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Linear Liftings Respecting Coordinates

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We establish the existence of linear liftings which respect coordinates for completed products of arbitrary many factors of complete probability spaces.

Key Words: lifting; linear lifting; product lifting; product measure; lifting respecting coordinates.

INTRODUCTION

The problem of the existence of liftings respecting coordinates for completed products of arbitrary many factors of complete probability spaces was raised by Burke [1] and Fremlin [2, problem 346$Z(a)$]. For finite products the positive solution was given by Burke [1]. Then Fremlin [2] proved the existence of such liftings in arbitrary products of Maharam homogeneous measure spaces. For arbitrary infinite products (even countable) the problem is still open, as far as we know. However, Fremlin [2, Theorem 346G, 346Xg] proved the existence of a lower density respecting coordinates in arbitrary products. Partial results are also contained in [7, 8].

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We cannot prove a general theorem on the existence of liftings respecting coordinates and so we have concentrated ourselves on linear liftings respecting coordinates (Definition in Section 2). Linear liftings are sufficient for the most important applications of lifting such as construction of vector valued densities and disintegrations.

It is easy to see that each lifting, linear lifting, or density respecting coordinates uniquely determines the coordinate liftings, linear liftings, and densities. In this paper we are interested not only in the existence problem, but we also describe a large class of linear liftings that can be marginals of linear liftings respecting coordinates. We call them admissible linear liftings.

It seems that all the procedures known for passing from linear liftings (resp. densities) to liftings (even those working for products with two factors from [6]) fail for liftings respecting coordinates.

Our method is totally different from Fremlin's method which is based on the Maharam structure theorem for measure algebras. As in the (now) standard proof of the existence of liftings we apply only transfinite induction so that the relation between Fremlin's method and ours is to some extent comparable to the relation of Maharam's and Ionescu Tulcea's proofs of the existence of a lifting.

1. PRELIMINARIES

For a given probability space (Ω, Σ, μ) a set $N \in \Sigma$ with $\mu(N) = 0$ is called a *μ-null set* and for $A, B \in \Sigma$ we write $A = B$ a.e. (μ) iff $A \triangle B$, the symmetric difference of A and B, is a μ-null set. The family of all μ-null members of Σ is denoted by Σ_0. The (Carathéodory) completion of (Ω, Σ, μ) will be denoted by $(\Omega, \hat{\Sigma}, \hat{\mu})$. If $\Xi \subset \Sigma$, then $\mu \hat{\restriction} \Xi$ will be the completion of $\mu \mid \Xi$. $\mathscr{L}^\infty(\mu)$ denotes the family of all bounded real-valued μ-measurable functions on (Ω, Σ, μ). Equivalent functions are not identified. The space of equivalence classes of functions that are μ-integrable (or bounded) is denoted by $L_1(\mu)$ (resp. by $L^\infty(\mu)$). The σ-algebra generated by a family $\mathscr{L}$ of sets is denoted by $\sigma(\mathscr{L})$. $\mathbf{N}$ and $\mathbf{R}$ stand for the natural numbers and the real numbers, respectively. If $M \subseteq \Omega$, then $M^c := \Omega \setminus M$. We use the notion of (*lower*) *density*, *linear lifting*, *lifting* in the sense of [5, Chap. III] and for any probability space (Ω, Σ, μ) we denote by $\vartheta(\mu)$, $\mathscr{G}(\mu)$, and $\Lambda(\mu)$ the system of all (lower) densities, linear liftings, and liftings, respectively. For each $\rho \in \Lambda(\hat{\mu})$ there exists exactly one (multiplicative) lifting $\tilde{\rho}$ (in the sense of [5, Chap. III] on $\mathscr{L}^\infty(\hat{\mu})$, such that $\tilde{\rho}(\chi_A) = \chi_{\rho(A)}$ for all $A \in \Sigma$ (χ_A denotes the characteristic function of A) and vice versa (see [5, pp. 35–36]). For simplicity we write $\rho = \tilde{\rho}$ throughout.

If γ is an ordinal, then we will identify it with the set $\{\alpha < \gamma\}$ of all ordinals less than γ. I will always be a nonempty set and if $(\Omega_i, \Sigma_i, \mu_i)_{i \in I}$

is a family of probability spaces then, for each $\emptyset \neq J \subseteq I$ we denote by $(\Omega_J, \Sigma_J, \mu_J)$ the product measure space $\bigotimes_{i\in J}(\Omega_i, \Sigma_i, \mu_i)$. $\hat{\bigotimes}_{i\in J}(\Omega_i, \Sigma_i, \mu_i)$ is the completion of $\bigotimes_{i\in J}(\Omega_i, \Sigma_i, \mu_i)$. We denote by I^* the collection of all nonempty proper subsets of I. The set $\{1, ..., n\}$ is denoted by $[n]$.

For a family $(\Omega_i, \Sigma_i, \mu_i)_{i\in I}$ of probability spaces and a probability space (Ω, Σ, μ) such that $\Omega = \Omega_I$, $\Sigma \supseteq \Sigma_I$, $\mu \mid \Sigma_I = \mu_I$, we call a (linear) lifting π for μ a *product-(linear) lifting* of the (linear) liftings ρ_i for μ_i $(i \in I)$, and we write $\pi \in \bigotimes_{i\in I} \rho_i$ if the equation

$$\pi([f_{i_1}, ..., f_{i_n}]) = [\rho_{i_1}(f_{i_1}), ..., \rho_{i_n}(f_{i_n})]$$

holds true for all $n \in \mathbf{N}$, $i_1, ..., i_n \in I$, and all $f_{i_k} \in \mathcal{L}^\infty(\mu_{i_k})$ $(k = 1, ..., n)$, where $[fi_1, ..., f_{i_n}]$ denotes the function $\bigotimes_{i\in I} g_i$, with $g_{i_k} = f_{i_k}$ $(k = 1, ..., n)$ and $g_i = \chi_{\Omega_i}$ if $i \in I \backslash \{i_1, ..., i_n\}$. If $I = \{1, ..., n\}$ then we write $\pi \in \rho_1 \otimes \cdots \otimes \rho_n$.

