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Radon–Nikodym Derivatives of Finitely Additive Interval Measures Taking Values in a Banach Space with Basis

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Abstract Let X be a Banach space with a Schauder basis $\{e_n\}$, and let $\Phi(I) = \sum_{n=1}^{\infty} e_n \int_I f_n(t)dt$ be a finitely additive interval measure on the unit interval $[0, 1]$, where the integrals are taken in the sense of Henstock–Kurzweil. Necessary and sufficient conditions are given for Φ to be the indefinite integral of a Henstock–Kurzweil–Pettis (or Henstock, or variational Henstock) integrable function $f : [0, 1] \to X$.

Keywords Henstock–Kurzweil integral, Henstock–Kurzweil–Pettis integral, Henstock integral, variational Henstock integral, Pettis integral

MR(2000) Subject Classification 28B05, 26A39, 46G10

1 Introduction

This paper deals with the problem of representing a vector-valued finitely additive interval measure as an integral. When the measure is countably additive there is abundant literature (see, e.g., [1] and the reference there) describing the topic, while there are few results if the measure is purely additive. In [2] and [3] we studied the last problem for Banach spaces possessing the Radon–Nikodym or the weak Radon–Nikodym property. Here we go on in such investigation and consider the case of finitely additive interval measures taking values in a Banach space with basis.

It has been proven in [4] that if (Ω, Σ, μ) is a complete probability space, X is a Banach space with a Schauder basis $\{e_n\}$ and ν is an X-valued (countably additive) measure of the form $\nu(E) = \sum_n e_n \int_E f_n \, d\mu$, then ν has a Pettis integrable Radon–Nikodym derivative if and only if the series $\sum_n f_n e_n$ is a.e. convergent in the norm topology of X. We present here analogues of that result in case of finitely additive interval measures. More precisely, we take into account a finitely additive interval measure Φ defined on subintervals of $[0, 1]$ and being of the form $\Phi(I) = \sum_{n=1}^{\infty} e_n \int_I f_n(t)dt$ (I is an interval and the integrals are taken in the sense of Henstock–Kurzweil) and we look for conditions guaranteeing the representation of Φ in the

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form $\Phi(I) = \int_I f(t)dt$, where the integral is now the Henstock–Kurzweil–Pettis integral. As one could expect, in the finitely additive case the situation is much more complicated, and the ordinary replacing, in the result of Lipecki and Musiał [4], of the measure ν by Φ does not provide a correct characterization (see Example 3.1). In Section 3 we present a proper generalization of the Lipecki–Musiał result for the Henstock–Kurzweil–Pettis integral (see Theorem 3.9 and Corollary 3.11). We obtain also some sufficient conditions on the target Banach space X guaranteeing, for a finitely additive interval measure Φ, the existence of a Henstock–Kurzweil–Pettis integrable Radon–Nikodym derivative (see Proposition 3.6 and Theorem 3.13). In Section 4 we characterize finitely additive interval measures possessing a variational Henstock (or a Henstock) integrable Radon–Nikodym derivative (see Theorem 4.3 and Theorem 4.4). We do not investigate the representation for the Riemann integral (cf. [5]) but we are going to do it in future.

2 Basic Facts

Throughout this paper, if not specified explicitly, X is a Banach space with a normalized base $\{e_n\}$ and $\{e_n^*\}$ is the associated biorthogonal sequence in the dual space X^* of X. A basis $\{e_n\}$ is said to be *shrinking* if the closed linear subspace of X^* spanned by the sequence $\{e_n^* : n \in \mathbb{N}\}$ coincides with X^*.

The unit interval $[0, 1]$ of the real line is equipped with the usual topology and the Lebesgue measure λ. We denote by $\mathcal{L}$ the family of all Lebesgue measurable subsets of $[0, 1]$ and by $\mathcal{I}$ the collection of all closed subintervals of the interval $[0, 1]$. For $E \in \mathcal{L}$ the symbol $|E|$ denotes the Lebesgue measure of E.

A *partition in* $[0, 1]$ is a collection of pairs $\mathcal{P} = \{(I_1, t_1), \ldots, (I_p, t_p)\}$, where $I_1, \ldots, I_p$ are non-overlapping subintervals of $[0, 1]$ and $t_i \in I_i$, $i = 1, \ldots, p$. If $\bigcup_{i=1}^p I_i = [0, 1]$, we say that $\mathcal{P}$ is a *partition of* $[0, 1]$. Given a subset E of $[0, 1]$, we say that the partition $\mathcal{P}$ is *anchored on* E if $t_i \in E$ for each $i = 1, \ldots, p$. A *gauge* on $[0, 1]$ is a positive function on $[0, 1]$. For a given gauge δ on $[0, 1]$, we say that a partition $\mathcal{P}$ is δ*-fine* if $I_i \subset (t_i - \delta(x_i), t_i + \delta(x_i))$, $i = 1, \ldots, p$.

Definition 2.1 *A function $f\colon [0, 1] \to X$ is said to be Henstock integrable (briefly H-integrable) on $[0, 1]$ if there exists $w \in X$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[0, 1]$ such that*

$$\left\| \sum_{i=1}^{p} f(t_i)|I_i| - w \right\| < \varepsilon, \tag{2.1}$$

for each δ-fine partition $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ of $[0, 1]$.

We set $(H)\int_0^1 f dt := w$. It is known that if $f\colon [0, 1] \to X$ is H-integrable on $[0, 1]$ and $I \in \mathcal{I}$, then also $f\chi_I$ is H-integrable on $[0, 1]$. We say in such a case that f is H-integrable on I. We call the finitely additive interval function $F(I) := (H)\int_I f dt$ the H-*primitive* of f. By $H([0, 1], X)$ we denote the set of all H-integrable functions $f\colon [0, 1] \to X$. In case of $X = \mathbb{R}$ we will rather use the name of Henstock–Kurzweil instead of Henstock only and we will denote by $(\mathrm{HK})\int_I f dt$ the corresponding integral and by $\mathrm{HK}[0, 1]$ the space of all HK-integrable functions $f\colon [0, 1] \to \mathbb{R}$ (we identify functions that are equal a.e.).

We recall that the Alexiewicz norm in HK[0, 1] is defined by

$$\|f\|_A := \sup_{0<s\leq 1} \left| (\mathrm{HK}) \int_0^s f(t)dt \right|.$$

The conjugate space of HK[0, 1] is linearly isometric to the space BV[0, 1] of functions of bounded variation (see [6]). The weak topology of HK[0, 1] will be denoted by $\sigma(\mathrm{HK}, \mathrm{BV})$.

A family $\{f_\alpha\}_{\alpha\in A}$ of functions $f_\alpha\colon [0, 1] \to X$ is said to be *Henstock equiintegrable* (briefly H-*equiintegrable*) if, in the definition of H-integrability, for each $\varepsilon > 0$ it is possible to find a gauge δ satisfying inequality (2.1) for all the functions of the family.

A generalization of the classical notion of Pettis integral is that of Henstock–Kurzweil–Pettis integral where the Henstock–Kurzweil integral takes the place of the Lebesgue one (see [7–10]).

