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Vitali Type Convergence Theorems for Banach Space Valued Integrals

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Abstract Let (Ω, Σ, μ) be a complete probability space and let X be a Banach space. We introduce the notion of scalar equi-convergence in measure which being applied to sequences of Pettis integrable functions generates a new convergence theorem. We also obtain a Vitali type $\mathcal{I}$-convergence theorem for Pettis integrals where $\mathcal{I}$ is an ideal on $\mathbb{N}$.

Keywords Convergence theorems for integrals, Pettis integral, scalar equi-convergence in measure, $\mathcal{I}$-convergence

MR(2010) Subject Classification 28A20, 28B05, 40A10, 40A30, 46G10

1 Introduction

Convergence theorems for integrals play an important role in analysis. The purpose of this paper is to obtain some new results of that kind for Banach space valued functions.

The classical Vitali convergence theorem for real functions is our starting point (see e.g. [12]). In [16, Theorem 1] (see also [15, 17]), one can find its generalization to sequences of Pettis integrable functions that produce weakly convergent sequences of the corresponding integrals. To achieve stronger results, we introduce a new metric on the space of scalarly measurable Banach space valued functions. From a formal point of view, this is simply the convergence in measure that is uniform on the unit ball of the conjugate Banach space. To the best of our knowledge, such a convergence has not been concerned so far. As a result, most known convergence theorems for integrals produce only weakly convergent sequences of integrals. We show (in Section 2) that the new metric determined by the scalar equi-convergence in measure produces sequences of integrals that are convergent in the norm topology of the Banach space and the corresponding integrands are convergent in the Pettis norm. Our Theorem 2.4 improves essentially some former results obtained in [14, 20]. In Section 3, we discuss new convergence integral theorems for convergence generated by ideals on $\mathbb{N} = \{1, 2, \dots\}$.

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Throughout the paper, (Ω, Σ, μ) stands for a complete probability space and λ denotes the Lebesgue measure on $[0,1]$. A family $\mathcal{F}$ of real-valued Lebesgue integrable functions on Ω is said to be *uniformly integrable* if $\sup\{\int_\Omega |f| d\mu \colon f \in \mathcal{F}\} < \infty$ and for every $\varepsilon > 0$ there is $\delta > 0$ such that $\int_A |f| d\mu < \varepsilon$ for all $f \in \mathcal{F}$ and $A \in \Sigma$ with $\mu(A) < \delta$. Throughout the paper, X is a Banach space with its dual X^* and $B(X) := \{x \in X \colon \|x\| \le 1\}$. If $f \colon \Omega \to X$, then $\mathcal{Z}_f := \{x^* f \colon \|x^*\| \le 1\}$.

2 Scalar Equi-convergence in Measure and a Vitali Type Theorem

For basic terminology from the theory of integral for vector-valued functions, we refer the reader to [3, 15, 23]. A scalarly measurable function $f \colon \Omega \to X$ is called *Pettis integrable* if $x^* f \in L_1(\Omega, \mu)$ for all $x^* \in X^*$, and if for each $E \in \Sigma$ there is $\nu_f(E) \in X$ such that $x^* \nu_f(E) = \int_E x^* f d\mu$ for all $x^* \in X^*$. Then $\nu_f(E)$ is called the *Pettis integral* of f over E with respect to μ. The Pettis integral is more general than the Bochner integral, usually treated as a counterpart of Lebesgue integral for X-valued strongly measurable functions.

The space of X-valued scalarly μ-integrable functions (scalarly equivalent functions are identified) can be endowed with a norm defined by $\|f\|_P := \sup_{\|x^*\| \le 1} \int_\Omega |x^* f| \, d\mu$. The space of X-valued Pettis integrable functions with the norm $\|\cdot\|_P$ is denoted by $\mathbb{P}(\mu, X)$. It is known that in general the normed space $\mathbb{P}(\mu, X)$ is not complete. An equivalent norm on $\mathbb{P}(\mu, X)$ can be defined by $|||f||| = \sup_{E \in \Sigma} \left\| \int_E f \, d\mu \right\|$. It follows from this fact that the convergence in the Pettis norm coincides with the uniform convergence of the integrals on the σ-algebra Σ. For a survey on the Pettis integral see [15] or [17].

A sequence (f_n) of X-valued scalarly measurable functions is called *scalarly convergent in measure* to a scalarly measurable function $f \colon \Omega \to X$ if for each $x^* \in X^*$ the sequence $(x^* f_n)$ is convergent in measure to $x^* f$. Recall the following Vitali-type theorem for Pettis integral due to Musiał [16, Theorem 1] (see also [17, Theorem 8.1] and [15, Theorem 5.2]).

