PETTIS INTEGRATION

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1 PRELIMINARIES.

(Ω, Σ, μ) – a fixed complete probability space,
$\mathcal{N}(\mu)$ – the collection of all μ-null sets,
Σ_μ^+ – the family of all sets of positive μ -measure.
λ– the Lebesgue measure on the real line or an interval;
$\mathcal{P}(\Omega)$ – the family of all subsets of Ω,
$\sigma(\mathcal{A})$ – the σ -algebra generated by $\mathcal{A} \subset \mathcal{P}(\Omega)$.
X, Y – Banach spaces, X^*, Y^* – their topological conjugate spaces. $B(X)$ – the closed unit ball of X. The value of a functional $x^* \in X^*$ on an element $x \in X$ is denoted by $x^*(x), x^*x$ or by $\langle x^*, x\rangle$.
A set function $\nu : \Sigma \to X$ is an X-*valued measure* if it is countably additive in the norm topology:

$$\lim_n \left\| \nu\left(\bigcup A_k\right) - \sum_{k=1}^n \nu(A_k) \right\| = 0$$

for each pairwise disjoint $A_1, A_2, \ldots \in \Sigma$.
Variation of ν is a measure $|\nu| : \Sigma \to [0, \infty)$ defined by

$$|\nu|(E) := \sup\left\{ \sum_{i=1}^n \|v(E_i)\| \colon E \supset \bigcup_{i=1}^n E_i \ \&\ i \neq j \Rightarrow E_i \cap E_j = \emptyset \right\}$$

2 MEASURABLE FUNCTIONS.

Definition 2.1 A function $f : \Omega \to X$ is called *simple* if there exist $x_1, ..., x_n$ in X and $E_1, ..., E_n \in \Sigma$ such that

$$f = \sum_{i=1}^n x_i \, \mathcal{X}_{E_i} \ .$$

A function $f : \Omega \to X$ is called *strongly measurable* if there exists a sequence of simple functions $f_n : \Omega \to X$ with $\lim_n \|f_n(\omega) - f(\omega)\| = 0$ μ – a.e..
A strongly measurable $f : \Omega \to X$ is called *Bochner integrable* if there exists a sequence of simple functions $f_n : \Omega \to X$ with

$$\lim_n \int_\Omega \|f_n(\omega) - f(\omega)\| \, d\mu = 0.$$

Then $\nu_f(E) := \lim_n \int_E f_n \, d\mu$ is the Bochner integral of F on $E \in \Sigma$ and $|\nu_f|$ is a finite measure.

Proposition 2.2 *A strongly measurable $f : \Omega \to X$ is Bochner integrable if and only if ν_f is of finite variation if and only if $\int_\Omega \|f\| d\mu < \infty$.*

Definition 2.3 A function $f : \Omega \to X$ is said to be scalarly measurable, if $x^* f$ is measurable for each $x^* \in X^*$. $f : \Omega \to X^*$ is said to be weak* scalarly measurable, if xf is measurable for each $x \in X$.

Theorem 2.4 *(Pettis' measurability theorem) A function $f : \Omega \to X$ is strongly measurable if and only if*

(i) f is scalarly measurable, and

(ii) f is essentially separably valued, i.e. there exists $E \in \mathcal{N}(\mu)$ such that $f(\Omega \setminus E)$ is a separable subset of X.

Definition 2.5 A function $f : \Omega \to X$ is *scalarly bounded* provided there is $M > 0$ such that for each $x^* \in X^*$ the inequality $|x^* f| \leq M\|x^*\|$ holds μ –a.e.
$f : \Omega \to X^*$ is *weak*-scalarly bounded* provided there is $M > 0$ such that for each $x \in X$ the inequality $|xf| \leq M\|x\|$ holds μ –a.e.

Theorem 2.6 (Dinculeanu, A. and C. Ionescu-Tulcea) *If $F \in L_1^*(\mu, X)$, then there exists a weak* scalarly bounded and weak* scalarly measurable function $h : \Omega \to X^*$ such that*

(i) $\langle F, f \rangle = \int_\Omega \langle h(\omega), f(\omega) \rangle \, d\mu(\omega)$;

(ii) $\|F\| = \sup_{\omega \in \Omega} \|h(\omega)\|$.

Proof. One has to apply lifting.

3 PETTIS INTEGRAL.

Definition 3.1 A function $f : \Omega \to X$ is *scalarly integrable* if $x^*f \in L_1(\mu)$ for every $x^* \in X^*$.
$f : \Omega \to X^*$ is *weak***-scalarly integrable* if $xf \in L_1(\mu)$, for every $x \in X$.