Definition 2.2 *A function $f\colon [0, 1] \to X$ is said to be scalarly Henstock–Kurzweil integrable (briefly scalarly* HK-*integrable) on* [0, 1] *if, for each $x^* \in X^*$, the function x^*f is Henstock–Kurzweil integrable on* [0, 1]. *A scalarly Henstock–Kurzweil integrable function $f\colon [0, 1] \to X$ is said to be Henstock–Kurzweil–Dunford integrable (briefly* HKD-*integrable) on* [0, 1] *if, for each interval $I \in \mathcal{I}$, there exists a vector $\Psi(I) \in X^{**}$ such that, for every $x^* \in X^*$,*

$$x^*\Psi(I) = (\mathrm{HK}) \int_I x^*f(t)dt. \tag{2.2}$$

If $\Psi(I) \in X$, for each $I \in \mathcal{I}$, then f is said to be Henstock–Kurzweil–Pettis integrable (briefly HKP-*integrable) on* [0, 1]. *We call $\Psi(I)$ the Henstock–Kurzweil–Pettis integral of f over I and we write* $(\mathrm{HKP})\int_I f(t)dt := \Psi(I)$. *Moreover by the symbol* $\mathrm{HKP}([0, 1], X)$ *we denote the family of all X-valued Henstock–Kurzweil–Pettis integrable functions on* [0, 1].

It follows from [11, Theorem 3] that a function $f\colon [0, 1] \to X$ is HKD-integrable if and only if f is scalarly Henstock–Kurzweil integrable (Gamez and Mendoza consider there the Denjoy–Khintchine integral, but the proof works also for the Henstock–Kurzweil integral).

In the following we will identify a finitely additive interval function $\Phi\colon \mathcal{I} \to X$ with the corresponding finitely additive (vector) measure defined in the field generated by $\mathcal{I}$ and will say simply "interval measure" instead of "finitely additive interval measure".

We say that an interval measure $\Phi\colon \mathcal{I} \to X$ is *continuous* if the pointwise function $\Phi(a) := \Phi([0, a])$ is continuous. $\Phi\colon \mathcal{I} \to X$ is called *scalarly continuous* if $x^*\Phi$ is continuous for each $x^* \in X^*$.

Definition 2.3 *Given an additive interval function $\Phi\colon \mathcal{I} \to X$, a gauge δ and a set $E \subset [0, 1]$, we define*

$$\mathrm{Var}(\Phi, \delta, E) = \sup\left\{ \sum_{i=1}^{p} \|\Phi(I_i)\| : \{(I_i, t_i) : i = 1, \ldots, p\} \ \delta\text{-fine partition anchored on } E \right\}.$$

Then we set

$$\mathrm{V}_\Phi(E) = \inf\{\mathrm{Var}(\Phi, \delta, E) : \delta \text{ gauge on } E\}.$$

We call V_Φ the *variational measure generated by* Φ. V_Φ is known to be a metric outer measure in [0, 1] (see [12]). In particular, V_Φ restricted to Borel subsets of [0, 1] is a measure. We say that the variational measure V_Φ is *absolutely continuous* if for $E \in \mathcal{L}$, $|E| = 0$ implies $V_\Phi(E) = 0$. In such a case we write $V_\Phi \ll \lambda$.

3 Henstock–Kurzweil–Pettis Integrable Derivatives of an Interval Measure

We start with an example of a finitely additive interval measure $\Phi\colon \mathcal{I} \to X$ such that $\Phi(I) = \sum_{n=1}^{\infty} e_n \int_I f_n(t)dt$, $\sum_{n=1}^{\infty} f_n e_n$ is a.e. convergent, and there exists $x^* \in X^*$ with

$$\int_0^1 x^* \sum_{n=1}^{\infty} f_n(t)e_n dt \neq \sum_{n=1}^{\infty} x^*(e_n) \int_0^1 f_n(t)dt.$$

This will show that the direct replacing of ν by Φ in [4] is impossible.

Example 3.1 Let $a \neq 0$ be an arbitrary real number and let $I_n := (2^{-n}, 2^{-n+1})$ for every $n \in \mathbb{N}$. Define

$$f_1 := \frac{a}{|I_1|}\chi_{I_1},$$

and

$$f_n := -\frac{a}{|I_{n-1}|}\chi_{I_{n-1}} + \frac{a}{|I_n|}\chi_{I_n},$$

for $n > 1$. We have $\sum_n |f_n(t)| < \infty$, for every $t \in [0,1]$. So $f(t) := \sum_n f_n(t)e_n$ defines an l_1-valued function.

We have (HK)$\int_0^1 f_n(t)dt = 0$, for $n > 1$ and (HK)$\int_0^1 f_1(t)dt = a$. Notice also that for a fixed $I \in \mathcal{I}$, we have $\int_I f_n(t)dt = 0$, for sufficiently large $n \in \mathbb{N}$, and so we can define $\Phi\colon \mathcal{I} \to l_1$ by formula

$$\Phi(I) := \sum_{n=1}^{\infty} e_n \int_I f_n(t)dt.$$

Take $x^* = (1,1,1,\ldots)$. Then $x^* \in l_\infty$, and

$$x^* f(t) = \sum_{n=1}^{\infty} f_n(t) = 0 \quad \text{for every } t \in [0,1].$$

Moreover,

$$\sum_{n=1}^{\infty} x^*(e_n) \int_0^1 f_n(t)dt = \sum_{n=1}^{\infty} \int_0^1 f_n(t)dt = a \neq \int_0^1 x^* f(t)dt = 0.$$

Remark 3.2 In [13, Theorem 40 and Example 41], Gordon proved that a separable Banach space X is weakly sequentially complete if and only if each X-valued scalarly HK-integrable function is also HKP-integrable. Since l_1 is weakly sequentially complete, it follows that the function f in Example 3.1 is not scalarly HK-integrable.

Example 3.1 motivates the hypothesis of scalar HK-integrability of $\sum_{n=1}^{\infty} f_n e_n$, in the next propositions.

We begin with necessary and sufficient conditions for an interval measure to be represented as a Henstock–Kurzweil–Pettis integral of a Banach space valued function that are almost obvious but their verification may be difficult.

We need the following proposition, proved in [14] by a different method.

Proposition 3.3 *Let g and g_n, $n = 1,2,\ldots,$ be real valued* HK*-integrable functions defined on $[0,1]$, and let Ψ and Ψ_n, $n = 1,2,\ldots$, be the* HK*-primitives of g and g_n, respectively. Then $\{g_n\}$ is $\sigma(\mathrm{HK},\mathrm{BV})$-convergent to g if and only if*

(i) $\sup_n \|g_n\|_A < +\infty$;

(ii) $(\mathrm{HK})\int_I g_n(t)dt \to (\mathrm{HK})\int_I g(t)dt$, *for each* $I \in \mathcal{I}$.