Theorem 2.1 ([16]) *Let f_n, $n \in \mathbb{N}$, be Pettis integrable functions from Ω to X such that $\bigcup_{n \in \mathbb{N}} \mathcal{Z}_{f_n}$ is uniformly integrable and (f_n) is scalarly convergent in measure to $f \colon \Omega \to X$. Then f is Pettis integrable and $\int_E f_n \to \int_E f$ weakly for each $E \in \Sigma$.*

We introduce a stronger notion of scalar equi-convergence in measure. Namely, we say that a sequence of scalarly measurable functions $f_n \colon \Omega \to X$, $n \in \mathbb{N}$, is *scalarly equi-convergent in measure* to a scalarly measurable function $f \colon \Omega \to X$, if for every $\delta > 0$, we have

$$\lim_n \sup_{\|x^*\| \le 1} \mu\{t \in \Omega \colon |x^* f_n(t) - x^* f(t)| > \delta\} = 0.$$

It is well known that each sequence of real valued functions that is convergent in $L_1(\Omega, \mu)$ is also convergent in measure to the same limit. Similarly, if a sequence of scalarly integrable functions $f_n \colon \Omega \to X$, $n \in \mathbb{N}$, is convergent to a scalarly integrable function $f \colon \Omega \to X$ in the Pettis norm, then it is scalarly equi-convergent in measure to $f \colon \Omega \to X$. Also it easily follows that the convergence in measure of strongly measurable functions yields the scalar equi-convergence in measure. In fact, we are able to say much more.

If f, $g \colon \Omega \to X$ are scalarly measurable functions, we define

$$|f|_\mu := \inf \Big\{ \lambda > 0 \colon \sup_{\|x^*\| \le 1} \mu\{|x^* f| \ge \lambda\} \le \lambda \Big\}.$$

and $d_\mu(f,g) := |f-g|_\mu$. One can check that d_μ is a translation invariant pseudometric on the linear space of X-valued scalarly measurable functions. Moreover, convergence in d_μ is equivalent to the scalar equi-convergence in measure.

Implications between several kinds of convergence are listed in the following lemma.

Lemma 2.2 *Let $f_n\colon \Omega \to X$, $n \in \mathbb{N}$, and $f\colon \Omega \to X$ be scalarly measurable functions. We then have* (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (D), *where*

(A) (f_n) *is μ-a.e. convergent in the norm topology of X to f;*

(B) $\forall\, \delta > 0$, $\lim_n \mu_*\{\|f_n - f\| > \delta\} = 0$ (μ_* *is the inner measure induced by* μ);

(C) (f_n) *is scalarly equi-convergent in measure to f;*

(D) (f_n) *is scalarly convergent in measure to f.*

Proof (A) $\Rightarrow$ (B) Given $\delta > 0$ and $n \in \mathbb{N}$, let

$$A_n^\delta := \{t \in \Omega\colon \|f_n(t) - f(t)\| > \delta\}.$$

Suppose that there exists $\delta > 0$ with $\limsup_n \mu_*(A_n^\delta) = a > 0$. If $B_n^\delta \subseteq A_n^\delta$ is a measurable kernel of A_n^δ, then there is an increasing sequence (n_k) of integers such that $\mu(B_{n_k}^\delta) > a/2$ for all $k \in \mathbb{N}$. Let

$$B^\delta := \limsup_k B_{n_k}^\delta = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} B_{n_k}^\delta .$$

Then $\mu(B^\delta) = \lim_m \mu\left(\bigcup_{k=m}^\infty B_{n_k}^\delta\right) \geq a/2$. It follows that, if $t \in B^\delta$, then the inequality $\|f_{n_k}(t) - f(t)\| > \delta$ holds true for infinitely many k's. That contradicts the μ-a.e. convergence of the sequence (f_n).

(B) $\Rightarrow$ (C) Let A_n^δ be defined as before. Then

$$A_n^\delta = \bigcup_{\|x^*\| \leq 1} \{t \in \Omega\colon |x^* f_n(t) - x^* f(t)| > \delta\}$$

and so

$$\mu_*(A_n^\delta) \geq \mu\{|x^* f_n - x^* f| > \delta\} \quad \text{for each } x^* \in B(X^*).$$

Consequently,

$$\mu_*(A_n^\delta) \geq \sup_{\|x^*\| \leq 1} \mu\{|x^* f_n - x^* f| > \delta\},$$

which yields the scalar equi-convergence of (f_n) in measure to f. □

None of the implications in Lemma 2.2 is reversible. Indeed, it is well known that (B) is essentially weaker than (A) for $X = \mathbb{R}$. The remaining reversions will be discussed below. Even in case of strongly measurable functions, the condition (C) is essentially weaker than (B) but as further results show, it is strong enough to guarantee a very strong convergence of the corresponding integrals. The following example is based on an idea of Fremlin (private communication); see also Remark 2.5 below.

