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Existence of linear liftings with invariant sections in product measure spaces

By

K. Musiał, W. Strauss and N. D. Macheras

Dedicated to Prof. Dr. D. Kölzow on the occasion of his 75th birthday

Abstract. We prove that on a complete product of two arbitrary probability spaces, one of which is endowed with a particular linear lifting (we call it admissible), there exists a linear lifting possessing the property that all its sections determined by the marginal space without lifting are invariant under corresponding, given a priori, marginal linear lifting. If both marginal spaces are endowed with linear liftings (at least one of them to be admissible) then the lifting in the product space may have additional product properties. We prove also that for non-atomic marginal spaces there exist no marginal linear liftings and no linear lifting in the product possessing all its sections invariant under both marginal linear liftings.

Introduction. It follows from a result of Talagrand in [7] that given a complete finite measure space (Ω, Σ, μ) there exists a lifting (called consistent) ρ on (Ω, Σ, μ) such that there exists a lifting π on the complete direct product $(\Omega, \Sigma, \mu) \widehat{\otimes} (\Omega, \Sigma, \mu)$ satisfying the equality

(C) $$\pi(f \otimes g) = \rho(f) \otimes \rho(g)$$

Talagrand proved also in [8] (assuming CH), that there exist non-consistent liftings for arbitrary $f, g \in \mathcal{L}^\infty(\mu)$. Then Macheras and Strauss proved in [2] that given complete probability spaces (Ω, Σ, μ), (Θ, T, ν) and a fixed lifting ρ on (Ω, Σ, μ), one can find liftings σ on (Θ, T, ν) and π on $(\Omega, \Sigma, \mu) \widehat{\otimes} (\Theta, T, \nu)$ satisfying the equality

(P) $$\pi(f \otimes g) = \rho(f) \otimes \sigma(g) \quad \text{for all} \quad f \in \mathcal{L}^\infty(\mu), g \in \mathcal{L}^\infty(\nu).$$

In [4] we have proven that for an arbitrary pair of complete probability spaces (Ω, Σ, μ), (Θ, T, ν) and a fixed lifting ρ on (Ω, Σ, μ) there exists a lifting σ on (Θ, T, ν) and a lifting π in the product space satisfying (P) and

(S) $$[\pi(f)]_\omega = \sigma([\pi(f)]_\omega) \quad \text{for all} \quad f \in \mathcal{L}^\infty(\mu \widehat{\otimes} \nu),\ \omega \in \Omega.$$

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Moreover, we have proven also that in case of non-purely atomic measure spaces there exist no liftings ρ, σ and π satisfying (S) and

$$[\pi(f)]^\theta = \rho([\pi(f)]^\theta) \quad \text{for all} \quad f \in \mathcal{L}^\infty(\mu \widehat{\otimes} \nu), \theta \in \Theta.$$

Here we investigate the same problem for the collection of all linear liftings, which form much larger class of objects than liftings. As the existence of linear liftings ρ, σ and π satisfying (P) and (S) follows from [4], we concentrate ourselves on the question of the existence of a linear lifting π on $\mathcal{L}^\infty(\mu \widehat{\otimes} \nu)$ satisfying (P) and (S) when marginal linear liftings ρ and σ are given a priori. We single out the class of "admissible linear liftings" on (Θ, T, ν) which has the property that if σ is an admissible linear lifting on (Θ, T, ν) then for each linear lifting ρ on (Ω, Σ, μ) there exists π satisfying (P) and (S) for almost all $\omega \in \Omega$ (see Proposition 2.4). If ν is separable, then (S) may be satisfied everywhere (Theorem 2.2). The proofs are independent of [4].

In case of non-atomic probability spaces however it turns out that the results of [4] cannot be improved. There exist no linear liftings ρ for μ, σ for ν, and π for $\mu \widehat{\otimes} \nu$ such that all sections of $\pi(f)$ are invariant with respect to ρ and σ, respectively, for every bounded measurable function f (see Theorem 3.2). The proof of this fact heavily depends on the results of [4] concerning densities.

1. Preliminaries. For a given probability space (Ω, Σ, μ) the family of all μ-null sets is denoted by Σ_0. The (Carathéodory) completion of (Ω, Σ, μ) will be denoted by $(\Omega, \widehat{\Sigma}, \widehat{\mu})$. $\mathcal{L}^\infty(\mu)$ denotes the family of all bounded real-valued μ-measurable functions on (Ω, Σ, μ). The equivalence class of all μ-measurable functions in $\mathcal{L}^\infty(\mu)$, that are μ-a.e. equal to f, will be denoted by $f^\bullet$ and the space of equivalence classes-by $L^\infty(\mu)$. Equivalent functions are not identified. The space of equivalence classes of functions that are μ-integrable is denoted by $L_1(\mu)$. The σ-algebra generated by a family $\mathcal{L}$ of sets is denoted by $\sigma(\mathcal{L})$. $\mathbb{N}$ and $\mathbb{R}$ stand for the natural numbers and the real numbers respectively. If $M \subseteq \Omega$, then $M^c := \Omega \backslash M$. We use the notion of *(lower) density, linear lifting, lifting* in the sense of [1] (see also [5]), and for any probability space (Ω, Σ, μ) we denote by $\vartheta(\mu)$, $\mathcal{G}(\mu)$, and by $\Lambda(\mu)$ the system of all (lower) densities, linear liftings, and liftings, respectively.

We denote by $(\Omega \times \Theta, \Sigma \otimes T, \mu \otimes \nu)$ the product probability space of the probability spaces (Ω, Σ, μ) and (Θ, T, ν). By $(\Omega \times \Theta, \Sigma \widehat{\otimes} T, \mu \widehat{\otimes} \nu)$ will be denoted its (Carathéodory) completion. If $\eta \subseteq T$ is a sub-σ-algebra then $\Sigma \otimes_0 \eta := \sigma((\Sigma \otimes T)_0 \cup (\Sigma \otimes \eta))$, $\Sigma \widehat{\otimes}_0 \eta$ is the completion of $\Sigma \otimes_0 \eta$ with respect to $(\mu \otimes \nu)|(\Sigma \otimes_0 \eta)$ and $\widehat{(\mu \otimes \nu)|}(\Sigma \otimes_0 \eta)$ is the completion of $(\mu \otimes \nu)|(\Sigma \otimes_0 \eta)$.

$E_\eta(g)$ denotes a version of the conditional expectation of a function $g \in \mathcal{L}^\infty(\nu)$ with respect to a sub-σ-algebra $\eta \subseteq T$.

If f is a function defined on $\Omega \times \Theta$ and $(\omega, \theta) \in \Omega \times \Theta$ is fixed, then we use the ordinary notation f_ω, f^θ for the functions obtained from f by fixing ω and θ respectively. In a similar way the sections E_ω, E^θ of a set $E \subseteq \Omega \times \Theta$ are defined.

All the densities used in this paper are lower densities so that for simplicity we will use the word "density" instead of "lower density".

We call a $\pi \in \mathcal{G}(\mu \widehat{\otimes} \nu)$ a *product* (*linear lifting*) of the linear liftings $\rho \in \mathcal{G}(\mu)$ and $\sigma \in \mathcal{G}(\nu)$ (and we write $\pi \in \rho \otimes \sigma$), if

$$\pi(f \otimes g) = \rho(f) \otimes \sigma(g) \quad \text{for all} \quad f \in \mathcal{L}^{\infty}(\mu) \quad \text{and} \quad g \in \mathcal{L}^{\infty}(\nu).$$

Corresponding definitions for liftings and densities can be found in [4].

