to any upper sum, even if they are based on different partitions. Let then $L(P_1, f)$ be the lower sum based on the partition P_1, and $U(P_2, f)$ be the upper sum based on the partition P_2. Let P^* be the common refinement of both partitions P_1 and P_2, that is

$$P^* = P_1 \cup P_2.$$

Let us denote the point of respective partitions as follows: $P_1 = \{a = x_0, \dots, x_n = b\}$, $P_2 = \{a = y_0, \dots, y_k = b\}$ and $P^* = \{a = z_0, \dots, z_m = b\}$. From the definition of P^* it follows, that each point x_i and each point y_j are also points of P^*. Let us notice then, that each subinterval $[x_i, x_{i+1}]$ of the partition P_1 and each subinterval $[y_j, y_{j+1}]$ of the partition P_2 is a union of some subintervals of the refined partition P^*. It follows, that

$$L(P_1, f) \le L(P^*, f) \le U(P^*, f) \le U(P_2, f).$$

The two outside inequalities are a consequence of the fact that P^* is a refinement of both P_1 and P_2, while the middle inequality is a simple observation that the lower sum is no greater than the upper sum, if they are based on the same partition. □

We then have the following corollary:

Corollary 10.3. *If for each $\epsilon > 0$ there exists a partition P such, that*

$$U(P, f) - L(P, f) < \epsilon, \tag{10.2}$$

then $f(x)$ is integrable, and for such partition P we have estimates

$$U(P, f) - \epsilon < \int_a^b f(x)\,dx < L(P, f) + \epsilon. \tag{10.3}$$

Proof. From the definition of the lower and upper integrals we have, for arbitrary partition P

$$\overline{\int_a^b f(x)\,dx} - \underline{\int_a^b f(x)\,dx} \leq U(P,f) - L(P,f).$$

If the condition (10.2) is satisfied, then

$$0 \leq \overline{\int_a^b f(x)\,dx} - \underline{\int_a^b f(x)\,dx} < \epsilon.$$

Since this is satisfied for any $\epsilon > 0$, the difference must be zero. The function $f(x)$ is thus integrable. On the other hand

$$\int_a^b f(x)\,dx \geq L(P,f) > U(P,f) - \epsilon,$$

and similarly for the other inequality in (10.3). □

We can now prove the following fundamental theorem:

Theorem 10.4. *If the function $f(x)$ is continuous on $[a,b]$, then it is integrable in the sense of Riemann on $[a,b]$.*

Proof. We first show that $f(x)$ satisfies the following condition:

$$\forall\,\epsilon > 0 \quad \exists\,\delta > 0 \quad \forall\,x, y \in [a,b] \quad |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon. \tag{10.4}$$

Let us observe, that this condition is stronger than just continuity at each point of the domain $[a,]$. The condition for continuity at a point x_0 allows one to match δ for given ϵ and given x_0. However in the above condition δ only depends on given ϵ, and should be good enough for all points in $[a,b]$. A function that satisfies (10.4) is called "uniformly continuous" on $[a,b]$, for obvious reasons. Therefore, using this language, we will now show, that a function that is continuous on an interval $[a,b]$ (including endpoints) is necessarily uniformly continuous on $[a,b]$, that is satisfies (10.4).

To appreciate the difference between continuity and uniform continuity let us consider a function $f(x) = \frac{1}{x}$ on he interval $(0,1]$. We know, that this function is continuous at each point of the interval $(0,1]$. It is not, however uniformly continuous, that is it does not satisfy (10.4). This is easy to observe. Let $\delta > 0$ be arbitrary, and let $n \in \mathbf{N}$ be also arbitrary, $n > 4$. Let $x = \frac{\delta}{n}$ and $y = x + \frac{\delta}{2}$. Then $|x - y| = \delta/2 < \delta$, while

$$f(x) - f(y) = \frac{1}{x} - \frac{1}{y} = \frac{n}{\delta} - \frac{1}{\frac{\delta}{n} + \frac{\delta}{2}} = \frac{n}{\delta}\left(1 - \frac{2}{n+2}\right) > \frac{n}{2\delta},$$

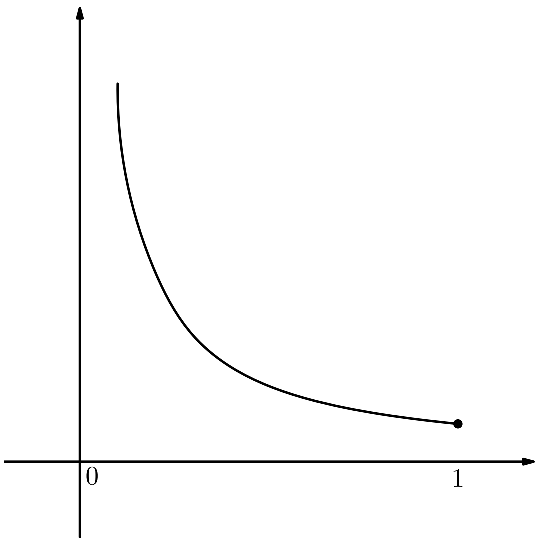

Figure 10.4: A function that is continuous but not uniformly continuous.

since for $n > 4$ we have $\frac{2}{n+2} < \frac{1}{2}$. We thus see, that regardless of δ the difference $|f(x) - f(y)|$ can be arbitrarily large, despite $|x - y| < \delta$. Thus, as we see, a function continuous at every point of its domain might not be uniformly continuous. In the situation we have in this theorem it is crucial, that the function in question is continuous on an interval, which is finite and contains endpoints. Thus, returning to the proof recall, that the function $f(x)$ is by assumption continuous on the interval $[a, b]$. Let us assume that $f(x)$ is not uniformly continuous, that is the condition (10.4) does not hold:

$$\exists\, \epsilon_0 > 0 \quad \forall\, \delta > 0 \quad \exists\, x, y \in [a, b],\ |x - y| < \delta \,\wedge\, |f(x) - f(y)| \geq \epsilon_0. \tag{10.5}$$

We will apply the above to $\delta = \frac{1}{n}$, $n = 1, 2, \dots$. For each n we thus obtain a pair of numbers $x_n, y_n \in [a, b]$ satisfying $|x_n - y_n| < \frac{1}{n}$, and $|f(x_n) - f(y_n)| \geq \epsilon_0$. We know, that since $\{x_n\} \subset [a, b]$ we can extract a subsequence $\{x_{n_k}\}$ converging to some $x_0 \in [a, b]$. As a consequence, the subsequence $\{y_{n_k}\}$ also must converge to x_0:

$$x_{n_k} - \frac{1}{n_k} < y_{n_k} < x_{n_k} + \frac{1}{n_k}.$$

Thus, by the continuity of $f(x)$ we have $f(x_{n_k}) \to f(x_0)$ and $f(y_{n_k}) \to f(x_0)$, therefore $f(x_{n_k}) - f(y_{n_k}) \to 0$, which contradicts the condition $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon_0 > 0$. Since we arrived at a contradiction the assumption (10.5) must be false, and thus its converse, (10.4) must hold.

We will show the integrability using Corollary 10.3. Let $\epsilon > 0$ be arbitrary, and let $\delta > 0$ be given by (10.4), but for some $\epsilon' < \frac{\epsilon}{b-a}$. Let $n \in \mathbf{N}$ be given by $n = [\frac{b-a}{\delta}] + 1$. We subdivide the interval $[a, b]$ into n subintervals of equal length using partition points

$$P = \left\{ x_i = a + (b-a)\,\frac{i}{n};\ i = 0, 1, \ldots, n \right\}.$$

Observe, that the length of each of the subintervals, which is equal to $(b-a)/n$ is smaller than δ, since $n > \frac{(b-a)}{\delta}$. Thus, if $x, y \in [x_i, x_{i+1}]$, then $|x - y| \leq \delta$ so $|f(x) - f(y)| < \epsilon'$. The extremes of $f(x)$ over $[x_i, x_{i+1}]$ must also satisfy $M_i - m_i \leq \epsilon' < \frac{\epsilon}{(b-a)}$. It follows, that

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i) - \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i) \\ &= \frac{b-a}{n} \sum_{i=0}^{n-1} (M_i - m_i) \\ &< \frac{b-a}{n} \cdot \frac{\epsilon}{b-a} \cdot n \\ &= \epsilon. \end{aligned}$$

Since ϵ was arbitrary, then from Corollary 10.3 we conclude that $f(x)$ is integrable. □

Remark: The above proof can be strengthened somewhat, and can be used to show that if $f(x)$ has only finitely many discontinuities in $[a, b]$ (and is bounded) then it is still integrable.

Riemann sums

Suppose we have a function $f(x)$ on the interval $[a, b]$, a partition of this interval $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$, and suppose in each of the subintervals of the partition we have a chosen point t_i:

$$t_i \in [x_i, x_{i+1}], \quad i = 0, 1, \ldots, n-1.$$

Let us construct a sum

$$R = \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i). \tag{10.6}$$

A sum like this is called a Riemann sum. It depends on a particular partition, an also on the choice of points t_i. Let us observe, that we always have

$$L(P, f) \leq R \leq U(P, f),$$

if the Riemann sum is based on the partition P. This follows from the fat that $t_i \in [x_i, x_{i+1}]$, $i = 0, 1, \dots, n-1$, and

$$m_i = \inf\{f(x) :\ x \in [x_i, x_{i+1}]\} \le f(t_i) \le \sup\{f(x) :\ x \in [x_i, x_{i+1}]\} = M_i.$$

For a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ we define its diameter $d(P)$:

$$d(P) = \max\{(x_{i+1} - x_i);\ i = 0, \dots, n-1\}.$$

We then have the following theorem:

Theorem 10.5. *Let the function $f(x)$ be continuous on $[a, b]$, and suppose we have a sequence of partitions $\{P_n\}$ of the interval $[a, b]$ such, that the diameters converge to zero: $d(P_n) \to 0$ as $n \to \infty$. Let R_n be a sequence of Riemann sums based on partitions P_n. In other words for each partition P_n we have chosen points $t_i \in [x_i, x_{i+1}]$, and a sum is formed as in* (10.6)*. Then*

$$\lim_{n\to\infty} R_n = \int_a^b f(x)\, dx.$$

Remark: This theorem gives us the freedom to interpret the integral as a limit of sums. Very often as t_i we pick the left or right endpoint of the subinterval $[x_i, x_{i+1}]$, or perhaps its midpoint, not worrying about where the function assumes its maximal and minimal values. But be careful: the function $f(x)$ must be continuous.

Proof of the theorem. Similarly as in the proof of Theorem 10.4 we observe, that the function $f(x)$ continuous on $[a, b]$ must be uniformly continuous, that is mus satisfy (10.4). Pick arbitrary $\epsilon > 0$ and let $\delta > 0$ be given by (10.4) for some $\epsilon' < \frac{\epsilon}{(b-a)}$ (similarly as in the proof of theorem 10.4), and let $n_0 \in \mathbf{N}$ be sufficiently large, so that

$$\forall\ n \ge n_0 \quad d(P_n) < \delta.$$

Then for $n \ge n_0$ we have

$$U(P_n, f) - L(P_n, f) < \epsilon.$$

From (10.3) we have

$$\int_a^b f(x)\, dx - \epsilon < L(P_n, f) \le \int_a^b f(x)\, dx,$$

and

$$\int_a^b f(x)\, dx \le U(P_n, f) < \int_a^b f(x)\, dx + \epsilon,$$

that is

$$\left| L(P_n, f) - \int_a^b f(x)\, dx \right| < \epsilon, \qquad \left| U(P_n, f) - \int_a^b f(x)\, dx \right| < \epsilon.$$

Since the ϵ was arbitrary, and the above inequalities hold for all $n \geq n_0$, thus

$$\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f) = \int_a^b f(x)\, dx.$$

On the other hand, ans we know

$$L(P_n, f) \leq R_n \leq U(P_n, f),$$

so also

$$\lim_{n\to\infty} R_n = \int_a^b f(x)\, dx.$$

□

Example: we will compute the following limit:

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+3}} + \frac{1}{\sqrt{n+6}} + \frac{1}{\sqrt{n+9}} + \cdots + \frac{1}{\sqrt{7\,n}} \right) \frac{1}{\sqrt{n}}.$$

We will try to transform this expression to recognize it as a Riemann sum for some function, interval, partition, and choice of points t_i.

$$\begin{aligned} &\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+3}} + \frac{1}{\sqrt{n+6}} + \frac{1}{\sqrt{n+9}} + \cdots + \frac{1}{\sqrt{7\,n}} \right) \frac{1}{\sqrt{n}} \\ &\quad = \sum_{i=0}^{2\,n} \frac{1}{\sqrt{n+3\,i}} \frac{1}{\sqrt{n}} \\ &\quad = \sum_{i=0}^{2\,n} \frac{\sqrt{n}}{\sqrt{n+3\,i}} \frac{1}{n} \\ &\quad = \sum_{i=0}^{2\,n} \frac{1}{\sqrt{1+3\,\frac{i}{n}}} \frac{1}{n} \\ &\quad = \frac{1}{n} + \frac{1}{n} \sum_{i=1}^{2\,n} \frac{1}{\sqrt{1+3\,\frac{i}{n}}} \end{aligned}$$

Now everything becomes clear: This is the Riemann sum for function $f(x) = \frac{1}{\sqrt{1+3\,x}}$, for interval $[0, 2]$, uniform partition in to $2n$ subintervals of equal length $\frac{1}{n}$, and points t_i being the right endpoints of the subintervals. Outside of the sum we threw the term $\frac{1}{n}$, which is not part of the Riemann sum, but converges to zero nevertheless. Since the terms of our sequence were identified as Riemann sums, and the associated partitions have diameters converging to zero, then the sequence converges to the definite integral

$$\int_0^2 \frac{dx}{\sqrt{1+3\,x}}.$$

Soon we will see, how to easily compute this definite integral.

Theorem 10.6. *(i) If the functions $f(x)$, $f_1(x)$ and $f_2(x)$ are integrable on $[a,b]$ and c is a constant, then $(f_1+f_2)(x)$ and $cf(x)$ are also integrable, and*

$$\int_a^b (f_1(x)+f_2(x))\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx$$
$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx.$$

(ii) If the functions $f_1(x)$ and $f_2(x)$ are integrable on $[a,b]$ and $f_1(x) \le f_2(x)$ then

$$\int_a^b f_1(x)\,dx \le \int_a^b f_2(x)\,dx. \tag{10.7}$$

(iii) If the function $f(x)$ is integrable on $[a,b]$ and $a < c < b$, then $f(x)$ is also integrable on each of the subintervals $[a,c]$ and $[c,b]$, and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{10.8}$$

Also the other way around: if $f(x)$ is integrable on intervals $[a,c]$ and$[c,b]$ ($a < c < b$), then it is also integrable on $[a,b]$, and (10.8) holds.

(iv) If the function $f(x)$ is integrable on $[a,b]$, then $|f(x)|$ is also integrable on $[a,b]$, and

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Remark: Part (ii) can be strengthened somewhat, and we can prove, that if actually $f_1(x) < f_2(x)$ in all but a finite number of points of $[a,b]$ ($a < b$), then the inequality (10.7) is actually sharp. The proof basically remains the same.