Example 2.3 Let $\{e_k\colon k \in \mathbb{N}\}$ be the standard normalized basis of l_2, and for each $n \in \mathbb{N}$ and $k \in \{1, \ldots, 2^n\}$, denote by I_{nk} the interval $[\frac{k-1}{2^n}, \frac{k}{2^n}]$. For each $n \in \mathbb{N}$, we define the function $f_n\colon [0,1] \to l_2$ by the formula

$$f_n(t) := \sum_{k=1}^{2^n} e_{2^n + k} \chi_{I_{nk}}(t) .$$

Let us fix $y = (\beta_n)_n \in B(l_2)$. Then $\langle y, f_n(t)\rangle = \sum_{k=1}^{2^n} \beta_{2^n+k}\chi_{I_{nk}}(t)$ and so $|\langle y, f_n(t)\rangle| \le \sup\{|\beta_k|\colon 2^n + 1 \le k \le 2^{n+1}\} \to 0$ as $n \to \infty$. It follows that for each functional y, the sequence $(\langle y, f_n\rangle)_n$ is uniformly convergent to zero. Notice however that we have $\|f_n(t)\| = 1$ for each n and $t \in [0,1]$. Consequently, the sequence does not fulfill the condition (B). It is however scalarly equi-convergent to zero in measure. To see this, for fixed $\delta \in (0,1)$ and $n \in \mathbb{N}$, denote by $p(n,\delta,y)$ the number of terms β_{2^n+k}, $1 \le k \le 2^n$, such that $|\beta_{2^n+k}| > \delta$. We have then $p(n,\delta,y) < \delta^{-2}$. Consequently,

$$\begin{aligned}\lambda\{t \in [0,1]\colon |\langle y, f_n(t)\rangle| > \delta\} &= \lambda\left\{t \in [0,1]\colon \left|\sum_{k=1}^{2^n} \beta_{2^n+k}\chi_{I_{nk}}(t)\right| > \delta\right\}\\ &\le \lambda\left\{t \in [0,1]\colon \sum_{k=1}^{2^n} |\beta_{2^n+k}|\chi_{I_{nk}}(t) > \delta\right\}\\ &\le p(n,\delta,y)2^{-n} \le \frac{1}{\delta^2 2^n}\end{aligned}$$

and so, the sequence $(f_n)_n$ is scalarly equi-convergent in measure to zero.

One may observe that the above proof works properly, with obvious changes, in case of any l_p, $1 < p < \infty$, and c_0. □

In virtue of Lemma 2.2 and Example 2.3, the following theorem is an essential improvement of the result due to Rodríguez [20, Theorem 2.8] (moreover, our proof seems to be more elementary). It improves also [14, Corollary 5.3].

Theorem 2.4 *Let functions $f_n\colon \Omega \to X$, $n \in \mathbb{N}$, be Pettis integrable and let $f\colon \Omega \to X$ be scalarly measurable. The following conditions are equivalent*:

(a) (f_n) *is scalarly equi-convergent in measure to f and $\bigcup_n \mathcal{Z}_{f_n}$ is uniformly integrable*;

(b) f *is Pettis integrable and* $\lim_n \|f_n - f\|_P = 0$.

In particular, (a) *implies that* $\lim_n \left\|\int_E f_n\,d\mu - \int_E f\,d\mu\right\| = 0$ *uniformly with respect to* $E \in \Sigma$.

Proof A nontrivial part of (b)⇒(a) was proved in [20, Theorem 2.8] — it suffices to apply Nikodým and Vitali–Hahn–Saks theorems for vector measures (see [3]). So, let us prove (a)⇒(b). The scalar integrability of f and the uniform integrability of $\mathcal{Z}_f$ are consequence of the uniform integrability of the collection $\bigcup_n \mathcal{Z}_{f_n}$. The Pettis integrability of f follows from Theorem 2.1. Now, fix $\varepsilon > 0$ and pick $\delta > 0$ such that $\int_A |x^*f_n - x^*f|d\mu < \varepsilon$ for all $n \in \mathbb{N}$, $\|x^*\| \le 1$ and $A \in \Sigma$ with $\mu(A) < \delta$. By the assumption of the scalar equi-convergence, pick $k \in \mathbb{N}$ such that $\sup_{\|x^*\|\le 1} \mu(\{t \in \Omega\colon |x^*f_n(t) - x^*f(t)| > \varepsilon\}) < \delta$ for all $n \ge k$. Then for all $n \ge k$, we have