2. Existence of linear liftings with one-sided lifting invariant sections. It has been observed already by Talagrand [8] that not all liftings have good product properties. The same holds true in the case of our investigations concerning linear liftings. Therefore we are going to separate a wide class of linear liftings posessing properties suitable for our purposes.

Definition 2.1 ([3]). Let (Θ, T, ν) be a probability space. A linear lifting $\sigma \in \mathcal{G}(\nu)$ is called an *admissible linear lifting* if it can be constructed with the help of the transfinite induction in the way described below.

(A) Let **d** be the smallest cardinal with the property, that there exists a collection $\mathcal{M} \subset T$ of cardinality **d** such that $\sigma(\mathcal{M})$ is dense in T in the pseudometric generated by ν. Let $\mathcal{M} = \langle M_\alpha \rangle_{\alpha<\kappa}$ be numbered by ordinals less than κ, where κ is the first ordinal of the cardinality **d**. Denote by η_0 the σ-algebra $\sigma(T_0)$ and for each $1 \leqq \alpha \leqq \kappa$ denote by η_α the σ-algebra generated by the family $\{M_\gamma : \gamma < \alpha\} \cup \eta_0$. We may assume that $M_\alpha \notin \eta_\alpha$ for each α. Notice, that all the measures $\nu|\eta_\alpha$ are complete. For each limit $\gamma \leqq \kappa$ of countable cofinality we fix an increasing sequence of ordinals $\gamma_n < \gamma$ which is cofinal with γ and a free ultrafilter $\mathcal{U}_\gamma$ on $\mathbb{N}$.

(B) For the algebra η_0 and $g \in \mathcal{L}^{\infty}(\nu|\eta_0)$ we define $\sigma_0 \in \mathcal{G}(\nu|\eta_0)$ by setting $\sigma_0(g) = a$, if $g = a$ a.e. (ν).

(C) If γ is a limit ordinal of uncountable cofinality, then $\eta_\gamma = \bigcup_{\alpha<\gamma} \eta_\alpha$ and we define $\sigma_\gamma \in \mathcal{G}(\nu|\eta_\gamma)$ by setting

$$\sigma_\gamma(g) := \sigma_\alpha(g) \quad \text{if} \quad g \in \mathcal{L}^{\infty}(\nu|\eta_\alpha).$$

(D) If γ is of countable cofinality, then we put for simplicity $\sigma_n := \sigma_{\gamma_n}$ and $\eta_n := \eta_{\gamma_n}$ for all $n \in \mathbb{N}$. Then $\eta_\gamma = \sigma(\bigcup_{n\in\mathbb{N}} \eta_n)$ and we define σ_γ by setting

$$\sigma_\gamma(h) := \lim_{n\in\mathcal{U}_\gamma} \sigma_n(E_{\eta_n}(h)) \quad \text{for} \quad h \in \mathcal{L}^{\infty}(\nu|\eta_\gamma).$$

Using the arguments of the proof of Theorem 2 in [1, Chapter IV, Section 1] and assuming the completeness of (Θ, T, ν), we get $\sigma_\gamma \in \mathcal{G}(\nu|\eta_\gamma)$ and $\sigma_\gamma|\mathcal{L}^{\infty}(\nu|\eta_\alpha) = \sigma_\alpha$ for each $\alpha < \gamma$. If (Θ, T, ν) is not complete, then we assume the T-measurability of every $\sigma_\gamma(h)$.

(E) Let now $\gamma = \beta + 1$. To simplify the notations let $M := M_\beta$. It then follows that

$$\mathcal{L}^{\infty}(\nu|\eta_\gamma) = \{g\chi_M + h\chi_{M^c} : g, h \in \mathcal{L}^{\infty}(\nu|\eta_\beta)\}.$$

Put

$$W_1 := \text{ess inf}\{B \in \eta_\beta : M \subseteqq B \text{ a.e. } (\nu|\eta_\beta)\},$$
$$W_2 := \text{ess inf}\{B \in \eta_\beta : M^c \subseteqq B \text{ a.e. } (\nu|\eta_\beta)\},$$

and

$$\sigma_\gamma(g\chi_M + h\chi_{M^c}) := \chi_M\sigma_\beta(g\chi_{W_1} + h\chi_{W_1^c}) + \chi_{M^c}\sigma_\beta(h\chi_{W_2} + g\chi_{W_2^c})$$

if $g, h \in \mathcal{L}^\infty(\nu|\eta_\beta)$. It follows that $\sigma_\gamma \in \mathcal{G}(\nu|\eta_\gamma)$ and $\sigma_\gamma|\mathcal{L}^\infty(\nu|\eta_\beta) = \sigma_\beta$.

(F) We define $\sigma \in \mathcal{G}(\nu)$ just by setting $\sigma = \sigma_\kappa$.

Throughout the collection of all admissible linear liftings on (Θ, T, ν) will be denoted by $A\mathcal{G}(\nu)$ and each $\sigma \in A\mathcal{G}(\nu)$ will be considered together with a fixed collection of the elements involved into the above construction, without any additional remarks. In case of $\mathbf{d} = \aleph_0$ we need only one ultrafilter, which we denote by $\mathcal{U}$.

By converting the above definition into an inductive proof, we see that $A\mathcal{G}(\nu) \neq \emptyset$ for every complete probability space (Θ, T, ν). It should be mentioned also that the above construction, which leads to admissible linear liftings, is the standard, well known, construction of linear liftings.

Theorem 2.2. *Let (Θ, T, ν) be a complete separable probability space and (Ω, Σ, μ) a complete probability space. Then for each $\rho \in \mathcal{G}(\mu)$ and each $\sigma \in A\mathcal{G}(\nu)$ there exists a $\varphi \in \mathcal{G}(\mu \widehat{\otimes} \nu)$ such that $\varphi \in \rho \otimes \sigma$ and*

$$[\varphi(f)]_\omega = \sigma([\varphi(f)]_\omega)$$

for every $f \in \mathcal{L}^\infty(\mu\widehat{\otimes}\nu)$ and every $\omega \in \Omega$.

Proof. Let there be given a $\rho \in \mathcal{G}(\mu)$ and $\sigma \in A\mathcal{G}(\nu)$ and choose the sequence $\mathcal{M} = \langle M_n\rangle_{n=0}^\infty \subset T$ as well as the σ-subalgebras η_n in T, the free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and the sequence $\langle\sigma_n\rangle$ of linear liftings $\sigma_n \in \mathcal{G}(\nu|\eta_n)$ according to the Definition 2.1. Note that $\Sigma \mathbin{\widehat{\otimes}_0} \eta_{n+1} = \sigma(\Sigma \mathbin{\widehat{\otimes}_0} \eta_n \cup \{\Omega \times M_n\})$.