We say that a (linear) lifting π for μ_I *respects coordinates*, if $\pi(f)$ is determined by coordinates in J whenever $f \in \mathcal{L}^\infty(\mu_I)$ is determined by coordinates in $J \subseteq I$ (cf. [2, Definition 346A]), that is, if $f = g \otimes \chi_{\Omega_{J^c}}$ for some $g \in \mathcal{L}^\infty(\mu_J)$.

If (Ω, Σ, μ) is a probability space and I is a nonempty set, we write μ^I for the product measure on Ω^I and Σ^I for its domain. A (linear) lifting ρ for μ is *consistent*, if for every $n \in \mathbf{N}$ there exists a (linear) lifting $\rho^{[n]}$ for $\mu^{[n]}$ such that

$$\rho^{[n]}(f_1 \otimes \cdots \otimes f_n) = \rho(f_1) \otimes \cdots \otimes \rho(f_n)$$

for all $f_i \in \mathcal{L}(\mu)$ (see Talagrand [9]).

2. ADMISSIBLE LINEAR LIFTINGS

Talagrand's paper [9] seems to be the first one, where a certain compatibility for products and liftings appears. It has been observed already by Talagrand [10] (see also [6]) that not all liftings have adequate properties from the product point of view. The same holds true in the case of linear liftings. Therefore we separate a wide class of linear liftings, the class of all *admissible linear liftings*, possessing properties suitable for our purposes.

Definition 2.1. Let (Θ, T, ν) be a complete probability space. A linear lifting $\tau \in \mathcal{G}(\nu)$ is called an *admissible linear lifting* if it can be constructed with the help of the transfinite induction in the way described below.

(A) Let $\mathbf{d}$ be the smallest cardinal with the property that there exists a collection $\mathcal{M} \subset T$ such that $\sigma(\mathcal{M})$ is dense in T in the pseudometric generated by ν. Let $\mathcal{M} = (M_\alpha)_{\alpha<\kappa}$ be numbered by ordinals less than κ, where κ is the first ordinal of the cardinality $\mathbf{d}$. Denote by η_0 the σ-algebra $\sigma(T_0)$ and for each $1 \leqslant \alpha \leqslant \kappa$ denote by η_α the σ-algebra generated by the family $\{M_\gamma : \gamma < \alpha\} \cup \eta_0$. We may assume that $M_\alpha \notin \eta_\alpha$ for each α. Notice that all the measures $\nu \mid \eta_\alpha$ are complete.

For each $\gamma \leqslant \kappa$ of countable cofinality, we fix an increasing sequence (γ_n^γ) of ordinals that is cofinal with γ. Moreover, for each such γ, we fix also a free ultrafilter $\mathscr{U}_\gamma$ on $\mathbf{N}$.

(B) For the algebra η_0 and $g \in \mathscr{L}^\infty(\nu \mid \eta_0)$ we define $\tau_0(g) = a$, if $g = a$ a.e. (ν). We have then $\tau_0 \in \mathscr{G}(\nu \mid \eta_0)$.

(C) If γ is a limit ordinal of uncountable cofinality, then $\eta_\gamma = \bigcup_{\alpha<\gamma} \eta_\alpha$ and we define $\tau_\gamma \in \mathscr{G}(\nu \mid \eta_\gamma)$ by setting

$$\tau_\gamma(g) := \tau_\alpha(g) \qquad \text{if} \quad g \in \mathscr{L}^\infty(\nu \mid \eta_\alpha).$$

(D) If γ is of countable cofinality, then we put for simplicity $\tau_n := \tau_{\gamma_n^\gamma}$ and $\eta_n := \eta_{\gamma_n^\gamma}$ for all $n \in \mathbf{N}$. Then $\eta_\gamma = \sigma(\bigcup_{n\in\mathbf{N}} \eta_n)$ and we can define τ_γ by setting

$$\tau_\gamma(h) := \lim_{n \in \mathscr{U}_\gamma} \tau_n[E_{\eta_n}(h)] \qquad \text{for} \quad h \in \mathscr{L}^\infty(\nu \mid \eta_\gamma),$$

where E_{η_n} denotes the conditional expectation with respect to η_n. Using the arguments of the proof of Theorem 2 in [5, Chap. IV, Sect. 1], we get $\tau_\gamma \in \mathscr{G}(\nu \mid \eta_\gamma)$ and $\tau_\gamma \mid \mathscr{L}^\infty(\nu \mid \eta_\alpha) = \tau_\alpha$ for each $\alpha < \gamma$.

(E) Let now $\gamma = \beta + 1$. To simplify the notations let $M := M_\beta$. It then follows that

$$\mathscr{L}^\infty(\nu \mid \eta_\gamma) = \{g\chi_M + h\chi_{M^c} : g, h \in \mathscr{L}^\infty(\nu \mid \eta_\beta)\}.$$

Put

$$M_1 := \text{ess inf}\{B \in \eta_\beta : M \subseteq B \text{ a.e. } (\nu)\},$$
$$M_2 := \text{ess inf}\{B \in \eta_\beta : M^c \subseteq B \text{ a.e. } (\nu)\},$$

i.e., M_1 and M_2 are η_β-measurable covers of M and M^c, respectively. Then put

$$\tau_\gamma(g\chi_M + h\chi_{M^c}) := \chi_M \tau_\beta(g\chi_{M_1} + h\chi_{M_1^c}) + \chi_{M^c} \tau_\beta(g\chi_{M_2^c} + h\chi_{M_2})$$

if $g, h \in \mathscr{L}^\infty(\nu \mid \eta_\beta)$. It follows that $\tau_\gamma \in \mathscr{G}(\nu \mid \eta_\gamma)$ and $\tau_\gamma \mid \mathscr{L}^\infty(\nu \mid \eta_\beta) = \tau_\beta$.

(F) If $\tau = \tau_\kappa$, then it is said to be admissible.

Throughout the collection of all admissible linear liftings on (Θ, T, ν) will be denoted by $A\mathscr{G}(\nu)$ and each $\tau \in A\mathscr{G}(\nu)$ will be considered together with all elements involved into the above construction without any additional remarks. In particular each sequence (γ_n^γ) and the ultrafilter $\mathscr{U}_\gamma$ will be fixed and said to be the sequence and the ultrafilter associated with γ.