$$\begin{aligned}\|f_n - f\|_P &= \sup_{\|x^*\|\le 1} \int_\Omega |x^*f_n - x^*f|d\mu\\ &\le \sup_{\|x^*\|\le 1} \int_{\{|x^*f_n - x^*f| > \varepsilon\}} |x^*f_n - x^*f|d\mu\\ &\quad + \sup_{\|x^*\|\le 1} \int_{\{|x^*f_n - x^*f| \le \varepsilon\}} |x^*f_n - x^*f|d\mu < 2\varepsilon,\end{aligned}$$

which completes the proof. □

Remark 2.5 A construction similar to that from Example 2.3 can be repeated in any infinite-dimensional Banach space X. Indeed, it was demonstrated in [4, Example 1] that Dvoretsky's theorem can be used to obtain a uniformly bounded sequence of strongly measurable functions $f_n\colon [0,1] \to X$ converging to 0 in the Pettis norm (hence also scalarly equi-convergent in measure) such that $\|f_n(t)\| \geq 1$ for every $t \in [0,1]$.

Our next example shows that, if the sequence $(f_n)_n$ in Theorem 2.4 is only scalarly convergent in measure to f, then the integrals may fail to converge in the Pettis norm. It also shows that implication (C)$\Rightarrow$(D) in Lemma 2.2 is not reversible.

Example 2.6 Recall that a Banach space has the *Schur property* if every weakly convergent sequence is convergent in norm. Let X be a Banach space without the Schur property and let (x_n) be a sequence of elements of X that is weakly convergent to zero but not in norm. Without loss of generality, one may assume that $\inf_n \|x_n\| = \delta > 0$. Then let, for each $n \in \mathbb{N}$, a function $f_n\colon [0,1] \to X$ be defined by $f_n(t) = x_n$ for all $t \in [0,1]$. The sequence (f_n) is scalarly pointwise convergent to zero, but it can be checked (by the use of the Hahn–Banach theorem) that it is not scalarly equi-convergent to zero in Lebesgue measure. We have $\lim_n \int_E x^* f_n \, d\lambda = \lambda(E) \lim_n x^*(x_n) = 0$ for every $x^* \in X^*$, but $\left\|\int_E f_n \, d\lambda\right\| \geq \delta\lambda(E)$ for all $n \in \mathbb{N}$ and every measurable set E of positive measure. □

The above example leads to the question, suggested to us by the Referee, whether conditions (C) and (D) are equivalent in Banach spaces with the Schur property. This seems nontrivial and interesting. At the moment, we cannot answer that question but we are going to be engaged upon it.

3 Vitali Type $\mathcal{I}$-convergence Theorems

There are several kinds of convergence that generalize the usual convergence for sequences of points in a metric space. Among them, the notion of $\mathcal{I}$-convergence (for a given ideal $\mathcal{I}$ of subsets of $\mathbb{N}$) is quite interesting and useful in several settings. We will always assume that an ideal $\mathcal{I}$ of subsets of $\mathbb{N}$ contains Fin (the family of all finite subsets of $\mathbb{N}$) and $\mathcal{I}$ is different from the power set $\mathcal{P}(\mathbb{N})$. Then we say that $\mathcal{I}$ is an *ideal on* $\mathbb{N}$. Since $\mathcal{P}(\mathbb{N})$ can be identified with the Cantor space $\{0,1\}^{\mathbb{N}}$ (via the mapping $A \mapsto \chi_A$), we may consider ideals on $\mathbb{N}$ as subsets of the Cantor space. For many examples of ideals on $\mathbb{N}$, see [5, 11].

If $\mathcal{I}$ is an ideal on $\mathbb{N}$, we say (cf. [11, 19]) that a sequence $(x_n)_{n\in\mathbb{N}}$ of points in a metric space (Y,ρ) is *$\mathcal{I}$-convergent* to $x \in Y$, if for every $\varepsilon > 0$, there is a set $A \in \mathcal{I}$ such that $\rho(x_n, x) < \varepsilon$ for all $n \in \mathbb{N}\backslash A$. We then write $x_n \to_{\mathcal{I}} x$ or $\mathcal{I}\text{-}\lim_n x_n = x$. Note that usual convergence (that is, Fin-convergence) implies $\mathcal{I}$-convergence, while the converse is not true in general. Of course, one can formulate the definition of $\mathcal{I}$-convergence using filters instead of ideals. In particular, if $\mathcal{I} := \mathcal{I}_d$ is the ideal of sets $A \subseteq \mathbb{N}$ with asymptotic density zero (that is, $A \in \mathcal{I}_d$ if and only if $d(A) = 0$, where $d(A) := \limsup_{n\to\infty}(|A \cap \{1,\ldots,n\}| \cdot n^{-1})$ and $|B|$ denotes the cardinality of a set B), we obtain the *statistical convergence* of (x_n) introduced by Steinhaus and Fast [6, 22], and studied by several authors (see [1, 2]).