Definition 3.2 A scalarly integrable $f : \Omega \to X$ is *Pettis integrable* if for each $E \in \Sigma$ there exists $\nu_f(E) \in X$ such that

$$x^*\nu_f(E) = \int_E x^* f \, d\mu \,, \quad \text{for each } x^* \in X^* \,.$$

The set function $\nu_f : \Sigma \to X$ is called the *Pettis –integral* of f with respect to μ, and $\nu_f(E)$ is called the *Pettis –integral* of f over $E \in \Sigma$ with respect to μ. We use the notations: $\boldsymbol{(P)\int_E f\,d\mu := \nu_f(E)}$.

Theorem 3.3 *If f is Pettis integrable, then ν_f is a μ-continuous measure (Pettis 1938) of σ–finite variation (Rybakov 1968).* □

Definition 3.4 We say that two weakly measurable functions $f, g : \Omega \to X$ are *scalarly (or weakly) equivalent* if $x^*f = x^*g$ μ– a.e. for each $x^* \in X^*$. Two weak* measurable functions $f, g : \Omega \to X^*$ are *weak***-scalarly equivalent* if $xf = xg$ μ– a.e. for each $x \in X$. Two strongly measurable functions f and g are *equivalent* if $f = g$ a.e..

Denote by $\mathbf{P}(\mu, X)$ the set of all X-valued Pettis integrable functions (we identify functions that are scalarly equivalent). One can define a norm on $\mathbf{P}(\mu, X)$ by

$$|\!|\!| f |\!|\!| = \sup\left\{\int_\Omega |\langle x^*, f\rangle| d\mu : \, x^* \in B(X^*)\right\}$$

It has been shown by Thomas [1974], that **if μ is not purely atomic and X is infinite dimensional, then $\mathbf{P}(\mu, X)$ is non–complete**. This fact is one of the most serious difficulties in investigation of Pettis integrability.

Proposition 3.5 *(Gelfand) Each weak***–scalarly integrable $f : \Omega \to X^*$ is weak***–integrable (or Gelfand integrable), that is for each $E \in \Sigma$ there exists $\nu_f(E) \in X^*$ such that*

$$x\nu_f(E) = \int_E xf \, d\mu \,, \quad \textit{for each } x \in X.$$

Proof. For a fixed $E \in \Sigma$ define $T : X \to L_1(\mu)$ by $T(x) = \langle x, f\rangle \chi_E$. It is easily seen that T has closed graph and hence –in virtue of Banach's Closed Graph Theorem – T is continuous. Thus,

$$\left|\int_E \langle x, f\rangle d\mu\right| \le \int_E \left|\langle x, f\rangle\right| d\mu = \|T(x)\| \le \|T\| \, \|x\|$$

and so the mapping $x \to \int_E \langle x, f\rangle d\mu$ is a continuous linear functional on X and defines and element $\nu_f(E) \in X^*$ satisfying the required in the definition equality. □

Example 3.6 *A scalarly integrable function that is not Pettis integrable.*
Define $f : (0,1] \to c_0$ by

$$f(t) = (2\chi_{(2^{-1},1]}(t), 2^2\chi_{(2^{-2},2^{-1}]}(t), ..., 2^n\chi_{(2^{-n},2^{-n+1}]}(t), ...).$$

If $x^* = (\alpha_1, \alpha_2, ...) \in l_1 = c^*$, then

$$x^* f = \sum_{n=1}^{\infty} \alpha_n 2^n \chi_{(2^{-n},2^{-n+1}]}$$

and

$$\int_0^1 |x^* f| d\lambda \leq \sum_{n=1}^{\infty} |\alpha_n| < \infty \ .$$

But

$$w^* - \int_{(0,1]} f \, d\lambda = (1, 1, 1, ..., 1, ..) \neq c_0$$

and so f is not λ–Pettis integrable □

It is one of the main problems in the theory of vector integration to find conditions guaranteeing the existence of the Pettis integral.

Theorem 3.7 *Let $f : \Omega \to X$ be scalarly integrable. Then the following are equivalent:*

(a) f is Pettis integrable ,

(b) $T_f : X^* \to L_1(\mu)$, defined by $T_f(x^*) = x^* f$, is weak*–weakly continuous.

The next theorem describes an essential difference between Bochner and Pettis integrable functions.