Proof of the theorem. (i) Suppose a partition P of the interval $[a, b]$ is given. Then

$$L(P, f_1) + L(P, f_2) \leq L(P, f_1 + f_2) \leq U(P, f_1 + f_2) \leq U(P, f_1) + U(P, f_2).$$

So

$$U(P, f_1 + f_2) - L(P, f_1 + f_2) \leq U(P, f_1) - L(P, f_1) + U(P, f_2) - L(P, f_2). \quad (10.9)$$

Since $f_1(x)$ and $f_2(x)$ were integrable, then for any $\epsilon > 0$ there exist partitions P_1 and P_2 such, that

$$U(P_1, f_1) - L(P_1, f_1) < \epsilon/2, \qquad U(P_2, f_2) - L(P_2, f_2) < \epsilon/2.$$

If P^* is the common refinement of partitions P_1 and P_2, then the inequalities also hold for P^*, and so from (10.7)

$$U(P^*, f_1 + f_2) - L(P^*, f_1 + f_2) < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, then $f_1(x) + f_2(x)$ is integrable (Corollary 10.3) and in addition

$$\begin{aligned}
\int_a^b (f_1(x) + f_2(x))\, dx &\leq U(P^*, f_1 + f_2) \\
&\leq U(P^*, f_1) + U(P^*, f_2) \\
&\leq \int_a^b f_1(x)\, dx + \epsilon/2 + \int_a^b f_2(x)\, dx + \epsilon/2 \\
&= \int_a^b f_1(x)\, dx + \int_a^b f_2(x)\, dx + \epsilon.
\end{aligned}$$

Again, since $\epsilon > 0$ was arbitrary, then

$$\int_a^b (f_1(x) + f_2(x))\, dx \leq \int_a^b f_1(x)\, dx + \int_a^b f_2(x)\, dx.$$

We can prove the converse inequality in a similar way, using $L(P^*, f_1 + f_2)$.

Now, let $c > 0$. Then, of course $L(P, cf) = c\, L(P, f)$ and $U(P, cf) = c\, U(P, f)$. Therefore

$$U(P, cf) - L(P, cf) = c\, (U(P, f) - L(P, f)).$$

Similarly, if $c < 0$ then $L(P, cf) = c\, U(P, f)$ and $U(P, cf) = c\, L(P, f)$, and so

$$U(P, cf) - L(P, cf) = c\, (L(P, f) - U(P, f)) = |c|\, (U(P, f) - L(P, f)).$$

In both cases for $\epsilon > 0$ we find a partition P such that

$$U(P,f) - L(P,f) < \frac{\epsilon}{|c|} \ \Rightarrow\ U(P,cf) - L(P,cf) < \epsilon.$$

Of course, if $c = 0$ then $cf(x) \equiv 0$, so it is integrable, and the integral is zero. In each case we obtain the claim. Observe, that as a corollary of the above we also obtain

$$\int_a^b (f_1(x) - f_2(x))\,dx = \int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx.$$

(ii) We have

$$\int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx = \int_a^b (f_2(x) - f_1(x))\,dx. \tag{10.10}$$

It is easy to see, that if $f(x) \geq 0$ for each $x \in [a,b]$ then for each partition P the lower sum $L(P,f) \geq 0$, and so if the function $f(x)$ is integrable, its integral must be ≥ 0. So, if $f_1(x) \leq f_2(x)$ for each $x \in [a,b]$, then the quantity (10.10) is ≥ 0, and we obtain

$$\int_a^b f_1(x)\,dx \leq \int_a^b f_2(x)\,dx.$$

(iii) Let $\epsilon > 0$, and let P be the partition of $[a,b]$ such, that

$$U(P,f) - L(P,f) < \epsilon \tag{10.11}$$

Let us add the point c to the partition P, and call the refinement that we obtain P^*. Since P^* is a refinement of P, then (10.11) remains true for P^*. Let P_1 and P_2 be the parts of the partition P^* which fall into $[a,c]$ and $[c,b]$ respectively. P_1 and P_2 are thus partitions of the intervals $[a,c]$ and $[c,b]$. Observe that

$$L(P^*,f) = L(P_1,f) + L(P_2,f), \quad \text{and} \quad U(P^*,f) = U(P_1,f) + U(P_2,f).$$

Plugging this to (10.11) we obtain

$$(U(P_1,f) - L(P_1,f)) + (U(P_2,f) - L(P_2,f)) = U(P^*,f) - L(P^*,f) < \epsilon.$$

Each of the quantities on the left hand side is nonnegative, so each separately is less than ϵ. Since $\epsilon > 0$ was arbitrary, then the function $f(x)$ is integrable on both intervals $[a, c]$ and $[c, b]$. Using the estimate in Corollary 10.3 we also obtain

$$\begin{aligned}\int_a^b f(x)\,dx &< U(P^*, f) = U(P_1, f) + U(P_2, f) \\ &< \int_a^c f(x)\,dx + \epsilon + \int_c^b f(x)\,dx + \epsilon \\ &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx + 2\epsilon.\end{aligned}$$

The above holds for any $\epsilon > 0$, so we must have

$$\int_a^b f(x)\,dx \leq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

The inequality in the other direction can be proved in the same way, using the lower sums, and the estimate from Corollary 10.3. We must thus have the equality of integrals.

(iv) Let $c = \pm 1$, depending on the sign of the integral. Then

$$\left|\int_a^b f(x)\,dx\right| = c\int_a^b f(x)\,dx = \int_a^b c\,f(x)\,dx \leq \int_a^b |f(x)|\,dx,$$

since $c\,f(x) \leq |c\,f(x)| = |f(x)|$. □

So far we have constructed the Riemann integral, but have no effective way to compute these integrals. Moreover, so far we have not seen any relations between the definite and indefinite integrals, that are suggested by the similarities in terminology and notation. The following two theorems address both issues. They show the relation between integration and differentiation, and provide way to effectively compute definite integrals.

Theorem 10.7. *Let the function $f(x)$ be integrable on the interval $[a, b]$. For $x \in [a, b]$ we let*

$$F(x) = \int_a^x f(t)\,dt.$$

Then the function $F(x)$ is continuous on $[a, b]$ and differentiable at each point x_0 where the integrand $f(x)$ is continuous. In every such point we have

$$F'(x_0) = f(x_0).$$

Proof. Since $f(x)$ is integrable, then, in particular, it must be bounded $|f(x)| \leq M$. Thus, for any $x, y \in [a, b]$, $x < y$ we must have the estimate

$$|F(y) - F(x)| = \left| \int_x^y f(t)\, dt \right| \leq \int_x^y |f(t)|\, dt \leq M(y - x).$$

It follows, that $F(x)$ is continuous, and even uniformly continuous on $[a, b]$ ($\delta = \epsilon/M$ is always sufficient). Now, let $x_0 \in (a, b)$ be the point of continuity of $f(x)$. Suppose $\epsilon > 0$ is arbitrary, and let $\delta > 0$ be such, that for $|t - x_0| < \delta$ we have

$$|f(t) - f(x_0)| < \epsilon.$$

Observe, that since $f(x_0)$ is a constant, independent of t, we may write

$$f(x_0) = \frac{1}{h} \int_{x_0}^{x_0+h} f(x_0)\, dt,$$

for any $h > 0$ such that $[x_0, x_0 + h] \subset [a, b]$. We have used the obvious observation, that for a constant function the Riemann integral is easy to compute from the definition:

$$\int_{x_0}^{x_0+h} f(x_0)\, dt = f(x_0) \int_{x_0}^{x_0+h} dt = f(x_0) \cdot h.$$

For $0 < h < \delta$ we can then write

$$\begin{aligned}
\left| \frac{F(x_0 + h) - F(x_0)}{h} - f(x_0) \right| &= \left| \frac{1}{h} \int_{x_0}^{x_0+h} f(t)\, dt - f(x_0) \right| \\
&= \left| \frac{1}{h} \int_{x_0}^{x_0+h} f(t)\, dt - \frac{1}{h} \int_{x_0}^{x_0+h} f(x_0)\, dt \right| \\
&= \left| \frac{1}{h} \int_{x_0}^{x_0+h} \big(f(t) - f(x_0)\big)\, dt \right| \\
&\leq \frac{1}{h} \int_{x_0}^{x_0+h} |f(t) - f(x_0)|\, dt \\
&\leq \frac{1}{h} \cdot h \cdot \epsilon \\
&= \epsilon.
\end{aligned}$$

Similarly, for $-\delta < h < 0$

$$\left| \frac{F(x_0 + h) - F(x_0)}{h} - f(x_0) \right| = \left| \frac{-1}{h} \int_{x_0+h}^{x_0} f(t)\, dt - f(x_0) \right|$$

$$
\begin{aligned}
&= \left| \frac{-1}{h} \int_{x_0+h}^{x_0} f(t)\,dt - \frac{-1}{h} \int_{x_0+h}^{x_0} f(x_0)\,dt \right| \\
&= \frac{1}{|h|} \left| \int_{x_0+h}^{x_0} \big(f(t) - f(x_0)\big)\,dt \right| \\
&\le \epsilon.
\end{aligned}
$$

We see, that the limit

$$\lim_{h\to 0} \frac{F(x_0+h) - F(x_0)}{h}$$

exists, and is equal to $f(x_0)$. □

From the above theorem we get the following corollary, for which we have been waiting ever since we begun to talk about integrals:

Corollary 10.8. *A function, continuous on an interval $[a, b]$, has in this interval an antiderivative (an indefinite integral).*

The following theorem is the main tool to compute definite integrals. The theorem itself is simple, and quite obvious, and is known as the "fundamental theorem of calculus".

Theorem 10.9 (Fundamental theorem of calculus)**.** *If a function $f(x)$ is integrable over an interval $[a, b]$ (in the sense of Riemann), and if there exists an antiderivative $F(x)$ to $f(x)$, that is*

$$F'(x) = f(x) \qquad x \in (a, b),$$

(which means that $f(x)$ is integrable in the sense of Definition 9.2), then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)|_a^b\,.$$

Let us observe the symbol $F(x)|_a^b$, it denotes the increase of the function $F(x)$ between a and b, and we will use it in the future.

Proof. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be an arbitrary partition of the interval $[a, b]$. For each subinterval $[x_i, x_{i+1}]$ of the partition we use the mean value theorem, and so there exists $t_i \in (x_i, x_{i+1})$ such that

$$f(t_i) = \frac{F(x_{i+1}) - F(x_i)}{x_{i+1} - x_i}, \qquad i = 0, \ldots, n-1.$$

Therefore

$$\sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} \big(F(x_{i+1}) - F(x_i)\big) = F(x_n) - F(x_0) = F(b) - F(a).$$

For each partition P we thus have the inequalities

$$L(P, f) \leq F(b) - F(a) \leq U(P, f).$$

$F(b) - F(a)$ is thus a number between the lower and upper integrals of the function $f(x)$ on $[a, b]$. Since the function is integrable, then this number must be equal to the integral. □

Limits of integration

The definite integral has been defined for intervals $[a, b]$, for $a < b$. The lower limit of integration was smaller than the upper limit. We will need to extend this definition. Let us introduce the following notation. If $a < b$ then

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx,$$

and for any c

$$\int_c^c f(x)\,dx = 0.$$

With this notation the formula (10.8) holds regardless of mutual relations between the numbers a, b, c

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \qquad \forall\ a, b, c,$$

Provided all of the integrals exist. The proof reduces to considering cases.

Integration by parts

Theorem 10.9 gives us the following formula for definite integrals. If $F'(x) = f(x)$ and $G'(x) = g(x)$ on an interval $[a, b]$, then

$$\int_a^b f(x)G(x)\,dx = F(x)G(x)\Big|_a^b - \int_a^b F(x)g(x)\,dx.$$

This formula holds whenever any of the two integrals exist, (both then exist).

Example:

$$\int_1^e \log x\,dx = \int_1^e x' \log x\,dx = x \log x\Big|_1^e - \int_1^e x \cdot \frac{1}{x}\,dx = e - x\Big|_1^e = e - e + 1 = 1.$$

Integration by substitution

If the function $f(x)$ is differentiable on the interval $[a, b]$, then

$$\int_a^b g(f(x))\, f'(x)\, dx = \int_{f(a)}^{f(b)} g(y)\, dy, \tag{10.12}$$

and, as before, the formula holds if any of the two integrals exist, which implies that both exist.

Example: In the following integral we will let $g(x) = \sin x$ $f(x) = x^2$

$$\int_0^\pi x \sin x^2\, dx = \frac{1}{2} \int_0^\pi \sin(x^2)\, 2 \cdot x\, dx =$$

$$= \frac{1}{2} \int_{0^2}^{\pi^2} \sin y\, dy = -\frac{1}{2} \cos y|_0^{\pi^2} = \frac{1 - \cos(\pi^2)}{2}.$$

Often we use the above formula in the following way, using the formula (10.12) "backwards":

$$\int_4^9 \frac{\sqrt{x}}{\sqrt{x} - 1}\, dx = \left\{ x = (t + 1)^2 \;\Rightarrow\; dx = 2(t + 1)\, dt \right\} =$$

$$= \int_{\sqrt{4}-1}^{\sqrt{9}-1} \frac{t + 1}{t} \cdot 2 \cdot (t + 1)\, dt = 2 \int_1^2 \frac{t^2 + 2t + 1}{t}\, dt.$$

The last integral can be easily computed, finding the antiderivative. The notation $dx = 2(t + 1)\, dt$ customarily stands for $\frac{dx}{dt} = 2(t + 1)$. Let us observe, that the above calculation is justified, and follows from formula (10.12). It is enough to observe, that the function $x = (t + 1)^2$ is invertible on the interval $[1, 2]$, and the inverse is the function $t = \sqrt{x} - 1$ on the interval $[4, 9]$. Sometimes, applying the substitution in this way one falls into a trap. For example

$$\int_{-2}^2 x^2\, dx = \left\{ x^2 = t \;\Rightarrow\; 2x\, dx = dt \right\} = \int_4^4 \frac{1}{2} \sqrt{t}\, dt = 0,$$

although we know that the integral on the left is equal to $\frac{16}{3} > 0$. In actual problems the situation might not be as obvious, so it is always worthy to check the computations carefully, especially in the situation, when the substituted function is not invertible.

Chapter 11

Applications of integrals

Many physical, "real life" intuitively obvious quantities can be described as limits of sums. Such a limit can often be interpreted as a limit of Riemann sums of some function. In such case a physical quantity can be interpreted as a definite integral of some function. Such an integral can then be computed using one of the known methods of integration. We will discuss some examples.

Arclength

Let a function $f(x)$, defined on the interval $[a, b]$ be continuous, differentiable, and let its derivative again be continuous on (a, b), with finite limits at the endpoints a, b. We will compute the length of the arc which is the graph of the function $f(x)$, that is the length of the curve $\{(x, f(x));\ x \in [a, b]\}$. The length of the arc is defined as the limit of the lengths of the broken lines, approximating appropriately the curve. In other words, we will choose a number of points along our curve, and then connect the adjacent points with a line segment. We obtain a broken line, whose length we compute. Then we increase the number of points along the curve, and compute the length again. We obtain a sequence of lengths, which should have a limit, provided the knots of the consecutive broken lines get progressively closer. Such a limit can obviously be interpreted as the length of the curve. Such length might not exist. In the case we consider, that is the curve being the graph of sufficiently regular function the length exists, and can be expressed as an integral.