By converting the above definition into an inductive proof, we get

PROPOSITION 2.2. *$A\mathscr{G}(v) \neq \emptyset$ for every complete probability space (Ω, T, v).*

THEOREM 2.3. *Let (Θ, T, v) be a complete probability space. If $\tau \in A\mathscr{G}(v)$ then for each complete probability space (Ω, Σ, μ) and for each $\rho \in \mathscr{G}(\mu)$ there exists a $\varphi \in \mathscr{G}(\mu \hat{\otimes} v)$ such that*

$$\varphi(g \otimes h) = \rho(g) \otimes \tau(h) \qquad \textit{for all} \quad g \in \mathscr{L}^\infty(\mu) \quad \textit{and} \quad h \in \mathscr{L}^\infty(v)\,.$$

If $(\Omega, \Sigma, \mu) = \hat{\bigotimes}_{i \in I}(\Omega_i, \Sigma_i, \mu_i)$, $\rho_i \in \mathscr{G}(\mu_i)$, and $\rho \in \bigotimes_{i \in I} \rho_i$ respects coordinates, then φ can be chosen to respect coordinates also.

Proof. Since the first part of the theorem is a particular case of the second one (for a one element I) we assume at once that $(\Omega, \Sigma, \mu) = (\Omega, \hat{\Sigma}_I, \hat{\mu}_I)$.

Let there be given a $\rho \in \mathscr{G}(\mu)$ respecting coordinates and such that $\rho \in \bigotimes_{i \in I} \rho_i$. Then, let there be given a $\tau \in A\mathscr{G}(v)$ all together with other elements involved into the construction of $\tau \in A\mathscr{G}(v)$. In particular the family $\mathscr{M} = (M_\alpha)_{\alpha < \kappa}$, the σ-subalgebras $(\eta_\alpha)_{\alpha < \kappa}$, and the sequences (γ_n^γ) associated with limit ordinals γ of countable cofinality are fixed.

Using the transfinite induction, we shall be constructing now a transfinite sequence $(\varphi_\alpha)_{\alpha \leqslant \kappa}$ with $\varphi_\alpha \in \mathscr{G}(\mu \otimes v \upharpoonright \Sigma \otimes \eta_\alpha)$ and such that

$$\varphi_\alpha(g \otimes h) = \rho(g) \otimes \tau_\alpha(h) \qquad \text{for all} \quad g \in \mathscr{L}^\infty(\mu), h \in \mathscr{L}^\infty(v \mid \eta_\alpha), \tag{1}$$

$$\varphi_\beta \mid \mathscr{L}^\infty(\mu \otimes v \upharpoonright \Sigma \otimes \eta_\alpha) = \varphi_\alpha \qquad \text{for all} \quad \alpha < \beta \leqslant \kappa\,, \tag{2}$$

and

$$\varphi_\alpha \qquad \text{respects coordinates for all} \quad \alpha \leqslant \kappa. \tag{3}$$

We have

$$\mathscr{L}^\infty(\mu \otimes v \mid \Sigma \otimes \eta_0) = \{f : f = g \otimes 1_\Theta \text{ a.e. } (\mu \otimes v) \text{ for some } g \in \mathscr{L}^\infty(\mu)\}.$$

For each $f \in \mathscr{L}^\infty(\mu \otimes v \mid \Sigma \otimes \eta_0)$ define

$$\varphi_0(f) := \rho(g) \otimes 1_\Theta \qquad \text{if} \quad f = g \otimes 1_\Theta \text{ a.e. } (\mu \otimes v) \qquad \text{for} \quad g \in \mathscr{L}^\infty(\mu).$$

It can be easily seen that φ_0 can be extended in the obvious way to a $\varphi_0 \in \mathscr{G}(\mu \otimes v \upharpoonright \Sigma \otimes \eta_0)$ respecting coordinates and satisfying the condition (1).

Assume now that given $\gamma \leqslant \kappa$, a system (φ_α) satisfying the required conditions (1)–(3), has been constructed for all $\alpha < \gamma$.

We have to distinguish three cases.

(A) $\gamma = \beta + 1$. To simplify the notations let $M := M_\beta$. It then follows that

$$\mathscr{L}^\infty(\mu \otimes \nu \mid \Sigma \otimes \eta_\gamma) = \{ G\chi_{\Omega \times M} + H\chi_{\Omega \times M^c} : G, H \in \mathscr{L}^\infty(\mu \otimes \nu \mid \Sigma \otimes \eta_\beta) \}.$$

Put

$$E_1 := \text{ess inf}\{ E \in \Sigma \otimes \eta_\beta : \Omega \times M \subseteq E \text{ a.e.}(\mu \otimes \nu) \},$$
$$E_2 := \text{ess inf}\{ E \in \Sigma \otimes \eta_\beta : \Omega \times M^c \subseteq E \text{ a.e.}(\mu \otimes \nu) \},$$

and

$$\begin{aligned} \varphi_\gamma(G\chi_{\Omega \times M} + H\chi_{\Omega \times M^c}) := {} & \chi_{\Omega \times M} \varphi_\beta(G\chi_{E_1} + H\chi_{E_1^c}) \\ & + \chi_{\Omega \times M^c} \varphi_\beta(G\chi_{E_2^c} + H\chi_{E_2}) \end{aligned} \tag{$*$}$$

if $G, H \in \mathscr{L}^\infty(\nu \otimes \mu \mid \Sigma \otimes \eta_\beta)$. It follows that the obvious extension of φ_γ to $\mathscr{L}^\infty(\mu \otimes \nu \lceil \Sigma \otimes \eta_\gamma)$, denoted again by φ_γ, belongs to $\mathscr{G}(\mu \otimes \nu \lceil \Sigma \otimes \eta_\gamma)$.

