CALCULUS
PROBLEMS LIST 4
18.10.2011

(1) Prove the inequality: $2^k < (k+1)!$ for each natural $k \geq 2$.

(2) Prove the Bernoulli's inequality: for $x > -1$ and any $n \in \mathbf{N}$

$$(1+x)^n \geq 1 + nx.$$

(3) Show that for $x > 0$ and any $n \in \mathbf{N}$ we have

$$(1+x)^n > 1 + \frac{n(n-1)}{2}x^2.$$

(4) Prove, that for any $n \in \mathbf{N}$ the following inequalities hold

$$\text{(a)} \qquad \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n,$$

$$\text{(b)} \qquad \sum_{\substack{k=1\\ \text{k-odd}}}^{n} \binom{n}{k} = \sum_{\substack{k=0\\ k\text{-even}}}^{n} \binom{n}{k}.$$

(5) Show, that for any natural number n we have the inequality $\binom{2n}{n} < 4^n$.

(6) Prove, that for any number $a \in \mathbf{R}$ or $a \in \mathbf{C}$ satisfying the condition $|a| < 1$ we have $\lim_{n\to\infty} a^n = 0$.

(7) Find the limits:
(a) $\lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right)^n$, (b) $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$.

(8) Find the limits of sequences:
(a) $a_n = \sqrt[n]{2^n + 3^n}$, (b) $a_n = \sqrt[n]{2^n + 3^n + 5^n}$.

(9) For which real α does the limit

$$\lim_{n\to\infty} \sqrt[3]{n + n^\alpha} - \sqrt[3]{n}$$

exist? Find this limit for those α for which it exists.

(10) Compute the limits:
(a) $\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2}$, (b) $\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$.

(11) Compute the limits of sequences:
(a) $a_n = \frac{\sin^2 n}{n}$, (b) $a_n = \sqrt[n]{\log n}$,
(c) $a_n = \frac{1}{n^2} \log\left(1 + \frac{(-1)^n}{n}\right)$.

(12) Prove, that if $a_n \xrightarrow{n\to\infty} g$ then the sequence of absolute values $\{|a_n|\}$ is also convergent, and

$$\lim_{n\to\infty} |a_n| = |g|.$$

Show that the above theorem does not hold the other way around, that is find a sequence $\{a_n\}$ which is not convergent, even though $\{|a_n|\}$ does converge.

(13) Prove, that if $|a_n| \xrightarrow{n\to\infty} 0$ then $\{a_n\}$ also converges to 0.

(14) Prove, that if sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_n \leq b_n$ and are convergent, then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

(15) The sequence a_n is given in the following way: $a_1 = 0$, $a_2 = 1$, and

$$a_{n+2} = \frac{a_n + a_{n+1}}{2}, \quad \text{for } n = 1, 2, \ldots.$$

Show that

$$\lim_{n\to\infty} a_n = \frac{2}{3}.$$

(16) Show that if $a_n \xrightarrow{n\to\infty} 0$ and the sequence $\{b_n\}$ is bounded, then

$$\lim_{n\to\infty} (a_n \cdot b_n) = 0.$$

(17) Show that if $a_n > 0$ for all $n \in \mathbf{N}$ and $a_n \xrightarrow{n\to\infty} 0$ then

$$\lim_{n\to\infty} \frac{1}{a_n} = \infty$$

(improper limit).

(18) Given is a sequence $\{b_n\}$, about which it is known, that

$$\forall\ \epsilon > 0\ \forall\ n \geq 10/\epsilon \quad |b_n + 2| < \epsilon.$$

Find M such that

$$\forall\ n \in \mathbf{N} \quad |b_n| < M,$$

n_1 such that

$$\forall\ n \geq n_1 \quad b_n < 0,$$

n_2 such that

$$\forall\ n \geq n_2 \quad b_n > -3,$$

and n_3 such that

$$\forall\ n \geq n_3 \quad |b_n - 2| > \frac{1}{10}.$$

(19) Let $a_n = \dfrac{\sqrt{n^2+n}}{n}$ and $\epsilon = \dfrac{1}{100}$. Find $n_0 \in \mathbf{N}$ such, that for $n \geq n_0$ we have $|a_n - 1| < \epsilon$.