LIFTINGS IN PRODUCT SPACES

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1 Existence of liftings.

Definition 1 Let $(X, \mathfrak{A}, P)$ be a probability space. $\mathfrak{A}_0 = \{A \in \mathfrak{A} : P(A) = 0\}$. A mapping $\rho : \mathfrak{A} \to \mathfrak{A}$ is said to be a $\boxed{\boxed{\text{lifting}}}$ if it satisfies the following properties:

(L1) $\rho(A) \overset{P}{=} A$;

(L2 if $A \overset{P}{=} B$ **then** $\rho(A) = \rho(B)$;

(L3) $\rho(A^c) = [\rho(A)]^c$;

(L4) $\rho(A \cap B) = \rho(A) \cap \rho(B)$;

(L5) $\rho(\emptyset) = \emptyset$ **and** $\rho(X) = X$.

A map $\rho : \mathfrak{A} \to \mathfrak{A}$ satisfying the conditions

$$(L1), (L2), (L4) \text{ and } (L5)$$

is called a $\boxed{\boxed{\text{density}}}$ on $(X, \mathfrak{A}, P)$.

Example 1 Let λ be the Lebesgue measure on $\mathbf{R}^n$, where $n \geq 1$. If $\mathfrak{A}$ is the family of Lebesgue measurable sets, we put for each $E \in \mathfrak{A}$

$$\rho(E) := \left\{ x \in \mathbf{R}^n : \lim_{\delta \searrow 0} \frac{\lambda(E \cap B(x, \delta))}{\lambda(B(x, \delta))} = 1 \right\},$$

where $B(x, \delta) = \{y \in \mathbf{R}^n : \|x - y\| \leq \delta\}$.

Lemma 1 ***Let $\mathfrak{C} \supset \mathfrak{A}_0$ be a sub$-\sigma-$algebra of $\mathfrak{A}$ and let τ_0 be a lower density on $\mathfrak{C}$. If $V \in \mathfrak{A} \setminus \mathfrak{C}$, then there exists a lower density τ on $\sigma(\mathfrak{C} \cup \{V\})$ such that $\tau|\mathfrak{C} = \tau_0$.***

Proof. Let
$$M_1 := \operatorname{ess\,inf}\{F \in \mathfrak{C} : V \subseteq F\}$$
and
$$M_2 := \operatorname{ess\,inf}\{F \in \mathfrak{C} : V^c \subseteq F\}\,.$$
If $W \in \sigma(\mathfrak{C} \cup \{V\})$, then there exist $A, B \in \mathfrak{C}$ such that $W = (A \cap V) \cup (B \cap V^c)$. We set

$$\begin{aligned} \boldsymbol{\tau}[(\boldsymbol{A} \cap \boldsymbol{V}) \cup (\boldsymbol{B} \cap \boldsymbol{V}^c)] :=& \\ & [\boldsymbol{V} \cap \boldsymbol{\tau}_0[(\boldsymbol{A} \cap \boldsymbol{M}_1) \cup (\boldsymbol{B} \cap \boldsymbol{M}_1^c)]] \\ \cup\ & [\boldsymbol{V}^c \cap \boldsymbol{\tau}_0[(\boldsymbol{A} \cap \boldsymbol{M}_2) \cup (\boldsymbol{B} \cap \boldsymbol{M}_2^c)]]\,. \end{aligned}$$

Lemma 2 *Let $(\mathfrak{C}_n)_{n=1}^{\infty}$ be an increasing sequence of sub$-\sigma-$algebras of $\mathfrak{A}$ and let $(\tau_n)_{n=1}^{\infty}$ be a sequence of densities (i.e. τ_n is a lower density on $\mathfrak{C}_n$). If $\tau_{n+1}|\mathfrak{C}_n = \tau_n$ for every $n \in \mathbf{N}$, then there exists a lower density τ on $\mathfrak{C} := \sigma\Big(\bigcup_{n=1}^{\infty} \mathfrak{C}_n\Big)$ such that $\tau|\mathfrak{C}_n = \tau_n$ for every $n \in \mathbf{N}$.*