If M_1, M_2 are defined according to Definition 2.1, then we have by [6]

$$E_1 = \Omega \times M_1 \qquad \text{and} \qquad E_2 = \Omega \times M_2 \text{ a.e. } (\mu \hat{\otimes} \nu).$$

Consequently, if $f \in \mathscr{L}^\infty(\mu)$ and $u = g\chi_M + h\chi_{M^c} \in \mathscr{L}^\infty(\nu \mid \eta_\gamma)$ with $g, h \in \mathscr{L}^\infty(\nu \mid \eta_\beta)$ then

$$\begin{aligned} \varphi_\gamma(f \otimes u) &= \chi_{\Omega \times M} \varphi_\beta[(f \otimes g)\chi_{E_1} + (f \otimes h)\chi_{E_1^c}] \\ &\qquad + \chi_{\Omega \otimes M^c} \varphi_\beta[(f \otimes g)\chi_{E_2^c} + (f \otimes h)\chi_{E_2}] \\ &= \chi_{\Omega \times M} \varphi_\beta[f \otimes (g\chi_{M_1} + h\chi_{M_1^c})] + \chi_{\Omega \times M^c} \varphi_\beta[f \otimes (g\chi_{M_2^c} + h\chi_{M_2})] \\ &= \rho(f) \otimes [\chi_M \tau_\beta(g\chi_{M_1} + h\chi_{M_1^c}) + \chi_{M^c} \tau_\beta(g\chi_{M_2^c} + h\chi_{M_2})] \\ &= \rho(f) \otimes \tau_\gamma(u). \end{aligned}$$

To show that φ_γ respects coordinates, we set $I_\gamma := I \cup \{\gamma\}$, $\Omega_\gamma := \Theta$, $\Sigma_\gamma := \eta_\gamma$ and $\rho_\gamma := \tau_\gamma$. For all $J \in I_\gamma^*$ we set also $J^c := I_\gamma \backslash J$.

We have to distinguish two cases.

(1°) $\gamma \notin J$. If $f = g \otimes \chi_{\Omega_{I \setminus J}} \otimes \chi_\Theta$ with $g \in \mathscr{L}^\infty(\hat{\mu}_J)$, then

$$\varphi_\gamma(f) = \rho(g \otimes \chi_{\Omega_{I \setminus J}}) \otimes \chi_\Theta = (g^* \otimes \chi_{\Omega_{I \setminus J}}) \otimes \chi_\Theta = g^* \otimes \chi_{\Omega_{J^c}}$$

and $g^* \in \mathbf{R}^{\Omega_J}$.

(2°) $\gamma \in J$. Then $J^c = I \backslash J$, $J = K \cup \{\gamma\}$, and $K \cup J^c = I$ disjointly for a subset K of I. Let $f = u \otimes \chi_{\Omega_{J^c}}$ with some $u \in \mathscr{L}^\infty(\mu_K \otimes \nu \mid \Sigma_K \otimes \eta_\gamma)$. Since

$$\Sigma_K \otimes \eta_\gamma = \Sigma_K \otimes \sigma(\eta_\beta \cup \{M\}) = \sigma[(\Sigma_K \otimes \eta_\beta) \cup \{\Omega_K \times M\}],$$

we have $u = g\chi_{\Omega_K \times M} + h\chi_{\Omega_K \times M^c}$ for $g, h \in \mathscr{L}^\infty(\mu_K \otimes \nu \mid \Sigma_K \otimes \eta_\beta)$ and so

$$f = (g \otimes \chi_{\Omega_{J^c}})\, \chi_{\Omega \times M} + (h \otimes \chi_{\Omega_{J^c}})\, \chi_{\Omega \times M^c}.$$

Consequently,

$$\begin{aligned} \varphi_\gamma(f) = {} & \chi_{\Omega \times M} \varphi_\beta[(g \otimes \chi_{\Omega_{J^c}})\, \chi_{E_1} + (h \otimes \chi_{\Omega_{J^c}})\, \chi_{E_1^c}] \\ & + \chi_{\Omega \times M^c} \varphi_\beta[(g \otimes \chi_{\Omega_{J^c}})\, \chi_{E_2^c} + (h \otimes \chi_{\Omega_{J^c}})\, \chi_{E_2}]. \end{aligned}$$

If

$$\begin{aligned} f_1 &:= (g \otimes \chi_{\Omega_{J^c}})\, \chi_{E_1} + (h \otimes \chi_{\Omega_{J^c}})\, \chi_{E_1^c} \\ &= (g \otimes \chi_{\Omega_{J^c}})\, \chi_{\Omega \times M_1} + (h \otimes \chi_{\Omega_{J^c}})\, \chi_{\Omega \times M_1^c} \\ &= (g\chi_{\Omega_K \times M_1} + h\chi_{\Omega_K \times M_1^c}) \otimes \chi_{\Omega_{J^c}} \end{aligned}$$

then it follows from the inductive assumption that $\varphi_\beta(f_1) = G \otimes \chi_{\Omega_{J^c}}$ for some $G \in \mathbf{R}^{\Omega_J}$, and in the same way we get for the second term $H \otimes \chi_{\Omega_{J^c}}$ for some $H \in \mathbf{R}^{\Omega_J}$, i.e., $\varphi_\gamma(f) = (\chi_{\Omega_{K \times M}} \cdot G + \chi_{\Omega_{M \times M^c}} \cdot H) \otimes \chi_{\Omega_{J^c}}$. This means that φ_γ respects the coordinates of f.

(B) γ is of countable cofinality. For simplicity put $\tau_n := \tau_{\gamma_n^\gamma}$, $\varphi_n := \varphi_{\gamma_n^\gamma}$ and $\eta_n := \eta_{\gamma_n^\gamma}$ for all $n \in \mathbf{N}$. Then

$$\Sigma \otimes \eta_\gamma = \sigma\left(\bigcup_{n \in \mathbf{N}} \Sigma \otimes \eta_n\right).$$

Taking the ultrafilter $\mathscr{U}_\gamma$ associated with γ and setting

$$\varphi_\gamma(f) := \lim_{n \in \mathscr{U}_\gamma} \varphi_n[E_{\Sigma \otimes \eta_n}(f)] \qquad \text{for} \quad f \in \mathscr{L}^\infty(\mu \otimes \nu \mathbin{\hat{\restriction}} \Sigma \otimes \eta_\gamma),$$

we get $\varphi_\gamma \in \mathscr{G}(\mu \otimes \nu \mathbin{\hat{\restriction}} \Sigma \otimes \eta_\gamma)$ and $\varphi_\gamma \mid \mathscr{L}^\infty(\mu \otimes \nu \mathbin{\hat{\restriction}} \Sigma \otimes \eta_\alpha) = \varphi_\alpha$ for each $\alpha < \gamma$ (see the arguments of the proof of Theorem 2 in [5, Chap. IV, Sect. 1]).