Proposition 3.8 *Let $f : \Omega \to X$ be represented in the form $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$ with pairwise disjoint $E_n \in \Sigma, n \in \mathbf{N}$. Then:*

(i) f is Pettis integrable if and only if $\sum_{n=1}^{\infty} x_n \mu(E_n)$ is unconditionally convergent.

(ii) f is Bochner integrable if and only if $\sum_{n=1}^{\infty} x_n \mu(E_n)$ is absolutely convergent.

In each case $\int_E f \, d\mu = \sum_{n=1}^{\infty} x_n \mu(E \cap E_n)$.

Remark 3.9 If X is an infinite dimensional Banach space, then there exists an X–valued strongly measurable function that is Pettis but not Bochner integrable. Indeed, Dvoretzky–Rogers's theorem guarantees the existence of an unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ such that $\sum_{n=1}^{\infty} \|x_n\| = \infty$. It is enough to take $\Omega = \mathbf{N}, \Sigma = \mathcal{P}(\mathbf{N})$ and to define μ by $\mu(\{n\}) = 2^{-n}$ for each n. The function $f : \mathbf{N} \to X$ given by $f(n) = 2^n x_n$ is suitable for our purpose. Observe, that the variation of ν_f is σ–finite but not finite.

The following result gives a necessary and sufficient condition for the Pettis integrability of strongly measurable function.

Theorem 3.10 *A strongly measurable and scalarly integrable* $f : \Omega \to X$ *is Pettis integrable if and only if the set* $\{x^*f : x^* \in B(X^*)\}$ *is relatively weakly compact in* $L_1(\mu)$ *(= uniformly integrable).*

When all scalarly integrable and strongly measurable X–valued functions are Pettis integrable? It follows from Example 3.6 that such spaces cannot contain c_0. It turns out that this is also the sufficient condition.

Theorem 3.11 (Dimitrov (1971), Diestel (1974)) *If* X *does not contain any isomorphic copy of* c_0*, then each strongly measurable and scalarly integrable* X*–valued function is Pettis integrable.*

Proof. Each strongly measurable function $f : \Omega \to X$ is the form $f = g + \sum_{n=1}^{\infty} x_n \chi_{E_n}$, where g is bounded (hence Bochner integrable) and, the sets $E_n \in \Sigma$ are pairwise disjoint.
We have then

$$\sum_{n=1}^{\infty} |x^*(x_n)|\mu(E_n) = \int_{\Omega} |x^*f| d\mu < \infty$$

This means that the series $\sum_{n=1}^{\infty} x_n \, \mu(E_n)$ is weakly unconditionally Cauchy. Since X does not contain any isomorphic copy of c_0 it is norm convergent. Thus, Proposition 3.8 yields the Pettis integrability of f. □

4 CRITERIA FOR PETTIS INTEGRABILITY.

Theorem 4.1 (Talagrand (1984)) *Let* X *be an arbitrary Banach space and* $f : \Omega \to X$ *be scalarly integrable. Then* $f \in \mathbf{P}(\mu, X)$ *if and only if there is a WCG space* $Y \subseteq X$ *such that* $x^*f = 0$ *a.e. for each* $x^* \in Y^{\perp}$ *and* $T_f : X^* \to L_1(\mu)$*, defined by* $T_f(x^*) = x^*f$*, is weakly compact.*

A scalarly integrable function may be Pettis non-integrable even if it is strongly measurable. What can be said in the situation of scalarly bounded functions? That not all such functions are Pettis integrable was already known to Phillips.

Example 4.2 (CH) Under the continuum hypothesis Sierpiński [?] constructed a set $B \subset [0,1] \times [0,1]$ with the following properties: (1) For each $t \in [0,1]$ the set $\{s \in [0,1] : (s,t) \in B\}$ is at most countable,
(2) For each $s \in [0,1]$ the set $\{t \in [0,1] : (s,t) \notin B\}$ is at most countable.
Consider $([0,1], \mathcal{L}, \lambda)$ and define $f : [0,1] \to l_\infty[0,1]$ by

$$[f(s)](t) = \chi_B(s,t)$$

f is scalarly measurable and bounded but f is not Pettis λ–integrable □.

5 CONVERGENCE THEOREMS.

Theorem 5.1 (Geitz, Musiał [14]) *Let $f : \Omega \to X$ be a function satisfying the following two conditions:*

(α) There exists a sequence of Pettis integrable functions $f_n : \Omega \to X$ such that $\lim_n x^* f_n = x^* f$ in measure, for each $x^* \in X^*$,

(β) There exists $h \in L_1(\mu)$ such that for each $x^* \in B(X^*)$ and each $n \in \mathbf{N}$, the inequality $|x^* f_n| \leq h$ holds a.e. (the exceptional set depends on x^*).