In our case any broken line with knots on the graph of $f(x)$ over the interval $[a, b]$ determines a partition of the interval $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. Points of the partition are the projections onto the axis OX of the knots of the broken line. The length of such broken line, associated

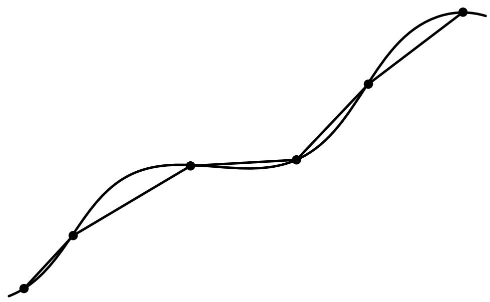

Figure 11.1: Approximating a curve with a broken line.

with the partition P is given by the formula

$$L_n = \sum_{i=0}^{n-1} \sqrt{(x_{i+1} - x_i)^2 + (f(x_{i+1} - f(x_i))^2} =$$

$$= \sum_{i=0}^{n-1} (x_{i+1} - x_i) \sqrt{1 + \left(\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}\right)^2}.$$

The function $f(x)$ is differentiable in each of the subintervals $[x_i, x_{i+1}]$, and so from the mean value theorem in each of these subintervals there is a point t_i such that

$$\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = f'(t_i).$$

We thus have

$$L_n = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \sqrt{1 + f'(t_i)^2}.$$

The length of the broken line is thus a Riemann sum of a continuous function $\sqrt{1 + f'(x)^2}$. Increasing the number of knots of the broken line leads to the refinement of the associated partition, and if the knot to knot distance tends to zero, then also the diameter of the partitions tends to zero. Thus, using Theorem 10.5 The Riemann sums converge to the integral

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx. \tag{11.1}$$

The integral L represents the length of the graph of $f(x)$. As we have mentioned earlier, a curve might not have the length. As we have justified now, a graph of a function which is differentiable, and has continuous derivative does in fact have length, and this length is given by the integral (11.1).

Example: $f(x) = \cosh x = \frac{e^x+e^{-x}}{2}$, $x \in [-1, 1]$. The graph of $f(x)$ is the so-called catenary curve. A flexible non-expandable rope (a chain is a good example) attached on its ends, and hanging freely will assume the shape of the graph of $\cosh x$, of course appropriately adjusted vertically and horizontally. This shape is considered to be structurally strong. For example, the famous "Gateway arch" in St. Louis on the banks of the Mississippi River has the shape of the catenary curve (upside down).

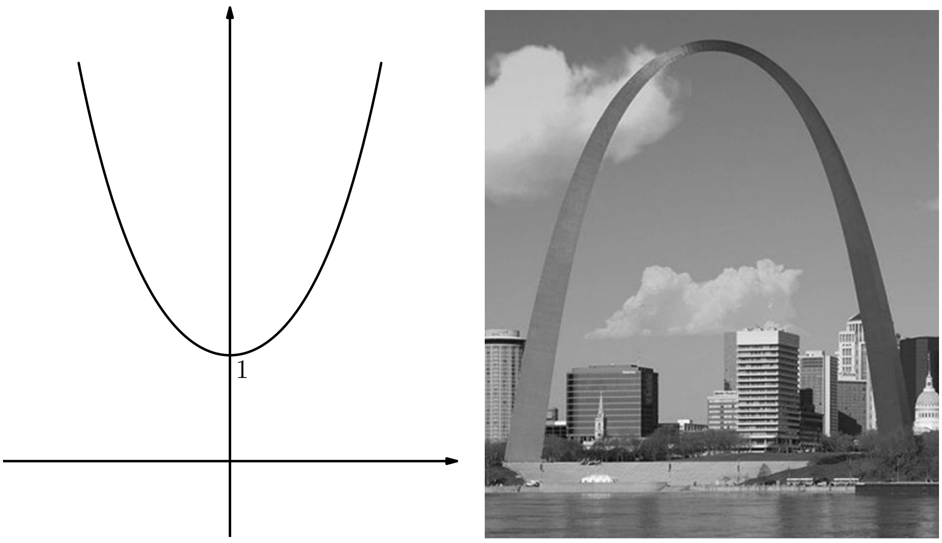

Figure 11.2: The catenary curve and the arch in St. Louis.

For our function $\cosh x$ we have

$$f'(x) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

We also know, and this is easy to verify, that $\sinh' x = \cosh x$ and that these "hyperbolic" functions satisfy the "hyperbolic unity"

$$\cosh^2 x - \sinh^2 x = 1.$$

We can then compute the length of the graph

$$L = \int_{-1}^{1} \sqrt{1 + f'(x)^2}\, dx = \int_{-1}^{1} \sqrt{1 + \sinh^2 x}\, dx = \int_{-1}^{1} \sqrt{\cosh^2 x}\, dx =$$
$$= \int_{-1}^{1} \cosh x\, dx = \sinh x|_{-1}^{1} = \frac{e^1 - e^{-1}}{2} - \frac{e^{-1} - e^1}{2} = e - \frac{1}{e}.$$

The volume of the solid of revolution around OX

Suppose we have a function $f(x)$ on an interval $[a, b]$, continuous and non-negative. Let us revolve the region under the graph of $f(x)$ around the OX axis. What we obtain is the co-called solid of revolution

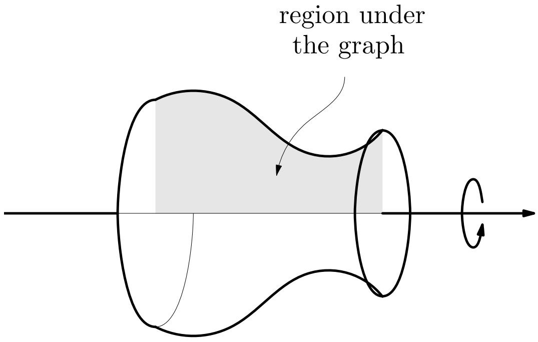

Figure 11.3: The solid of revolution.

The volume of such solid can be approximated by the combined volume of cylinders, obtained by the revolution around OX rectangles over subintervals of $[a, b]$.

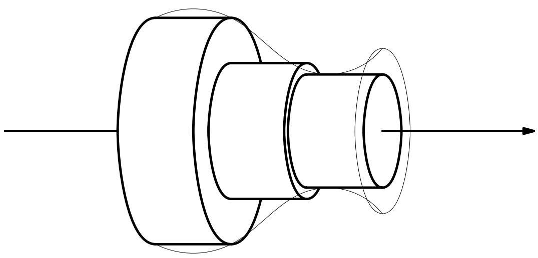

Figure 11.4: Approximating a solid with cylinders.

Let us pick a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of the interval $[a, b]$. Let, for $i = 0, \ldots, n-1$

$$m_i = \inf\{f(x);\ x_i \leq x \leq x_{i+1}\}, \qquad M_i = \sup\{f(x);\ x_i \leq x \leq x_{i+1}\}.$$

Let us consider the "slice" of the solid of revolution which is around the subinterval $[x_i, x_{i+1}]$. The cylinder with radius m_i is completely contained in this slice, while the cylinder with radius M_i completely contains our slice. It follows, that the volume of such slice (let us denote it by V_i) must be a number between the volumes of these two cylinders, that is

$$(x_{i+1} - x_i)\, \pi\, m_i^2 \leq V_i \leq (x_{i+1} - x_i)\, \pi\, M_i^2.$$

The volume V of the entire solid of revolution, which is comprised of all the "slices" thus satisfies

$$\sum_{i=0}^{n-1}(x_{i+1}-x_i)\,\pi\,m_i^2 \leq V \leq \sum_{i=0}^{n-1}(x_{i+1}-x_i)\,\pi\,M_i^2. \tag{11.2}$$

The sums on the left and right of the above double inequality are the lower and upper sums of the function $\pi f^2(x)$, for the partition P. Since the inequalities (11.2) hold for all partitions, and the function $\pi f^2(x)$ is integrable (since it is continuous), thus V must be equal to the integral

$$V = \pi \int_a^b f^2(x)\,dx.$$

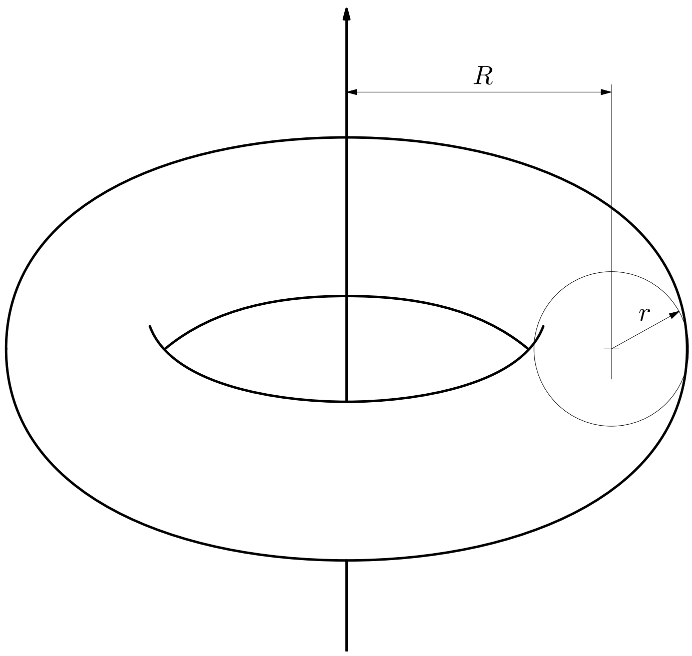

Figure 11.5: A torus.

Example: Let us consider a torus with its major (larger) radius R and the minor radius r $(0 < r < R)$. A torus like that can be represented as a solid of revolution of the circle

$$x^2 + (y-R)^2 \leq r^2 \tag{11.3}$$

around the OX axis. The region (11.3) is not a region under the graph of a non-negative function, but we can present it as a difference of two such regions, and thus present the torus as a difference of two solids of revolution, whose volumes we know how to compute using the integrals. The larger body is obtained by revolving the region bounded by the upper semicircle, and the smaller solid is obtained by revolving the region bounded by the lower semicircle. The upper and the lower semicircles are the graphs of functions

$$f_1(x) = R + \sqrt{r^2 - x^2}, \qquad f_2(x) = R - \sqrt{r^2 - x^2}, \quad -r \le x \le r.$$

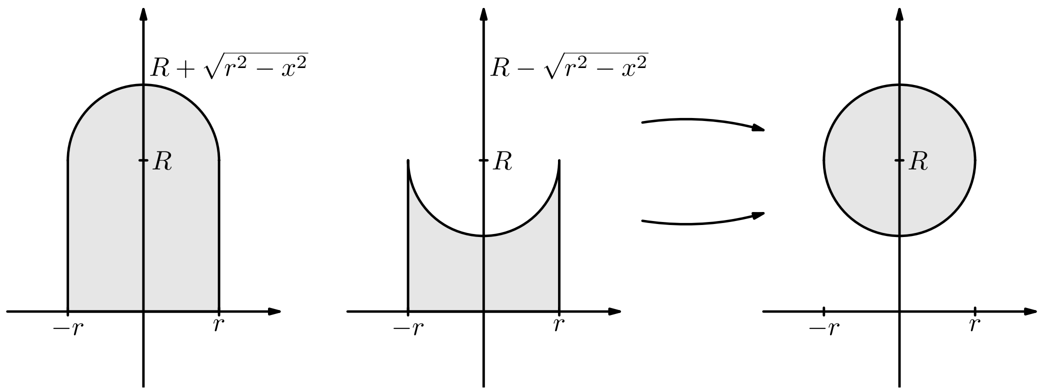

Figure 11.6: The region (11.3) as a difference of two regions.

We thus have the formula for the volume of the torus:

$$\begin{aligned} V &= \pi \int_{-r}^{r} f_1^2(x)\,dx - \pi \int_{-r}^{r} f_2^2(x)\,dx \\ &= \pi \int_{-r}^{r} (f_1^2(x) - f_2^2(x))\,dx \\ &= \pi \int_{-r}^{r} (f_1(x) - f_2(x))(f_1(x) + f_2(x))\,dx \\ &= \pi \int_{-r}^{r} 2\sqrt{r^2 - x^2} \cdot 2 \cdot R\,dx \\ &= 4R\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx. \end{aligned}$$

The last integral can be computed using the substitution $x = \sin t$, but let us quickly observe, that the graph of the integrand is the upper semicircle with center at $(0, 0)$ and with radius r. The integral, which is the area under the graph, is thus half the surface area of a disc with radius r, which comes down to $\frac{\pi r^2}{2}$. We have thus obtained the following formula for the volume of a torus:

$$V = 2\,\pi^2\,R\,r^2.$$

The surface area of the solid of revolution around OX

Let us now consider the surface area of the solid of revolution described in the previous section. Let us assume in addition that the function $f(x)$ is differentiable, and its derivative is continuous on (a, b), and has finite one-sided limits at the end-points a, b (to compute the volume it sufficed that $f(x)$ be continuous). Let us again consider the partition $P = \{a = x_0 < \cdots < x_n = b\}$ of the interval $[a, b]$, and a "slice" of the solid of revolution around the subinterval $[x_i, x_{i+1}]$. The lateral surface of this slice can be approximated by the lateral surface of a cone (not cylinder), obtained by revolving the region under the secant to the graph around the OX axis.

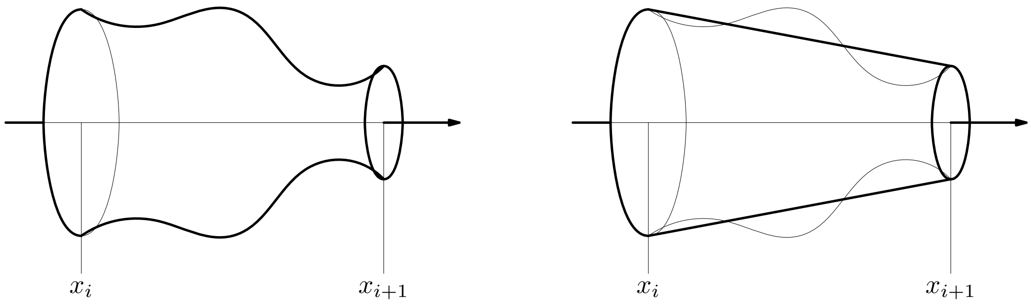

Figure 11.7: A cone approximating the solid of revolution.

The cone we obtain has the radii of its bases equal to $f(x_i)$ and $f(x_{i+1})$, and it has the height $x_{i+1} - x_i$. As we know from geometry the area of the lateral surface of such cone is equal to the length of the cone's "element" multiplied by the mean of the circumferences of bases.