Proof. For each $E \in \mathfrak{C}$ we set
$$\tau(E) := \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \tau_m\Big(\Big\{x \in X : E_{\mathfrak{C}_m}(\chi_E)(x) \geq 1 - 1/2^k\Big\}\Big)\,.$$
($E_{\mathfrak{C}_m}$ is the conditional expectation operator on $L_\infty(\mu)$ with respect to $\mathfrak{C}_m$, i.e.
$$\forall\, f \in L_\infty(\mu)\ \forall\, A \in \mathfrak{C}_m\ \int_A f\, d\mu = \int_A E_{\mathfrak{C}_m}(f)\, d\mu).$$
□

Theorem 1 *For an arbitrary $(X, \mathfrak{A}, P)$ there exists a lower density τ on $\mathfrak{A}$.*

Proof. Let $\mathcal{D}$ be the smallest cardinal with the property, that there exists a collection $\mathcal{M} \subset \mathfrak{A}$ such that $\sigma(\mathcal{M})$ is dense in $\mathfrak{A}$ in the pseudometric generated by P. Let $\mathcal{M} = (M_\alpha)_{\alpha<\kappa}$ be numbered by ordinals less than κ, where κ is the first ordinal of the cardinality $\mathcal{D}$. Denote by $\mathfrak{C}_0$ the σ - algebra $\sigma(\mathfrak{A}_0)$ and for each $1 \leq \alpha \leq \kappa$ denote by $\mathfrak{C}_\alpha$ the σ - algebra generated by the family $\{M_\gamma : \gamma < \alpha\} \cup \mathfrak{C}_0$. We assume that $M_\alpha \notin \mathfrak{C}_\alpha$ for each α. It is clear that without loss of generality, we may do so.
We shall be constructing the final density inductively. τ_0 will be the only existing density on $(X, \mathfrak{A}_0, P|\mathfrak{A}_0)$, i.e.
$$\tau_0(B) = \begin{cases} \emptyset & \text{if } \ B \in \mathfrak{A}_0 \\ X & \quad B \notin \mathfrak{A}_0 \end{cases}$$

Assume that for each $\alpha < \gamma \leq \kappa$ a density τ_α on $\mathfrak{C}_\alpha$ is already constructed. We assume, that $\alpha < \beta < \gamma$ yields $\tau_\beta|\mathfrak{C}_\alpha = \tau_\alpha$. We have to separate three cases.

A) $\gamma \leq \kappa$ is a limit ordinal of uncountable cofinality. Then $\mathfrak{C}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{C}_\alpha$ and we define $\tau_\gamma \in \vartheta(\mu|\mathfrak{C}_\gamma)$ by setting

$$\tau_\gamma(B) := \tau_\alpha(B) \quad \textbf{if} \quad B \in \mathfrak{C}_\alpha \quad \textbf{and} \quad \alpha < \gamma\,.$$

B) There exists an increasing sequence (γ_n^γ) of ordinals that is cofinal to $\gamma \leq \kappa$.

For simplicity put $\tau_n := \tau_{\gamma_n^\gamma}$ and $\mathfrak{C}_n := \mathfrak{C}_{\gamma_n^\gamma}$ for all $n \in \mathbf{N}$. Then $\mathfrak{C}_\gamma = \sigma(\cup_{n\in\mathbf{N}}\mathfrak{C}_n)$ and we can define τ_γ by setting for each $B \in \mathfrak{C}_\gamma$

$$\tau_\gamma(B) := \bigcap_{k\in\mathbf{N}} \bigcup_{n\in\mathbf{N}} \bigcap_{m\geq n} \tau_m(\{E_{\mathfrak{C}_m}(\chi_B) > 1 - 1/2^k\})\,.$$

It follows from Lemma 2, that τ_γ is a density on $\mathfrak{C}_\gamma$ and $\tau_\gamma|\mathfrak{C}_n = \tau_n$ for each $n \in \mathbf{N}$.