Let $g \in \mathscr{L}^\infty(\mu)$ and $h \in \mathscr{L}^\infty(\nu \mid \eta_\gamma)$. Applying [6, Sect. 2, Lemma 1], and the inductive assumptions, we get

$$\begin{aligned}\varphi_\gamma(g\otimes h) &= \lim_{n\in\mathcal{U}_\gamma} \varphi_n[E_{\Sigma\otimes\eta_n}(g\otimes h)]\\ &= \lim_{n\in\mathcal{U}_\gamma} \varphi_n[g\otimes E_{\eta_n}(h)]\\ &= \lim_{n\in\mathcal{U}_\gamma} \rho(g)\otimes\tau_n[E_{\eta_n}(h)]\\ &= \rho(g)\otimes \lim_{n\in\mathcal{U}_\gamma} \tau_n[E_{\eta_n}(h)]\\ &= \rho(g)\otimes\tau_\gamma(h),\end{aligned}$$

i.e., (1) holds true also for φ_γ.

To prove that φ_γ respects coordinates, we consider the same two cases as in (A).

(1^0) $\gamma\notin J$. Put $K:=I\backslash J$, so that $J^c=K\cup\{\gamma\}$. Let $f=g\otimes\chi_{\Omega_{J^c}}$ with $g\in\mathcal{L}^\infty(\hat{\mu}_J)$. Since ρ respects coordinates and φ_γ is a product of ρ and τ_γ, we have

$$\begin{aligned}\varphi_\gamma(f)=\varphi_\gamma(g\otimes\chi_{\Omega_{J^c}})&=\varphi_\gamma(g\otimes\chi_{\Omega_K}\otimes\chi_\Theta)\\ &=\rho(g\otimes\chi_{\Omega_K})\otimes\chi_\Theta=g^*\otimes\chi_\Theta,\end{aligned}$$

for some $g^*\in\mathbf{R}^{\Omega_J}$.

(2^0) $\gamma\in J$. Then $J^c=I\backslash J$, $J=K\cup\{\gamma\}$, and $K\cup J^c=I$ disjointly for a subset K of I as well as $f=g\otimes\chi_{\Omega_{J^c}}$ with some $g\in\mathcal{L}^\infty(\mu_K\otimes\nu\mid\Sigma_K\otimes\eta_\gamma)$. By [6, Sect. 2, Lemma 1] we get

$$\begin{aligned}E_{\Sigma\otimes\eta_n}(f)=E_{\Sigma_K\otimes\eta_n\otimes\Sigma_{J^c}}(g\otimes\chi_{\Omega_{J^c}})&=E_{\Sigma_K\otimes\eta_n}(g)\otimes E_{\Sigma_{J^c}}(\chi_{\Omega_{J^c}})\\ &=E_{\Sigma_K\otimes\eta_n}(g)\otimes\chi_{\Omega_{J^c}}.\end{aligned}$$

Again by the inductive assumption

$$\varphi_n[E_{\Sigma\otimes\eta_n}(f)]=g_n\otimes\chi_{\Omega_{J^c}}$$

with some $g_n\in\mathbf{R}^{\Omega_J}$ for $n\in\mathbf{N}$. Consequently φ_γ respects coordinates.

(C) γ is a limit ordinal of uncountable cofinality. Then

$$\Sigma\otimes\eta_\gamma=\bigcup_{\alpha<\gamma}(\Sigma\otimes\eta_\alpha). \tag{4}$$

Setting

$$\varphi_\gamma(f):=\varphi_\alpha(f)\qquad\text{if}\quad f\in\mathcal{L}^\infty(\mu\otimes\nu\mid\Sigma\otimes\eta_\alpha)$$

and denoting the obvious extension of φ_γ from $\mathscr{L}^\infty(\mu\otimes\nu\mid\Sigma\otimes\eta_\gamma)$ to $\mathscr{L}^\infty(\mu\otimes\nu\,\widehat{\restriction}\,\Sigma\otimes\eta_\gamma)$ again by φ_γ we get unambiguously defined linear liftings $\varphi_\gamma\in\mathscr{G}(\mu\otimes\nu\,\widehat{\restriction}\,\Sigma\otimes\eta_\gamma)$ such that

$$\varphi_\gamma\mid\mathscr{L}^\infty(\mu\otimes\nu\,\widehat{\restriction}\,\Sigma\otimes\eta_\alpha)=\varphi_\alpha\qquad\text{for all}\quad\alpha<\gamma.$$

For all $g\in\mathscr{L}^\infty(\mu), h\in\mathscr{L}^\infty(\nu\mid\eta_\gamma)$ there exists an $\alpha<\gamma$ such that $h\in\mathscr{L}^\infty(\nu\mid\eta_\alpha)$, hence $g\otimes h\in\mathscr{L}^\infty(\mu\otimes\nu\mid\Sigma\otimes\eta_\alpha)$. This implies

$$\varphi_\gamma(g\otimes h)=\varphi_\alpha(g\otimes h)=\rho(g)\otimes\tau_\alpha(h)=\rho(g)\otimes\tau_\gamma(h),$$

i.e., (1) holds also true for φ_γ. It is easily seen that the conditions (2) and (3) are also satisfied.

We can define now a map φ on $\mathscr{L}^\infty(\mu\otimes\nu\,\widehat{\restriction}\,\Sigma\otimes T)$ satisfying conditions (1)–(3) just by setting $\varphi=\varphi_\kappa$. ∎

LEMMA 2.4. *Let* (Ω_i,Ξ_i,μ_i), $i=1,2,3$, *be probability spaces and let* $f:\Omega_1\times\Omega_2\times\Omega_3\to[0,1]$ *be a bounded* $\Xi_1\otimes\Omega_2\otimes\Xi_3$*-measurable function. Then there exists a* $\Xi_1\otimes\Omega_2\otimes\Omega_3$ *-measurable version of* $E_{\Xi_1\otimes\Xi_2\otimes\Omega_3}(f)$.