Given an ideal $\mathcal{I}$ on $\mathbb{N}$, we say that a set $M \subseteq \mathbb{N}$ is *$\mathcal{I}$-thick* if $\mathbb{N}\backslash M \in \mathcal{I}$. We also say that a sequence (n_k) of positive integers is $\mathcal{I}$-thick if the set of its values is $\mathcal{I}$-thick. Note that, if (x_n) is a sequence of elements of X and $\lim_{n\in M} x_n = x \in X$ for an $\mathcal{I}$-thick set M, then $x_n \to_{\mathcal{I}} x$.

We say that a property holds for $\mathcal{I}$-*almost all* $n \in \mathbb{N}$ if it holds for all n from an $\mathcal{I}$-thick set. We will use a simple fact that if $y_n \to_{\mathcal{I}} y$ in a metric space then there is a subsequence $y_{k_n} \to y$.

So far, there are only few results on $\mathcal{I}$-convergence theorems for integrals. We are going to fill up this gap partially. Considering a sequence of functions $f_n\colon \Omega \to X$, $n \in \mathbb{N}$, and a limit function $f\colon \Omega \to X$, it is natural to distinguish two kinds of $\mathcal{I}$-convergence, pointwise and uniform. They are defined (respectively) as follows (cf. [1]): $f_n \to_{\mathcal{I}} f$ whenever $f_n(t) \to_{\mathcal{I}} f(t)$ for all $t \in \Omega$, and $f_n \rightrightarrows_{\mathcal{I}} f$ whenever $\sup_{t\in\Omega} \|f_n(t) - f(t)\| \to_{\mathcal{I}} 0$.

If $\Omega = [0, 1]$ and $X = \mathbb{R}$, the classical convergence theorem for the Riemann integral was easily generalized by the use of uniform $\mathcal{I}$-convergence of (f_n) (see [1, 7]). As to the Lebesgue integral, the $\mathcal{I}$-version of the monotonic convergence theorem is a simple reformulation of the classical one. To obtain an $\mathcal{I}$-version of the Lebesgue dominated convergence theorem, one meets a difficulty concerned with the fact that the pointwise $\mathcal{I}$-limit of a sequence of measurable functions need not be measurable (cf. [10, Example 1]). However, if $\mathcal{I}$ is an analytic ideal, the measurability is preserved [13]. Kadets and Leonov [8] gave characterizations of filters $\mathcal{F}$ on $\mathbb{N}$ for which the Lebesgue dominated $\mathcal{F}$-convergence theorem holds. They wrote also another paper [9] touching dominated convergence theorems for filter convergence. Solecki [21] considered filters on $\mathbb{N}$ for which the respective version of the Fatou lemma is valid. So, it happens that the analogues of classical theorems hold only for some classes of ideals (filters).

Instead of looking for ideals with special properties, we propose an $\mathcal{I}$-convergence of functions that guarantees the $\mathcal{I}$-convergence for an arbitrary ideal $\mathcal{I}$. To infer new integral $\mathcal{I}$-convergence theorems, we will use $\mathcal{I}$-convergence in measure. Let f and f_n, $n \in \mathbb{N}$, be measurable functions from Ω to $\mathbb{R}$. By $f_n \to^{\mu} f$, we denote the convergence of (f_n) in measure μ to f. We say (cf. [1]) that (f_n) is $\mathcal{I}$-*convergent in measure* μ to f if the sequence $\mu(\{t \in \Omega\colon |f_n(t) - f(t)| \geq \varepsilon\})$, $n \in \mathbb{N}$, is $\mathcal{I}$-convergent to 0 for every $\varepsilon > 0$. We then write $f_n \to^{\mu}_{\mathcal{I}} f$. It is obvious that $f_n \to^{\mu} f$ implies $f_n \to^{\mu}_{\mathcal{I}} f$ but the converse need not hold (cf. [1]). If $\mathcal{I} = \mathcal{I}_d$, the theorem due to Fast and Steinhaus [6, 22] states that $f_n \to_{\mathcal{I}_d} f$ μ-a.e. implies $f_n \to^{\mu}_{\mathcal{I}_d} f$. The same authors proved that the converse holds under some additional conditions, however in general this implication is not true (cf. [1]). For a further discussion, see [10].

It is known that pointwise or almost everywhere $\mathcal{I}$-convergence is too weak to guarantee satisfactory convergence of the corresponding integrals (cf. Example 3.4). However, $\mathcal{I}$-convergence in measure behaves very well. In particular, the following Vitali type theorem (we present its proof just for completeness) holds true (cf. [18, II.5.6]).