We shall be constructing now a sequence $\langle\varphi_n\rangle_{n=0}^\infty$ of linear liftings $\varphi_n \in \mathcal{G}(\widehat{(\mu \otimes \nu)\lceil}(\Sigma \otimes_0 \eta_n))$, satisfying the following set of conditions:

(1) $$\varphi_n|\mathcal{L}^\infty(\widehat{(\mu \otimes \nu)\lceil}(\Sigma \otimes_0 \eta_{n-1})) = \varphi_{n-1} \text{ if } n \geqq 1;$$

(2) $$\varphi_n(g \otimes h) = \rho(g) \otimes \sigma_n(h) \text{ for all } g \in \mathcal{L}^\infty(\mu),\ h \in \mathcal{L}^\infty(\nu|\eta_n) \text{ and } n \geqq 0;$$

(3) $$[\varphi_n(f)]_\omega \in \mathcal{L}^\infty(\nu|\eta_n) \text{ for every } f \in \mathcal{L}^\infty(\widehat{(\mu \otimes \nu)\lceil}(\Sigma \otimes_0 \eta_n)),$$
$$\omega \in \Omega \text{ and } n \geqq 0.$$

For every $n \geqq 0$ and every $f \in \mathcal{L}^\infty(\widehat{(\mu \otimes \nu)\lceil}(\Sigma \otimes_0 \eta_n))$, there exists $N_{n,f} \in \Sigma_0$ such that for each $\omega \notin N_{n,f}$

(4) $$[\varphi_n(f)]_\omega = \sigma_n([\varphi_n(f)]_\omega);$$

(5) $$E_{\eta_{n-1}}([\varphi_n(f)]_\omega) = [\varphi_{n-1}(E_{\Sigma\widehat{\otimes}_0\eta_{n-1}}(f))]_\omega \quad \text{a.e. } (\nu)$$

for every $f \in \mathcal{L}^\infty(\widehat{(\mu \otimes \nu)\lceil}(\Sigma \otimes_0 \eta_n))$, $\omega \in \Omega$ and $n \geqq 1$;

(6) $$[\varphi_n(f)]^\theta = \rho([\varphi_n(f)]^\theta),$$

for every $n \geqq 0$, $f \in \mathcal{L}^\infty(\widehat{(\mu \otimes \nu)\lceil}(\Sigma \otimes_0 \eta_n))$ and $\theta \in \Theta$.

The proof is inductive. We start the induction by setting $\varphi_0(f) := \rho(g) \otimes \chi_\Theta$ if $f \in \mathcal{L}^\infty(\mu \otimes \nu\widehat{|}\Sigma \otimes_0 \eta_0)$ and $f = g \otimes \chi_\Theta$ a.e. $(\mu\widehat{\otimes}\nu)$.

Assume now that the conditions (1)-(6) are satisfied for some $n \geqq 1$ (or (2)-(4) and (6) in case of $n = 0$) and take $f \in \mathcal{L}^\infty((\mu \otimes \nu)\widehat{|}(\Sigma \otimes_0 \eta_{n+1}))$ with $f = G\chi_{\Omega\times M_n} + H\chi_{\Omega\times M_n^c}$ and $G, H \in \mathcal{L}^\infty((\mu \otimes \nu)\widehat{|}(\Sigma \otimes_0 \eta_n))$. We set then

$$\varphi_{n+1}(f) = \chi_{\Omega\times M_n}\varphi_n(G\chi_{V_1} + H\chi_{V_1^c}) + \chi_{\Omega\times M_n^c}\varphi_n(H\chi_{V_2} + G\chi_{V_2^c}), \tag{7}$$

where

$$V_1 := \operatorname{essinf}\{\mathrm{E} \in \Sigma \,\widehat{\otimes}_0\, \eta_\beta : \Omega \times \mathrm{M_n} \subseteqq \mathrm{E} \text{ a.e. } (\mu \,\widehat{\otimes}\, \nu|\eta_\beta)\},$$

$$V_2 := \operatorname{essinf}\{\mathrm{E} \in \Sigma \,\widehat{\otimes}_0\, \eta_\beta : \Omega \times \mathrm{M_n^c} \subseteqq \mathrm{E} \text{ a.e. } (\mu \,\widehat{\otimes}\, \nu|\eta_\beta)\},$$

and depend in fact on n. If W_1, W_2 are defined according to Definition 2.1 (E), then we have by [2] (Claim 3 in the proof of Theorem 3) and by the $\mu \,\widehat{\otimes}\, \nu$-density of $\Sigma \otimes \eta_n$ in $\Sigma \,\widehat{\otimes}_0\, \eta_n$,

$$V_1 = \Omega \times W_1 \quad \text{and} \quad V_2 = \Omega \times W_2 \quad \text{a.e. } ((\mu \otimes \nu)\widehat{|}(\Sigma \otimes_0 \eta_\beta)). \tag{8}$$

For simplicity we assume that the above equalities hold true everywhere. Hence for each (ω, θ)

$$[\varphi_{n+1}(f)]_\omega = \chi_{M_n}[\varphi_n(G\chi_{V_1} + H\chi_{V_1^c})]_\omega + \chi_{M_n^c}[\varphi_n(H\chi_{V_2} + G\chi_{V_2^c})]_\omega$$

and

$$\begin{aligned}&[\varphi_{n+1}(f)]^\theta \\ &\quad = \chi_{M_n}(\theta)[\varphi_n(G\chi_{V_1} + H\chi_{V_1^c})]^\theta + \chi_{M_n^c}(\theta)[\varphi_n(H\chi_{V_2} + G\chi_{V_2^c})]^\theta.\end{aligned}$$

The above two formulae prove the required measurability of all sections $[\varphi_{n+1}(f)]_\omega$ and the condition (6). Let $f \in \mathcal{L}^\infty(\mu)$ and $u = g\chi_{M_n} + h\chi_{M_n{}^c} \in \mathcal{L}^\infty(\nu|\eta_{n+1})$ with $g, h \in \mathcal{L}^\infty(\nu|\eta_n)$. We get (2) for φ_{n+1}:

$$\begin{aligned}\varphi_{n+1}(f \otimes u) &= \chi_{\Omega\times M_n}\varphi_n[(f \otimes g)\chi_{V_1} + (f \otimes h)\chi_{V_1^c}] \\ &\qquad + \chi_{\Omega\otimes M_n{}^c}\varphi_n[(f \otimes h)\chi_{V_2} + (f \otimes g)\chi_{V_2^c}] \\ &= \rho(f) \otimes [\chi_{M_n}\sigma_n(g\chi_{W_1} + h\chi_{W_1^c}) + \chi_{M_n{}^c}\sigma_n(h\chi_{W_2} + g\chi_{W_2^c})] \\ &= \rho(f) \otimes \sigma_{n+1}(u).\end{aligned}$$