Proof. We may assume that there is a bounded $\Xi_1\otimes\Xi_3$-measurable function g satisfying everywhere the equality $f(\omega_1,\omega_2,\omega_3)=g(\omega_1,\omega_3)$. Let a function h be given by the equality: $h(\omega_1):=\int_{\Omega_3}g(\omega_1,\omega_3)\,d\mu_3(\omega_3)$. Since g can be uniformly approximated by measurable simple functions, we get the Ξ_1-measurability of h. Then, applying the Fubini theorem (cf. [4, Theorem 21.12]), we have for each $D\in\Xi_1\otimes\Xi_2$

$$\begin{aligned}\int_{D\times\Omega_3}fd(\mu_1\otimes\mu_2\otimes\mu_3)&=\int_D\left(\int_{\Omega_3}g(\omega_1,\omega_3)\,d\mu_3\right)d(\mu_1\otimes\mu_2)\\&=\int_D h(\omega_1)\,d(\mu_1\otimes\mu_2)(\omega_1,\omega_2)\\&=\int_{D\times\Omega_3}hd(\mu_1\otimes\mu_2\otimes\mu_3)\,.\end{aligned}$$

This means that $h\otimes\chi_{\Omega_2}\otimes\chi_{\Omega_3}$ is a $\Xi_1\otimes\Omega_2\otimes\Omega_3$-measurable version of $E_{\Xi_1\otimes\Xi_2\otimes\Omega_3}(f)$. ∎

To some extent we can prescribe the marginals of linear lifting respecting coordinates.

THEOREM 2.5. *Let* $(\Omega_i,\Sigma_i,\mu_i)_{i\in I}$ *be a family of complete probability spaces. If* $i_0\in I$ *is fixed, then for each* $\tau_{i_0}\in\mathscr{G}(\mu_{i_0})$ *and for arbitrary* $\tau_i\in A\mathscr{G}(\mu_i)$ *with* $i\in I\setminus\{i_0\}$ *there exists a* $\varphi\in\mathscr{G}(\hat\mu_I)$ *such that* φ *respects coordinates and* $\varphi\in\bigotimes_{i\in I}\tau_i$.

Proof. Let κ be the first ordinal of the cardinality equal to card(I). Without loss of generality, we may assume that $I=\kappa$ and $i_0=0$. Put $(X_\gamma, T_\gamma, \nu_\gamma) := \bigotimes_{\alpha<\gamma}(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$ for $1 \leqslant \gamma \leqslant \kappa$.

We shall be constructing inductively linear liftings $\varphi_\gamma \in \mathscr{G}(\hat{\nu}_\gamma)$ respecting coordinates and such that

$$\varphi_\gamma \in \bigotimes_{\delta<\gamma} \tau_\delta \qquad \text{for} \quad 1 \leqslant \gamma \leqslant \kappa \tag{5}$$

and

$$\varphi_\gamma(g \circ f_{\alpha\gamma}) = \varphi_\alpha(g) \circ f_{\alpha\gamma}, \tag{6}$$

for all $g \in \mathscr{L}^\infty(\hat{\nu}_\alpha)$, if $f_{\alpha\gamma}$ are the canonical projections from X_γ onto X_α for $1 \leqslant \alpha \leqslant \gamma$.

To start the induction define $\varphi_1 := \tau_0$.

Suppose that for some $\gamma \leqslant \kappa$ and all $1 \leqslant \alpha < \gamma$ the linear liftings $\varphi_\alpha \in \mathscr{G}(\hat{\nu}_\alpha)$ respecting coordinates and satisfying (5) and (6) with γ replaced by arbitrary β, where $\alpha < \beta < \gamma$ are already known.

We have to distinguish three cases.

(A) $\gamma = \beta + 1$. By Theorem 2.3 there exists a linear lifting $\varphi_\gamma \in \mathscr{G}(\hat{\nu}_\gamma)$ respecting coordinates and satisfying the relations $\varphi_\gamma \in \bigotimes_{\alpha<\gamma} \tau_\alpha$ and $\varphi_\gamma \in \varphi_\beta \otimes \tau_\gamma$.

(B) γ is of countable cofinality. For each α with $1 \leqslant \alpha < \gamma$ consider the σ-algebras $T^*_\alpha := f^{-1}_{\alpha\gamma}(T_\alpha)$ and $\hat{T}^*_\alpha := f^{-1}_{\alpha\gamma}(\hat{T}_\alpha)$. Moreover, let $\nu^*_\alpha := \nu_\gamma \mid T^*_\alpha$ and let $\hat{\nu}^*_\alpha := \hat{\nu}_\gamma \mid \hat{T}^*_\alpha$. Clearly for $1 \leqslant \alpha \leqslant \beta < \gamma$, it holds true that

$$T^*_\alpha \subseteq T^*_\beta \qquad \text{and} \qquad \nu^*_\beta \,|\, T^*_\alpha = \nu^*_\alpha.$$

For each $\alpha < \gamma$ define a linear lifting $\varphi^*_\alpha \in \mathscr{G}(\hat{\nu}^*_\alpha)$ by means of

$$\varphi^*_\alpha(g^*) := \varphi_\alpha(g) \circ f_{\alpha\gamma},$$

where $g^* \in \mathscr{L}^\infty(\hat{\nu}^*_\alpha)$ and $g \in \mathscr{L}^\infty(\hat{\nu}_\alpha)$ with $g^* = g \circ f_{\alpha\gamma}$ a.e. $(\hat{\nu}^*_\alpha)$. It is easily seen that $\varphi^*_\beta \mid \mathscr{L}^\infty(\hat{\nu}^*_\alpha) = \varphi^*_\alpha$ for all α, β with $1 \leqslant \alpha \leqslant \beta < \gamma$.