C) If $\gamma = \beta + 1$ then, τ_γ is constructed with the help of Lemma 1.

Finally, we define τ just by setting $\tau = \tau_\kappa$. □

Definition 2 Each density constructed in the way described in the proof of Theorem 1 will be called an **admissible density**. The family of all admissible densities on $(X, \mathfrak{A}, P)$ will be denoted by $A\vartheta(P)$.

Theorem 2 ***(Traynor) Let*** $(X, \mathfrak{A}, P)$ ***be a complete measure space and let*** $\tau : \mathfrak{A} \to \mathfrak{A}$ ***be a lower density on*** $\mathfrak{A}$***, Then there exists a lifting*** ρ ***on*** $\mathfrak{A}$***, such that***

$$\tau(E) \subseteq \rho(E) \subseteq [\tau(E^c)]^c \tag{1}$$

for all $E \in \mathfrak{A}$***. Each such*** ρ ***is said to be generated by*** τ***.***

Definition 3 Each lifting constructed on a complete measure space $(X, \mathfrak{A}, P)$ in the manner described by Theorem 2 from an admissible density will be called an **admissibly generated lifting**. The collection of all admissibly generated liftings on $(X, \mathfrak{A}, P)$ will be denoted by $\boldsymbol{AG\Lambda(P)}$**.**

As a direct consequence of Theorems 1 and 2 we get

Theorem 3 ***On an arbitrary complete*** $(X, \mathfrak{A}, P)$ ***there exists an admissibly generated lifting.***

Definition 4 A mapping $\rho : \mathcal{L}_\infty(P) \to \mathcal{L}_\infty(P)$ is called a $\boxed{\boxed{\textbf{lifting}}}$ on $\mathcal{L}_\infty(P)$ if it possessess the following properties:

(L1) $\rho(f) \equiv f$ for every $f \in \mathcal{L}_\infty(P)$;

(L2) $f \equiv g \Longrightarrow \rho(f) = \rho(g)$;

(L3) ρ is linear and multiplicative;

(L4) $\rho(1) = 1$.

Notice that ρ may be considered as a mapping from $L_\infty(P)$ into $\mathcal{L}_\infty(P)$.

Proposition 1 ***If*** $\rho : \mathcal{L}_\infty(P) \to \mathcal{L}_\infty(P)$ ***is a lifting, then***

(L5) ***If*** $f \geq 0$ $P-$***a.e., then*** $\rho(f) \geq 0$ ***everywhere.***

(L6) ***If*** $f \leq g$ $P-$***a.e., then*** $\rho(f) \leq \rho(g)$ ***everywhere.***

(L7) $|\rho(f)| = \rho(|f|)$.

(L8) $\sup(\rho(f), \rho(g)) = \rho(\sup(f, g))$ ***and*** $\inf(\rho(f), \rho(g)) = \rho(\inf(f, g))$.

Proof. $(L5)$ **If** $f \geq 0$ $P-$**a.e., then** $f = (\sqrt{f})^2$ **and so** $\rho(f) = [\rho(\sqrt{f})]^2 \geq 0$. □

Proposition 2 ***Each lifting*** $\rho : \mathcal{L}_\infty(P) \to \mathcal{L}_\infty(P)$ ***uniquely determines a lifting*** $\rho' : \mathfrak{A} \to \mathfrak{A}$ ***satisfying for each*** $A \in \mathfrak{A}$ ***the equality*** $\rho(\chi_A) = \chi_{\rho'(A)}$***. And conversely, if*** $\rho' : \mathfrak{A} \to \mathfrak{A}$ ***is a lifting on*** $\mathfrak{A}$***, then there is a unique lifting*** $\rho : \mathcal{L}_\infty(P) \to \mathcal{L}_\infty(P)$ ***such that*** $\rho(\chi_A) = \chi_{\rho'(A)}$***, for every*** $A \in \mathfrak{A}$.