Theorem 3.1 *Let $f\colon \Omega \to \mathbb{R}$ be measurable and let $\{f_n\colon n \in \mathbb{N}\} \subseteq L_1(\Omega, \mu)$ be a uniformly integrable family. If $f_n \to^{\mu}_{\mathcal{I}} f$, then $f \in L_1(\Omega, \mu)$, $\int_\Omega |f_n - f| d\mu \to_{\mathcal{I}} 0$ and consequently, $\int_\Omega f_n d\mu \to_{\mathcal{I}} \int_\Omega f d\mu$.*

Proof Since $f_n \to^{\mu}_{\mathcal{I}} f$, we may select a subsequence $f_{n_k} \to^{\mu} f$ and thus, by the Vitali theorem, $f \in L_1(\Omega, \mu)$. Now, we may assume that $f = 0$. So, let $f_n \to^{\mu}_{\mathcal{I}} 0$. Since $\{f_n\colon n \in \mathbb{N}\}$ is uniformly integrable, fix $\varepsilon > 0$ and pick $\delta > 0$ such that $\int_A |f_n| d\mu < \varepsilon$ for all $n \in \mathbb{N}$ and $A \in \Sigma$ with $\mu(A) < \delta$. Since $f_n \to^{\mu}_{\mathcal{I}} 0$, select a $B \in \mathcal{I}$ such that $\mu(\{t \in \Omega\colon |f_n(t)| > \varepsilon\}) < \delta$ for all $n \in \mathbb{N}\backslash B$. Then for all $n \in \mathbb{N}\backslash B$, we have

$$\int_\Omega |f_n| d\mu = \int_{\{t\colon |f_n(t)|>\varepsilon\}} |f_n| d\mu + \int_{\{t\colon |f_n(t)|\leq\varepsilon\}} |f_n| d\mu < 2\varepsilon,$$

which completes the proof. □

Example 3.2 Among ideals on $\mathbb{N}$, there is an important class of P-ideals. For $A, B \subseteq \mathbb{N}$, we write $A \subseteq^* B$ whenever $A \backslash B$ is finite. We say that $\mathcal{I}$ is a P-*ideal* if whenever $A_n \in \mathcal{I}$, $n \in \mathbb{N}$, there is $A \in \mathcal{I}$ with $A_n \subseteq^* A$ for each $n \in \mathbb{N}$ (cf. [5]). For instance, $\mathcal{I}_d$ is a P-ideal. Note that P-ideals behave regularly with respect to $\mathcal{I}$-convergence since every $\mathcal{I}$-convergent sequence, restricted to some $\mathcal{I}$-thick set, is convergent in the usual way [11]. In particular, in the case of such an ideal, Theorem 3.1 follows directly from the classical Vitali theorem for sequences. In the general case, even if the proof of Theorem 3.1 is easy, the result itself is not any trivial consequence of the classical Vitali theorem.

In order to present an example of that phenomenon, we recall a known example of an ideal on $\mathbb{N}$ which is not a P-ideal (cf. [11]). Let $\{\Delta_j : j \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into pairwise disjoint infinite sets, and let $\mathcal{I}$ be the ideal of all sets that are contained in unions of finitely many Δ_j's. Consider a sequence $f_n : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$, given by $f_n := \chi_{[0,1/j]}$, $n \in \Delta_j$, $j \in \mathbb{N}$. Then $\{f_n : n \in \mathbb{N}\} \subseteq L_1([0,1], \lambda)$ is a uniformly integrable family with $f_n \to_{\mathcal{I}}^{\lambda} 0$ and $f_n \not\to^{\lambda} 0$. Also, $\int_{[0,1]} f_n \, d\lambda \to_{\mathcal{I}} 0$ but $\int_{[0,1]} f_{n_k} \, d\lambda \not\to 0$ for every $\mathcal{I}$-thick sequence (n_k). Indeed, this follows from $\int_{[0,1]} f_n \, d\lambda = 1/j$ for all $n \in \Delta_j$, $j \in \mathbb{N}$. □

From Theorem 3.1, we infer the following version of the Lebesgue dominated convergence theorem.

Corollary 3.3 *Let $f : \Omega \to \mathbb{R}$ and $f_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, be measurable functions such that $f_n \to_{\mathcal{I}}^{\mu} f$ and $|f_n(t)| \le g(t)$ for all $n \in \mathbb{N}$, $t \in \Omega$, where $g \in L_1(\Omega, \mu)$. Then $f \in L_1(\Omega, \mu)$, $\int_\Omega |f_n - f| d\mu \to_{\mathcal{I}} 0$ and consequently, $\int_\Omega f_n d\mu \to_{\mathcal{I}} \int_\Omega f d\mu$.*

The next example shows that the assumption $f_n \to_{\mathcal{I}}^{\mu} f$ cannot be replaced by $f_n \to_{\mathcal{I}} f$.