Let $(\gamma_n)_{n\in\mathbf{N}}$ be an increasing sequence of ordinals cofinal with γ. For simplicity put $\tau_n := \tau_{\gamma_n}$, $\varphi_n := \varphi_{\gamma_n}$, $\nu_n := \nu_{\gamma_n}$, and $T_n := T_{\gamma_n}$ for all $n \in \mathbf{N}$. Then for each $\alpha < \gamma$ there exists $n \in \mathbf{N}$ such that $T^*_\alpha \subseteq T^*_n$. Clearly $\varphi^*_n \mid \mathscr{L}^\infty(\hat{\nu}^*_\alpha) = \varphi^*_\alpha$, $\varphi^*_{n+1} \mid \mathscr{L}^\infty(\hat{\nu}^*_n) = \varphi^*_n$ and $T_\gamma = \sigma(\bigcup_{n\in\mathbf{N}} T^*_n)$. Thus, if $\mathscr{V}$ is a free ultrafilter on $\mathbf{N}$, then we can set

$$\varphi_\gamma(f) := \lim_{n\in\mathscr{V}} \varphi^*_n\,[E_{T^*_n}(f)] \qquad \text{for each} \quad f \in \mathscr{L}^\infty(\nu_\gamma).$$

It follows immediately that $\varphi_\gamma \in \mathscr{G}(\hat{\nu}_\gamma)$, and $\varphi_\gamma \mid \mathscr{L}^\infty(\hat{\nu}_n^*) = \varphi_n^*$ for all $n \in \mathbf{N}$. Hence $\varphi_\gamma \mid \mathscr{L}^\infty(\hat{\nu}_\alpha^*) = \varphi_\alpha^*$ for all $1 \leqslant \alpha < \gamma$ and so the condition (6) is satisfied. Since $\varphi_\alpha \in \bigotimes_{\delta < \alpha} \tau_\delta$, if $1 \leqslant \alpha < \gamma$, the condition (5) is also fulfilled.

We have to show yet that φ_γ respects coordinates. To do it, take a nonempty set $J \subset \gamma$ with $J \neq \gamma$ and assume that $f = g \otimes \chi_{\Omega_{J^c}}$, where $g \in \mathscr{L}^\infty(\mu_J)$ and $J^c := \gamma \backslash J$. Notice that for each $n \in \mathbf{N}$ we have the equality

$$\gamma = (\gamma_n \cap J) \cup (\gamma_n \cap J^c) \cup \gamma_n^c,$$

where $\gamma_n^c := \gamma \backslash \gamma_n$. Let $\Xi_1 := \Sigma_{\gamma_n \cap J}$, $\Xi_2 := \Sigma_{\gamma_n \cap J^c}$, and $\Xi_3 := \Sigma_{\gamma_n^c}$. Then $T_n^* = \Xi_1 \otimes \Xi_2 \otimes \Omega_{\gamma_n^c}$.

Applying Lemma 2.4 to f, we see that the function $E_{T_n^*}(f)$ can be assumed to be $\Xi_1 \otimes \Omega_{\gamma_n^c \cup J^c}$-measurable. Since φ_n respects coordinates, we get the measurability of $\varphi_n^*[E_{T_n^*}(f)]$ with respect to $\hat{\Xi}_1 \otimes \Omega_{\gamma_n^c \cup J^c}$. In particular the function $\varphi_n^*[E_{T_n^*}(f)]$ is $\hat{\Sigma}_J \times \Omega_{J^c}$-measurable. Consequently, the function $\varphi_\gamma(f)$ is also $\hat{\Sigma}_J \times \Omega_{J^c}$-measurable. This proves that φ_γ respects coordinates.

(C) Assume that γ is of uncountable cofinality. In this case $T_\gamma = \bigcup_{1 \leqslant \alpha < \gamma} T_\alpha^*$, where T_α^* is defined as in (B), and $1 \leqslant \alpha \leqslant \beta < \gamma$ implies $T_\alpha^* \subseteq T_\beta^*$. The symbols $\hat{T}_\alpha^*$, ν_α, and $\hat{\nu}_\alpha^*$ have the same meaning as in (B).

Now define for each $1 \leqslant \alpha < \gamma$ a linear lifting $\varphi_\alpha^* \in \mathscr{G}(\hat{\nu}_\alpha^*)$, by

$$\varphi_\alpha^*(g^*) := \varphi_\alpha(g) \circ f_{\alpha\gamma}$$

for each $g^* \in \mathscr{L}^\infty(\hat{\nu}_\alpha^*)$ and $g \in \mathscr{L}^\infty(\hat{\nu}_\alpha)$ with $g^* = g \circ f_{\alpha\gamma}$ a.e. $(\hat{\nu}_\alpha^*)$. Since $\varphi_\beta^* \mid \mathscr{L}^\infty(\hat{\nu}_\alpha^*) = \varphi_\alpha^*$ for all α, β with $1 \leqslant \alpha \leqslant \beta < \gamma$, it follows that one can define $\varphi_\gamma \in \mathscr{G}(\hat{\nu}_\gamma)$ by setting $\varphi_\gamma(f) = \varphi_\alpha^*(g)$ for each $f \in \mathscr{L}^\infty(\nu_\gamma)$, where $g \in \mathscr{L}^\infty(\hat{\nu}_\alpha^*)$ is ν_γ-equivalent to f.

Clearly φ_γ respects coordinates and $\varphi_\gamma \mid \mathscr{L}^\infty(\hat{\nu}_\alpha^*) = \varphi_\alpha^*$ for arbitrary $1 \leqslant \alpha < \gamma$. The relation $\varphi_\gamma \in \bigotimes_{\beta < \gamma} \tau_\beta$ is a direct consequence of the inductive assumption about each φ_α, with $\alpha \in \gamma$.

We can define now $\varphi \in \mathscr{G}(\hat{\mu}_I)$ possessing all the required properties just by setting $\varphi := \varphi_\kappa$. ∎

As far as the proof of the last theorem goes, some comments, explaining why we have to restrict ourselves to linear liftings, seem to be relevant. The main difficulties to overcome in an inductive proof for a (linear) lifting are concentrated on the inductive steps (A) and (B) of the proof.