Proof The "only if" part is obvious. To prove the "if" part, remark that the functions Ψ_n, $n = 1, 2, \ldots$, are continuous and by (i) they are also uniformly bounded. Then, applying the integration by parts formula (see [15]), for each function h of bounded variation, by (ii), we get

$$\lim_n (\mathrm{HK})\int_0^1 hg_n\,dt = \lim_n \left[\Psi_n(1)\,h(1) - \int_0^1 \Psi_n dh\right] = \Psi(1)h(1) - \int_0^1 \Psi dh$$
$$= (\mathrm{HK})\int_0^1 hg\,dt. \qquad \square$$

Theorem 3.4 *Let* $\Phi\colon \mathcal{I} \to X$ *be an interval measure such that for every* $I \in \mathcal{I}$ *and every* $n \in \mathbb{N}$ *the scalar interval measure* $e_n^*\Phi$ *has a representation* $e_n^*\Phi(I) = (\mathrm{HK})\int_I f_n(t)dt$. *Then, the following are equivalent*:

(j) *There exists* $f \in \mathrm{HKP}([0,1], X)$ *such that*

$$\Phi(I) = (\mathrm{HKP})\int_I f(t)dt, \quad \text{for every } I \in \mathcal{I};$$

(jj) *The series* $\sum_{n=1}^\infty f_n e_n$ *is a.e. convergent to a function* $f\colon [0,1] \to X$ *in the norm topology of* X *and for every* $x^* \in X^*$ *the series* $\sum_{n=1}^\infty f_n x^*(e_n)$ *is* $\sigma(\mathrm{HK}, \mathrm{BV})$*-convergent to* x^*f;

(jjj) *There exists* $f\colon [0,1] \to X$ *such that for every* $x^* \in X^*$ *the series* $\sum_{n=1}^\infty f_n x^*(e_n)$ *is* $\sigma(\mathrm{HK}, \mathrm{BV})$*-convergent to* x^*f.

In either case $f = \sum_{n=1}^\infty f_n e_n$ *and the series is convergent a.e..*

Proof (j)⇒(jj) For $I \in \mathcal{I}$, set $\Phi_n(I) := (\mathrm{HK})\int_I f_n(t)dt$. Then $\Phi(I) = \sum_{n=1}^\infty \Phi_n(I)\,e_n$ and the series is norm convergent. That is,

$$\lim_n \left\| (\mathrm{HKP})\int_I f(t)dt - (\mathrm{HKP})\int_I \sum_{k=1}^n f_k(t)e_k dt \right\| = 0.$$

Consequently, we have, for each $x^* \in X^*$,

$$\lim_n \left| (\mathrm{HK})\int_I x^*f(t)dt - (\mathrm{HK})\int_I \sum_{k=1}^n f_k(t)x^*(e_k)dt \right| = 0. \tag{3.1}$$

Let $W = \{\Phi(I)\colon I \in \mathcal{I}\}$. Then, for each x^* we have $\sup\{|x^*\Phi(I)| : I \in \mathcal{I}\} \le \|x^*f\|_A < \infty$ and so, due to the Banach–Steinhaus Theorem, the set W is bounded. Set $M = \sup\{\|x\| : x \in W\}$. As a consequence we get, for each $x^* \in B(X^*)$, $n \in \mathbb{N}$, and $I \in \mathcal{I}$,

$$\left\| (\mathrm{HK})\int_I \sum_{k=1}^n f_k(t)x^*(e_k)dt \right\|_A \le M. \tag{3.2}$$

Taking into account (3.1), (3.2) and Proposition 3.3, we get the $\sigma(\mathrm{HK}, \mathrm{BV})$-convergence of $\sum_{n=1}^\infty f_n x^*(e_n)$ to x^*f.

The a.e. convergence of the series $\sum_{n=1}^\infty f_n e_n$ is a direct consequence of the equalities

$$\int_I e_n^* f(t)dt = e_n^*\Phi(I) = (\mathrm{HK})\int_I f_n(t)dt, \quad I \in \mathcal{I},\ n \in \mathbb{N}.$$

(jj)⇒(jjj) We set $f = \sum_{n=1}^\infty f_n e_n$.

(jjj)⇒(j) For a fixed $I \in \mathcal{I}$ and $x^* \in X^*$, we have

$$\begin{aligned} x^*\Phi(I) &= \lim_{n\to\infty} x^*\Big(\sum_{k=1}^{n} \Phi_k(I)e_k\Big) = \lim_{n\to\infty} \sum_{k=1}^{n} \Phi_k(I)x^*(e_k) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \Big[(\mathrm{HK})\int_I f_k dt\Big]\, x^*(e_k) = \lim_{n\to\infty} (\mathrm{HK})\int_I \sum_{k=1}^{n} f_k x^*(e_k) dt \\ &= \lim_{n\to\infty} (\mathrm{HK})\int_I x^*\Big(\sum_{k=1}^{n} f_k e_k\Big) dt = (\mathrm{HK})\int_I x^* f dt. \end{aligned}$$

□

Remark 3.5 Taking into account [8, Theorem 3], we can add to the equivalent conditions in Theorem 3.4 the following statement:

(jv) The series $\sum_{n=1}^{\infty} f_n e_n$ is a.e. convergent to a scalarly HK-integrable function $f\colon [0,1] \to X$ in the norm topology of X and the topology $\sigma(\mathrm{HK}, \mathrm{BV})$ coincides on the set $Z_f := \{x^* f : \|x^*\| \le 1\}$ with the topology of convergence in measure.

Proposition 3.6 *Assume that X is weakly sequentially complete or $\{e_n\}$ is shrinking. Let $\Phi\colon \mathcal{I} \to X$ be an interval measure such that for every $I \in \mathcal{I}$ and every $n \in \mathbb{N}$ the scalar interval measure $e_n^*\Phi$ has the representation $e_n^*\Phi(I) = (\mathrm{HK})\int_I f_n(t)dt$.*

If the series $\sum_{n=1}^{\infty} f_n e_n$ is a.e. convergent in the norm topology of X to a scalarly HK-*integrable function $f\colon [0,1] \to X$, then $f \in \mathrm{HKP}([0,1], X)$ and*

$$\Phi(I) = (\mathrm{HKP})\int_I f(t)dt, \quad \textit{for every } I \in \mathcal{I}. \tag{3.3}$$

Conversely, if there exists f satisfying equality (3.3)*, then the series $\sum_{n=1}^{\infty} f_n e_n$ with $f_n = e_n^*(f), n \in \mathbb{N}$, is a.e. convergent in the norm topology of X and $f = \sum_{n=1}^{\infty} f_n e_n$ a.e..*

Proof Assume the weak sequential completeness of X. According to [13, Theorem 40] the function f is Henstock–Kurzweil–Pettis integrable. Let $\Psi\colon \mathcal{I} \to X$ be its Henstock–Kurzweil–Pettis integral.

We have

$$e_n^*\Psi(I) = (\mathrm{HK})\int_I e_n^* f(t)dt = (\mathrm{HK})\int_I f_n(t)dt = e_n^*\Phi(I), \quad \text{for every } n \in \mathbb{N}.$$

Therefore,

$$\Phi(I) = (\mathrm{HKP})\int_I f(t)dt, \quad \text{for every } I \in \mathcal{I}.$$

Assume now that $\{e_n\}$ is shrinking. As remarked before, by [11, Theorem 3] and the assumption, it follows that the function f is Henstock–Kurzweil–Dunford integrable; that is, there exists an interval function $\Psi\colon \mathcal{I} \to X^{**}$ such that

$$x^*\Psi(I) = (\mathrm{HK})\int_I x^* f(t)dt, \quad \text{for every } x^* \in X^* \text{ and } I \in \mathcal{I}.$$

In particular,

$$e_n^*\Psi(I) = (\mathrm{HK})\int_I e_n^* f(t)dt = (\mathrm{HK})\int_I f_n(t)dt = e_n^*\Phi(I), \quad \text{for every } n \in \mathbb{N}.$$

Since the sequence $\{e_n^*\}$ separates points of X^{**}, we have the equality $\Phi = \Psi$. This completes the proof of the direct part.

