CALCULUS
PROBLEMS LIST
17.10.2013

(1) Find the first 10 terms and the limit of the sequence $\{a_n\}$ given by the formula: $a_n = \frac{(-1)^n}{n^2}$.

(2) What are the values taken by the sequence: $a_n = \sin\frac{n\pi}{2}$?
And the sequence $a_n = \cos\frac{n\pi}{3} + i\,\sin\frac{n\pi}{3}$?

(3) The Fibonacci sequence is defined inductively in the following way: $F_1 = F_2 = 1$, and then $F_{n+2} = F_{n+1} + F_n$ for $n = 1, 2, 3, \ldots$. Compute terms of this sequence numbered from 3 till 12. Prove, that for every natural number n the following inequality holds: $F_{n+2} \cdot F_n - F_{n+1}^2 = (-1)^{n+1}$.

(4) Using **only** the definition, prove the convergence of the following sequences, by finding their limits:
(a) $a_n = \frac{1}{n^2}$, (b) $a_n = \frac{(-1)^n}{n}$,
(c) $a_n = \left(\frac{2}{3}\right)^n$, (d) $a_n = \frac{n+2}{n-1}$, $n \geq 2$,
(e) $a_n = \frac{1}{1+\sqrt{n}}$, (f) $a_n = \frac{3n^3 - 2n^2 - 7n + 5}{4n^3 + n - 6}$.

(5) Prove that if x is a real number with the decimal expansion

$$\beta, \alpha_1\alpha_2\cdots,$$

then the sequence given by the formula

$$a_n = \beta, \alpha_1 \cdots \alpha_n$$

is convergent to x (, is the decimal point and $\beta \in \mathbf{Z}$).

(6) Prove that the limit of the sum (difference, quotient) of convergent sequences is the sum (difference, quotient) of their limits. Of course, in the case of the quotient we assume that the sequence in the denominator had non-zero terms and its limit is different from zero.

(7) Check the monotonicity of the sequences:
(a) $a_n = n + \frac{1}{n}$, (b) $a_1 = 3,\ a_{n+1} = a_n^2 - 2$,
(c) $a_n = \sqrt[n]{n!}$, (d) $a_n = \sqrt[n]{2^n + 3^n}$
(e) $a_n = \frac{2^n}{n!}$, (f) $a_1 = 1,\ a_{n+1} = \frac{a_n}{1+a_n}$.

(8) Find the limits (perhaps improper) of the sequences:
(a) $a_n = \frac{7n + (\sqrt[3]{n}\sqrt[6]{n})^5\sqrt{9n+1}}{11n^3 + 7n + 3}$, (b) $a_n = \sqrt{n^2+n} - n$,
(c) $a_n = \frac{\sin n}{n}$, (d) $a_n = r^n,\ r > 1$,
(e) $a_n = \sqrt[n]{r},\ 0 < r < 1$, (f) $a_n = 2^n - \frac{1}{n}$,

(g) $a_n = \dfrac{\sqrt[3]{n^2+n}}{n+2}$, (h) $a_n = \dfrac{1+2+4+\cdots+2^n}{1+3+9+\cdots+3^n}$,

(i) $a_n = \dfrac{1-2+3-4+5-6+\cdots-2n}{\sqrt{n^2+2}}$, (j) $a_n = \dfrac{1+2+\cdots+n}{n^2}$,

(k) $a_n = \dfrac{1+3+9+\cdots+3^n}{3^n}$, (l) $a_n = \sqrt{3^n+2^n}\sqrt{3^n+1}$,

(m) $a_n = \sqrt[(n^2)]{n}$, (n) $a_n = \sqrt[n]{n^2}$,

(o) $a_n = n(\sqrt{n^2+7}-n)$, (p) $a_n = \dfrac{n^2+n+1}{(n+\sin n)^2}$,

(q) $a_n = \dfrac{n^2+1}{n^3+1} + \dfrac{n^2+2}{n^3+2} + \dfrac{n^2+3}{n^3+3} + \cdots + \dfrac{n^2+n}{n^3+n}$,

(r) $a_n = \dfrac{1}{n^2} + \dfrac{1}{n^2+1} + \dfrac{1}{n^2+2} + \cdots + \dfrac{1}{(n+1)^2}$,

(s) $a_n = \dfrac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+7}-\sqrt{n}}$, (t) $a_n = r^n,\ -1 < r < 1$.