Then $f \in \mathbf{P}(\mu, X)$ and

$$\lim_n \int_E f_n \, d\mu = \int_E f \, d\mu$$

weakly in X, for all $E \in \Sigma$.

6 THE RADON–NIKODYM PROPERTY.

A Banach space X is said to have the Radon-Nikodym property if for each finite complete measure space (Ω, Σ, μ) and each μ-continuous X-valued measure $\nu : \Sigma \to X$ of finite variation there exists a strongly measurable function $f : \Omega \to X$ such that

$$\forall\ A \in \Sigma\ \nu(A) = (B) \int_A f \, d\mu.$$

There are several characterizations of Banach spaces possessing RNP. Here are some of them:

Theorem 6.1 (Stegall [1975]) *X^* has RNP iff each separable subspace of X has separable dual.*

Theorem 6.2 (Huff, Morris [1975]) *X^* has RNP iff X^* has the Krein-Milmann property (= if $K \subset X^*$ is closed, convex and bounded, then $K = \overline{conv(extK)}$.*

Theorem 6.3 *For a fixed Banach space X the following conditions are equivalent:*

(i) X has RNP;

(ii) Every function $f; [0,1] \to X$ of bounded variation is differentiable a.e. (i.e. $\lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$ exists for almost all $t_0 \in [0,1]$, in the norm topology);

(iii) Every nonempty closed bounded subset of X contains an extreme point of its closed convex hull;

(iv) For any finite measure space (Ω, Σ, μ) and each uniformly integrable martingale of Bochner integrable functions $f_n : \Omega \to X$ with $\sup_n \int \|f_n\| \, d\mu < \infty$, there exists a Bochner integrable $f : \Omega \to X$ such that $\lim_n \int \|f_n - f\| \, d\mu = 0$.

If $X = Y^*$, then we can add the following:

(iv) For any finite measure space (Ω, Σ, μ) and for any $1 < p < \infty$ $L_p(\mu, Y)^* = L_q(\mu, X)$, where $1/p + 1/q = 1$.

7 THE WEAK RADON–NIKODYM PROPERTY.

Which X have the property, that each X–valued measure of σ–finite variation is a Pettis integral ? The following theorem is the starting point for the whole theory. Its proof makes use of the lifting, however, in the case of a separable X, it can be done without it. In the case of a measure of finite variation the theorem is a consequence of a representation theorem of A. and C.Ionescu–Tulcea [8]. Explicitely it was first stated by Dinculeanu [5]. The σ–finite case was proved by Rybakov [17].

Theorem 7.1 *If $\nu : \Sigma \to X^*$ is a μ-continuous measure of σ–finite variation, then there exists a weak* scalarly integrable function $f : \Omega \to X^*$ such, that*

$$\langle x, \nu(E)\rangle = \int_E \langle x, f\rangle \, d\mu$$

for each $x \in X$ and each $E \in \Sigma$.

Proof. Assume that there exists M such that $|\nu|(E) \le M\mu(E)$, for every $E \in \Sigma$. According to the classical RN theorem for each $x \in X$ there exists $f_x \in L_1(\mu)$ such that

$$\forall\, E \in \Sigma\ \langle x, \nu(E)\rangle = \int_E f_x \, d\mu.$$

Clearly $|f_x| \le M\|x\|$ a.e., for each x separately. Let ρ be an arbitrary lifting on $L_\infty(\mu)$. Define $f : \Omega \to X^*$ by

$$\langle x, f(\omega)\rangle := \rho(f_x)(\omega).$$

This is possible, because for each fixed ω the mapping $X \ni x \longrightarrow \rho(f_x)(\omega)$ is linear and bounded: $|\rho(f_x)(\omega)| \le M\|x\|$. We have then

$$\boldsymbol{\forall\, x \in X\ \forall\, E \in \Sigma\ \langle x, \nu(E)\rangle = \int_E \langle x, f\rangle \, d\mu\,.}$$

$\square$

Definition 7.2 (Musiał, Studia Math. 64(1979)) X has the $WRNP$ if for each complete probability space (Ω, Σ, μ) and each X–valued continuous measure of σ–finite variation there exists $f \in \mathbf{P}(\mu, X)$ such that

$$\langle x^*, \nu(E)\rangle = \int_E \langle x^*, f\rangle \, d\mu$$

for each $x^* \in X^*$ and $E \in \Sigma$.