Assume now that the equality (3.3) holds true. Then the a.e. convergence of the series $\sum_{n=1}^{\infty} f_n e_n$ is a direct consequence of the equalities

$$\int_I e_n^* f(t)dt = e_n^*\Phi(I) = (\mathrm{HK})\int_I f_n(t)dt, \quad I \in \mathcal{I}, \ n \in \mathbb{N}. \tag{3.4}$$

Thus the proof is complete. □

In the next part of this section we obtain an integral representation of interval measures by means of conditions on the variational measures of their primitives. The result is a direct generalization of the main theorem from [4].

It has been proven in [8] that if $f : [0,1] \to X$ is HKP-integrable, then the set $\mathcal{Z}_f := \{x^*f : \|x^*\| \leq 1\}$ is weakly compact in HK$[0,1]$. It turns out that in case of an arbitrary $\Phi : \mathcal{I} \to X$ the following result holds true:

Lemma 3.7 *Let X be an arbitrary separable Banach space and let $\Phi : \mathcal{I} \to X$ be such that $V_{x^*\Phi} \ll \lambda$ for every $x^* \in X^*$. Let $T_\Phi : X^* \to \mathrm{HK}[0,1]$ be defined by $T_\Phi(x^*) := (x^*\Phi)'$. Then T_Φ is sequentially weak*-weakly continuous on X^* and the set $T_\Phi(B(X^*))$ is weakly compact in* HK$[0,1]$.

Proof Notice first that T_Φ is a linear bounded operator. Indeed, $\|T_\Phi(x^*)\|_A = \|(x^*\Phi)'\|_A < \infty$, because the function $(x^*\Phi)'$ is HK-integrable. The Banach–Steinhauss theorem yields the boundedness of T_Φ.

In order to check the weak*-weak continuity of T_Φ let $\{x_n^*\}$ be a sequence of points from $B(X^*)$ that is weak* converging to x_0^*. Then for each $I \in \mathcal{I}$ (cf. [15]), we have

$$\lim_n (\mathrm{HK})\int_I (x_n^*\Phi)'(t)dt = \lim_n \langle x_n^*, \Phi(I)\rangle = \langle x_0^*, \Phi(I)\rangle = (\mathrm{HK})\int_I (x_0^*\Phi)'(t)dt.$$

Due to the boundedness of T_Φ, we have also $\sup_n \|(x_n^*\Phi)'\|_A < \infty$. It follows from Proposition 3.3 that the sequence $\{(x_n^*\Phi)'\}$ is weakly convergent to $(x_0^*\Phi)'$. □

Proposition 3.8 *Let X be an arbitrary Banach space and Φ be an X-valued interval measure such that $V_{x^*\Phi} \ll \lambda$, for every $x^* \in X^*$. Assume that there is a decomposition $[0,1] = \bigcup_k H_k$ into measurable sets of positive measure such that $V_{x^*\Phi}(H_k) < \infty$ for every $k \in \mathbb{N}$ and every $x^* \in X^*$ and, for every $k \in \mathbb{N}$, the function $x^* \to V_{x^*\Phi}(H_k)$ is sequentially weak*-continuous.*

If $f : [0,1] \to X$ is a scalarly measurable function, then the set

$$K = \left\{x^* \in X^* : x^*f \in \mathrm{HK}[0,1] \ \text{and} \ x^*\Phi(I) = (\mathrm{HK})\int_I x^*f(t)dt, \ \forall\, I \in \mathcal{I}\right\}$$

is weak* sequentially closed.

Proof It is obvious that $K \neq \emptyset$. Notice first that if $x^* \in K$, then $(x^*\Phi)' = x^*f$ a.e. (cf. [15]). Let $\{x_n^*\} \subset K$ be such that $x_n^* \to x_0^*$ in the w^*-topology. We may assume, without loss of generality, that all x_n^*, $n = 0, 1, 2, \ldots$, belong to $B(X^*)$. By hypothesis $V_{x_0^*\Phi} \ll \lambda$, and so there exists $g \in \mathrm{HK}[0,1]$ such that $x_0^*\Phi(I) = (\mathrm{HK})\int_I g\, dt$, for all $I \in \mathcal{I}$ (cf. [16]).

By the assumption and by [17, Corollary 3], we have, for each $k \in \mathbb{N}$,

$$\lim_n \int_{H_k} |x_n^*f(t) - g(t)|dt = \lim_n V_{(x_n^* - x_0^*)\Phi}(H_k) = 0.$$

Hence, there is a subsequence $\{x^*_{k,n_m}\}_m$ of $\{x^*_n\}$ with $\lim_m x^*_{k,n_m} f = g$, a.e. on H_k. It follows that $g = x^*_0 f$ a.e. and so $x^*_0 f \in \mathrm{HK}[0,1]$. By Lemma 3.7, the sequence $(x^*_n f)$ is $\sigma(\mathrm{HK},\mathrm{BV})$-convergent to $x^*_0 f$. This yields $x^*_0 \in K$ and so K is weak* sequentially closed. □

Theorem 3.9 *Let $\Phi\colon \mathcal{I} \to X$ be an interval measure such that for every $n \in \mathbb{N}$ the scalar interval measure $e^*_n\Phi$ has the representation $e^*_n\Phi(I) = (\mathrm{HK})\int_I f_n(t)dt$, $I \in \mathcal{I}$. Then the following two conditions are equivalent*:

(A) (a) *the series $\sum_{n=1}^\infty f_n e_n$ is a.e. convergent to a function $f\colon [0,1] \to X$ in the norm topology of X*;

(b) $V_{x^*\Phi} \ll \lambda$, *for every $x^* \in X^*$*;

(c) *there is a decomposition $[0,1] = \bigcup_k H_k$ into measurable sets of positive measure such that $V_{x^*\Phi}(H_k) < \infty$ for every $k \in \mathbb{N}$ and every $x^* \in X^*$ and, for every $k \in \mathbb{N}$ the function $x^* \to V_{x^*\Phi}(H_k)$ is sequentially weak*-continuous.*

(B) *There exists $f \in \mathrm{HKP}([0,1],X)$ such that*

$$\Phi(I) = (\mathrm{HKP})\int_I f(t)dt, \quad \text{for every } I \in \mathcal{I}. \tag{3.5}$$

Proof (A)⇒(B) If the function $x^* \to V_{x^*\Phi}(H_k)$ is weak*-continuous, for every $k \in \mathbb{N}$, then we have $K \supset Q := \{\sum_{i=1}^m a_i e^*_i : (a_1,\dots,a_m) \in \mathbb{R}^m, m \in \mathbb{N}\}$, where K is defined as in Proposition 3.8. But Q is sequentially weak* dense in X^*, and so we may apply Proposition 3.8 to get the required HKP-integrability of f and the equality (3.5).