For any $f \in \mathcal{L}^\infty((\mu \otimes \nu)\widehat{|}(\Sigma \otimes_0 \eta_{n+1}))$ there exist $G, H \in \mathcal{L}^\infty((\mu \otimes \nu)\widehat{|}(\Sigma \otimes_0 \eta_n))$ such that $f = G\chi_{\Omega\times M_n} + H\chi_{\Omega\times M_n^c}$. By the inductive assumption and the Fubini theorem, there exists $N_f \in \Sigma_0$ such that

$$\begin{aligned}&[\varphi_n(G\chi_{V_1} + H\chi_{V_1^c})]_\omega = \sigma_n([\varphi_n(G\chi_{V_1} + H\chi_{V_1^c})]_\omega) \\ &\quad = \sigma_n([G\chi_{V_1} + H\chi_{V_1^c}]_\omega) \quad \text{a.e. } (\nu), \\ &[\varphi_n(H\chi_{V_2} + G\chi_{V_2^c})]_\omega = \sigma_n([\varphi_n(H\chi_{V_2} + G\chi_{V_2^c})]_\omega) \\ &\quad = \sigma_n([H\chi_{V_2} + G\chi_{V_2^c}]_\omega) \quad \text{a.e. } (\nu)\end{aligned}$$

and $[\varphi_n(f)]_\omega = f_\omega$ a.e. (ν) for all $\omega \notin N_f$. Applying condition (8) we have for each $\omega \in \Omega\backslash N_f$

$$\begin{aligned}
&[\varphi_{n+1}(f)]_\omega \\
&\quad = \chi_{M_n}[\varphi_n(G\chi_{V_1} + H\chi_{V_1^c})]_\omega + \chi_{M_n{}^c}[\varphi_n(H\chi_{V_2} + G\chi_{V_2^c})]_\omega \\
&\quad = \chi_{M_n}\sigma_n([\varphi_n(G\chi_{V_1} + H\chi_{V_1^c})]_\omega) + \chi_{M_n{}^c}\sigma_n([\varphi_n(H\chi_{V_2} + G\chi_{V_2^c})]_\omega) \\
&\quad = \chi_{M_n}\sigma_n(G_\omega\chi_{[V_1]_\omega} + H_\omega\chi_{[V_1^c]_\omega}) + \chi_{M_n{}^c}\sigma_n(H_\omega\chi_{[V_2]_\omega} + G_\omega\chi_{[V_2^c]_\omega}) \\
&\quad = \chi_{M_n}\sigma_n(G_\omega\chi_{W_1} + H_\omega\chi_{W_1^c}) + \chi_{M_n{}^c}\sigma_n(H_\omega\chi_{W_2} + G_\omega\chi_{W_2^c}) \\
&\quad = \sigma_{n+1}(f_\omega),
\end{aligned}$$

i.e. (4) holds true for $\varphi_{n+1}(f)$ with $\omega \notin N_f$.

It follows from (7) that we have $((\mu \otimes \nu)\widehat{|}(\Sigma \otimes_0 \eta_n))$-a.e.

$$\begin{aligned}
E_{\Sigma\widehat{\otimes}_0\eta_n}(f) &= E_{\Sigma\widehat{\otimes}_0\eta_n}(\varphi_{n+1}(f)) \\
&= E_{\eta_n}(\chi_{M_n})\varphi_n(G\chi_{V_1} + H\chi_{V_1^c}) + [1 - E_{\eta_n}(\chi_{M_n})]\varphi_n(H\chi_{V_2} + G\chi_{V_2^c})
\end{aligned}$$

and so setting for the simplicity $m_n = \sigma_n(E_{\eta_n}(\chi_{M_n}))$ we have also $((\mu \otimes \nu)\widehat{|}(\Sigma \otimes_0 \eta_n))$-a.e.

$$\varphi_n(E_{\Sigma\widehat{\otimes}_0\eta_n}(f)) = m_n\varphi_n(G\chi_{V_1} + H\chi_{V_1^c}) + (1 - m_n)\varphi_n(H\chi_{V_2} + G\chi_{V_2^c}).$$

According to the Fubini theorem, there exists a set $L_{n,f} \in T_0$ such that for each $\theta \in \Theta\backslash L_{n,f}$, we have μ-a.e.

$$\begin{aligned}
&[\varphi_n(E_{\Sigma\widehat{\otimes}_0\eta_n}(f))]^\theta = m_n(\theta)[\varphi_n(G\chi_{V_1} + H\chi_{V_1^c})]^\theta \\
&\quad + [1 - m_n(\theta)][\varphi_n(H\chi_{V_2} + G\chi_{V_2^c})]^\theta.
\end{aligned}$$

Since ρ is a linear lifting and the condition (6) is satisfied, the above equality holds true everywhere on Ω when $\theta \in \Theta\backslash L_{n,f}$. Hence we get for each $\omega \in \Omega$

$$\begin{aligned}
&[\varphi_n(E_{\Sigma\widehat{\otimes}_0\eta_n}(f))]_\omega = m_n[\varphi_n(G\chi_{V_1} + H\chi_{V_1^c})]_\omega \\
&\quad + (1 - m_n)[\varphi_n(H\chi_{V_2} + G\chi_{V_2^c})]_\omega
\end{aligned} \tag{9}$$

and the equality holds true $(\nu|\eta_n)$−a.e. Now, applying (9), we are able to prove for all ω the equality

$$E_{\eta_n}([\varphi_{n+1}(f)]_\omega) = [\varphi_n(E_{\Sigma\widehat{\otimes}_0\eta_n}(f))]_\omega \quad \text{a.e. } (\nu).$$

To do it, let us fix $\omega \in \Omega$ and $D \in \eta_n$. We have the following equalities:

$$\begin{aligned}
&\int_D [\varphi_{n+1}(f)]_\omega(\theta)d\nu(\theta) \\
&\quad = \int_D \chi_{M_n}[\varphi_n(G\chi_{V_1} + H\chi_{V_1^c})]_\omega d\nu(\theta)
\end{aligned}$$

$$
\begin{aligned}
&+\int_D (1-\chi_{M_n})[\varphi_n(H\chi_{V_2}+G\chi_{V_2^c})]_\omega d\nu(\theta)\\
&=\int_D E_{\eta_n}(\chi_{M_n}[\varphi_n(G\chi_{V_1}+H\chi_{V_1^c})]_\omega)d\nu(\theta)\\
&\quad+\int_D E_{\eta_n}((1-\chi_{M_n})[\varphi_n(H\chi_{V_2}+G\chi_{V_2^c})]_\omega)d\nu(\theta)\\
&=\int_D E_{\eta_n}(\chi_{M_n})[\varphi_n(G\chi_{V_1}+H\chi_{V_1^c})]_\omega d\nu(\theta)\\
&\quad+\int_D [1-E_{\eta_n}(\chi_{M_n})][\varphi_n(H\chi_{V_2}+G\chi_{V_2^c})]_\omega d\nu(\theta)\\
&=\int_D m_n[\varphi_n(G\chi_{V_1}+H\chi_{V_1^c})]_\omega d\nu(\theta)\\
&\quad+\int_D (1-m_n)[\varphi_n(H\chi_{V_2}+G\chi_{V_2^c})]_\omega d\nu(\theta)\\
&=\int_D [\varphi_n(E_{\Sigma\widehat{\otimes}\eta_n}(f))]_\omega d\nu,
\end{aligned}
$$

where the last equality follows from (9). Thus, the condition (5) is fulfilled for $n+1$.

Following the definition

$$\sigma(h)=\lim_{n\in\mathcal{U}}\sigma_n(E_{\eta_n}(h))\quad\text{for}\quad h\in\mathcal{L}^\infty(\nu),$$

we put

$$\widetilde{\varphi}(f):=\lim_{n\in\mathcal{U}}\varphi_n(E_{\Sigma\widehat{\otimes}_0\eta_n}(f))\quad\text{for}\quad f\in\mathcal{L}^\infty(\mu\,\widehat{\otimes}\,\nu).$$

According to [1], we have $\widetilde{\varphi}\in\mathcal{G}(\mu\,\widehat{\otimes}\,\nu)$ and $\widehat{\widetilde{\varphi}|\Sigma}\ \otimes_0\ \eta_n=\varphi_n$ for all n.