Theorem 4 $(X, \mathfrak{A}, P)$ ***- complete. If*** $\{E_t : t \in T\}$ ***is an arbitrary family of sets such that*** $E_t \subseteq \rho(E_t)$ ***for each*** $t \in T$***, then*** $\bigcup_{t\in T} E_t \in \mathfrak{A}$ ***and*** $\bigcup_{t\in T} E_t \subseteq \rho\Big(\bigcup_{t\in T} E_t\Big)$. ***More generally, if*** $\{f_t : t \in T\} \subseteq \mathcal{L}_\infty(P)$ ***is a uniformly bounded family of functions, such that*** $f_t \leq \rho(f_t)$ ***for every*** $t \in T$ ***, then*** $\sup_{t\in T} f_t$ ***is a measurable function and*** $\sup_{t\in T} f_t \leq \rho(\sup_{t\in T} f_t)$.

Proof. Let $\{E_t : t \in T\}$ be a family of sets satisfying for each $t \in T$ the inclusion $E_t \subseteq \rho(E_t)$. Moreover let Ξ be the collection of all at most countable subsets of T. Since P is bounded there is a real number a such that $a = \sup_{\alpha\in\Xi} P(\bigcup_{t\in\alpha} E_t)$. Let $E_{t_n}, n \in \mathbf{N}$ be such that $a = P\Big(\bigcup_n E_{t_n}\Big)$. Let $E := \bigcup_n E_{t_n}$.
Notice now that for an arbitrary $t \in T$ $P(E \setminus E_t) = 0$ and so the inclusions $E_t \subseteq \rho(E_t) \subseteq \rho(E)$ hold true. Consequently,

$$E \subseteq \bigcup_{t\in T} E_t \subseteq \rho(E).$$

This proves the measurability of the set $\bigcup_{t\in T} E_t$ and the required inclusion.
The proof of the function part of the theorem is based on a similar idea. □

2 Product densities and liftings.

Let $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ be probability spaces.
The completion of P is denoted by $\widehat{P}$ and $\widehat{\mathfrak{A}}$ is the P-completion of $\mathfrak{A}$.
$\boxed{P \otimes Q}$ **– the direct product of P and Q on the product σ-algebra $\boxed{\mathfrak{A} \otimes \mathfrak{B}}$.**

$\boxed{P \widehat{\otimes} Q}$ **– the completion of $P \otimes Q$.**

$\boxed{\mathfrak{A} \widehat{\otimes} \mathfrak{B}}$ **– the completion of $\mathfrak{A} \otimes \mathfrak{B}$ with respect to $P \otimes Q$.**

$\boxed{\Lambda(P)}$ **– the set of all liftings on $(X, \mathfrak{A}, P)$.**

$\boxed{\vartheta(P)}$ **– the set of all densities on $(X, \mathfrak{A}, P)$.** The σ-algebra generated by a family $\boldsymbol{\mathcal{L}}$ of sets is denoted by $\boldsymbol{\sigma(\mathcal{L})}$. **N** and **R** stand for the natural numbers and the real numbers respectively.
If $\boldsymbol{M \subseteq \mathfrak{A}}$, then $\boldsymbol{M^c := X \setminus M}$.

Definition 5 We call $\pi \in \Lambda(P \widehat{\otimes} Q)$ a **product lifting** of the liftings $\rho \in \Lambda(P)$ for and $\sigma \in \Lambda(Q)$ (we write then $\pi \in \rho \otimes \sigma$) if the equation

$$\pi(A \times B) = \rho(A) \times \sigma(B)$$

holds true for all $A \in \mathfrak{A}$ and all $B \in \mathfrak{B}$. We use a similar definition for densities instead of liftings.

Assuming the continuum hypothesis Talagrand proved that there exists a lifting ξ on $([0,1], \mathfrak{L}, \lambda)$ such that no lifting η on the unit square satisfies the relation $\eta \in \xi \otimes \xi$.