(B)⇒(A) Assume that the equality (3.5) holds. Then conditions (a) follows by (3.4), and condition (b) by [16]. It remains to prove condition (c). For each $k \in \mathbb{N}$, let $H_k := \{t \in [0,1] : k-1 \le \|f(t)\| < k\}$. The separability of X yields the strong measurability of f and the measurability of each set H_k. Let $\{x^*_n\}$ be an arbitrary sequence of functionals with $x^*_n \to x^*_0$ in the w^*-topology. For arbitrary $k \in \mathbb{N}$ and $n \in \mathbb{N}$ (see [17, Corollary 3]), we have

$$V_{(x^*_n - x^*_0)\Phi}(H_k) = \int_{H_k} |x^*_n f(t) - x^*_0 f(t)| dt$$

and the integrands are bounded by $2k$. It follows from the Lebesgue dominated convergence theorem that, for each $k \in \mathbb{N}$,

$$\lim_n V_{(x^*_n - x^*_0)\Phi}(H_k) = 0.$$

Since $|V_{x^*_n\Phi}(H_k) - V_{x^*_0\Phi}(H_k)| \le V_{(x^*_n - x^*_0)\Phi}(H_k)$, we obtain the required continuity. □

Remark 3.10 The assumption (a) in Theorem 3.9 can be replaced by any of the following two conditions:

(a′) There exists a scalarly HK-integrable function f such that for each $x^* \in X^*$ the series $\sum_{n=1}^\infty f_n x^*(e_n)$ is a.e. convergent to $x^* f$;

(a″) There exists a scalarly HK-integrable function f such that for each $x^* \in X^*$ the series $\sum_{n=1}^\infty f_n x^*(e_n)$ is convergent in measure to $x^* f$.

As a particular case of Theorem 3.9, we obtain [4, Theorem 2] for the Lebesgue measure.

Corollary 3.11 ([4]) *Let $\nu : \mathcal{L} \to X$ be a countably additive measure such that $e^*_n\nu(E) = \int_E f_n(t)dt$, for every $E \in \mathcal{L}$ and $n \in \mathbb{N}$. If the series $\sum_{n=1}^\infty f_n e_n$ is a.e. convergent to a function*

$f\colon[0,1]\to X$ in the norm topology of X (or in measure for every $x^*\in X^*$), then f is Pettis integrable and

$$(P)\int_E f(t)dt=\sum_n e_n\int_E f_n(t)dt,\quad \textit{for every } E\in\mathcal{L}.$$

Proof Let $\nu:\mathcal{L}\to X$ be a countably additive measure such that $e_n^*\nu(E)=\int_E f_n(t)dt$ for every $E\in\mathcal{L}$ and $n\in\mathbb{N}$, and let us assume that the condition (a) of Theorem 3.9 is fulfilled. By the hypothesis, we have

$$\nu(E)=\sum_{n=1}^{\infty}e_n\int_E f_n(t)dt,\quad \text{for every } E\in\mathcal{L}.$$

Then $\nu\ll\lambda$. Therefore, for each $x^*\in X^*$, $x^*\nu$ is a finite measure and $x^*\nu\ll\lambda$. Moreover, by the finiteness of $x^*\nu$, we infer that it is of bounded variation. Therefore, $x^*\nu$ is a Lebesgue primitive, for each $x^*\in X^*$.

Let now $\Phi:\mathcal{I}\to X$ be defined by $\Phi(I)=\nu(I)$. By [2, Lemma 3.3], we have $V_{x^*\Phi}(E)\le|x^*\nu|(E)$, for every $E\in\mathcal{L}$ (where $|x^*\nu|$ denotes the ordinary variation of the measure $x^*\nu$).

Hence, the condition (b) of Theorem 3.9 is fulfilled.

Consider now a new norm on X defined by

$$|\!|\!|x|\!|\!|:=\sup_n\left\|\sum_{k=1}^{n}\alpha_k e_k\right\|,\quad \text{if } x=\sum_{k=1}^{\infty}\alpha_k e_k. \tag{3.6}$$

It is well known that the norm $|\!|\!|\cdot|\!|\!|$ is monotone and equivalent to the initial norm of X (cf. [18, p. 250]). Let $|\cdot|$ be the norm on X^* determined by $|\!|\!|\cdot|\!|\!|$.

For each $k\in\mathbb{N}$, let $H_k:=\{t\in[0,1]:k-1\le|\!|\!|f(t)|\!|\!|<k\}$, where $f=\sum_{n=1}^{\infty}f_ne_n$. Then $[0,1]=\bigcup_k H_k$. Moreover, the separability of X yields the strong measurability of f and the measurability of each set H_k.

Since $|x^*f(t)|<|x^*|k$, for each $t\in H_k$, we have $x^*f\in L^1(H_k)$ for each $x^*\in X^*$ and each $k\in\mathbb{N}$. Moreover, by the Lebesgue Dominated Convergence Theorem, we have

$$\int_{E\cap H_k}x^*f(t)dt=\lim_p\int_{E\cap H_k}\sum_{n=1}^{p}x^*(e_n)f_n(t)dt=\lim_p\sum_{n=1}^{p}x^*(e_n)\int_{E\cap H_k}f_n(t)dt=x^*\nu(E\cup H_k),$$

for each $x^*\in X^*$ and $E\in\mathcal{L}$, and, applying once again [2, Lemma 3.3], we obtain

$$V_{x^*\Phi}(H_k)\le|x^*\nu|(H_k)=\int_{H_k}|x^*f(t)|dt.$$

So

$$\int_0^1|x^*f(t)|dt=\sum_1^{\infty}\int_{H_k}|x^*f(t)|dt=\sum_1^{\infty}|x^*\nu|(H_k)=|x^*\nu|([0,1])<\infty$$

for each $x^*\in X^*$.

Let now $\{x_n^*\}$ be an arbitrary sequence of functionals with $x_n^*\to x_0^*$ in the w^*-topology.

Then

$$|V_{x_n^*\Phi}(H_k)-V_{x_0^*\Phi}(H_k)|\le V_{(x_n^*-x_0^*)\Phi}(H_k)\le\int_{H_k}|x_n^*f(t)-x_0^*f(t)|dt,$$

and, applying once again the Lebesgue Dominated Convergence Theorem, we get the sequential weak*-continuity of the function $x^* \to V_{x^*\Phi}(H_k)$. Consequently, it follows from Theorem 3.9 that f is HKP-integrable and

$$x^*\nu(I) = (\mathrm{HK})\int_I x^* f(t)dt, \quad \text{for each } I \in \mathcal{I}.$$

Since each function x^*f is Lebesgue integrable, the right-hand side of the above equality extends uniquely to a measure defined on $\mathcal{L}$. As the left-hand side is the restriction of the measure $x^*\nu$, we obtain the equality $x^*\nu(E) = \int_E x^* f(t)dt$, for every $E \in \mathcal{L}$. This proves the Pettis integrability of f and the expected equality $\nu(E) = (P)\int_E f dt$. □

Since we have not applied any part of [4] in the proof of Theorem 3.9, our result presents a new proof of [4, Theorem 2] in case of the Lebesgue measure.