If one does not remember this formula, then it can be derived in the following way. Let us slice the cone along its element, and let us flatten the lateral surface. The area should be preserved. What we obtain is a sector of the circle. We can deduce the appropriate formula analysing this sector. In our case of the cone around the subinterval $[x_i, x_{i+1}]$ the mean circumference, that is the circumference in the middle of the height is equal to

$$2\pi \frac{f(x_{i+1}) + f(x_i)}{2},$$

and the length of the "element" is equal to

$$\sqrt{(x_{i+1} - x_i)^2 + (f(x_{i+1}) - f(x_i))^2}.$$

The combined lateral surface of all cones approximating our solid of revolution is therefore given by the formula

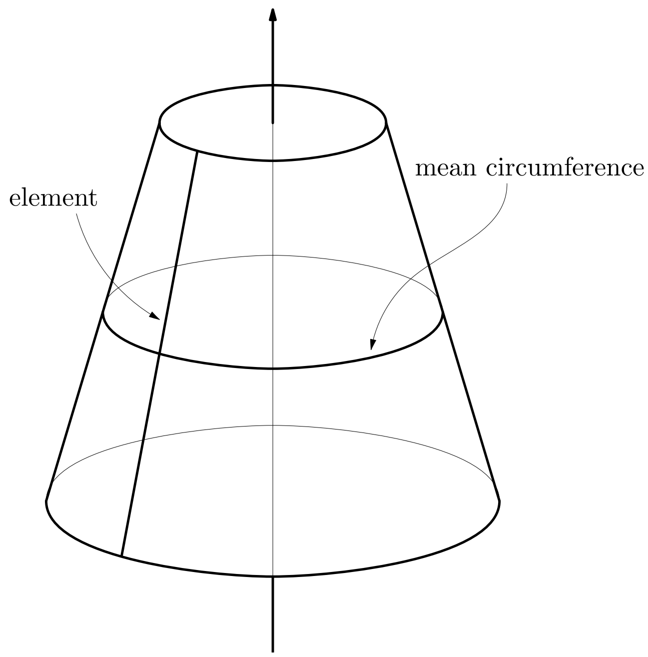

Figure 11.8: The cone's element and the mean circumference.

$$S_n = \sum_{i=0}^{n-1} 2\pi \left(\frac{f(x_{i+1}) + f(x_i)}{2} \right) \times$$
$$\times (x_{i+1} - x_i) \cdot \sqrt{1 + \left(\frac{f(x_{i+1}) + f(x_i)}{x_{i+1} - x_i} \right)^2}. \quad (11.4)$$

Using the mean value theorem we can write this sum as

$$\sum_{i=0}^{n-1} 2\pi \left(\frac{f(x_{i+1}) + f(x_i)}{2} \right) \cdot (x_{i+1} - x_i) \cdot \sqrt{1 + f'(t_i)^2}, \quad (11.5)$$

for appropriately chosen points $t_i \in (x_i, x_{i+1})$. This is not immediately a Riemann sum for any function, so we have to resort to one more approximation. Since $f(x)$ is uniformly continuous (this was discussed at length in the proof of Theorem 10.4) then for each $\epsilon > 0$

$$\left| \frac{f(x_{i+1}) + f(x_i)}{2} - f(t_i) \right| < \epsilon$$

provided the diameter of the partition P is sufficiently small. From our assumptions it also follows that $f'(x)$ is bounded, and so the sum (11.4), which is equal to (11.5) can be replaced by the sum

$$\sum_{i=0}^{n-1} 2\pi\, f(t_i)\cdot(x_{i+1}-x_i)\cdot\sqrt{1+f'(t_i)^2}, \tag{11.6}$$

with error converging to 0 as the diameter of the partition tends to 0. The sum (11.6) is a Riemann sum of a continuous function $2\pi f(x)\sqrt{1+f'(x)^2}$, and so as the diameters of the partition converge to zero, these converge to the integral of this function. Thus the area S of the lateral surface of the solid of revolution is given by

$$S = 2\pi\int_a^b f(x)\,\sqrt{1+f'(x)^2}\,dx.$$

Example: Let us compute the surface of the torus whose volume we have computed in the previous section. We know, that the torus can be presented as the solid obtained by the revolution of the disc

$$x^2+(y-R)^2\le r^2$$

around the OX axis, and so its surface is the union of the lateral surfaces of solids of revolution of two regions under the graphs of the upper and lower semicircles:

$$S = 2\pi\int_{-r}^{r} f_1(x)\sqrt{1+f_1'(x)^2}\,dx + 2\pi\int_{-r}^{r} f_2(x)\sqrt{1+f_2'(x)^2}\,dx,$$

where, as before

$$f_1(x)=R+\sqrt{r^2-x^2},\qquad f_2(x)=R-\sqrt{r^2-x^2}.$$

We thus have

$$f_1'(x)=\frac{1}{2}\,\frac{1}{\sqrt{r^2-x^2}}\cdot(-2\,x)=\frac{-x}{\sqrt{r^2-x^2}},$$

and similarly

$$f_2'(x)=\frac{x}{\sqrt{r^2-x^2}}.$$

Both derivatives only differ by a sign, so we have

$$\sqrt{1+f_1'(x)^2}=\sqrt{1+f_2'(x)^2}=$$

$$= \sqrt{1 + \frac{x^2}{r^2 - x^2}} = \sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}} = \frac{r}{\sqrt{r^2 - x^2}}.$$

Finally, we obtain

$$\begin{aligned}
S &= 2\,\pi \int_{-r}^{r} \left((R + \sqrt{r^2 - x^2})\, \frac{r}{\sqrt{r^2 - x^2}} + (R - \sqrt{r^2 - x^2})\, \frac{r}{\sqrt{r^2 - x^2}} \right) dx \\
&= 4\,\pi\, R\, r \int_{-r}^{r} \frac{dx}{\sqrt{r^2 - x^2}} \\
&= 4\,\pi\, R \int_{-r}^{r} \frac{dx}{\sqrt{1 - (\frac{x}{r})^2}} \\
&= 4\,\pi\, R\, r \int_{-r}^{r} \frac{\frac{1}{r}\, dx}{\sqrt{1 - (\frac{x}{r})^2}} \qquad y = \frac{x}{r} \\
&= 4\,\pi\, R\, r \int_{-1}^{1} \frac{dy}{\sqrt{1 - y^2}} \\
&= 4\,\pi\, R\, r\ \arcsin y\big|_{-1}^{1} \\
&= 4\,\pi^2\, R\, r.
\end{aligned}$$

The surface area of the torus is the main circumference $2\pi R$ times the minor circumference $2\pi r$.

Chapter 12

Improper integrals

The definite integral has been introduced for functions which are bounded, and for finite intervals $[a, b]$. Now we extend this definition to functions which are not necessarily bounded, and to infinite intervals. Such integrals are called improper integrals. Let us first consider the case of a function which is not bounded on the interval $[a, b]$, but is bounded and integrable on every subinterval $[c, b]$, $a < c < b$. We can compute the integral over each of these subintervals and ask, what happens with these integrals as $c \to a^+$. If the limit

$$g = \lim_{c \to a^+} \int_c^b f(x)\, dx, \tag{12.1}$$

exists we say that the function $f(x)$ is integrable in the improper sense on the interval $[a, b]$, or that the improper integral of $f(x)$ on $[a, b]$ converges. The limit g is, of course, denoted by

$$\int_a^b f(x)\, dx = g = \lim_{c \to a^+} \int_c^b f(x)\, dx,$$

and is called the improper integral of $f(x)$ on $[a, b]$. Similarly we can define the improper integral when the function $f(x)$ has "singularity" on the right end of the interval of integration (that is is not bounded around this end-point, and usually is not even defined there). The improper integral exists (converges) if $f(x)$ is integrable on each subinterval $[a, c]$, where $a < c < b$, and the limit exists

$$g = \lim_{c \to b^-} \int_a^c f(x)\, dx. \tag{12.2}$$

Remark: If the function $f(x)$ is integrable on $[a, b]$ to begin with, then the limits (12.1) and (12.2) obviously exist, and are equal to the integral in the normal (proper) sense. The improper integral is therefore an extension of the normal definition of the definite integral.

The improper integral can also be defined in situations where the function $f(x)$ has "singularities" on bot ends of the interval of integration $[a, b]$, or at one or few points inside the interval. To define such integral we first split the integral $[a, b]$ into subintervals so that the function $f(x)$ has only one "singularity", and at only one end of the subinterval. Then the improper integral is defined as a sum of improper integrals over each of the subintervals of the partition, provided each of these improper integrals exists separately. For example, if we want to check the existence of the improper integral of the function $f(x) = \frac{1}{x}$ on the interval $[-1, 1]$, we consider the convergence of two improper integrals

$$\int_{-1}^{0} \frac{dx}{x} \quad \text{and} \quad \int_{0}^{1} \frac{dx}{x}, \tag{12.3}$$

and if both of these integrals converges (both are improper) we say thet the improper integral on the interval $[-1, 1]$ exists. Let us observe, that in this particular case neither of the above integrals converges (example (b).

Examples: (a) Let us consider $f(x) = \frac{1}{\sqrt{x}}$ on the interval $[0, 1]$. The function is continuous on $(0, 1]$, but has singularity (sometimes we say it "explodes") at 0. We thus check

$$\int_{\epsilon}^{1} \frac{dx}{\sqrt{x}} = \int_{\epsilon}^{1} x^{-\frac{1}{2}}\, dx = \left.\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right|_{\epsilon}^{1} = 2\,(1 - \sqrt{\epsilon}) \xrightarrow{\epsilon \to 0^+} 2,$$

so the improper integral converges, and

$$\int_{0}^{1} \frac{dx}{x} = 2.$$

(b) Let us consider the function $f(x) = \frac{1}{x}$ on the interval $[-1, 1]$. This function has the sole singularity at 0, which is the interior point of the interval of integration. We have to check the convergence of each of the improper integrals (12.3) separately. First let us check the integral over $[0, 1]$

$$\int_{\epsilon}^{1} \frac{dx}{x} = \log x|_{\epsilon}^{1} = 0 - \log \epsilon = \log \frac{1}{\epsilon} \xrightarrow{\epsilon \to 0^+} \infty.$$

There is no need to check the other integral from (12.3) (which does not converge either). Since one of the integrals does not converge (12.3) the integral over the entire interval $[-1, 1]$, by definition, fails to exist.

The second type of improper integrals arises on infinite intervals. Suppose the function $f(x)$ is integrable on all intervals of the form $[a, M]$, for some a and every $M > a$. If the limit

$$g = \lim_{M\to\infty} \int_a^M f(x)\, dx,$$

exists, then we say that $f(x)$ is integrable in the improper sense on $[a, \infty)$ and we write

$$\int_a^\infty f(x)\, dx = g = \lim_{M\to\infty} \int_a^M f(x)\, dx.$$

Similarly we define the improper integral on the interval $(-\infty, b]$:

$$\int_{-\infty}^b f(x)\, dx = \lim_{M\to-\infty} \int_M^b f(x)\, dx,$$

provided the integrals on the right hand side exist (in the proper sense), and the limit exists. Finally the integral on the entire real axis $(-\infty, \infty)$ is defined as the sum

$$\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^b f(x)\, dx + \int_b^\infty f(x)\, dx,$$

provided both integrals on the right exist independently. Let us observe, that this definition does not depend on the point b, which is chosen to separate the half-axes.

Finally, we can combine both types of improper integrals, and define an integral on an infinite interval of a function that has singularities at a finite number of points. To define such integral we proceed as follows. The interval of integration is divided into subintervals so that the function in each of the subintervals has only one singularity, only at one endpoint, and so that in the eventual infinite subintervals the function has no singularities. Then we check the convergence of the resulting improper integrals separately on each of the subintervals. For example, let us consider the function $f(x) = \frac{1}{x^2}$ on the entire real axis $(-\infty, \infty)$. To determine the existence of this improper integral we have to independently check the convergence of each of the following 4 improper integrals

$$\int_{-\infty}^{-1} \frac{dx}{x^2}, \qquad \int_{-1}^{0} \frac{dx}{x^2}, \qquad \int_{0}^{1} \frac{dx}{x^2}, \qquad \int_{1}^{\infty} \frac{dx}{x^2}.$$

In this case the first and the last of the integrals converge, while the second and third do not. Hence the improper integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2} \qquad \text{does not exist.}$$

Examples: (a) The functions $\frac{\sin x}{x}$ and $\frac{|\sin x|}{x}$ are continuous on $[0, \infty)$ (the value at 0 is set at 1). The first is integrable in the improper sense, while the second is not.

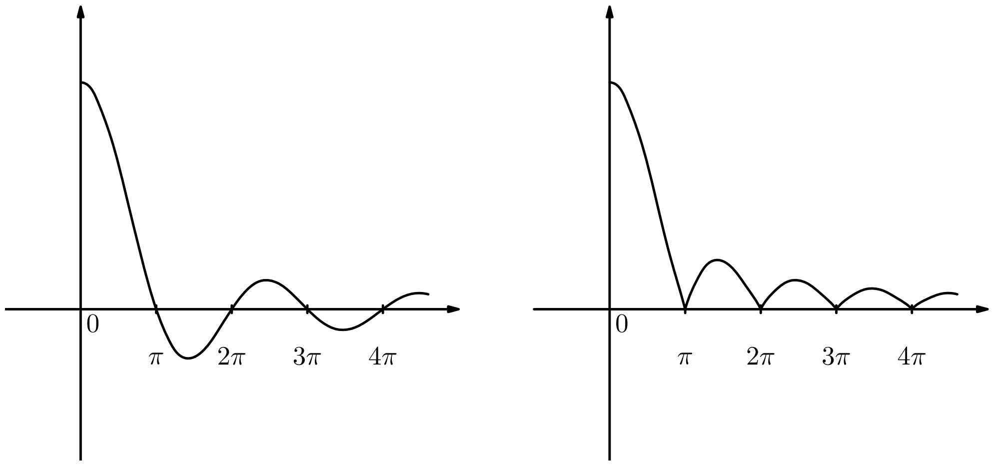

Figure 12.1: Functions $\frac{\sin x}{x}$ and $\frac{|\sin x|}{x}$.