Theorem 5 ***[MMS, Fund. Math. 166(2000)] Let*** $(X, \mathfrak{A}, P)$ ***and*** $(Y, \mathfrak{B}, Q)$ ***be complete. For each*** $\rho \in AG\Lambda(P)$ ***and each*** $\sigma \in \Lambda(Q)$***, there exists*** $\pi \in \Lambda(P \widehat{\otimes} Q)$ ***such that the following conditions are satisfied:***

(i) $\pi(A \times B) = \rho(A) \times \sigma(B)$
for all $A \in \mathfrak{A}$ ***and*** $B \in \mathfrak{B}$;

(ii) $[\pi(E)]^y = \rho\Big([\pi(E)]^y\Big)$
for all $y \in Y$ ***and*** $E \in \mathfrak{A} \widehat{\otimes} \mathfrak{B}$.

Proposition 3 Let $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ be probability spaces and densities $\rho \in \vartheta(P)$, $\sigma \in \vartheta(Q)$ and $\pi \in \vartheta(P \widehat{\otimes} Q)$ be such that for each $E \in \mathfrak{A} \widehat{\otimes} \mathfrak{B}$ and each (x, y)

$$[\pi(E)]_x = \sigma\Big(\,[\pi(E)]_x\Big) \text{ and } [\pi(E)]^y = \rho\left([\pi(E)]^y\right).$$

Then either P or Q is atomic.

Proposition 4 ***Let*** $(X, \mathfrak{A}, P)$ ***be complete non–atomic and let*** $(Y, \mathfrak{B}, Q)$ ***be complete, non–atomic and perfect . If*** ρ, σ ***and*** π ***are liftings satisfying***

(i) $\pi(A \times B) = \rho(A) \times \sigma(B)$
for all $A \in \mathfrak{A}$ ***and*** $B \in \mathfrak{B}$;

(ii) $[\pi(E)]^y = \rho\Big([\pi(E)]^y\Big)$
for all $y \in Y$ ***and*** $E \in \mathfrak{A}\widehat{\otimes}\mathfrak{B}$,

then for each $x \in X$ ***there exists*** $E \in \mathfrak{A}\widehat{\otimes}\mathfrak{B}$ ***such that*** $[\pi(E)]_x$ ***is non-measurable.***

3 Densities and liftings in product spaces.

Definition 6 Assume that $R : \mathfrak{A} \otimes \mathfrak{B} \to [0, 1]$ **(or** $R : \mathfrak{A}\widehat{\otimes}\mathfrak{B} \to [0, 1]$**) is a probability measure with marginals** P **and** Q**. A product regular conditional probability** $\{S_y : y \in Y\}$ **on** $\mathfrak{A}$ **with respect to** $\mathfrak{B}$ **is a collection of probability measures** $\{S_y : y \in Y\}$ **on** $\mathfrak{A}$ **such that**

(D1) $\forall\, A \in \mathfrak{A}\ \ y \to S_y(A)$ **is** $\mathfrak{B}$**-measurable;**

(D2) $R(A \times B) = \int_B S_y(A)\, dQ(y)$

for every $A \in \mathfrak{A}$ **and** $B \in \mathfrak{B}$**.**

Theorem 6 ***[MMS, JMAA 335(2007)] Let*** R ***be a probability measure defined on*** $\mathfrak{A} \otimes \mathfrak{B}$***. If*** $R \ll P \otimes Q$***, then there exists a product r.c.p.*** $\{S_y : y \in Y\}$ ***on*** $\mathfrak{A}$ ***with respect to*** $\mathfrak{B}$ ***which is absolutely continuous with respect to*** P ***and there exist*** $\xi \in \Lambda(\widehat{R})$ ***and a collection of liftings*** $\{\xi_y \in \Lambda(\widehat{S}_y) : y \in Y\}$ ***such that for every*** $\boldsymbol{E} \in \boldsymbol{\mathfrak{A}\widehat{\otimes}_R\mathfrak{B}}$

$$\boxed{[\boldsymbol{\xi}(\boldsymbol{E})]^{\boldsymbol{y}} = \boldsymbol{\xi_y}\big([\boldsymbol{\xi}(\boldsymbol{E})]^{\boldsymbol{y}}\big)} \quad \textit{for all} \quad y \in Y\,.$$