In the next theorem, we present a characterization of HKP-differentiable interval measures with values in spaces not containing c_0. It turns out that under such restriction the assumptions may be essentially weaker. For readers' convenience we recall the Romanowski lemma:

Lemma 3.12 ([15, Lemma 5.18]) *Let $\mathcal{F}$ be a family of open intervals in $(0,1)$ and assume that $\mathcal{F}$ has the following properties*:

(1) *if (α,β) and (β,γ) belong to $\mathcal{F}$, then (α,γ) belongs to $\mathcal{F}$*;

(2) *if (α,β) belongs to $\mathcal{F}$, then every open interval in (α,β) belongs to $\mathcal{F}$*;

(3) *if (α,β) belongs to $\mathcal{F}$ for every interval $[\alpha,\beta] \subset (c,d)$, then (c,d) belongs to $\mathcal{F}$*;

(4) *if all the intervals contiguous to the perfect set $E \subset [0,1]$ belong to $\mathcal{F}$, then there exists an interval I in $\mathcal{F}$ such that $I \cap E \neq \emptyset$.*

Then $\mathcal{F}$ contains the interval $(0,1)$.

Theorem 3.13 *Assume that $c_0 \not\subseteq X$. Let $\Phi\colon \mathcal{I} \to X$ be an interval measure such that for every $I \in \mathcal{I}$ and every $n \in \mathbb{N}$ the scalar set function $e_n^*\Phi$ has the representation $e_n^*\Phi(I) =$ $(\mathrm{HK})\int_I f_n(t)dt$. Then the following two conditions are equivalent*:

(C) (a) *the series $\sum_{n=1}^{\infty} f_n e_n$ is a.e. convergent in the norm topology of X to a scalarly* HK-*integrable function $f\colon [0,1] \to X$*;

(b) *Φ is scalarly continuous.*

(D) *There exists $f \in \mathrm{HKP}([0,1],X)$ such that*

$$\Phi(I) = (\mathrm{HKP})\int_I f(t)dt, \quad \textit{for every } I \in \mathcal{I}. \tag{3.7}$$

Proof (C)⇒(D) Let $\mathcal{F}$ be the collection of all open intervals K in $[0,1]$ such that f is HKP-integrable on each closed subinterval I of K and

$$e_n^*\Phi(I) = (\mathrm{HK})\int_I e_n^* f\, dt, \tag{3.8}$$

for every $I \in \mathcal{I}$ and for every $n \in \mathbb{N}$.

$\mathcal{F}$ is not empty. In fact, by hypothesis (a), there exists $J \subset [0,1]$ such that $f|_J$ is Dunford integrable (see [13, Theorem 33]). Since $c_0 \not\subseteq X$ and X is separable, then by [13, Theorem 22], $f|_J$ is Pettis integrable. Set $\hat{\Phi}(K) = (P)\int_K f dt$ for $K \in \mathcal{I}$, $K \subset J$. For every $n \in \mathbb{N}$, we have

$$e_n^*\hat{\Phi}(K) = \int_K e_n^* f dt = \int_K f_n dt.$$

Then $\hat{\Phi}(K) = \Phi(K)$. This means that $J \in \mathcal{F}$, hence $\mathcal{F}$ is not empty.

Now we prove that, by the hypothesis (b), if $(a, b) \in \mathcal{F}$, then f is HKP-integrable in $[a, b]$ and (3.7) is satisfied for each subinterval I of $[a, b]$. In fact, for each $I \in \mathcal{I}$, let x_I^{**} be the HKD-integral of f on I. Then for every $x^* \in X^*$, we have

$$x^*(x^{**}_{(a,b)}) = (\mathrm{HK})\int_a^b x^* f(t)dt = \lim_n\, (\mathrm{HK})\int_{a+1/n}^{b-1/n} x^* f(t)dt$$
$$= \lim_n x^*\Phi[a + 1/n, b - 1/n] = x^*\Phi[a, b].$$

Since a weak* convergent sequence may have only one limit point, we have $x^{**}_{(a,b)} = \Phi[a, b] \in X$, and f is HKP-integrable in $[a, b]$.

So to get the thesis it is enough to show that $(0, 1) \in \mathcal{F}$. To this end we are going to apply Romanovski's lemma to the family $\mathcal{F}$. Properties (1)–(3) of Romanovski's lemma are easily verified. Now we are going to prove property (4). Let E be a perfect subset of $[0, 1]$ such that each interval in $(0, 1)$ contiguous to E belongs to $\mathcal{F}$. Since f is scalarly HK-integrable, then f is HKD-integrable on $[0, 1]$ (see [11]). Therefore f is also Denjoy–Dunford integrable on $[0, 1]$. Hence by [13, Theorem 33] there exists an interval $[u, v]$ with $u, v \in E$ and $E \cap (u, v) \neq \emptyset$ such that f is Dunford integrable in $[u, v]$. Since $c_0 \not\subseteq X$ and X is separable, by [13, Theorem 22], f is Pettis integrable in $[u, v]$ and equality (3.8) is satisfied for each interval $K \subset (u, v)$. Therefore (u, v) belongs to $\mathcal{F}$ and also property (4) of the Romanovski Lemma is satisfied. Then $(0, 1) \in \mathcal{F}$ and the proof of the direct part is complete.

(D)⇒(C) follows as in Proposition 3.6. □

Theorem 3.13 can be also interpreted as a necessary and sufficient condition for integrability of a function with values in a Banach space not containing any isomorphic copy of c_0. In such a case, it can be considered as a generalization of a result of Gordon from [13], where he proved that each scalarly Denjoy-integrable function with values in a weakly sequentially complete Banach space is Denjoy–Pettis integrable and presented a counterexample for c_0-valued functions.

Corollary 3.14 *Assume that $c_0 \not\subseteq X$ and $f : [0, 1] \to X$ is a scalarly* HK*-integrable function with $f = \sum_{n=1}^{\infty} f_n e_n$. If there exists an interval measure $\Phi : \mathcal{I} \to X$ such that $e_n^*\Phi(I) = (\mathrm{HK})\int_I e_n^* f(t)dt$, for every $n \in \mathbb{N}$, then f is* HKP*-integrable and*

$$\Phi(I) = (\mathrm{HKP})\int_I f(t)dt, \quad \textit{for every } I \in \mathcal{I}.$$

4 Other Integrals

In this section we obtain some representations of interval measures by means of the Radon–Nikodym derivatives which are integrable in a stronger sense than in the Henstock–Kurzweil–Pettis one.

We recall the definition of the variational Henstock integral.