Let au pick arbitrary $M > 0$, and consider

$$\int_0^{2\pi M} \frac{\sin x}{x}\, dx = \int_0^{2\pi [M]} \frac{\sin x}{x}\, dx + \int_{2\pi[M]}^{2\pi M} \frac{\sin x}{x}\, dx.$$

As $M \to \infty$ the second integral on the right hand side has limit 0. This is easy to observe, sine the length of the interval of integration does not exceed 2π, and the integrand is bounded in the absolute value by $\frac{1}{2\pi[M]}$ which tends to 0 as $M \to \infty$. So, we have

$$\lim_{M\to\infty} \int_0^{2\pi M} \frac{\sin x}{x}\, dx = \lim_{M\to\infty} \int_0^{2\pi [M]} \frac{\sin x}{x}\, dx, \tag{12.4}$$

and the existence of one of these limits implies the existence of the other. We now consider the limit on the right hand side, and we will show that it exists. Let us decompose the integral:

$$\int_0^{2\pi [M]} \frac{\sin x}{x}\, dx = \sum_{k=0}^{[M]-1} \int_{2\pi k}^{2\pi(k+1)} \frac{\sin x}{x}\, dx. \tag{12.5}$$

Let us examine the terms of the sum:

$$\int_{2\pi k}^{2\pi(k+1)} \frac{\sin x}{x}\, dx = \int_{2\pi k}^{2\pi k+\pi} \frac{\sin x}{x}\, dx + \int_{2\pi k+\pi}^{2\pi(k+1)} \frac{\sin x}{x}\, dx$$

$$= \int_0^\pi \frac{\sin x}{x + 2k\pi}\, dx - \int_0^\pi \frac{\sin x}{x + (2k+1)\pi}\, dx,$$

where we have used the fact that $\sin(x+2k\pi) = \sin(x)$ and $\sin(x+(2k+1)\pi) = -\sin(x)$. Continuing, we obtain

$$\int_{2\pi k}^{2\pi(k+1)} \frac{\sin x}{x}\, dx = \int_0^\pi \sin x \left(\frac{1}{x + 2k\pi} - \frac{1}{x + (2k+1)\pi} \right) dx$$

Observe, that the integral is positive, since the integrand is positive within the interval of integration. The sums (12.5) thus have positive terms, so they converge as $M \to \infty$ when they are bounded. To show this boundedness, let us examine the last integral. For $k > 0$ we have

$$\begin{aligned} \int_0^\pi \sin x \left(\frac{1}{x + 2k\pi} - \frac{1}{x + (2k+1)\pi} \right) dx &= \\ = \int_0^\pi \sin x \frac{\pi}{(x + 2k\pi) \cdot (x + (2k+1)\pi)}\, dx &\leq \\ \leq \frac{\pi}{4\pi^2 k^2} \int_0^\pi \sin x\, dx &= \frac{1}{2\pi k^2}, \end{aligned}$$

while for $k = 0$ we may estimate brutally

$$\int_0^{2\pi} \frac{\sin x}{x}\, dx \leq \int_0^{2\pi} \left| \frac{\sin x}{x} \right| dx \leq 2\pi.$$

We thus arrive at

$$\begin{aligned} \int_0^{2\pi[M]} \frac{\sin x}{x}\, dx &\leq 2\pi + \sum_{k=1}^{[M]-1} \frac{1}{2\pi k^2} \\ &= 2\pi + \frac{1}{2\pi} \sum_{k=1}^{[M]-1} \frac{1}{k^2} \\ &< 2\pi + \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{1}{k^2}. \end{aligned}$$

As we have already mentioned the left hand side in a non-decreasing function of M, and is bounded, so it has a limit as $M \to \infty$. The limits in (12.4) thus exist, and so the improper integral

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

converges. It can be proved (but this requires completely different tools) that this integral is equal to $\sqrt{\frac{\pi}{2}}$.This integral is important, and appears in applications. Its existence relies on the fact that the "hills" of the sin function appear alternately above and below the OX axis, and their areas cancel for most part. The combined area of all these "hills" is infinite, which we will now show. If we apply the absolute value to the integrand the improper integral is no longer convergent. To see this, let us pick any $M > 0$, and let us compute

$$\begin{aligned}\int_0^{\pi M} \frac{|\sin x|}{x}\,dx &\geq \int_0^{\pi[M]} \frac{|\sin x|}{x}\,dx \\ &= \sum_{k=0}^{[M]-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\,dx \\ &= \sum_{k=0}^{[M]-1} \int_0^{\pi} \frac{\sin x}{x+k\pi}\,dx \\ &> \sum_{k=0}^{[M]-1} \frac{1}{k\pi+\pi} \int_0^{\pi} \sin x\,dx \\ &= \sum_{k=1}^{[M]} \frac{2}{k\pi}.\end{aligned}$$

As $M \to \infty$ then $[M] \to \infty$ and so

$$\int_0^{\pi M} \frac{|\sin x|}{x}\,dx \to \infty,$$

thus the improper integral indeed diverges.

(b) We will show that the improper integral of the function $f(x) = e^{-x^2}$ on the entire real axis $(-\infty, \infty)$ converges. We first show, that the improper integral on the positive half-axis $[0, \infty)$ exists. Let $M > 0$. Since the integrand e^{-x^2} is positive, the integral

$$\int_0^M e^{-x^2}\,dx \tag{12.6}$$

increases with M, so the limit as $M \to \infty$ exists, if the above integrals have a common bound from above, for all M.

$$\int_0^M e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^M e^{-x^2}\,dx$$

$$
\begin{aligned}
&\leq \int_0^1 e^{-x^2}\,dx + \int_1^M e^{-M}\,dx \\
&= \int_0^1 e^{-x^2}\,dx - e^{-x}\Big|_1^M \\
&= \int_0^1 e^{-x^2}\,dx + e^{-1} - e^{-M} \\
&< \int_0^1 e^{-x^2}\,dx + e^{-1}.
\end{aligned}
$$

The integrals (12.6) form an increasing and bounded function of M, so they have a limit as $M \to \infty$. We now take up the other improper integral, on the negative half-axis. We will simply use the parity of the integrand. Let $M > 0$.

$$
\int_{-M}^0 e^{-x^2}\,dx = -\int_M^0 e^{-x^2}\,dx = \int_0^M e^{-x^2}\,dx.
$$

Thus, the improper integral on $(-\infty, 0]$ exists precisely when the improper integral on $[0, \infty)$ exists, which we have already shown. Since both improper integrals exist we conclude that the improper integral on the entire real axis exists. It can be shown that

$$
\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi},
$$

but, again, this requires additional tools. The function e^{-x^2} is the so-called Gauss function, and is one of the most important functions in mathematics and in applications.

(c) Let $f(x) = \frac{1}{x}$, $x \geq 1$. If we revolve the graph of this function around the OX axis we obtain an infinite "funnel". We will compute the volume of this infinite funnel and its surface area. The funnel is infinite, and thus fits well into our study of integrals on infinite intervals. Both the volume of the funnel and its surface area is clearly the limit of volumes and areas of funnels with their "mouth" truncated further and further to the right. We thus see, that these quantities are expressed by improper integrals.

$$
V = \pi \int_1^\infty \frac{1}{x^2}\,dx, \qquad S = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \left(\frac{1}{x}'\right)^2}\,dx.
$$

Let us compute these integrals.

$$
V = \lim_{M\to\infty} \pi \int_1^M \frac{1}{x^2}\,dx = \pi \lim_{M\to\infty} -\frac{1}{x}\Big|_1^M = \pi \lim_{M\to\infty} \left(-\frac{1}{M} + 1\right) = \pi.
$$

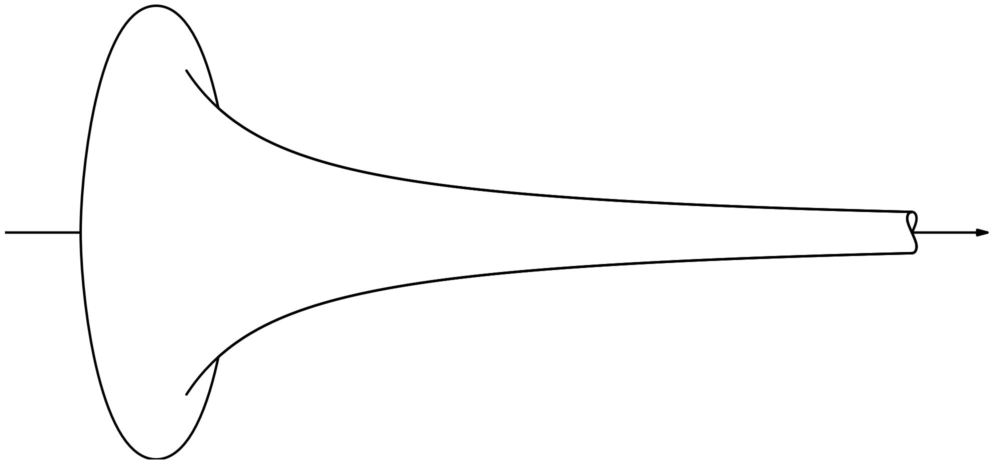

Figure 12.2: Infinite "funnel".

We thus see, that the improper integral converges, and the volume of the infinite funnel is finite, and equals to π. Let us now compute the lateral surface area.

$$2\pi \int_1^M \frac{1}{x} \sqrt{1 + \left(\frac{1}{x}'\right)^2}\, dx = 2\pi \int_1^M \frac{1}{x} \sqrt{1 + \frac{1}{x^4}}\, dx \geq$$

$$\geq 2\pi \int_1^M \frac{1}{x}\, dx = 2\pi \log x|_1^M = 2\pi \log M \xrightarrow{M\to\infty} \infty.$$

We see, that this improper integral does not exist. So, the lateral surface area of the funnel is infinite. Let us end with the following observation. The funnel's surface has infinite area, but can still be painted with finite amount of paint. How to do this? You need π liters of paint, which you pour into the funnel. The inside – of infinite area – will obviously all get painted.

Chapter 13

Wallis' and Stirling's formulas

We will use this opportunity to prove the important Stirling's formula. This formula is used in applications, in statistics for example, to approximately compute the factorial. The factorial only appears to be easy to compute. In practice computing factorial from the definition is impossible, it involves too many operations.

The Wallis' formula

We first compute the following formula, which is known as the Wallis' formula:

$$\pi = \lim_{n\to\infty} \frac{1}{n} \left(\frac{2 \cdot 4 \cdot \dots \cdot 2n}{1 \cdot 3 \cdot \dots \cdot (2n-1)} \right)^2 = \lim_{n\to\infty} \frac{1}{n} \left(\frac{(2n)!!}{(2n-1)!!} \right)^2 .$$

The above Wallis' formula will be used as a step in the proof of Stirling's formula. We have, for $n \geq 2$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^n x \, dx &= \int_0^{\frac{\pi}{2}} \sin x \, \sin^{n-1} x \, dx \\
&= \int_0^{\frac{\pi}{2}} (-\cos x)' \, \sin^{n-1} x \, dx \\
&= -\cos x \, \sin^{n-1} x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, (n-1) \, \sin^{n-2} x \, \cos x \, dx \\
&= (n-1) \int_0^{\frac{\pi}{2}} \cos^2 x \, \sin^{n-2} x \, dx \\
&= (n-1) \int_0^{\frac{\pi}{2}} (1 - \sin^2 x) \, \sin^{n-2} x \, dx
\end{aligned}$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x\,dx.$$

We deduce that

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx.$$

Iterating this we get

$$\int_0^{\frac{\pi}{2}} \sin^{2k} x\,dx = \frac{1\cdot 3\cdot 5\cdot \dots \cdot (2k-1)}{2\cdot 4\cdot 6\cdot \dots \cdot (2k)}\int_0^{\frac{\pi}{2}} dx = \frac{(2k-1)!!}{(2k)!!}\cdot\frac{\pi}{2},$$
$$\int_0^{\frac{\pi}{2}} \sin^{2k+1} x\,dx = \frac{2\cdot 4\cdot 6\cdot \dots \cdot (2k)}{3\cdot 5\cdot 7\cdot \dots \cdot (2k+1)}\int_0^{\frac{\pi}{2}} \sin x\,dx = \frac{(2k)!!}{(2k+1)!!},$$

that is

$$\frac{\pi}{2} = \frac{\int_0^{\frac{\pi}{2}} \sin^{2k} x\,dx}{\frac{(2k-1)!!}{(2k)!!}} = \frac{(2k)!!}{(2k-1)!!}\,\frac{\int_0^{\frac{\pi}{2}} \sin^{2k} x\,dx}{\frac{(2k+1)!!}{(2k)!!}\int_0^{\frac{\pi}{2}} \sin^{2k+1} x\,dx} =$$
$$= \left(\frac{(2k)!!}{(2k-1)!!}\right)^2 \frac{1}{(2k+1)}\,\frac{\int_0^{\frac{\pi}{2}} \sin^{2k} x\,dx}{\int_0^{\frac{\pi}{2}} \sin^{2k+1} x\,dx}. \quad (13.1)$$

Let us observe, that we have

$$0 < \int_0^{\frac{\pi}{2}} \sin^{2k+1} x\,dx \le \int_0^{\frac{\pi}{2}} \sin^{2k} x\,dx \le \int_0^{\frac{\pi}{2}} \sin^{2k-1} x\,dx,$$

so

$$1 \le \frac{\int_0^{\frac{\pi}{2}} \sin^{2k} x\,dx}{\int_0^{\frac{\pi}{2}} \sin^{2k+1} x\,dx} \le \frac{\int_0^{\frac{\pi}{2}} \sin^{2k-1} x\,dx}{\int_0^{\frac{\pi}{2}} \sin^{2k+1} x\,dx} =$$
$$= \frac{(2(k-1))!!}{(2k-1)!!}\cdot\frac{(2k+1)!!}{(2k)!!} = \frac{2k+1}{2k}.$$

The expressions on the beginning and on the end of these inequalities both tend to 1, so from the 3 sequence theorem we see, that the fraction of integrals in (13.1) also tends to 1, and thus

$$\frac{\pi}{2} = \lim_{k\to\infty}\frac{1}{(2k+1)}\left(\frac{(2k)!!}{(2k-1)!!}\right)^2 = \frac{1}{2}\lim_{k\to\infty}\frac{1}{k}\left(\frac{(2k)!!}{(2k-1)!!}\right)^2.$$

We have thus obtained the announced Wallis' formula. Now we will use it in the proof of the Stirling's formula.

The Stirling's formula

The Stirling's formula is the following:

$$\lim_{n\to\infty} \frac{n!\, e^n}{\sqrt{2\,\pi\, n}\, n^n} = 1.$$

Let

$$a_n = \frac{n!\, e^n}{\sqrt{n}\, n^n}.$$

Obviously, $a_n > 0$, and we will also show, that the sequence $\{a_n\}$ is decreasing. First let us observe, that

$$\frac{a_n}{a_{n+1}} = \frac{n!\, e^n}{\sqrt{n}\, n^n} \cdot \frac{\sqrt{n+1}\,(n+1)^{n+1}}{(n+1)!\, e^{n+1}} = \frac{1}{e}\left(\frac{n+1}{n}\right)^{n+\frac{1}{2}} \tag{13.2}$$

We want to show, that the above quantity is greater than 1. Introduce $\frac{1}{x}$ for n, and consider the function being the logarithm of the expression (13.2), multiplied by e.

$$f(x) = \left(\frac{1}{x} + \frac{1}{2}\right) \log(1+x), \quad x > 0. \tag{13.3}$$

We will show, that the function $f(x)$ is always greater than 1, so the function $e^{f(x)}$ is always greater than e, and in particular is greater than e at points of the form $x = \frac{1}{n}$. This obviously implies that the expression (13.2) is greater than 1 for every $n = 1, 2, 3, \dots$, that is the sequence $\{a_n\}$ is decreasing. Let us thus return to the function (13.3), and let us show that $f(x) > 1$ for $x > 0$. This is a typical exercise involving analysing the function. Firstly we have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x}\log(1+x) + \frac{1}{2}\lim_{x\to 0^+} \log(1+x) = 1$$

(the first limit we computed in the past, we use the de l'Hôspital's rule to show that it is equal to 1, while the second limit is simply the continuity of the logarithm). We now compute the derivative, and show that $f(x)$ increases on $x > 0$.

$$f'(x) = -\frac{1}{x^2}\log(1+x) + \left(\frac{1}{x} + \frac{1}{2}\right)\frac{1}{1+x}.$$

We want to show, that for $x > 0$ the above is positive, that is

$$-\frac{1}{x^2}\log(1+x) + \left(\frac{1}{x} + \frac{1}{2}\right)\frac{1}{1+x} > 0,$$

$$-(1+x)\,\log(1+x)+x+\frac{x^2}{2}>0.$$

Consider the auxiliary function

$$g(x)=-(1+x)\log(1+x)+x+\frac{x^2}{2}.$$

We have $g(0)=0$, and $g'(x)=-\log(1+x)-1+1+x=x-\log(1+x)>0$, (the last inequality is simply $e^x>1+x$ for $x>0$). Function $g(x)$ is thus increasing, and since it "takes off" form 0, it must be greater than 0 for $x>0$. We thus have $f'(x)>0$ so indeed $f(x)$ is increasing, and so

$$f(x)>\lim_{t\to 0^+} f(t)=1.$$

We have thus shown that $f(x)$ is greater than 1, and so, as was announced earlier, the expression (13.2) is greater than 1 for all $n\in\mathbf{N}$, and so the sequence $\{a_n\}$ is decreasing. A sequence that is decreasing, with positive terms must be convergent. Let us denote its limit by g.