Proof. Let

$$\begin{aligned}\Phi :\ &= \ \{\overline{\varphi} \in \vartheta(\widehat{R}) : \forall y \in Y\, \forall E \in \mathfrak{A}\widehat{\otimes}_R\mathfrak{B} \\ &\qquad [\overline{\varphi}(E)]^y \subseteq \xi_y\Big([\overline{\varphi}(E)]^y\Big)\ \&\ \psi(E) \subseteq \overline{\varphi}(E)\,\}\end{aligned}$$

Notice first that $\Phi \neq \emptyset$ **(One constructs** $\psi \in \Phi$**).**
We consider Φ **with inclusion as the partial order:** $\overline{\varphi}_1 \leq \overline{\varphi}_2$ **if** $\overline{\varphi}_1(E) \subseteq \overline{\varphi}_2(E)$ **for each** $E \in \mathfrak{A}\widehat{\otimes}_R\mathfrak{B}$**.**
Φ **contains a maximal element** ξ **that satisfies the required section property.**
$\square$

Theorem 7 ***[MMS, Ann. Prob. 32(2004)]***
Assume that $\mathfrak{A}$ contains a countably generated σ-algebra which is dense in $\mathfrak{A}$ with respect to P. Then there exist $\sigma_y \in \Lambda(\widehat{S}_y)$ for all $y \in Y$ and $\pi \in \Lambda(\widehat{R})$ such that

$$[\pi(E)]^y = \sigma_y\Big([\pi(E)]^y\Big)$$

for all $y \in Y$ and $E \in \mathfrak{A}\widehat{\otimes}_R\mathfrak{B}$.

Proof. **According to Proposition ?? there exist $\tau_y \in \vartheta(\widehat{S}_y)$ for all $y \in Y$ and $\psi \in \vartheta(\widehat{R})$ such that for all $E \in \mathfrak{A}\widehat{\otimes}_R\mathfrak{B}$**

$$[\psi(E)]^y = \tau_y\Big([\psi(E)]^y\Big) \qquad \text{for all } y \in Y\ , \tag{2}$$

and

$$\widehat{S}_y\Big([\psi(E)]^y \cup [\psi(E^c)]^y\Big) = 1 \quad \text{for all} \quad y \in Y\ . \tag{3}$$

We take now for each $y \in Y$ a lifting $\sigma_y \in \Lambda(\widehat{S}_y)$ such that $\tau_y \subseteq \sigma_y$ and define $\pi \in \vartheta(\widehat{R})$ by setting for each $E \in \mathfrak{A}\widehat{\otimes}_R\mathfrak{B}$ and each $y \in Y$

$$[\pi(E)]^y = \sigma_y\Big([\psi(E)]^y\Big)\ . \tag{4}$$

Since $\psi(E) \subseteq \pi(E)$ for all $E \in \mathfrak{A}\widehat{\otimes}_R\mathfrak{B}$ we get $\widehat{R}$-measurability of $\pi(E)$ and $\pi(E) \stackrel{\widehat{R}}{=} E$.
In order to prove that π is a lifting it suffices to show that we have always $\pi(E^c) = [\pi(E)]^c$. But this is a consequence of (3) and (4) as we get for each y

$$\begin{aligned} [\pi(E^c)]^y &= \sigma_y\Big([\psi(E^c)]^y\Big) = \sigma_y\Big[\Big([\psi(E)]^y\Big)^c\Big] \\ &= \Big[\sigma_y\Big([\psi(E)]^y\Big)\Big]^c = \Big([\pi(E)]^y\Big)^c . \end{aligned}$$