It is an easy consequence of the inductive assumption (5) that for each $\omega\in\Omega$ and for each $f\in\mathcal{L}^\infty(\mu\,\widehat{\otimes}\,\nu)$ the sequence

$$\langle[\varphi_n(E_{\Sigma\widehat{\otimes}_0\eta_n}(f))]_\omega\rangle_{n=1}^\infty$$

is a martingale. Due to the Martingale Convergence Theorem the above sequence is for each $\omega\in\Omega$ ν-a.e. convergent, and so

$$[\widetilde{\varphi}(f)]_\omega=\lim_{n\to\infty}[\varphi_n(E_{\Sigma\,\widehat{\otimes}_0\,\eta_n}(f))]_\omega\quad\text{a.e. }(\nu).$$

Consequently $[\widetilde{\varphi}(f)]_\omega$ is a measurable function and again from the Martingale Convergence Theorem we have for all ω

$$E_{\eta_n}([\widetilde{\varphi}(f)]_\omega)=[\varphi_n(E_{\Sigma\,\widehat{\otimes}_0\,\eta_n}(f))]_\omega\qquad\text{a.e. }(\nu).$$

These facts imply that for each $\omega \notin \bigcup_n N_{n,f}$

$$[\widetilde{\varphi}(f)]_\omega = \lim_{n\in\mathcal{U}} [\varphi_n(E_{\Sigma\,\widehat{\otimes}_0\,\eta_n}(f))]_\omega = \lim_{n\in\mathcal{U}} \sigma_n(E_{\eta_n}([\widetilde{\varphi}(f)]_\omega)) = \sigma([\widetilde{\varphi}(f)]_\omega).$$

Setting $[\varphi(f)]_\omega = \sigma([\widetilde{\varphi}(f)]_\omega)$ for each $\omega \in \Omega$, we get the σ-invariance of all Ω-sections of $\varphi(f)$. It is also easy to check the product property of φ. □

Remark 2.3. If (Θ, T, ν) and (Ω, Σ, μ) are not necessarily complete but they admit the existence of linear liftings σ and ρ and the admissibility of σ, then the lifting $\varphi \in \mathcal{G}(\mu \,\widehat{\otimes}\, \nu)$ takes in fact its values in a proper subspace of $\mathcal{L}^\infty(\mu \,\widehat{\otimes}\, \nu)$. The functions from this space have all their Ω-sections T-measurable.

Proposition 2.4. *Let (Θ, T, ν) be a complete probability space. If $\sigma \in A\mathcal{G}(\nu)$ then for each complete probability space (Ω, Σ, μ) and each $\rho \in \mathcal{G}(\mu)$ there exists $\varphi \in \mathcal{G}(\mu\widehat{\otimes}\nu)$ such that $\varphi \in \rho \otimes \sigma$ and for every $f \in \mathcal{L}^\infty(\mu\widehat{\otimes}\nu)$ there exists a set $N_f \in \Sigma_0$ such that*

$$[\varphi(f)]_\omega = \sigma([\varphi(f)]_\omega) \quad \textit{for every} \quad \omega \in \Omega \setminus N_f.$$

Proof. Let there be given a $\rho \in \mathcal{G}(\mu)$ and a $\sigma \in A\mathcal{G}(\nu)$ together with other elements involved into the construction of $\sigma \in A\mathcal{G}(\nu)$. In particular the family $\mathcal{M} = \langle M_\alpha \rangle_{\alpha<\kappa}$, the σ-subalgebras $\langle \eta_\alpha \rangle_{\alpha<\kappa}$ and the sequences $\langle \gamma_n \rangle$ cofinal with limit ordinals $\gamma \leqq \kappa$ are fixed. Using transfinite induction, we shall be constructing now a transfinite sequence $\langle \varphi_\alpha \rangle_{\alpha\leqq\kappa}$ with $\varphi_\alpha \in \mathcal{G}((\mu \otimes \nu)\widehat{|}(\Sigma \otimes_0 \eta_\alpha))$ such that:

$$\varphi_\alpha(g \otimes h) = \rho(g) \otimes \sigma_\alpha(h) \quad \text{for all} \quad g \in \mathcal{L}^\infty(\mu),\ h \in \mathcal{L}^\infty(\nu|\eta_\alpha); \tag{10}$$

$$\varphi_\gamma | \mathcal{L}^\infty((\mu \otimes \nu)\widehat{|}(\Sigma \otimes_0 \eta_\alpha)) = \varphi_\alpha \quad \text{for all} \quad \alpha < \gamma \leqq \kappa; \tag{11}$$

if $f \in \mathcal{L}^\infty(\mu \otimes \nu \widehat{|} \Sigma \otimes_0 \eta_\alpha)$ then there exists $N_f^\alpha \in \Sigma_0$ such that for every $\omega \notin N_f^\alpha$ we have

$$[\varphi_\alpha(f)]_\omega \in \mathcal{L}^\infty(\nu|\eta_\alpha) \quad \text{and} \quad [\varphi_\alpha(f)]_\omega = \sigma_\alpha([\varphi_\alpha(f)]_\omega). \tag{12}$$

We start the induction defining the linear lifting $\varphi_0 \in \mathcal{G}((\mu \otimes \nu)\widehat{|}(\Sigma \otimes_0 \eta_0))$ exactly as in Theorem 2.2.

Assume now, that we are given $\gamma \leqq \kappa$ with a system $\langle \varphi_\alpha \rangle_{\alpha<\gamma}$ satisfying the conditions (10), (11) and (12) ((10) and (12) in case of $\gamma = 0$).

We distinguish three cases.

A) γ is a limit ordinal of uncountable cofinality. Then $\Sigma \,\widehat{\otimes}_0\, \eta_\gamma = \bigcup_{\alpha<\gamma} (\Sigma \,\widehat{\otimes}_0\, \eta_\alpha)$. Setting

$$\varphi_\gamma(f) := \varphi_\alpha(f) \quad \text{if} \quad f \in \mathcal{L}^\infty((\mu \otimes \nu)\widehat{|}(\Sigma \otimes_0 \eta_\alpha))$$

we get unambiguously defined linear liftings $\varphi_\gamma \in \mathcal{G}((\mu \otimes \nu)\widehat{|}(\Sigma \otimes_0 \eta_\gamma))$ satisfying all required conditions.

B) γ is of countable cofinality. For simplicity put $\sigma_n := \sigma_{\gamma_n}$, $\varphi_n := \varphi_{\gamma_n}$ and $\eta_n := \eta_{\gamma_n}$ for all $n \in \mathbb{N}$. Then

$$\Sigma \mathbin{\widehat{\otimes}_0} \eta_\gamma = \sigma\Big(\bigcup_{n\in\mathbb{N}} \Sigma \mathbin{\widehat{\otimes}_0} \eta_n\Big).$$

Taking the ultrafilter $\mathcal{U}_\gamma$ and setting

$$\varphi_\gamma(f) := \lim_{n\in\mathcal{U}_\gamma} \varphi_n(E_{\Sigma \widehat{\otimes}_0 \eta_n}(f)) \quad \text{for} \quad f \in \mathcal{L}^\infty((\mu \otimes \nu)\widehat{\restriction}(\Sigma \otimes_0 \eta_\gamma))$$

we get $\varphi_\gamma \in \mathcal{G}((\mu \otimes \nu)\widehat{\restriction}(\Sigma \otimes_0 \eta_\gamma))$ (see the arguments of the proof of Theorem 2 in [1, Chapter IV, Section 1]).