$$g=\lim_{n\to\infty} a_n.$$

Clearly, since the terms of the sequence are positive, we must have $g\geq 0$. We will show, that this inequality is actually sharp: $g>0$. We will do this by showing that the terms of the sequence are not only positive, but are all greater than some strictly positive number. We have

$$\log\frac{a_n}{a_{n+1}}=\big(n+\frac{1}{2}\big)\,\log\big(1+\frac{1}{n}\big)-1. \tag{13.4}$$

We will need the following inequality:

$$\log\left(1+\frac{1}{n}\right)\leq\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right). \tag{13.5}$$

In Fig. 13.1 the area of the region under the graph, from n to $n+1$ is the integral of the function $\frac{1}{x}$, on the interval $[n,n+1]$, that is $\log(n+1)-\log n=\log(1+\frac{1}{n})$. On the other hand, the area of the trapezium is equal to $\frac{1}{2}(\frac{1}{n}+\frac{1}{n+1})$. The function $\frac{1}{x}$ is convex: $(\frac{1}{x})''=\frac{2}{x^3}>0$. The graph thus lies under any secant between the section points, and therefore the region under the graph is contained within the trapezium. So, the area under the graph is no larger that the area of the trapezium, and this is exactly the estimate (13.5). Plugging (13.5) to (13.4) we get

$$\log\frac{a_n}{a_{n+1}}\leq\frac{1}{2}\left(n+\frac{1}{2}\right)\left(\frac{1}{n}+\frac{1}{n+1}\right)-1$$

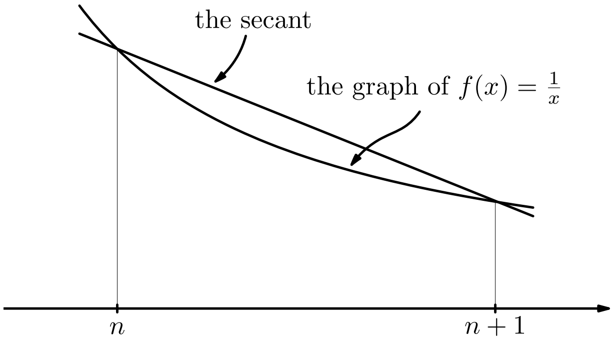

Figure 13.1: The estimate (13.5).

$$= \frac{1}{2}\left(1 + \frac{n}{n+1} + \frac{1}{2\,n} + \frac{1}{2\,n+2}\right) - 1$$
$$= \frac{1}{4\,n} - \frac{1}{4\,n+4}.$$

Adding the above estimates together for $n = 1, \ldots, k-1$ we get

$$\log\frac{a_1}{a_k} = \log\frac{a_1}{a_2} + \log\frac{a_2}{a_3} + \cdots + \log\frac{a_{k-1}}{a_k}$$
$$\leq \sum_{i=1}^{k-1} \frac{1}{4}\left(\frac{1}{i} - \frac{1}{i+1}\right)$$
$$= \frac{1}{4}\left(1 - \frac{1}{k}\right)$$
$$< \frac{1}{4}.$$

So, we have obtained

$$\frac{a_1}{a_k} < e^{\frac{1}{4}} \;\Rightarrow\; a_k > a_1\,e^{-\frac{1}{4}} = e^{\frac{3}{4}},$$

because $a_1 = e$. All terms of the sequence are thus greater than $e^{\frac{3}{4}}$, and so also $g \geq e^{\frac{3}{4}} > 0$. Now we only need some more manipulations.

$$a_n^2 = \frac{(n!)^2\,e^{2\,n}}{n\,n^{2\,n}}, \qquad a_{2n} = \frac{(2\,n)!\,e^{2\,n}}{\sqrt{2\,n}\,(2\,n)^{2\,n}},$$

so

$$\frac{a_n^2}{a_{2n}\sqrt{2}} = \frac{(n!)^2}{n\,n^{2\,n}\sqrt{2}} \cdot \frac{\sqrt{2\,n}\,(2\,n)^{2\,n}}{(2\,n)!} = \frac{(n!)^2\,2^{2\,n}}{(2\,n)!\sqrt{n}}.$$

Notice the relation with the Wallis' formula, which can be written as

$$\frac{1}{\sqrt{n}} \frac{(2\,n)!!}{(2\,n-1)!!} \;\to\; \sqrt{\pi}.$$

Also observe the following

$$(2\,n)!! = 2 \cdot 4 \cdot \cdots \cdot 2\,n = 2^n\, n!,$$
$$(2\,n-1)!! = 1 \cdot 3 \cdot \cdots \cdot (2\,n-1) = \frac{(2\,n)!}{(2\,n)!!} = \frac{(2\,n)!}{2^n\, n!}.$$

Putting these together

$$\frac{1}{\sqrt{n}} \frac{(2\,n)!!}{(2\,n-1)!!} = \frac{1}{\sqrt{n}} \frac{2^n\, n!\, 2^n\, n!}{(2\,n)!} = \frac{(n!)^2\, 2^{2\,n}}{(2\,n)!\sqrt{n}} = \frac{a_n^2}{a_{2n}\,\sqrt{2}}.$$

Finally,

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{a_n^2}{a_{2n}\sqrt{2}} = \frac{g^2}{g\,\sqrt{2}} \;\Rightarrow\; g^2 - \sqrt{2\,\pi}\, g = 0 \;\Rightarrow\; g = \sqrt{2\,\pi},$$

since $g > 0$. Let us observe, that we have this way proved the Stirling's formula.

Chapter 14

Numerical integration

There are a few typical algorithms for numerical integration. We will now describe the method of trapezia, the Simpson's method, and, as a curiosity the Monte Carlo method. In each case we approximate the area of the region under the graph.

The method of trapezia

We want to compute the integral of a function $f(x)$ on an interval $[a, b]$. Function $f(x)$ is approximated by a linear function, which agrees with $f(x)$ on endpoints, and the the integral of $f(x)$ is approximated by the integral of this linear function. Let $y_0 = f(a)$ and $y_1 = f(b)$. The linear function which at points a and b assumes values y_0 and y_1 respectively is given by the formula

$$w(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}\,(x - x_0),$$

and its integral on $[a, b]$ is given by

$$\int_a^b w(x)\,dx = (b-a)\,\frac{y_0 + y_1}{2}.$$

On Fig. 14.1 we see why this method is called the method of trapezia. If the function $f(x)$ has bounded second derivative on $[a, b]$, $|f''(x)| \leq M$ for $x \in (a, b)$, then the error in the method of trapezia is no greater than

$$R \leq \frac{M(b-a)^3}{12}.$$

To increase the accuracy of the approximation of the interval we divide the interval into n subintervals $a = x_0 < x_1 < \cdots < x_n = b$, we let $y_i = f(x_i)$

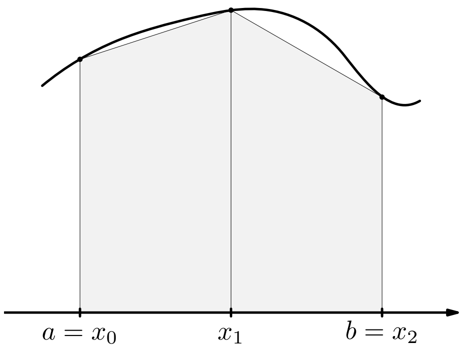

Figure 14.1: The method of trapezia, $n = 2$.

and we write

$$S = \frac{b-a}{2\,n} \sum_{i=0}^{n-1} (y_i + y_{i+1}) = \frac{b-a}{2\,n}\,(y_0 + 2\,y_1 + 2\,y_2 + \cdots + 2\,y_{n-1} + y_n).$$

The Simpson's method

This method is similar the the method of trapezia. The integrand $f(x)$ is approximated by a quadratic function, which agrees with $f(x)$ on the end-points, and in the mid-point of the interval of integration $[a, b]$. Let $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$, and let $y_i = f(x_i)$ for $i = 0, 1, 2$. The quadratic function we are talking about has the form

$$w(x) = \alpha\,x^2 + \beta\,x + \gamma,$$

and satisfies $w(x_i) = y_i$, $i = 0, 1, 2$, that is satisfies the following 3 conditions

$$\begin{aligned} \alpha\,a^2 + \beta\,a + \gamma &= y_0, \\ \alpha\left(\frac{a+b}{2}\right)^2 + \beta\,\frac{a+b}{2} + \gamma &= y_1, \\ \alpha\,b^2 + \beta\,b + \gamma &= y_2. \end{aligned}$$

We do not have to compute α, β nor γ we are going to express the integral of $w(x)$ in terms of y_0, y_1 and y_2.

$$\begin{aligned} \int_a^b w(x)\,dx &= \int_a^b (\alpha\,x^2 + \beta\,x + \gamma)\,dx \\ &= \left(\alpha\,\frac{x^3}{3} + \beta\,\frac{x^2}{2} + \gamma\,x\right)\Big|_a^b \end{aligned}$$

$$
\begin{aligned}
&= \alpha\,\frac{b^3}{3} + \beta\,\frac{b^2}{2} + \gamma\,b - \alpha\,\frac{a^3}{3} - \beta\,\frac{a^2}{2} - \gamma\,a \\
&= \frac{\alpha}{3}(b^3 - a^3) + \frac{\beta}{2}(b^2 - a^2) + \gamma\,(b-a) \\
&= \frac{b-a}{6}(2\alpha b^2 + 2\alpha ab + 2\alpha a^2 + 3\beta b + 3\beta a + 6\gamma) \\
&= \frac{b-a}{6}(\alpha a^2 + \beta a + \gamma + \alpha b^2 + \beta b + \gamma + \alpha(b^2 + 2ab + a^2) + 2\beta(b+a) + 4\gamma) \\
&= \frac{b-a}{6}\left(y_0 + y_2 + 4\,\alpha\left(\frac{b+a}{2}\right)^2 + 4\,\beta\frac{b+a}{2} + 4\,\gamma\right) \\
&= \frac{b-a}{6}(y_0 + 4\,y_1 + y_2).
\end{aligned}
$$

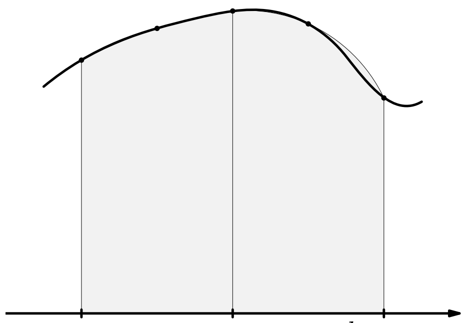

Figure 14.2: The Simpson's method, $n = 2$.

If the function $f(x)$ that we integrate has bounded derivative of order 4, $|f^{(4)}(x)| \leq M$ for $x \in (a, b)$, then the error of the Simpson's method is no greater than

$$
R \leq \frac{M(b-a)^5}{180}.
$$

Again, in applications we subdivide the interval $[a, b]$ into subintervals of equal length to increase accuracy. Suppose we have n such subintervals, and let $a = x_0 < x_1 < \cdots < x_{2n} = b$ be the partition points, and also the midpoints of subintervals. Let $y_i = f(x_i)$ and write

$$
\begin{aligned}
S &= \frac{b-a}{6\,n} \sum_{i=0}^{n-1} (y_{2i} + 4\,y_{2i+1} + y_{2i+2}) \\
&= \frac{b-a}{6\,n}\,(y_0 + y_{2n} + 2\,(y_2 + y_4 + \cdots + y_{2n-2}) + 4\,(y_1 + y_3 + \cdots + y_{2n-1})).
\end{aligned}
$$

The method of trapezia and the Simpson's method are the generic methods of choice in computing integrals numerically. We also sketch a completely different approach, known as the Monte Carlo method.

Monte Carlo

Suppose we have a non-negative, bounded function $f(x)$ on $[a, b]$. We establish the upper bound, say $f(x) \leq M$. We then generate in random n points (x_i, y_i) in the rectangle $[a, b] \times [0, M]$ (we say that we randomly "toss" n points into the rectangle). The probability distribution should be uniform (that is the probability that a point falls into any given region should be proportional to its area), and all the random numbers x_i, y_i, $i = 1, \ldots, n$ should be generated independently (they should be independent random variables). Then we count all the instances in which $y_i < f(x_i)$. These are the points "tossed", which happened to fall into the region under the graph of $f(x)$. Suppose we have counted m of them. Then the proportion $\frac{m}{n}$ should be the same as the proportion of the area under the graph to the entire area of the target recatngle

$$\frac{m}{n} \simeq \frac{\int_a^b f(x)\, dx}{(b-a) \cdot M}.$$

This is the Monte Carlo method for computing the integral.

Chapter 15

Function sequences and series

Let $f_n(x)$, $n = 1, 2, \dots$ be functions defined on some set E. We say that they for a function sequence on E. Let us observe, that for each fixed point $x \in E$ we have a usual, numerical sequence $\{f_n(x)\}$. Such sequence may converge, or it might not. If for each $x \in E$ there exists a limit $\lim_{n\to\infty} f_n(x)$, we say that the function sequence $\{f_n(x)\}$ is pointwise convergent on E. Similarly, if for each $x \in E$ the ordinary series $\sum_{n=1}^{\infty} f_n(x)$ converges, we say that the function series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise on E.

Our goal is to investigate the possibility of interchanging the order of analytic operations on function sequences and series. For example, whether a function series can be differentiated term by term. In other words, whether we can "enter" with the derivative "under" the summation sign. For finite sums we can, but how about series?

Examples: (a) Consider the series

$$\sum_{n=1}^{\infty} n\, q^n.$$

This series converges for $|q| < 1$ (we may check with d'Alembert's criterion)), but can we compute its sum? Let us write

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{for } x \in (-1, 1).$$

The derivative of the function $f(x)$ is easy to compute: $f'(x) = \frac{1}{(1-x)^2}$. If we could differentiate the series $\sum_{n=0}^{\infty} x^n$ term by term, we would obtain

$$\frac{1}{(1-x)^2} = f'(x) = \sum_{n=0}^{\infty} (x^n)' = \sum_{n=1}^{\infty} n\, x^{n-1} = \frac{1}{x} \sum_{n=1}^{\infty} n\, x^n.$$

Thus we would have

$$\sum_{n=1}^{\infty} n\, q^n = \frac{q}{(1-q)^2}, \quad \text{for } |q| < 1.$$

(b) Suppose we are looking for a function $f(x)$ for which

$$f'(x) = \alpha\, f(x). \tag{15.1}$$

Let us try to find a function $f(x)$ like this in the form of a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$. If we could differentiate a series like this term by term, we would have, plugging this into the equation (15.1)

$$f'(x) = \big(\sum_{n=0}^{\infty} a_n x^n\big)' = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, x^n = \sum_{n=0}^{\infty} \alpha\, a_n\, x^n.$$

We see, that it is enough to find coefficients a_n, which satisfy the recurrence relation

$$(n+1)\, a_{n+1} = \alpha\, a_n, \quad \text{for } n = 0, 1, \dots .$$

This recurrence relation can be solved easily:

$$a_{n+1} = \alpha\, \frac{a_n}{n+1} \;\Rightarrow\; a_n = \alpha^n\, \frac{a_0}{n!}.$$

We would then have a solution

$$f(x) = \sum_{n=0}^{\infty} a_0\, \frac{\alpha^n}{n!}\, x^n = a_0\, e^{\alpha\, x}. \tag{15.2}$$

Let us observe, that even though we do not know for a while whether the above reasoning is correct that is whether in the above situation we may indeed differentiate a power series term by term, nevertheless the function given by (15.2) indeed satisfies the equation (15.1)

(c) Let the function series be given by

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}.$$

Observe, that for each fixed $x \in \mathbf{R}$ the sequence converges to zero: $f_n(x) \to 0$, as $n \to \infty$. The terms of the sequence are differentiable functions, and $f_n'(x) = \sqrt{n} \cos nx$. The sequence of derivatives does not converge to the zero function. For example, $f_n'(0) = \sqrt{n} \nrightarrow 0$. We see, that in this case the limit of derivatives is not the derivative of the limit.