Remark 7.3 $L_1[0,1]$ and c_0 are examples of a Banach spaces without the WRNP.

Theorem 7.4 *X^* has the weak Radon–Nikodym property if and only if X contains no isomorphic copy of l_1.*

Given a directed set $(\Pi, \leq)$, a family of σ–algebras $\Sigma_\pi \subseteq \Sigma$, and functions $f_\pi \in \mathbb{P}((\Omega, \Sigma_\pi, \mu|\Sigma_\pi); X)$ with $\pi \in \Pi$, the system $\{f_\pi, \Sigma_\pi; \pi \in \Pi\}$ is a *martingale* if $\pi \leq \rho$ yields $\Sigma_\pi \subseteq \Sigma_\rho$ and

$$\int_E f_\rho \, d\mu = \int_E f_\pi \, d\mu, \qquad \text{for every } E \in \Sigma_\pi.$$

The martingale is bounded if there is $M > 0$ such, that for each $x^* \in X^*$ and each $\pi \in \Pi$ the inequality $|\langle x^*, f_\pi \rangle| \leq M\|x^*\|$ holds μ–a.e.. The martingale is convergent in $\mathbb{P}(\mu, X)$ if there is $f \in \mathbb{P}(\mu, X)$ such that $\lim_\pi \| f_\pi - f \| = 0$. The collection of all finite Σ–partitions of Ω into sets of positive measure is denoted by Π_Σ. We order it in the following way: $\pi_1 \leq \pi_2$ if each element of π_1 is, except for a null set, a union of element of π_2.
The following theorem is a martingale characterization of the WRNP. 9.4

Theorem 7.5 (Musiał (1980)) *The following conditions are equivalent:*

(i) X has the WRNP,

(ii) Given any (Ω, Σ, μ) and any bounded martingale $\{f_n, \Sigma_n; n \in \mathbb{N}\}$ of X–valued Pettis μ–integrable (simple) functions, then $\{f_n, \Sigma_n; n \in \mathbb{N}\}$ is convergent in $\mathbb{P}(\mu, X)$.

Basic books:
J. Diestel, J.J. Uhl – *Vector measures*, **Math. Surveys 15(1977).**

K. Musiał, *Topics in the theory of Pettis integration*, **Rend. Istit. Mat. Univ. Trieste 23 (1991), 177–262.**
K. Musiał, *Pettis integral*, **Handbook of Measure Theory I, 531–586. Elsevier, Amsterdam (2002)**

Example 7.6 *A weakly measurable function that is not strongly measurable but is weakly equivalent to a strongly measurable function.*

Let $V \subset [0,1]$ be a Vitali set and let $\{e_t : t \in [0,1]\}$ be the canonical basis for the nonseparable Hilbert space $l_2([0,1])$. Define $f_V : [0,1] \to l_2([0,1])$ by
$\boldsymbol{f_V(t) = e_t}$ whenever $t \in V$ and $f_V(t) = 0$ otherwise.
It is a consequence of the Riesz Representation Theorem that $x^*f = 0$ λ–a.e. for each $x^* \in l_2([0,1])^*$ (i.e. f is weakly λ–equivalent to the zero function). On the other hand, if $E \in \mathcal{N}(\lambda)$, then $f_V(V \setminus E)$ is nonseparable. Since $\lambda^*(V) > 0$, f_V is not

essentially separably valued, and so – in virtue of the Pettis theorem f_V is not strongly λ–measurable.
Since $\|f_V(t)\| = \chi_V(t)$ also the function $\|f_V\| : t \to \|f_V(t)\|$ is not measurable. □
The fact that a weakly measurable function may have nonmeasurable norm causes a lot of troubles in the theory of weakly measurable functions. □

Example 7.7 *(Hagler) A weakly measurable function that is not scalarly equivalent to a strongly measurable one. Let (A_n) be a sequence of nonempty subintervals of [0.1], such that :*

(i) $A_1 = [0, 1]$

(ii) $A_n = A_{2n} \cup A_{2n+1}$ *for each* $n \in \mathbf{N}$,

(iii) $A_i \cap A_j = 0$ *if* $i \neq j$ *and* $2^n \leq i,\ j \leq 2^{n+1}$,

(iv) $\lim_n \lambda(A_n) = 0$.

Define $f : [0, 1] \to l_\infty$ *by* $\boldsymbol{f(t) = (\chi_{A_n}(t))}$ *for* $t \in [0, 1]$. □

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