In case of an increasing sequence of σ-algebras (step (B)) the well-known ultrafilter device applies to linear liftings respecting coordinates as well but there seems to be no chance to convert the resulting linear liftings into liftings respecting coordinates by any existing method (very likely this cannot be done). The existing formula of [5] for the successor ordinal (step (A)) is limited to liftings and cannot work for linear liftings in general, because the linear lifting preserves lattice operations if and only if it is a lifting.

But in the above proof we were able to overcome the latter difficulty by applying the formula (∗) from the proof of Theorem 2.3 which allows respectability of coordinates to pass through in this step and work for linear liftings in general. This formula can be found in Graf and von Weizsäcker (see [3, Lemma 2]).

Until now there were no applications of that formula since for complete probability spaces the existence of a lifting can be proved. For this reason the above proof is in some sense new even when considered only as an existence proof of a linear lifting.

The following results are immediate consequences of Proposition 2.2 and Theorem 2.5.

COROLLARY 2.6. *Let $(\Omega_i, \Sigma_i, \mu_i)_{i\in I}$ be a family of complete probability spaces. If $i_0 \in I$ is fixed, then for each $\tau_{i_0} \in \mathscr{G}(\mu_{i_0})$ there exist $\tau_i \in \mathscr{G}(\mu_i)$ with $i \in I\setminus\{i_0\}$ and a $\varphi \in \mathscr{G}(\hat{\mu}_I)$ such that φ respects coordinates and $\varphi \in \bigotimes_{i\in I} \tau_i$.*

COROLLARY 2.7. *Let (Ω, Σ, μ) be a complete probability space. Then for each $\tau \in A\mathscr{G}(\mu)$ and for an arbitrary nonempty index set I there exists a linear lifting $\tau^I \in \mathscr{G}(\hat{\mu}^I)$ respecting coordinates and satisfying for each finite non-empty set $J \subseteq I$ the condition*

$$\tau^I\left(\bigotimes_{i\in J} f_i \circ p_J\right) = \bigotimes_{i\in J} \tau(f_i) \circ p_J,$$

where all $f_i \in \mathscr{L}^\infty(\mu)$ and p_J is the canonical projection of Ω^I onto Ω^J. In particular, each admissible linear lifting is consistent.

Question 2.8. Is each consistent linear lifting admissible ?

For any complete probability space (Ω, Σ, μ) and $\rho \in \mathscr{G}(\mu)$ one can define (according to [5, p. 36]), a lower density $\bar{\rho} \in \vartheta(\mu)$, by setting

$$\bar{\rho}(A) := \{\omega \in \Omega : \rho(\chi_A)(\omega) = 1\} \qquad \text{for} \quad A \in \Sigma.$$

COROLLARY 2.9. *Let $(\Omega_i, \Sigma_i, \mu_i)$, $i \in I$, be complete probability spaces and $\tau_i \in \mathscr{G}(\mu_i)$ for all $i \in I$. If $\varphi \in \mathscr{G}(\hat{\mu}_I)$ respects coordinates and $\varphi \in \bigotimes_{i\in I} \tau_i$, then*

(i) *$\bar{\varphi}$ respects coordinates;*

(ii) *$\bar{\varphi} \in \bigotimes_{i\in I} \bar{\tau}_i$.*

Proof. Ad (i). Since φ respects coordinates, for each $J \in I^*$ there exists a $\varphi_J \in \mathcal{G}(\hat{\mu}_J)$ such that for each $A \in \hat{\Sigma}_J$, we have

$$\varphi(\chi_A \otimes \chi_{\Omega_J^c}) = \varphi_J(\chi_A) \otimes \chi_{\Omega_J^c}.$$

Consequently, if $\omega = (\omega_J, \omega_{J^c}) \in \Omega_I = \Omega_J \times \Omega_{J^c}$, then $(\omega_j, \omega_{J^c}) \in \bar{\varphi}(A \times \Omega_{J^c})$ if and only if

$$\begin{aligned} 1 &= \varphi(\chi_A \otimes \chi_{\Omega_J^c})(\omega_J, \omega_{J^c}) \\ &= [\varphi_J(\chi_A) \otimes \chi_{\Omega_J^c}](\omega_J, \omega_{J^c}) \\ &= \varphi_J(\chi_A)(\omega_J)\, \chi_{\Omega_J^c}(\omega_{J^c}) \end{aligned}$$

which happens if and only if $\varphi_J(\chi_A)(\omega_J) = 1$. Consequently, $\omega \in \bar{\varphi}(A \times \Omega_{J^c})$ if and only if $\omega \in \bar{\varphi}(A) \times \chi_{\Omega_J^c}$, i.e., the condition (i) holds true.

Condition (ii) follows in a similar way. ∎

The following result is a particular case of Fremlin's Theorem 346G [2], proved in a different way.

COROLLARY 2.10. *For any family $(\Omega_i, \Sigma_i, \mu_i)_{i \in I}$ of complete probability spaces with product $(\Omega_I, \Sigma_I, \mu_I)$ there exists a density $\psi \in \vartheta(\hat{\mu}_I)$ respecting coordinates.*

Remark 2.11. Suppose we have $\varphi \in \mathcal{G}(\hat{\mu}_I)$ respecting coordinates and let $\bar{\varphi} \in \vartheta(\hat{\mu}_I)$ be the density obtained from φ. Let

$$\mathcal{G}(\varphi) := \{\xi \in \mathcal{G}(\hat{\mu}_I) : \forall E \in \Sigma\ \chi_{\bar{\varphi}(E)} \leqslant \xi(\chi_E) \leqslant \chi_{[\bar{\varphi}(E^c)]^c}\ \&\ \xi \text{ respects coordinates}\}.$$

Following [5] we can prove that $\mathcal{G}(\varphi)$ is convex and compact in $\mathbf{R}^{L^\infty(\mu_I) \times \Omega_I}$. Thus it has an extreme point. It is not however obvious that there is an extreme point of $\mathcal{G}(\varphi)$ which is extreme in the set

$$\{\xi \in \mathcal{G}(\hat{\mu}_I) : \forall E \in \Sigma\ \chi_{\bar{\varphi}(E)} \leqslant \xi(\chi_E) \leqslant \chi_{[\bar{\varphi}(E^c)]^c}\}.$$

(see [5]). Each such extreme point would be a lifting respecting coordinates.

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