(d) Let us consider the function sequence

$$f_n(x) = n\,x\,(1-x^2)^n, \quad \text{for} \quad 0 \le x \le 1.$$

This sequence has a limit at each point, and this limit is the zero function $f(x) \equiv 0$:

$$f_n(0) = 0, \quad \lim_{n\to\infty} f_n(x) = 0 \quad \text{for} \quad x \in (0,1].$$

On the other hand

$$\begin{aligned} \int_0^1 f_n(x)\,dx &= n \int_0^1 x\,(1-x^2)^n\,dx \\ &= -\frac{n}{2}\int_1^0 t^n\,dt \\ &= \frac{n}{2}\int_0^1 t^n\,dt \\ &= \frac{n}{2}\,\frac{t^{n+1}}{n+1}\bigg|_0^1 \\ &= \frac{n}{2\,n+2} \to \frac{1}{2}, \end{aligned}$$

even though $\int_0^1 0\,dx = 0$. In this case the integral of the limit is not equal to limit of integrals.

Examples (a) and (b) show that the interchange of the order of analytic operations like integrating a function series term by term can be useful, while examples (c) and (d) show that the issue is delicate, and sometimes such interchange is not possible. We will now investigate the issue in detail and, for example will show that the power series can indeed be differentiated term by term.

Definition 15.1. *A function sequence $\{f_n(x)\}$ converges uniformly to a function $f(x)$ on the set E, if*

$$\forall\, \epsilon > o \quad \exists\, n_0 \in \mathbf{N} \quad \forall\, n \ge n_0 \quad \forall\, x \in E \quad |f_n(x) - f(x)| < \epsilon,$$

In other words, not only the sequence is convergent at every point, that is pointwise, but n_0 can be chosen independently of $x \in E$. Similarly, the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on E, if the sequence of partial sums

$$s_n(x) = \sum_{i=1}^{n} f_i(x)$$

converges uniformly.

It is worthwhile to think a little about this definition. Uniform convergence on a set E means that the sequence converges at each point, and, additionally the rate of convergence is uniform at all points. Having a given $\epsilon > 0$ we can choose $n_0 \in \mathbf{N}$, which will be sufficient at all points $x \in E$.

Theorem 15.2. *A function sequence $\{f_n(x)\}$ is uniformly convergent on set E if and only if it uniformly satisfies the Cauchy's condition, that is when*

$$\forall\ \epsilon > 0 \quad \exists\ n_0 \in \mathbf{N} \quad \forall\ m, n \geq n_0 \quad \forall x \in E \quad |f_n(x) - f_m(x)| < \epsilon.$$

Proof. If $f_n(x)$ converges uniformly to $f(x)$, then for $\epsilon > 0$ we can find $n_0 \in \mathbf{N}$ such, that $\forall\ m, n \geq n_0 \quad \forall\ x \in E$

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}, \quad |f_m(x) - f(x)| < \frac{\epsilon}{2}.$$

Then

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so we see, that the uniform convergence indeed implies the uniform Cauchy's condition. Now the other way. If the uniform Cauchy's condition is satisfied, then in particular we have a Cauchy's condition at each point $x \in E$. In that case at each point there exists a limit $f(x)$:

$$f(x) = \lim_{n \to \infty} f_n(x), \quad \forall\ x \in E.$$

We have to show that the function sequence converges to $f(x)$ not only pointwise, but uniformly. Let $\epsilon > 0$ be arbitrary, and $n_0 \in \mathbf{N}$ be such, that for $m, n \geq n_0$ and $x \in E$

$$|f_n(x) - f_m(x)| < \epsilon.$$

As $m \to \infty$ the the numerical sequence on the left converges to $|f_n(x) - f(x)|$, an so also

$$|f_n(x) - f(x)| < \epsilon.$$

Since the above holds for all $n \geq n_0$ and $x \in E$, while $\epsilon > 0$ was arbitrary, thus $f_n(x) \to f(x)$ uniformly. □

Theorem 15.3. *A limit of a uniformly convergent sequence of continuous functions is also a continuous function.*

Proof. Let $f_n(x) \to f(x)$ uniformly on the set E, and suppose all functions $f_n(x)$ are continuous on E. Let $x_0 \in E$, and let an arbitrary $\epsilon > 0$ be given. Then using the uniform convergence $f_n(x) \to f(x)$ there exists $n_0 \in \mathbf{N}$ such that

$$\forall\, n \geq n_0 \quad \forall\, x \in E \quad |f_n(x) - f(x)| < \frac{\epsilon}{3}.$$

Then, since the function $f_{n_0}(x)$ is continuous, there exists $\delta > 0$ such, that

$$\forall\, x \in E \quad |x - x_0| < \delta \;\Rightarrow\; |f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\epsilon}{3}.$$

Then, if $|x - x_0| < \delta$ we have

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_{n_0}(x)| + \\ &\quad + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

We have found the required $\delta > 0$, so the limit function $f(x)$ is continuous at x_0. □

Example: Let $f_n(x) = x^n$ on $[0, 1]$. Each of the functions $f_n(x)$ is continuous on the interval $[0, 1]$. It is easy to see, that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 : & x = 1 \\ 0 : & x < 1. \end{cases}$$

The limit of the given function sequence is not continuous (at the point 1), and so $f_n(x)$ cannot converge to $f(x)$ uniformly. This function sequence provides thus an example of a sequence converging pointwise, but not uniformly.

Theorem 15.4. *Let $\{f_n(x)\}$ be a sequence of functions which are integrable on $[a, b]$ in the sense of Riemann, and suppose $f_n(x) \to f(x)$ uniformly on $[a, b]$. Then $f(x)$ is also integrable on $[a, b]$ in the sense of Riemann, and*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx. \tag{15.3}$$

Proof. Let $\epsilon > 0$ be given. From the uniform convergence of the sequence $\{f_n(x)\}$ it follows, that there exists an $n_0 \in \mathbf{N}$ such, that

$$\forall\, n \geq n_0 \quad \forall\, x \in [a, b] \quad |f_n(x) - f(x)| < \epsilon' = \frac{\epsilon}{2\,(b - a)}.$$

It follows that

$$\forall\, n \geq n_0 \quad \forall\, x \in [a, b] \quad f_n(x) - \frac{\epsilon}{2\,(b - a)} < f(x) < f_n(x) + \frac{\epsilon}{2\,(b - a)},$$

and so, in particular the limit function $f(x)$ is bounded. Let us pick a partition P of the interval $[a,b]$, and then

$$U(P,f) \le U\Big(P, f_n + \frac{\epsilon}{2\,(b-a)}\Big),$$

and so

$$\overline{\int_a^b f(x)\,dx} \le \int_a^b \Big(f_n(x) + \frac{\epsilon}{2\,(b-a)}\Big)\,dx = \int_a^b f_n(x)\,dx + \frac{\epsilon}{2}.$$

Similarly,

$$L(P,f) \ge L\Big(P, f_n - \frac{\epsilon}{2\,(b-a)}\Big) \;\Rightarrow\; \underline{\int_a^b f_n(x)\,dx} \ge \int_a^b f_n(x)\,dx - \frac{\epsilon}{2}.$$

We thus have

$$0 \le \overline{\int_a^b f(x)\,dx} - \underline{\int_a^b f_n(x)\,dx} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since ϵ was arbitrary, then the upper and lower integrals coincide, and thus the function $f(x)$ is integrable in the sense of Riemann. Let us prove now the equality (15.3). Let, again, $\epsilon > 0$ be arbitrary, and let $n_0 \in \mathbf{N}$ be such, that

$$\forall\; n \ge n_0 \quad \forall\; x \in [a,b] \quad |f_n(x) - f(x)| < \frac{\epsilon}{(b-a)}.$$

Then

$$\begin{aligned}\Big|\int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx\Big| &= \\ &= \Big|\int_a^b \big(f(x) - f_n(x)\big)\,dx\Big| \le \int_a^b |f(x) - f_n(x)|\,dx \le \epsilon.\end{aligned}$$

Since the above estimate holds for every $n \ge n_0$ we obtain (15.3). □

Remark: The above theorem was proved for proper integrals. It is not necessarily true for improper integrals. For example, let

$$f_n(x) = \begin{cases} \frac{1}{n}\cos\left(\frac{x}{n}\right) : & |x| \le \frac{n\pi}{2} \\ 0 : & |x| > \frac{n\pi}{2} \end{cases}$$

Clearly, $f_n(x) \to 0$ uniformly on the entire real axis$(-\infty, \infty)$, but

$$\int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\frac{n\pi}{2}}^{\frac{n\pi}{2}} \frac{1}{n}\cos\left(\frac{x}{n}\right) dx = \{\frac{x}{n} = t\} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\,dt = \sin t|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2.$$

Therefore we see, that

$$\int_{-\infty}^{\infty} f_n(x)\,dx \nrightarrow \int_{-\infty}^{\infty} 0 \cdot dx = 0.$$

In the case of improper integrals, to pass to the limit under the integral sign we have to assume something extra than just the uniform convergence.

Corollary 15.5. *If the functions $f_n(x)$ are integrable in the sense of Riemann on the interval $[a, b]$, and*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \textit{uniformly on} \quad [a, b],$$

then

$$\int_a^b f(x)\,dx = \int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx.$$

The next theorem theorem provides conditions under which one can "enter" with differentiation "under" a limit.

Theorem 15.6. *Let $\{f_n(x)\}$ be a sequence of differentiable functions on an interval (a, b), such that the sequence of derivatives $\{f_n'(x)\}$ converges uniformly on (a, b). If the sequence $\{f_n(x)\}$ itself converges in at least one point $x_0 \in (a, b)$, then it converges uniformly to some function $f(x)$, differentiable on (a, b), and we have*

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

Proof. By assumption the sequence $\{f_n(x)\}$ converges at a point $x_0 \in (a, b)$. Let $\epsilon > 0$ and let $n_0 \in \mathbf{N}$ be such, that for all $m, n \geq n_0$ we have

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2},$$

(this is the Cauchy's condition for the sequence $\{f_n(x_0)\}$) and

$$|f_n'(x) - f_m'(x)| < \frac{\epsilon}{2\,(b-a)}, \quad x \in (a, b),$$

(this is the uniform Cauchy's condition for $\{f_n'(x)\}$). Next we fix some $m, n \geq n_0$, and introduce an auxiliary function

$$\Phi(x) = f_n(x) - f_m(x).$$

The function $\Phi(x)$ is clearly differentiable, and we apply to it the mean value theorem.

$$\begin{aligned}
|\Phi(x)| &= |\Phi(x) - \Phi(x_0) + \Phi(x_0)| \\
&\leq |\Phi(x) - \Phi(x_0)| + |\Phi(x_0)| \\
&< |\Phi'(\theta)| \cdot |x - x_0| + |\Phi(x_0)| \quad \text{(for some } \theta \text{ between } x \text{ and } x_0) \\
&< \frac{\epsilon}{2\,(b-a)}\,|x - x_0| + \frac{\epsilon}{2} \qquad (15.4) \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

The above estimate holds for all $m, n \geq n_0$ and for all $x \in (a, b)$. ϵ was arbitrary, so the function sequence $\{f_n(x)\}$ satisfies the uniform Cauchy's condition, so by Theorem 15.2 is uniformly convergent to some function $f(x)$. The limit function $f(x)$ as the uniform limit of a sequence of continuous functions is itself continuous. We will now show that it is also differentiable, and that its derivative is actually the limit of the sequence of derivatives $\{f_n'(x)\}$. Let us fix a point $x_1 \in (a, b)$ and consider two auxiliary functions

$$\varphi(x) = \begin{cases} \frac{f(x)-f(x_1)}{x-x_1} & : x \neq x_1, \\ A = \lim_{n\to\infty} f_n'(x_1) & : x = x_1, \end{cases} \qquad \varphi_n(x) = \begin{cases} \frac{f_n(x)-f_n(x_1)}{x-x_1} & : x \neq x_1, \\ f_n'(x_1) & : x = x_1. \end{cases}$$

Let us observe, that at each point $x \in (a, b)$ we have $\varphi_n(x) \to \varphi(x)$. Also, it follows immediately from the definition that functions $\varphi_n(x)$ are continuous everywhere, and the function $\varphi(x)$ is continuous at every point other than x_1. We now want to prove the continuity of $\varphi(x)$ at this one remaining point x_1. The continuity at x_1 would mean precisely that $f(x)$ is differentiable at x_1, and its derivative at that point is the limit of the derivatives of functions $f_n(x)$. Our goal now is to prove, that the convergence $\varphi_n(x) \to \varphi(x)$ is uniform on (a, b), and as a consequence the function $\varphi(x)$ must be continuous (as a uniform limit of continuous functions). Let $m, n \in \mathbf{N}$ be arbitrary, $x \neq x_1$ and let us compute

$$\varphi_n(x) - \varphi_m(x) = \frac{\big(f_n(x) - f_m(x)\big) - \big(f_n(x_1) - f_m(x_1)\big)}{(x - x_1)}$$

$$= \frac{\left(f_n'(\theta) - f_m'(\theta)\right)(x - x_1)}{(x - x_1)},$$

where in the numerator we used the mean value theorem for the function $\Phi(x) = f_n(x) - f_m(x)$, and θ is an intermediate point between x and x_1. According to (15.4) we thus have

$$|\varphi_n(x) - \varphi_m(x)| = |f_n'(\theta) - f_m'(\theta)| < \epsilon,$$

provided $n_0 \in \mathbf{N}$ is sufficiently large, $m, n \geq n_0$, and $x \neq x_1$. Therefore we see, that the sequence $\{\varphi_n(x)\}$ satisfies the uniform Cauchy's condition on the set $E = (a, b) \setminus \{x_1\}$, and is thus uniformly convergent on this set. The sequence is also convergent at the point x_1:

$$\varphi_n(x_1) = f_n'(x_1) \to A = \varphi(x_1). \tag{15.5}$$

Of course, since the sequence $\{\varphi_n(x)\}$ converges uniformly on $(a, b) \setminus \{x_1\}$ and additionally converges at the point x_1, then it is uniformly convergent on the entire interval (a, b). It follows immediately from the simple observation, that if a function sequence converges uniformly on set E_1 and also converges uniformly on the set E_2, then it also converges uniformly on the union $E_1 \cup E_2$. The interval (a, b) is a union of two sets $(a, b) \setminus \{x_1\}$ and an one-element set $\{x_1\}$. The uniform convergence on the first component has just been proved, and on an one-element set the pointwise convergence and the uniform convergence mean the same thing, that is (15.6).