Definition 4.1 *A function $f\colon [0, 1] \to X$ is said to be variationally Henstock integrable* (*briefly* vH*-integrable*) *if there is an additive interval function $F\colon \mathcal{I} \to X$ such that for each $\varepsilon > 0$ there* *exists a gauge δ with*

$$\sum_{i=1}^{p} \|f(t_i)|I_i| - F(I_i)\| < \varepsilon, \tag{4.1}$$

for each δ-fine partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ of $[0,1]$. By the symbol $\mathrm{vH}([0,1],X)$ *we denote the family of all X-valued variationally Henstock integrable functions f.*

Let us recall that for real valued functions the HK-integral coincides with the vH-integral. A family $\{f_\alpha\}_{\alpha\in A}$ of functions $f_\alpha\colon [0,1]\to X$ is said to be *variationally Henstock equiintegrable* (briefly vH-*equiintegrable*) if it is possible to find for each $\varepsilon>0$ a gauge δ satisfying inequality (4.1) for all the functions of the family.

Let us start with the obvious remark that if f is vH-integrable (resp. H-integrable) with respect to $\|\cdot\|$, then it is also vH-integrable (resp. H-integrable) with respect to any equivalent norm on X. In the sequel we assume that X is endowed with the monotone norm defined in (3.6).

We need the following version of [19, Theorem 4].

Theorem 4.2 *Let $\{f_n\in \mathrm{vH}([0,1],X)\}$ be a sequence of functions and let $N\subset[0,1]$ be a set with $|N|=0$ such that*

(i) *$f_n(t)\to f(t)$ for $t\in E$, where $E=[0,1]\setminus N$;*

(ii) *$\{f_n\}$ is pointwise bounded in N;*

(iii) *$\{f_n\}$ is* vH-*equiintegrable on* $[0,1]$.

Then $f\in \mathrm{vH}([0,1],X)$ and $\|f-f_n\|_A\to 0$.

Proof By proceeding, with easy changes, as in [19, Theorem 4] we have that the sequence $\{f_n\chi_E\}$ is pointwise convergent to $f\chi_E$ on $[0,1]$ and vH-equiintegrable on $[0,1]$. Then it is also H-equiintegrable on $[0,1]$. So, by [20, Theorem 1], $f\chi_E$ is H-integrable on $[0,1]$ and $\|f\chi_E - f_n\chi_E\|_A \to 0$. Then, by a simple passage to the limit in the formula of the vH-equiintegrability, we get the vH-integrability of $f\chi_E$ and then also of f. □

Theorem 4.3 *Let $\Phi\colon\mathcal{I}\to X$ be a continuous interval measure such that, for every $I\in\mathcal{I}$ and every $n\in\mathbb{N}$, the scalar interval measure $e_n^*\Phi$ has the representation $e_n^*\Phi(I)=(\mathrm{HK})\int_I f_n(t)dt$. Then the following conditions are equivalent:*

(E) *There exists $f\in\mathrm{vH}([0,1],X)$ such that*

$$\Phi(I) = (\mathrm{vH})\int_I f(t)dt, \quad \textit{for every } I\in\mathcal{I};$$

(F) *The series $\sum_{n=1}^{\infty} f_n(t)e_n$ is a.e. convergent and the sequence of functions $g_n := \sum_{i=1}^{n} f_i e_i$ is* vH-*equiintegrable.*

Proof (E)⇒(F) As f is vH-integrable, for every $n\in\mathbb{N}$ and for every $I\in\mathcal{I}$, we have $e_n^*\Phi(I) = (\mathrm{HK})\int_I e_n^* f(t)dt$, $e_n^* f(t) = f_n(t)$ a.e., and $f(t)=\sum_{n=1}^{\infty} f_n(t)\, e_n$ a.e.. Moreover, for every $\varepsilon>0$, there exists a gauge δ on $[0,1]$ such that

$$\sum_{i=1}^{p} \|f(t_i)|I_i| - \Phi(I_i)\| < \varepsilon, \tag{4.2}$$

for each δ-fine partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ of $[0,1]$.

Set $\Phi_n := e_n^*\Phi$ and $\Psi_n := \sum_{j=1}^n \Phi_j e_j$. It follows from the monotonicity of the norm that, for every $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{i=1}^p \|g_n(t_i)|I_i| - \Psi_n(I_i)\| &= \sum_{i=1}^p \left\| \sum_{j=1}^n [f_j(t_i)|I_i| - \Phi_j(I_i)]e_j \right\| \\ &\le \sum_{i=1}^p \left\| \sum_{j=1}^\infty [f_j(t_i)|I_i| - \Phi_j(I_i)]e_j \right\| \\ &= \sum_{i=1}^p \|f(t_i)|I_i| - \Phi(I_i)\| < \varepsilon.\end{aligned}$$

Consequently the functions g_n are vH-equiintegrable with respect to $\|\cdot\|$.

(F)⇒(E) Let us denote by $N \subset [0,1]$ the set of all points $t \in [0,1]$ such that the series $\sum_{n=1}^\infty f_n(t)e_n$ is not convergent and define $f(t) := \sum_{n=1}^\infty f_n(t)e_n$ for $t \in [0,1] \setminus N$, $f(t) := 0$ for $t \in N$. We are going to prove that f is vH-integrable. Set $\Phi_n = e_n^*\Phi$ and observe that, for every $n \in \mathbb{N}$, the function $G_n = \sum_{i=1}^n \Phi_i e_i$ is the vH-primitive of g_n. Since for every $t \in [0,1]$, for every $h \in \mathbb{R}$ and for every $n \in \mathbb{N}$, we have

$$\left\| \sum_{i=1}^n (\Phi_i(t) - \Phi_i(t+h))e_i \right\| \le \|\Phi(t) - \Phi(t+h)\|,$$

by the continuity of Φ, we get the equicontinuity of the functions $\{G_n\}$. Now we are going to show that the sequence $\{g_n\}$ is pointwise bounded in N. Indeed, fix $\varepsilon = 1$. According to the vH-equiintegrability of the sequence $\{g_n\}$, for each $t \in N$ let $\delta(t)$ be such that

$$\|g_n(t)|I| - G_n(I)\| < 1 \tag{4.3}$$

and

$$\|G_n(I)\| < 1, \tag{4.4}$$

for every $n \in \mathbb{N}$ and for every δ-fine interval I containing t. Fix one of such intervals and call it I.

So by (4.3) and (4.4), we infer

$$\|g_n(t)\| \le \frac{\|G_n(I)\| + 1}{|I|} < \frac{2}{|I|}.$$

Therefore, the sequence $\{g_n\}$ is pointwise bounded in N. Then, by applying Theorem 4.2, we get that $f \in \mathrm{vH}([0,1], X)$. □

Theorem 4.4 *Let $\Phi\colon \mathcal{I} \to X$ be a continuous interval measure such that, for every $I \in \mathcal{I}$ and every $n \in \mathbb{N}$, the scalar interval measure $e_n^*\Phi$ has the representation $e_n^*\Phi(I) = (\mathrm{HK})\int_I f_n(t)dt$. Then the following conditions are equivalent:*

(G) *There exists $f \in H([0,1], X)$ such that*

$$\Phi(I) = (H)\int_I f(t)dt, \quad \textit{for every } I \in \mathcal{I};$$

(H) *The series $\sum_{n=1}^\infty f_n(t)e_n$ is a.e. convergent and the sequence of functions $g_n := \sum_{i=1}^n f_i e_i$ is H-equiintegrable.*

Proof (G)⇒(H) If f is H-integrable, proceeding as in the case of the vH-integral, we obtain $f(t) = \sum_{n=1}^{\infty} f_n(t)e_n$ a.e. in $[0,1]$.