(9) Write out the formula for a sequence for which $a_1 = 1,\ a_2 = \frac{1}{2}$, and each consecutive term is the harmonic average of its neighbors:

$$\frac{1}{a_n} = \frac{1}{2}\Big(\frac{1}{a_{n-1}} + \frac{1}{a_{n+1}}\Big), \quad n \geq 2.$$

(10) Write out the formula for a sequence for which $a_1 = 1,\ a_2 = 2$, and each consecutive term is the geometric average of its neighbors:

$$a_n = \sqrt{a_{n-1}a_{n+1}}, \quad n \geq 2.$$

(11) Prove the inequality: $2^k < (k+1)!$ for each natural $k \geq 2$.

(12) Prove the Bernoulli's inequality: for $x > -1$ and any $n \in \mathbf{N}$

$$(1+x)^n \geq 1 + nx.$$

(13) Show that for $x > 0$ and any $n \in \mathbf{N}$ we have

$$(1+x)^n > 1 + \frac{n(n-1)}{2}x^2.$$

(14) Prove, that for any $n \in \mathbf{N}$ the following inequalities hold

$$\text{(a)} \quad \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n,$$

$$\text{(b)} \quad \sum_{\substack{k=1\\ \text{k-odd}}}^{n} \binom{n}{k} = \sum_{\substack{k=0\\ k\text{-even}}}^{n} \binom{n}{k}.$$

(15) Show, that for any natural number n we have the inequality $\binom{2n}{n} < 4^n$.

(16) Prove, that for any number $a \in \mathbf{R}$ or $a \in \mathbf{C}$ satisfying the condition $|a| < 1$ we have $\lim_{n\to\infty} a^n = 0$.

(17) Find the limits:

(a) $\lim_{n\to\infty} \big(1 + \frac{1}{n^2}\big)^n$, (b) $\lim_{n\to\infty} \big(1 - \frac{1}{n}\big)^n$.

(18) Find the limits of sequences:

(a) $a_n = \sqrt[n]{2^n+3^n}$, (b) $a_n = \sqrt[n]{2^n+3^n+5^n}$.

(19) For which real α does the limit

$$\lim_{n\to\infty} \sqrt[3]{n+n^\alpha} - \sqrt[3]{n}$$

exist? Find this limit for those α for which it exists.

(20) Compute the limits:

(a) $\displaystyle\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2}$, (b) $\displaystyle\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$.

(21) Compute the limits of sequences:

(a) $a_n = \dfrac{\sin^2 n}{n}$, (b) $a_n = \sqrt[n]{\log n}$,

(c) $a_n = \dfrac{1}{n^2}\log\big(1+\dfrac{(-1)^n}{n}\big)$.

(22) Prove, that if $a_n \xrightarrow{n\to\infty} g$ then the sequence of absolute values $\{|a_n|\}$ is also convergent, and

$$\lim_{n\to\infty} |a_n| = |g|.$$

Show that the above theorem does not hold the other way around, that is find a sequence $\{a_n\}$ which is not convergent, even though $\{|a_n|\}$ does converge.

(23) Prove, that if $|a_n| \xrightarrow{n\to\infty} 0$ then $\{a_n\}$ also converges to 0.

(24) Prove, that if sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_n \le b_n$ and are convergent, then

$$\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n.$$

(25) The sequence a_n is given in the following way: $a_1 = 0$, $a_2 = 1$, and

$$a_{n+2} = \frac{a_n + a_{n+1}}{2}, \quad \text{for } n = 1, 2, \ldots.$$

Show that

$$\lim_{n\to\infty} a_n = \frac{2}{3}.$$

(26) Show that if $a_n \xrightarrow{n\to\infty} 0$ and the sequence $\{b_n\}$ is bounded, then

$$\lim_{n\to\infty} (a_n \cdot b_n) = 0.$$

(27) Show that if $a_n > 0$ for all $n \in \mathbf{N}$ and $a_n \xrightarrow{n\to\infty} 0$ then

$$\lim_{n\to\infty} \frac{1}{a_n} = \infty$$

(improper limit).

(28) Given is a sequence $\{b_n\}$, about which it is known, that

$$\forall\, \epsilon > 0\ \forall\, n \ge 10/\epsilon \quad |b_n + 2| < \epsilon.$$

Find M such that

$$\forall\, n \in \mathbf{N} \quad |b_n| < M,$$

n_1 such that

$$\forall\, n \ge n_1 \quad b_n < 0,$$

n_2 such that

$$\forall\, n \ge n_2 \quad b_n > -3,$$

and n_3 such that

$$\forall\, n \ge n_3 \quad |b_n - 2| > \frac{1}{10}.$$

(29) Let $a_n = \frac{\sqrt{n^2+n}}{n}$ and $\epsilon = \frac{1}{100}$. Find $n_0 \in \mathbf{N}$ such, that for $n \geq n_0$ we have $|a_n - 1| < \epsilon$.