Let $g \in \mathcal{L}^\infty(\mu)$ and $h \in \mathcal{L}^\infty(\nu|\eta_\gamma)$. Applying [2] (Section 2, Lemma 1), and the inductive assumptions, we get (10) holds true for φ_γ:

$$\begin{aligned}
\varphi_\gamma(g \otimes h) &= \lim_{n\in\mathcal{U}_\gamma} \varphi_n(E_{\Sigma \widehat{\otimes}_0 \eta_n}(g \otimes h)) = \lim_{n\in\mathcal{U}_\gamma} \varphi_n(E_{\Sigma \otimes \eta_n}(g \otimes h)) \\
&= \lim_{n\in\mathcal{U}_\gamma} \varphi_n(g \otimes E_{\eta_n}(h)) = \lim_{n\in\mathcal{U}_\gamma} \rho(g) \otimes \sigma_n(E_{\eta_n}(h)) \\
&= \rho(g) \otimes \lim_{n\in\mathcal{U}_\gamma} \sigma_n(E_{\eta_n}(h)) = \rho(g) \otimes \sigma_\gamma(h).
\end{aligned}$$

According to [4, Lemma 2.1], if $n \in \mathbb{N}$, then

$$N_{fn} := \{\omega \in \Omega : [E_{\Sigma\widehat{\otimes}_0\eta_n}(f)]_\omega = E_{\eta_n}(f_\omega) \text{ a.e. } (\nu)\}^c \in \Sigma_0.$$

Applying the above property and the martingale convergence theorem, we get for all $\omega \notin \bigcup_n N^{\gamma_n}_{\varphi_\gamma(f)} \cup N_{fn}$

$$\begin{aligned}
[\varphi_\gamma(f)]_\omega &= [\varphi_\gamma(\varphi_\gamma(f))]_\omega = \lim_{n\in\mathcal{U}_\gamma} (\varphi_n[E_{\Sigma\widehat{\otimes}_0\eta_n}[\varphi_\gamma(f)]])_\omega \\
&= \lim_{n\in\mathcal{U}_\gamma} \sigma_n[(\varphi_n[E_{\Sigma\widehat{\otimes}_0\eta_n}[\varphi_\gamma(f)]])_\omega] = \lim_{n\in\mathcal{U}_\gamma} \sigma_n(E_{\eta_n}([\varphi_\gamma(f)]_\omega)) \\
&= \sigma_\gamma([\varphi_\gamma(f)]_\omega)
\end{aligned}$$

i.e. (12) holds true for φ_γ (in the last equality the admissibility of σ has been applied).

C) $\gamma = \beta + 1$.

To simplify the notations let $M := M_\beta$. It then follows that

$$\begin{aligned}
&\mathcal{L}^\infty((\mu \otimes \nu)\widehat{\restriction}(\Sigma \otimes_0 \eta_\gamma)) \\
&\quad = \{G\chi_{\Omega\times M} + H\chi_{\Omega\times M^c} : G, H \in \mathcal{L}^\infty((\mu \otimes \nu)\widehat{\restriction}(\Sigma \otimes_0 \eta_\beta))\}.
\end{aligned}$$

Put

$$V_1 := \operatorname{essinf}\{\mathrm{E} \in \Sigma \mathbin{\widehat{\otimes}_0} \eta_\beta : \Omega \times \mathrm{M} \subseteqq \mathrm{E} \text{ a.e. } (\mu \mathbin{\widehat{\otimes}} \nu|\eta_\beta)\},$$

$$V_2 := \operatorname{essinf}\{\mathrm{E} \in \Sigma \mathbin{\widehat{\otimes}_0} \eta_\beta : \Omega \times \mathrm{M}^c \subseteqq \mathrm{E} \text{ a.e. } (\mu \mathbin{\widehat{\otimes}} \nu|\eta_\beta)\},$$

and

$$\varphi_\gamma(G\chi_{\Omega\times M} + H\chi_{\Omega\times M^c}) :=$$
$$\chi_{\Omega\times M}\varphi_\beta(G\chi_{V_1} + H\chi_{V_1^c}) + \chi_{\Omega\times M^c}\varphi_\beta(H\chi_{V_2} + G\chi_{V_2^c}) \tag{13}$$

if $G, H \in \mathcal{L}^\infty((\mu \otimes \nu)\widehat{\rceil}(\Sigma \otimes_0 \eta_\beta))$.

It follows that $\varphi_\gamma \in \mathcal{G}((\mu \otimes \nu)\widehat{\rceil}(\Sigma \otimes_0 \eta_\gamma))$. If W_1, W_2 are defined according to Definition 2.1 (E), then we have by [2] and by the $\mu \widehat{\otimes} \nu$-density of $\Sigma \otimes \eta_\beta$ in $\Sigma \widehat{\otimes}_0 \eta_\beta$,

$$V_1 = \Omega \times W_1 \quad \text{and} \quad V_2 = \Omega \times W_2 \quad \text{a.e. } ((\mu \otimes \nu)\widehat{\rceil}(\Sigma \otimes_0 \eta_\beta)). \tag{14}$$

Validity of the product property (10) follows exactly as in Theorem 2.2. Also the proof of (12) imitates that from Theorem 2.2.

We can define now φ on $\mathcal{L}^\infty(\mu \widehat{\otimes} \nu)$ satisfying the conditions (10), (11), (12) just by setting $\varphi = \varphi_\kappa$. □

Remark 2.5. Clearly we would like to have in Proposition 2.4 the equality $[\varphi(f)]_\omega = \sigma([\varphi(f)]_\omega)$ for all $\omega \in \Omega$. Unfortunately, we do not know, whether the measurability of $[\varphi(f)]_\omega$ always takes place for all $\omega \in \Omega$, under the assumptions of Proposition 2.4. As Theorem 2.2 shows, with a little bit more precise induction, one can achieve this result if (Θ, T, ν) is separable. We prove in the next theorem that under the assumptions of Proposition 2.4 one can always have a linear lifting with Ω-sections lifting invariant but possibly without the product property.

Theorem 2.6. *Let (Θ, T, ν) be a complete probability space. If $\sigma \in A\mathcal{G}(\nu)$ then for each complete probability space (Ω, Σ, μ) there exists $\psi \in \mathcal{G}(\mu\widehat{\otimes}\nu)$ such that*

$$[\psi(f)]_\omega = \sigma([\psi(f)]_\omega)$$

for every $f \in \mathcal{L}^\infty(\mu\widehat{\otimes}\nu)$ and every $\omega \in \Omega$.

Moreover, if a lifting $\rho \in \Lambda(\mu)$ is given a priori, then ψ can be selected in such a way that the relation $\psi \in \rho \otimes \sigma$ also holds true.

Proof. By Proposition 2.4 there exists $\varphi \in \mathcal{G}(\mu \widehat{\otimes} \nu)$ such that for every $f \in \mathcal{L}^\infty(\mu \widehat{\otimes} \nu)$ there exists $N_f^1 \in \Sigma_0$ with

$$[\varphi(f)]_\omega = \sigma([\varphi(f)]_\omega) \qquad \text{for all} \quad \omega \in \Omega \setminus N_f^1. \tag{15}$$

Moreover, by [4] there exists $\pi \in \Lambda(\mu \widehat{\otimes} \nu)$ such that $[\pi(f)]_\omega \in \mathcal{L}^\infty(\nu)$ for every $\omega \in \Omega$ and $f \in \mathcal{L}^\infty(\mu \widehat{\otimes} \nu)$.