Moreover, for every $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that

$$\left\|\sum_{i=1}^{p}[f(t_i)|I_i| - \Phi(I_i)]\right\| < \varepsilon, \tag{4.5}$$

for each δ-fine partition $\{(I_1,t_1),\ldots,(I_p,t_p)\}$ of $[0,1]$.

Set $\Phi_n := e_n^*\Phi$ and $\Psi_n := \sum_{j=1}^{n}\Phi_j e_j$. It follows from the monotonicity of the norm that, for every $n \in \mathbb{N}$,

$$\begin{aligned}\left\|\sum_{i=1}^{p}[g_n(t_i)|I_i| - \Psi_n(I_i)]\right\| &= \left\|\sum_{i=1}^{p}\sum_{j=1}^{n}[f_j(t_i)|I_i| - \Phi_j(I_i)]e_j\right\| \\ &= \left\|\sum_{j=1}^{n}\sum_{i=1}^{p}[f_j(t_i)|I_i| - \Phi_j(I_i)]e_j\right\| \\ &\le \left\|\sum_{j=1}^{\infty}\sum_{i=1}^{p}[f_j(t_i)|I_i| - \Phi_j(I_i)]e_j\right\| \\ &= \left\|\sum_{i=1}^{p}\left[\sum_{j=1}^{\infty} f_j(t_i)e_j|I_i| - \sum_{j=1}^{\infty}\Phi_j(I_i)e_j\right]\right\| \\ &= \left\|\sum_{i=1}^{p}[f(t_i)|I_i| - \Phi(I_i)]\right\| < \varepsilon.\end{aligned}$$

Hence the functions g_n are H-equiintegrable with respect to $\|\cdot\|$.

(H)⇒(G) We may proceed as in the previous theorem, applying this time the uniform version of the Henstock Lemma (see [21, Lemma III 5.6] and the convergence theorem for H-equintegrable functions [19, Theorem 2]). □

Using the notion of equiintegrability we obtain the following sufficient condition for the HKP-integral.

Proposition 4.5 *Let $\Phi\colon \mathcal{I} \to X$ be a continuous interval measure such that, for every $I \in \mathcal{I}$ and every $n \in \mathbb{N}$, the scalar set function $e_n^*\Phi$ has the representation $e_n^*\Phi(I) = (\mathrm{HK})\int_I f_n(t)dt$. If the series $\sum_{n=1}^{\infty} f_n(t)e_n$ is a.e. convergent to a function f and the sequence of functions $g_n := \sum_{i=1}^{n} f_i e_i$ is scalarly H-equiintegrable (i.e. for every $x^* \in X^*$ the sequence $\{x^*g_n\}$ is* HK-*equiintegrable), then f is* HKP-*integrable and*

$$\Phi(I) = (\mathrm{HKP})\int_I f(t)dt, \quad \textit{for every } I \in \mathcal{I}.$$

Proof Let us denote by $N \subset [0,1]$ the set of all points $t \in [0,1]$ such that the series $\sum_{n=1}^{\infty} f_n(t)e_n$ is not convergent, and set $\Phi_n = e_n^*\Phi$. We observe that for every $n \in \mathbb{N}$ the function g_n is H-integrable and $G_n = \sum_{i=1}^{n}\Phi_i e_i$ is its H-primitive. Proceeding as in Theorem 4.3, we get that the sequence $\{g_n\}$ is pointwise bounded in N. Then by the convergence theorem for HK-equintegrable functions [19, Theorem 2], we obtain that for every $x^* \in X^*$ the series $\sum_{n=1}^{\infty} f_n x^*(e_n)$ is $\sigma(\mathrm{HK},\mathrm{BV})$-convergent to x^*f. So an application of Theorem 3.4 gives us the thesis. □

We do not know if the HK-equiintegrability of the sequence $\{f_n\}$ itself and the a.e. convergence of the series $\sum_{n=1}^{\infty} f_n(t)e_n$ guarantee the HKP-integrability of f. The next example shows that such HK-equiintegrability is not necessary.

Example 4.6 We are going to consider the example of Gamez–Mendoza [11]:

Given a sequence of intervals $A_n = [a_n, b_n] \subseteq [0,1]$ such that $a_1 = 0$, $b_n < a_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = 1$, we define $f\colon [0,1] \to c_0$ by

$$f(t) := \left(\frac{1}{2|A_{2n-1}|}\chi_{A_{2n-1}}(t) - \frac{1}{2|A_{2n}|}\chi_{A_{2n}}(t)\right)_{n=1}^{\infty}.$$

Moreover, for $n \in \mathbb{N}$, we set

$$f_n(t) := \frac{1}{2|A_{2n-1}|}\chi_{A_{2n-1}}(t) - \frac{1}{2|A_{2n}|}\chi_{A_{2n}}(t),$$

and take an arbitrary $0 < \varepsilon < 1/4$.

The function f takes values in c_0, it is HKP-integrable but the sequence $\{e_n^* f = f_n : n \in \mathbb{N}\}$ in not HK-equiintegrable. In fact, assume there is a gauge $\delta\colon [0,1] \to (0,1)$ such that, for each δ-fine partition $\{(I_i, t_i)\}_{1\le i\le p}$ and each $n \in \mathbb{N}$, we have

$$\left|\sum_{1\le i\le p} f_n(t_i)|I_i| - (\mathrm{HK})\int_0^1 f_n(t)dt\right| < \varepsilon. \tag{4.6}$$

Now let us define a new gauge δ' in the following way: if $t \in (a_n, b_n)$, then $(t - \delta'(t), t + \delta'(t)) \subset (a_n, b_n)$. If $t \in (b_n, a_{n+1})$, then $(t - \delta'(t), t + \delta'(t)) \subset (b_n, a_{n+1})$. If $t \in \{a_n, b_n\}$, then $\delta(t) < \frac{1}{2^{n+2}}|A_n|$. Moreover, let $\delta''(t) = \min\{\delta(t), \delta'(t)\}$ if $t < 1$ and $\delta''(1) = \delta(1)$. Let $n_0 \in \mathbb{N}$ be the first index such that $\delta''(1) \le a_{2n_0}$. If $\delta''(1) < a_{2n_0}$, then we leave $\delta''(1)$ as it is. If $\delta''(1) = a_{2n_0}$, then we redefine $\delta''(1)$ such that $b_{2n_0-1} \le \delta''(1) \le a_{2n_0}$.

Let $\{(I_1, t_1), \dots, (I_p, t_p)\}$ be a δ''-fine partition of $[0,1]$. Then $t_p = 1$ and

$$\left|\sum_{i=1}^{p} f_{n_0}(t_i)|I_i| - (\mathrm{HK})\int_0^1 f_{n_0}(t)dt\right| = \left|\sum_{i=1}^{p} f_{n_0}(t_i)|I_i|\right| = \frac{1}{2}\sum_{t_i\in A_{2n_0-1}} \frac{|I_i|}{|A_{2n_0-1}|} \ge 1/4,$$

in contradiction to (4.6). This proves that the sequence $\{f_n\}$ is not HK-equiintegrable.

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