As we have mentioned before, the uniform limit $\{\varphi(x)\}$, must be continuous, and in particular continuous at x_1. This means, that

$$\lim_{n\to\infty} f_n'(x_1) = A = \varphi(x_1) = \lim_{x\to x_1} \varphi(x) = \lim_{x\to x_1} \frac{f(x) - f(x_1)}{x - x_1} = f'(x_1).$$

The point $x_1 \in (a, b)$ was arbitrary, so we have shown that at each point $x \in (a, b)$ we have

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

This finishes the proof. □

We have the following immediate corollary.

Corollary 15.7. *Let the sequence $\{f_n(x)\}$ converge to $f(x)$ uniformly on the interval (a, b), and let $F_n'(x) = f_n(x)$, that is let $F_n(x)$ be the antiderivatives of functions $f_n(x)$. Let us assume additionally that for some $x_0 \in (a, b)$ the sequence $F_n(x_0)$ converges. Then the sequence of antiderivatives $\{F_n(x)\}$*

converges uniformly to some function $F(x)$, and the function $F(x)$ is the antiderivative of the function $f(x)$:

$$F'(x) = f(x)$$

We can formulate this in the notation of indefinite integrals. Let $f_n(x) \to f(x)$ uniformly on (a, b), and let the sequence

$$\int f_n(x)\,dx \tag{15.6}$$

converge at some point of the interval (a, b). Then the sequence (15.6) converges at every point of the interval (a, b) (even uniformly on (a, b)), and

$$\lim_{n\to\infty} \int f_n(x)\,dx = \int \lim_{n\to\infty} f(x)\,dx.$$

Let us stress, that the assumption that the sequence (15.6) converges in at least one point of the interval (a, b) is important, and in fact comes down to the choice of constants of integration for the sequence of indefinite integrals.

The following theorem is a very useful in practice criterion for uniform convergence.

Theorem 15.8 (The Weierstrass' criterion)**.** *If $|f_n(x)| \le a_n$ for $n = 1, 2, \ldots$ and $x \in E$, and if the series $\sum_{n=1}^{\infty} a_n$ converges, then the function series*

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly on the set E.

Proof. The sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$ converges, and therefore satisfies the Cauchy's condition:

$$\forall\, \epsilon > 0 \quad \exists\, n_0 \in \mathbf{N} \quad \forall\, m > n \ge n_0 \quad |s_m - s_n| = \sum_{k=n+1}^{m} a_k < \epsilon.$$

We have, for every $x \in E$

$$\left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x)\right| = \left|\sum_{k=n+1}^{m} f_k(x)\right| \le \sum_{k=n+1}^{m} |f_k(x)| \le \sum_{k=n+1}^{m} a_k < \epsilon.$$

The sequence of partial sums of the function series $\sum_{k=1}^{\infty} f_k(x)$ satisfies thus the uniform Cauchy's condition, and therefore the series converges uniformly. □

Power series

We will now apply the theorems that we proved above to he power series, which are a typical example of function series. We know, that a series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{15.7}$$

converges on the interval $(x_0 - R, x_0 + R)$ (convergence is not guaranteed on the end-points $x_0 \pm R$), where $R > 0$ is the radius of convergence. A series of the form (15.7) is called a power series about x_0. We have so far studied power series about $x_0 = 0$, and the entire theory applies to the current case, simply by the shift of variable $x \mapsto x - x_0$. We have

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}},$$

with $R = \infty$ if $\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|} = 0$, and $R = 0$ if $\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|} = +\infty$, in which case the series (15.7) only converges for $x = x_0$. A power series defines a function, its sum, with the domain being the interval of convergence of the series:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{15.8}$$

We now elaborate on the theorems we proved before on convergence of power series

Theorem 15.9. *1. A power series* (15.8) *converges uniformly on every closed subinterval* $[x_0 - r, x_0 + r]$ *contained inside the interval of convergence, that is* $r < R$*:*

$$[x_0 - r, x_0 + r] \subset (x_0 - R, x_0 + R).$$

2. The series of derivatives

$$\sum_{n=1}^{\infty} n\, a_n\, (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} (x - x_0)^n \tag{15.9}$$

has the same radius of convergence R *as the original series* (15.8)*, and therefore is also uniformly convergent on each closed interval* $[x_0 - r, x_0 + r]$*, for* $r < R$*.*

3. A power series can thus be differentiated and integrated term by term inside the interval of convergence $(x_0 - R, x_0 + R)$*.*

Proof. The proof in fact repeats the proof of Theorem 5.11. Let

$$s_n(x) = \sum_{k=0}^{n} a_k(x - x_0)^k$$

be the sequence of partial sums. Then for $x \in [x_0 - r, x_0 + r]$ we have

$$|a_k(x - x_0)^k| = |a_k|\, |x - x_0|^k \le |a_k|\, r^k. \tag{15.10}$$

Let us observe, that the series

$$\sum_{n=0}^{\infty} |a_n|\, r^n$$

converges, which follows from the Cauchy's criterion:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|\, r^n} = r \cdot \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \frac{r}{R} < 1.$$

Therefore, applying (15.10) and the Weierstrass' criterion we conclude that the power series (15.8) converges uniformly on the interval $[x_0 - r, x_0 + r]$. We have thus proved the first part of the theorem.
Second part: We have

$$\sqrt[n]{|a_{n+1}|(n+1)} = \sqrt[n]{|a_{n+1}|}\; \sqrt[n]{n+1}. \tag{15.11}$$

It is a simple observation, that the upper limit (finite or not) of the sequence (15.11) is the same as the upper limit $\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}$, so the radius of convergence of the series of derivatives (15.9) is the same as the radius of convergence R of the original series (15.8). The series of derivatives is therefore also uniformly convergent on every interval $[x_0 - r, x_0 + r]$, if $r < R$.
Third part: The term by term differentiation and integration of a power series inside the interval of convergence now follows from Theorems 15.4 and 15.6, from the fact that for each $x_1 \in (x_0 - R, x_0 + R)$ we can find $r < R$ such that $x_1 \in [x_0 - r, x_0 + r]$, and from already proved parts of the theorem. □

We have the following corollary.

Corollary 15.10. *A power series $\sum_{n=0}^{\infty} a_n\,(x - x_0)^n$ with a positive radius of convergence $R > 0$ defines on the interval $(x_0 - R, x_0 + R)$ a function which is differentiable infinitely many times*

$$f(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n. \tag{15.12}$$

We also have

$$f^{(n)}(x_0) = n!\, a_n.$$

Proof. Differentiability once follows from the above theorem, and then we reiterate the theorem, sice the series of derivatives is also a power series of the same type, with the same radius of convergence. Differentiating the series (15.12) n-times term by term we obtain, for $x \in (x_0 - R, x_0 + R)$

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k\,(k-1)\cdot \dots \cdot (k-n+1)\, a_k\, (x-x_0)^{k-n}.$$

Plugging in $x = x_0$ we get

$$f^{(n)}(x_0) = n\,(n-1)\cdot \dots \cdot 1\, a_n = n!\, a_n.$$

□

Corollary 15.11. *The Taylor series of the series* (15.12) *is the same series.*

Example: Let us develop into Taylor series the function $f(x) = \frac{1}{1-x}$ about the point $x_0 = \frac{1}{2}$. We can do it simply

$$\frac{1}{1-x} = \frac{1}{\frac{1}{2} - (1 - \frac{1}{2})} = 2\,\frac{1}{1 - 2(x - \frac{1}{2})} = \sum_{n=0}^{\infty} 2^{n+1} \left(x - \frac{1}{2}\right)^n.$$

We know that the power series on the right-hand side converges for $|x - \frac{1}{2}| < \frac{1}{2}$, and its sum is equal to $\frac{1}{1-x}$. In that case, according to the above corollary, the series on the right is the Taylor's series of the function on the left. There is no need to compute even one derivative.

Corollary 15.12. *If two power series with strictly positive radii of convergence*

$$\sum_{n=0}^{\infty} a_n\,(x-x_0)^n \quad \textit{and} \quad \sum_{n=0}^{\infty} b_n\,(x-x_0)^n$$

are equal in some interval $(x_0 - \epsilon, x_0 + \epsilon)$*, then they must be identical:*

$$a_n = b_n \quad n = 0, 1, \dots .$$

Examples: (a) Let $f(x) = \arctan x$. We will develop the function $f(x)$ into Maclaurin's series ($x_0 = 0$).

$$\begin{aligned} \arctan(x) &= \int \frac{dx}{1+x^2} \\ &= \int \sum_{n=0}^{\infty} \left(-x^2\right)^n\, dx \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\,dx \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2\,n+1} \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots .
\end{aligned}
$$

The above follows from the fact, that the power series $\sum_{n=0}^{\infty}(-x^2)^n$ can be integrated term by term. Choosing for the integrals the constants of integration equal to 0 (as in the above computations), the series of integrals converges, for example at the point $x_0 = 0$ to function $\arctan(x)$. Therefore it has to converge to $\arctan(x)$ everywhere insider the interval of convergence. Observe that the resulting series only converges in $(-1, 1]$, even though the domain of $\arctan(x)$ is the entire real axis. As a consequence we have the formula for the derivatives

$$
\arctan^{(n)}(x) = \begin{cases} (-1)^{\frac{n-1}{2}}(n-1)! & : n \text{ - odd} \\ 0 & : n \text{ - even.} \end{cases}
$$

Observe, again, that we did not have to compute any derivatives of $\arctan(x)$.

(b) In the same way we will find an expansion into Maclaurin's series of the function $f(x) = \log(1 + x)$.

$$
\begin{aligned}
\log(1+x) &= \int \frac{dx}{1+x} \\
&= \int \sum_{n=0}^{\infty}(-x)^n\,dx \\
&= \sum_{n=0}^{\infty}(-1)^n \int x^n\,dx \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots
\end{aligned}
$$

(c) The Taylor's series of a function may be convergent, but to a different function. This is a particularly rare pathology, but it may happen nevertheless. For example, let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & : x \neq 0 \\ 0 & : x = 0. \end{cases}$$

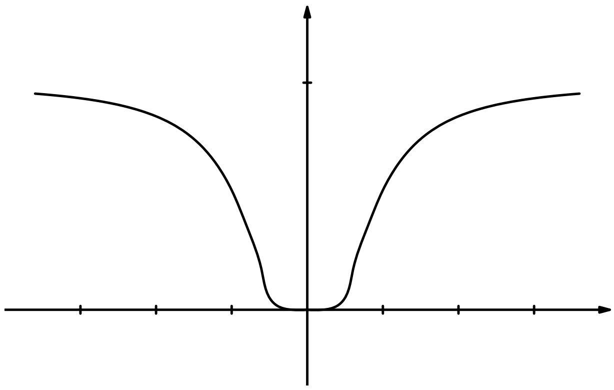

Figure 15.1: The function from example (c).

The function $f(x)$ is differentiable at every point. At points different from 0 it follows directly from the formula for $f(x)$, while at 0 it requires computation. Let us compute the limit of the differential fraction at 0, the left-hand and right-hand limits separately.

$$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x^2}} - 0}{x} = \lim_{y\to +\infty} \frac{e^{-y^2}}{\frac{1}{y}} = \lim_{y\to +\infty} \frac{y}{e^{y^2}} = \lim_{y\to +\infty} \frac{1}{2\,y\,e^{y^2}} = 0.$$

We similarly compute the left-hand limit, as $x \to 0^-$. The derivative $f'(0)$ thus exists, and is equal to 0. Outside zero, from the definition of $f(x)$ we have

$$f'(x) = \frac{2}{x^3}\, e^{-\frac{1}{x^2}}, \quad x \neq 0.$$

Similarly as the first derivative,using the de l'Hôspital's rule we check that $f''(0) = 0$. It is not hard to see, that the derivative of an arbitrary order $f^{(n)}(x)$, at $x \neq 0$ is a sum of terms of the form $\frac{1}{x^k}e^{-\frac{1}{x^2}}$, and we can prove inductively that $f^{(n)}(0)$ exists for every $n \in \mathbf{N}$, and is equal to 0. The function $f(x)$ is thus differentiable infinitely many times, and its Taylor's series at 0 (Maclaurin's series) is the zero series

$$0 + 0 \cdot x + 0 \cdot x^2 + \cdots = 0.$$

On the other hand clearly $f(x) \neq 0$ for $x \neq 0$, so the function nowhere, except for 0, equals to its Taylor's series.

(d) We will find the formula for the sum of the series $\sum_{n=1}^{\infty} n^2 x^n$. The interval of convergence of this series is, as can be easily checked, the interval $(-1, 1)$. We have

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^2\, x^n &= \sum_{n=1}^{\infty} (n+2)\,(n+1)\, x^n - \sum_{n=1}^{\infty} 3\, n\, x^n - \sum_{n=1}^{\infty} 2\, x^n \\
&= \sum_{n=1}^{\infty} \left(x^{n+2}\right)'' - 3 \sum_{n=1}^{\infty} (n+1)\, x^n + \sum_{n=1}^{\infty} x^n \\
&= \sum_{n=1}^{\infty} \left(x^{n+2}\right)'' - 3 \sum_{n=1}^{\infty} \left(x^{n+1}\right)' + \sum_{n=1}^{\infty} x^n \\
&= \left(\sum_{n=1}^{\infty} x^{n+2}\right)'' - 3 \left(\sum_{n=1}^{\infty} x^{n+1}\right)' + \sum_{n=1}^{\infty} x^n \\
&= \left(\sum_{n=3}^{\infty} x^{n}\right)'' - 3 \left(\sum_{n=2}^{\infty} x^{n}\right)' + \sum_{n=1}^{\infty} x^n \\
&= \left(x^3 \sum_{n=0}^{\infty} x^{n}\right)'' - 3 \left(x^2 \sum_{n=0}^{\infty} x^{n}\right)' + x \sum_{n=0}^{\infty} x^n \\
&= \left(\frac{x^3}{1-x}\right)'' - 3 \left(\frac{x^2}{1-x}\right)' + \frac{x}{1-x} \\
&= \frac{2\,x^3 - 6\,x^2 + 6\,x}{(1-x)^3} - \frac{6\,x - 3\,x^2}{(1-x)^2} + \frac{x}{1-x} \\
&= \frac{2\,x^3 - 5\,x^2 + 5\,x}{(1-x)^3}.
\end{aligned}
$$

Testing for uniform convergence

Let us write down simple facts, which in most cases will allow us to determine whether the convergence of a function sequence is uniform. Let $f_n(x) \to f(x)$ at each point $x \in E$.

(a) If $|f_n(x) - f(x)| \leq \alpha_n$ for each $x \in E$ and $\alpha_n \to 0$, then $f_n(x) \to f(x)$ uniformly on E.

(b) If there exists a sequence $\{x_n\} \subset E$ such that $|f_n(x_n) - f(x_n)|$ does not converge to 0, then $f_n(x)$ is not uniformly convergent to $f(x)$ on E.

(c) If $E = E_1 \cup E_2$ and $f_n(x) \to f(x)$ uniformly on both E_1 and E_2, then $f_n(x) \to f(x)$ uniformly on E. In practice this means, that the uniform convergence can be verified separately on subintervals. This might apply to situations, where the functions are defined by different formulas on different subintervals, or estimates are different on different subintervals.