Example 3.4 In [10, Theorem 2], Komisarski constructed an analytic P-ideal $\mathcal{I}$ such that, for every nonatomic finite measure space (Ω, Σ, μ) and for every two measurable functions $f, g : \Omega \to \mathbb{R}$, there exists a sequence (f_n) of measurable functions $f_n : \Omega \to \mathbb{R}$ with $f_n \to_{\mathcal{I}} f$ and $f_n \to_{\mathcal{I}}^{\mu} g$. It follows from the construction in [10] that for all $n \in \mathbb{N}$ and $t \in \Omega$ we have $f_n(t) \in \{f(t), g(t)\}$. Now, assume that the measure μ is nonatomic and fix two functions $f, g \in L_1(\Omega, \mu)$ with distinct integrals on Ω. Next, consider (f_n) as in the Komisarski construction. If $h = \max\{f, g\}$, we have $h \in L_1(\Omega, \mu)$ with $|f_n(t)| \le h(t)$ for all $n \in \mathbb{N}$ and $t \in \Omega$. Hence by Corollary 3.3, we obtain $\int_\Omega f_n d\mu \to_{\mathcal{I}} \int_\Omega g d\mu$. If the version of Corollary 3.3 with assumption $f_n \to_{\mathcal{I}} f$ were true, we would have $\int_\Omega f_n d\mu \to_{\mathcal{I}} \int_\Omega f d\mu$. However, this is impossible since $\int_\Omega f d\mu \ne \int_\Omega g d\mu$. □

Now, let us turn to vector integrals. Clearly, Theorem 3.1 and Corollary 3.3 have obvious counterparts for strongly measurable functions from Ω to X and their Bochner integrals. Then consider the Pettis integral. We need two definitions. The set $Y^\perp := \{x^* \in X^* : (\forall x \in Y)\, x^*(x) = 0\}$ is called the *annihilator* of $Y \subseteq X$. We say that a function $f : \Omega \to X$ is *determined* by a space $Y \subseteq X$ if $x^* \in Y^\perp$ yields $x^* f = 0$, μ-a.e..

Now, the following $\mathcal{I}$-version of Lebesgue dominated convergence theorem for the Pettis integral generalizes Corollary 3.3. It is also a generalization of [17, Theorem 8.2] and [15, Theorem 5.3].

Theorem 3.5 *Let $f : \Omega \to X$ be a scalarly measurable function and let $f_n : \Omega \to X$, $n \in \mathbb{N}$,* *be a sequence of Pettis integrable functions. Assume that the following conditions are satisfied*:

(i) *there exists* $g \in L_1(\Omega, \mu)$ *such that for each* $x^* \in B(X^*)$ *we have* $|x^* f_n| \leq g$, *μ-a.e. and for* $\mathcal{I}$*-almost all* $n \in \mathbb{N}$;

(ii) $x^* f_n \to_{\mathcal{I}}^{\mu} x^* f$ *for each* $x^* \in X^*$.

Then f *is Pettis* μ*-integrable and* $\int_\Omega |x^* f_n - x^* f| \, d\mu \to_{\mathcal{I}} 0$ *for every* $x^* \in X^*$. *In particular, for each* $E \in \Sigma$, *the sequence* $(\int_E f_n \, d\mu)_{n \in \mathbb{N}}$ *is weakly* $\mathcal{I}$*-convergent to* $\int_E f \, d\mu$.

Proof For each fixed $x^* \in X^*$, the family $\{x^* f_n : n \in \mathbb{N}\}$ fulfils the assumptions of Theorem 3.1. Consequently, $x^* f$ is integrable and $\int_\Omega |x^* f_n - x^* f| \, d\mu \to_{\mathcal{I}} 0$.

To prove the Pettis integrability of f, we apply (ii) to pick a subsequence $(x^* f_n)_{n \in M_0}$ convergent in measure to $x^* f$ and then a next subsequence $(x^* f_n)_{n \in M}$ convergent μ-a.e. to $x^* f$ (where M is an infinite subset of M_0). Condition (i) yields then the inequality $|x^* f| \leq g$, μ-a.e.. Since x^* is arbitrary, the integrability of g implies the uniform integrability of the set $\mathcal{Z}_f$.