We define now a mapping $\psi : \mathcal{L}^\infty(\mu \widehat{\otimes} \nu) \to \mathcal{L}^\infty(\mu \widehat{\otimes} \nu)$ by setting

$$[\psi(f)]_\omega := \sigma([\pi(f)]_\omega)$$

for each $f \in \mathcal{L}^\infty(\mu \widehat{\otimes} \nu)$ and each $\omega \in \Omega$. The idempotence of σ implies the desired formula in the assertion of the Theorem. To complete the first part of the proof we need to show yet the measurability of $\psi(f)$. But due to the Fubini theorem, we have $[\varphi(f)]_\omega \overset{\nu}{=} [\pi(f)]_\omega$ for $\omega \notin N_f^2 \in \Sigma_0$ and so it follows from (15) that $[\psi(f)]_\omega = \sigma([\pi(f)]_\omega) = \sigma([\varphi(f)]_\omega) = [\varphi(f)]_\omega$ for all $\omega \notin N_f^1 \cup N_f^2$. Consequently, $\psi(f) = \varphi(f)$ $\mu \widehat{\otimes} \nu$-a.e., what proves the measurability of $\psi(f)$.

Assume now that we are given a lifting $\rho \in \Lambda(\mu)$. Then following again [4], we can have $\pi \in \Lambda(\mu \widehat{\otimes} \nu)$ and $\widehat{\sigma} \in \Lambda(\nu)$ such that $\pi \in \rho \otimes \widehat{\sigma}$ and

$$\widehat{\sigma}([\pi(f)]_\omega) = [\pi(f)]_\omega \quad \text{for every} \quad \omega \in \Omega \quad \text{and} f \in \mathcal{L}^\infty(\mu \widehat{\otimes} \nu).$$

Define ψ as in the first part of the proof. If $f \in \mathcal{L}^\infty(\mu)$ and $g \in \mathcal{L}^\infty(\nu)$, then

$$\begin{aligned}[\psi(f \otimes g)]_\omega &= \sigma([\pi(f \otimes g)]_\omega) = \sigma(\widehat{\sigma}([\pi(f \otimes g)]_\omega) \\ &= \sigma(\widehat{\sigma}[\rho(f) \otimes \widehat{\sigma}(g)]_\omega) = \sigma(\widehat{\sigma}[\rho(f)(\omega) \cdot \widehat{\sigma}(g)]) \\ &= \rho(f)(\omega) \cdot \sigma(\widehat{\sigma}(\widehat{\sigma}(g))) = \rho(f)(\omega) \cdot \sigma(g).\end{aligned}$$

It follows that $\psi \in \rho \otimes \sigma$. □

There is an obvious question in Proposition 2.4 and Theorems 2.2 and 2.6 about measurability of all Θ-sections. Similarly, one can ask on lifting invariance of almost all Θ-sections. Unfortunately, such generalizations cannot be achieved with the methods presented in this paper.

Question 2.7. Is it possible to obtain in theorems 2.4 and 2.6 a linear lifting $\psi \in \mathcal{G}(\mu \widehat{\otimes} \nu)$ such that $[\psi(f)]^\theta$ is in addition measurable for all $f \in \mathcal{L}^\infty(\mu \widehat{\otimes} \nu)$ and all $\theta \in \Theta$?

Question 2.8. Given two complete probability spaces (Ω, Σ, μ) and (Θ, T, ν), does there exist a linear lifting $\psi \in \mathcal{G}(\mu \widehat{\otimes} \nu)$ such that $[\psi(f)]_\omega \in T$ and $[\psi(f)]^\theta \in \Sigma$ for all $f \in \mathcal{L}^\infty(\mu \widehat{\otimes} \nu)$, $\omega \in \Omega$ and $\theta \in \Theta$?

This question has a positive answer for densities instead of linear liftings (see [4]) but it remains open also for liftings. Also if one drops positivity of linear liftings, then the answer is affirmative (see Remark 2.9).

Subject to (CH) if we assume that the measure algebra of the product probability space $(\Omega \times \Theta, \Sigma \otimes T, \mu \otimes \nu)$ has cardinality less or equal to $\aleph_2$, then there is a lifting $\psi \in \Lambda(\mu \otimes \nu)$ which gives a positive answer to the last question (see e.g. [5, Theorem 2.7]).

Remark 2.9. For given probability space (Ω, Σ, μ) we denote by $\mathcal{V}(\mu)$ the system of all linear maps υ from $\mathcal{L}^\infty(\mu)$ into itself satisfying $\upsilon(1) = 1$ and the two basic properties of the lifting $\upsilon(f) = f$ a.e. (μ) and $\upsilon(f) = \upsilon(g)$ for all $f, g \in \mathcal{L}^\infty(\mu)$ with $f = g$ a.e. (μ). (Such maps should be called 'linear liftings', but cannot, since this naming has been reserved for the positive linear lifting already).

If two complete probability spaces (Ω, Σ, μ) and (Θ, T, ν) are given, then for all $\rho \in \mathcal{V}(\mu)$ and all $\tau \in \mathcal{V}(\nu)$ there exists a $\varphi \in \mathcal{V}(\mu \widehat{\otimes} \nu)$ such that $\varphi \in \rho \otimes \tau$ and for all $f \in \mathcal{L}^\infty(\mu \widehat{\otimes} \nu)$ and for all $\omega \in \Omega$ we have $[\varphi(f)]_\omega \in \mathcal{L}^\infty(\nu)$ and for all $\theta \in \Theta$ we have $[\varphi(f)]^\theta \in \mathcal{L}^\infty(\mu)$.

To see it choose Hamel bases $(e_i^\bullet)_{i\in I}$ and $(f_j^\bullet)_{j\in J}$ for $L^\infty(\mu)$ and $L^\infty(\nu)$, respectively. By [9, Exercise 39.2] the system $(e_i^\bullet \otimes f_i^\bullet)_{(i,j)\in I\times J}$ is a Hamel basis for $L^\infty(\mu)\otimes L^\infty(\nu)$. Next choose an algebraic complementary linear subspace $\mathcal{X}$ for $L^\infty(\mu)\otimes L^\infty(\nu)$ in $L^\infty(\mu\,\widehat{\otimes}\,\nu)$, a Hamel basis $(g_k^\bullet)_{k\in K}$ for $\mathcal{X}$, $\xi_k \in g_k^\bullet \cap \mathcal{L}^\infty(\mu\otimes\nu)$ for $k\in K$, and define

$$\varphi^\bullet\left(\sum_{(i,j)\in I\times J} a_{(i,j)} e_i^\bullet \otimes f_i^\bullet + \sum_{k\in K} a_k g_k^\bullet\right) := \sum_{(i,j)\in I\times J} a_{(i,j)} \rho(e_i)\otimes\tau(f_i) + \sum_{k\in K} a_k\xi_k,$$

where $0 \neq a_k \in \mathbb{R}$ and $0\neq a_{(i,j)} \in \mathbb{R}$ for at most finitely many a_k and $a_{(i,j)}$. Then put $\varphi := \varphi^\bullet \circ r^\infty$, where r^∞ is the canonical surjection from $\mathcal{L}^\infty(\mu\,\widehat{\otimes}\,\nu)$ onto $L^\infty(\mu\,\widehat{\otimes}\,\nu)$. Since $\varphi(f)\in\mathcal{L}^\infty(\mu\otimes\nu)$ it follows clearly that $[\varphi(f)]_\omega \in \mathcal{L}^\infty(\nu)$ and $[\varphi(f)]^\theta \in \mathcal{L}^\infty(\mu)$ for all $f\in\mathcal{L}^\infty(\mu\,\widehat{\otimes}\,\nu)$, all $\omega\in\Omega$ and all $\theta\in\Theta$. The relation $\varphi\in\rho\otimes\tau$ is clear by definition.