This proves that $\pi \in \Lambda(\widehat{R})$. □

4 Category product liftings

Throughout X, Y are topological spaces such that $X \times Y$ is a Baire space.
$\boldsymbol{\mathcal{M}(X)}$ – the σ-ideal of meager subsets of X. $\boldsymbol{X}$ is called a Baire space if every non-empty open set in $\boldsymbol{X}$ is non-meager.
$\boldsymbol{A \subseteq_{\mathcal{M}} B}$ denotes that $\boldsymbol{A \setminus B \in \mathcal{M}(X)}$.
$\boldsymbol{A =_{\mathcal{M}} B}$ denotes that $\boldsymbol{(A \setminus B) \cup (B \setminus A) \in \mathcal{M}(X)}$. $\boldsymbol{B \subseteq X}$ has the Baire property if there exists an open set U such that $\boldsymbol{B =_{\mathcal{M}} U}$.
$\boldsymbol{\mathfrak{B}_c(X)}$ – the σ-algebra of subsets of X possessing the Baire property.
DEFINITION. A map $\boldsymbol{\rho : \mathfrak{B}_c(X) \to \mathfrak{B}_c(X)}$ is a category density if

(L1) $\rho(A) =_{\mathcal{M}} A$;

(L2) if $A =_{\mathcal{M}} B$ then $\rho(A) = \rho(B)$;

(L3) $\rho(\emptyset) = \emptyset$ and $\rho(X) = X$;

(L4) $\rho(A \cap B) = \rho(A) \cap \rho(B)$.

If ρ satisfies also

(L5) $\rho(A^c) = [\rho(A)]^c$

then, it is called a **category lifting**.

$\Lambda(\mathcal{M}(X))$ – all liftings on $(X, \mathfrak{B}_c(X), \mathcal{M}(X))$.

$\vartheta(\mathcal{M}(X))$ – all densities on $(X, \mathfrak{B}_c(X), \mathcal{M}(X))$.

Y another space and let σ be lifting or density on $(Y, \mathfrak{B}_c(Y), \mathcal{M}(Y))$.
Consider now the product space
$(X \times Y, \mathfrak{B}_c(X \times Y), \mathcal{M}(X \times Y))$. If π is a lifting or density on $(X \times Y, \mathfrak{B}_c(X \times Y), \mathcal{M}(X \times Y))$, then we say that
π is a product of ρ and σ if

$$\pi(A \times B) = \rho(A) \times \sigma(B)$$

$\forall A \in \mathfrak{B}_c(X)$ and $\forall B \in \mathfrak{B}_c(Y)$. We write then $\pi \in \rho \otimes \sigma$.

An open set $U \subseteq X$ is **regular open** if it coincides with the interior of its closure.

EXAMPLE of a category density.
For each $E \in \mathfrak{B}_c(X)$ we denote by $\varphi_X(E)$ the regular open set equivalent to E.
$\varphi_X : \mathfrak{B}_c(X) \to \mathfrak{B}_c(X)$ defined in that way is called the **canonical density** on X.
THE BANACH CATEGORY THEOREM: In any topological space X, if A is a set which is covered by open sets U such that every $U \cap A$ is meager, then A is meager.

Proposition 5 . *Let X be a Baire space and let $\delta \in \vartheta(\mathcal{M}(X))$ be arbitrary. Then for each collection $\mathcal{C} \subseteq \mathfrak{B}_c(X)$ such that $C \subseteq \delta(C)$ for each $C \in \mathcal{C}$, we have*

$$\bigcup \mathcal{C} \in \mathfrak{B}_c(X) \quad \text{and} \quad \bigcup \mathcal{C} \subseteq \delta\left(\bigcup \mathcal{C}\right).$$