Then, it is a consequence of the Pettis integrability of each f_n and of [15, Theorem 4.5] that each f_n is determined by a WCG (weakly compactly generated) space. As a result, it follows from (ii) that also f is determined by a WCG subspace of X. Consequently, as $\mathcal{Z}_f$ is weakly relatively compact, [15, Theorem 4.5] yields the Pettis integrability of f. □

Condition (i) in the above theorem may be obviously replaced by the uniform integrability of the set $\bigcup_n \mathcal{Z}_{f_n}$ since such a uniform integrability yields the uniform integrability of $\mathcal{Z}_f$. It generalizes Theorem 2.1.

As the next result shows, the assumption concerning uniform integrability can be sometimes essentially weakened. The theorem is a generalization of [16, Theorem 1] (see also [17, Theorem 8.1] and [15, Theorem 5.2]).

Theorem 3.6 *Let* $f : \Omega \to X$ *be a scalarly measurable function and let* $f_n : \Omega \to X$, $n \in \mathbb{N}$, *be a sequence of Pettis integrable functions. Assume that* $\mathcal{I}$ *is a P-ideal and the following conditions are satisfied*:

(j) *each sequence* (x_n^*) *with terms in* $B(X^*)$ *contains a subsequence* $(x_{n_k}^*)$ *such that for an* $\mathcal{I}$*-thick set* $M \subseteq \mathbb{N}$ *the collection* $\{x_{n_k}^* f_m : m \in M,\ k \in \mathbb{N}\}$ *is uniformly integrable*;

(jj) $x^* f_n \to_{\mathcal{I}}^{\mu} x^* f$, *for each* $x^* \in X^*$.

Then f *is Pettis* μ*-integrable, and* $\int_\Omega |x^* f_n - x^* f| \, d\mu \to_{\mathcal{I}} 0$ *for every* $x^* \in X^*$. *In particular, for each* $E \in \Sigma$, *the sequence* $(\int_E f_n \, d\mu)_{n \in \mathbb{N}}$ *is weakly* $\mathcal{I}$*-convergent to* $\int_E f \, d\mu$.

Proof Fix $x^* \in X^*$. By (j) pick an $\mathcal{I}$-thick set $M_0 \subseteq \mathbb{N}$ such that $\{x^* f_m : m \in M_0\}$ is uniformly integrable. Since $\mathcal{I}$ is a P-ideal, the condition (jj) implies the existence of a further $\mathcal{I}$-thick set $M \subseteq M_0$ such that $(x^* f_n)_{n \in M}$ converges in measure to $x^* f$. So, by Theorem 3.1 we infer that $x^* f$ is integrable and $\int_\Omega |x^* f_n - x^* f| \, d\mu \to_{\mathcal{I}} 0$.

Now, if a sequence $(x_n^*)_{n \in \mathbb{N}}$ with terms in $B(X^*)$ is quite arbitrary, then (with the notation of (j)) the family $\{x_{n_k}^* f_m : m \in M,\ k \in \mathbb{N}\}$ is uniformly integrable. It follows directly from the Vitali convergence theorem that also the set $\{x_{n_k}^* f : k \in \mathbb{N}\}$ is uniformly integrable. Thus, each infinite subset of $\mathcal{Z}_f$ contains an infinite uniformly integrable subset. It follows that the set $\mathcal{Z}_f$ is uniformly integrable. The rest of the proof is the same as that of Theorem 3.5. □

Now, we will formulate a counterpart of Theorem 2.4. We say that a sequence of scalarly measurable functions $f_n \colon \Omega \to X$, $n \in \mathbb{N}$, is *$\mathcal{I}$-scalarly equi-convergent* in measure to a scalarly measurable function $f \colon \Omega \to X$ if for every $\delta > 0$ we have

$$\mathcal{I}\text{-}\lim_n \sup_{\|x^*\| \le 1} \mu\{t \in \Omega \colon |x^* f_n(t) - x^* f(t)| > \delta\} = 0.$$

Theorem 3.7 *Let $f_n \colon \Omega \to X$, $n \in \mathbb{N}$, be a sequence of Pettis integrable functions $\mathcal{I}$-scalarly equi-convergent in measure to a function $f \colon \Omega \to X$. If the family $\bigcup_n \mathcal{Z}_{f_n}$ is uniformly integrable, then f is Pettis integrable and $\mathcal{I}\text{-}\lim_n \|f_n - f\|_P = 0$. In particular,*

$$\mathcal{I}\text{-}\lim_n \left\| \int_E f_n \, d\mu - \int_E f \, d\mu \right\| = 0 \quad \textit{uniformly with respect to } E \in \Sigma\,.$$

Proof Pick a subsequence of (f_n) that is scalarly equi-convergent in measure to f. Theorem 2.4 yields then the Pettis integrability of f. The rest of the proof is analogous to that of Theorem 2.4, where we use an exceptional set $B \in \mathcal{I}$ instead of $\{1, \dots, k-1\}$. □

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