3. Existence of product linear liftings with all sections invariant with respect to the marginal linear liftings. Besides the two problems formulated at the end of the previous section there is yet another obvious question: Do there exist $\rho\in\mathcal{G}(\mu), \sigma\in\mathcal{G}(\nu)$ and $\varphi\in\mathcal{G}(\mu\,\widehat{\otimes}\,\nu)$ such that for each $f\in\mathcal{L}^\infty(\mu\,\widehat{\otimes}\,\nu)$ and each $(\omega,\theta)\in\Omega\times\Theta$

(F) $$\rho([\varphi(f)]^\theta) = [\varphi(f)]^\theta \quad\text{and}\quad \sigma([\varphi(f)]_\omega) = [\varphi(f)]_\omega.$$

We say that φ has (ρ,σ)-*invariant sections*, if the above condition holds true. A corresponding definition can be given for liftings and densities. In case when only the inequalities

$$\rho([\varphi(f)]^\theta) \leqq [\varphi(f)]^\theta \quad\text{and}\quad \sigma([\varphi(f)]_\omega) \leqq [\varphi(f)]_\omega$$

hold true, we say on (ρ,σ)-*sub-invariant sections*.

If $\delta\in\vartheta(\mu)$, $\tau\in\vartheta(\nu)$ and $\varphi\in\vartheta(\mu\,\widehat{\otimes}\,\nu)$ are such that for each $E\in\Sigma\,\widehat{\otimes}\,T$ and each $(\omega,\theta)\in\Omega\times\Theta$ we have

(SF) $$\delta([\varphi(E)]^\theta) \subseteqq [\varphi(E)]^\theta \quad\text{and}\quad \sigma([\varphi(E)]_\omega) \subseteqq [\varphi(E)]_\omega,$$

then φ is said to have (δ,τ)-sub-invariant sections.

It is our aim to prove now that the above question has in general a negative answer. In particular, it is so in the case of non-atomic measures μ and ν. To prove it we define (according to [1, page 36]), for $\rho\in\mathcal{G}(\mu)$ a lower density $\overline{\rho}$ by setting

$$\overline{\rho}(A) := \{\omega\in\Omega : \rho(\chi_A)(\omega) = 1\} \quad\text{for}\quad A\in\Sigma.$$

Lemma 3.1. *If there exist $\rho\in\mathcal{G}(\mu)$, $\sigma\in\mathcal{G}(\nu)$ and $\varphi\in\mathcal{G}(\mu\,\widehat{\otimes}\,\nu)$ possessing (ρ,σ)-sub-invariant sections, then the corresponding $\overline{\varphi}\in\vartheta(\mu\,\widehat{\otimes}\,\nu)$ has $(\overline{\rho},\overline{\sigma})$-sub-invariant sections.*

Proof. Notice first that if for an $E \in \Sigma\widehat{\otimes}T$ and an ω the inequality $[\varphi(\chi_E)]_\omega \geqq \sigma([\varphi(\chi_E)]_\omega)$ is satisfied, then

$$\overline{\sigma}([\overline{\varphi}(E)]_\omega) \subseteqq [\overline{\varphi}(E)]_\omega.$$

Indeed, since we have

$$\chi_{\overline{\varphi}(E)} \leqq \varphi(\chi_E) \leqq \chi_{[\overline{\varphi}(E^c)]^c} = 1 - \chi_{\overline{\varphi}(E^c)},$$

then

$$\sigma(\chi_{[\overline{\varphi}(E)]_\omega}) \leqq \sigma([\varphi(\chi_E)]_\omega) \leqq 1 - \sigma(\chi_{[\overline{\varphi}(E^c)]_\omega}).$$

If $\theta \in \overline{\sigma}([\overline{\varphi}(E)]_\omega)$, then $\sigma(\chi_{[\overline{\varphi}(E)]_\omega})(\theta) = 1$, and so it is a consequence of the above inequalities that $\sigma([\varphi(\chi_E)]_\omega)(\theta) = 1$. But in virtue of the assumption this implies $[\varphi(\chi_E)]_\omega(\theta) = 1$. It follows that $\theta \in [\overline{\varphi}(E)]_\omega$. Consequently, we have $\overline{\sigma}([\overline{\varphi}(E)]_\omega) \subseteqq [\overline{\varphi}(E)]_\omega$. In a similar way the inclusion $\overline{\sigma}([\overline{\varphi}(E)]^\theta) \subseteqq [\overline{\varphi}(E)]^\theta$ can be obtained. □

Now we are able to formulate the next essential result.

Theorem 3.2. *Let (Ω, Σ, μ) and (Θ, T, ν) be complete probability spaces. If there exist $\rho \in \mathcal{G}(\nu)$, $\sigma \in \mathcal{G}(\nu)$ and $\varphi \in \mathcal{G}(\mu \widehat{\otimes} \nu)$ possessing (ρ, σ)-(sub)invariant sections, then either μ or ν is purely atomic.*

Proof. According to (Lemma 3.1) $\overline{\varphi}$ has $(\overline{\rho}, \overline{\sigma})$-sub-invariant sections, but then it follows from Theorem 5 from [6] that either μ or ν is purely atomic. □

A consequence of the above theorem is that the linear lifting φ from Theorem 2.2 cannot have in general (ρ, σ)-invariant sections. In particular, it is so in the case of non-atomic probability measures μ and ν.

The following question remains open:

Question 3.3. Do there exist $\rho \in \mathcal{G}(\nu)$, $\sigma \in \mathcal{G}(\nu)$ and $\varphi \in \mathcal{G}(\mu \widehat{\otimes} \nu)$ such that ($\varphi \in \rho \otimes \sigma$ *and*) for each $f \in \mathcal{L}^\infty(\mu \widehat{\otimes} \nu)$ there exist $N_f \in \Sigma_0$ and $M_f \in T_0$ with the property that whenever $\omega \notin N_f$ and $\theta \notin M_f$ then

$$\rho([\varphi(f)]^\theta) = [\varphi(f)]^\theta \quad \text{and} \quad \sigma([\varphi(f)]_\omega) = [\varphi(f)]_\omega?$$

Do there exist densities instead of linear liftings with the above properties?

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K. Musiał
Institute of Mathematics
Wrocław University
Pl. Grunwaldzki 2/4
PL-50-384 Wrocław
Poland
musial@math.uni.wroc.pl

W. Strauss
Universität Stuttgart
Fachbereich Mathematik
Institut für Stochastik und Anwendungen
Abteilung für Finanz- und Versicherungsmathematik
Postfach 80 11 40
D-70511 Stuttgart
Germany
strauss@mathematik.uni-stuttgart.de

N. D. Macheras
University of Piraeus
Department of Statistics and Insurance Science
80 Karaoli and Dimitriou street
GR-185 34 Piraeus
Greece
macheras@unipi.gr