Proof. Let U be the regular open set in $\bigvee\{C^\bullet : C \in \mathcal{C}\}$, where $C^\bullet$ denotes the equivalence class of C in $\mathfrak{B}_c(X)$ and $\bigvee$ is the sup operation in the algebra $\mathfrak{B}_c(X)/\mathcal{M}(X)$.
For any $C \in \mathcal{C}$, we have $C^\bullet \leq U^\bullet$ and hence $C \subseteq_{\mathcal{M}} U$. This gives $C \subseteq \delta(C) \subseteq \delta(U)$ and hence

$$\bigcup \mathcal{C} \subseteq \delta(U).$$

It remains to check that $\delta(U) \setminus \bigcup \mathcal{C}$ is meager, or equivalently, that $U \setminus \bigcup \mathcal{C}$ is meager.
Note that if U_C denotes the regular open set equivalent to C, then $\bigcup\{U_C : C \in \mathcal{C}\}$ is a dense open subset of U. Moreover,

$$\begin{aligned} & U_C \cap (U \setminus \bigcup \mathcal{C}) \\ \subseteq \; & U_C \cap (U \setminus C) =_{\mathcal{M}} U_C \cap (U \setminus U_C) = \emptyset. \end{aligned}$$

Hence $U \setminus \bigcup \mathcal{C}$ has a meager trace on each U_C and thus, by the Banach Category Theorem, it has a meager trace on $\bigcup\{U_C : C \in \mathcal{C}\}$ and hence is meager. □

Proposition 6 . *Given* $\rho \in \vartheta(\mathcal{M}(X))$ *and* $\sigma \in \vartheta(\mathcal{M}(Y))$, *we set*

$$\boxed{\xi(E) := \bigcup\{\rho(A) \times \sigma(B) : A \times B \subseteq_{\mathcal{M}} E\}}$$

for every $E \in \mathfrak{B}_c(X \times Y)$.
Then $\xi \in \vartheta(\mathcal{M}(X \times Y))$ *and it satisfies the following conditions:*

(j) $\xi \in \rho \otimes \sigma$;

(jj) $[\xi(E)]_x \in \mathfrak{B}_c(Y)$ *and* $[\xi(E)]_x \subseteq \sigma([\xi(E)]_x)$ *for every* $E \in \mathfrak{B}_c(X \times Y)$ *and* $x \in X$;

(jjj) $[\xi(E)]^y \in \mathfrak{B}_c(X)$ *and* $[\xi(E)]^y \subseteq \rho([\xi(E)]^y)$ *for every* $E \in \mathfrak{B}_c(X \times Y)$ *and* $y \in Y$.

If $\xi_1 : \mathfrak{B}_c(X \times Y) \to \mathcal{P}(X \times Y)$ *is defined by*

$$\boxed{[\xi_1(E)]_x = \sigma([\xi(E)]_x)}$$

then, $\xi_1 \in \vartheta(\mathcal{M}(X \times Y))$ *and* $\xi_1 \in \rho \otimes \sigma$.

Theorem 8 . *[BMMS, Top. Appl. 153(2006)] Assume that* $X \times Y$ *is a Baire space. Then for arbitrary* $\rho \in \Lambda(\mathcal{M}(X))$ *and* $\sigma \in \Lambda(\mathcal{M}(Y))$, *there exists* $\pi_1 \in \Lambda(\mathcal{M}(X \times Y))$ *such that*

(α) $\boxed{\pi_1 \in \rho \otimes \sigma}$;

(β) $\boxed{[\boldsymbol{\pi_1}(\boldsymbol{E})]_x = \boldsymbol{\sigma}\Big([\boldsymbol{\pi_1}(\boldsymbol{E})]_x\Big)}$

for all $\boldsymbol{x} \in \boldsymbol{X}$ *and all* $\boldsymbol{E} \in \mathfrak{B}_c(\boldsymbol{X} \times \boldsymbol{Y})$ *.*

Proof. Let ξ be taken from Proposition 6 and let $\boldsymbol{\xi_1}$ be defined by $[\boldsymbol{\xi_1}(\boldsymbol{E})]_x = \boldsymbol{\sigma}([\boldsymbol{\xi}(\boldsymbol{E})]_x)$. Then, let

$$\begin{aligned}\boldsymbol{\Phi} := \{\varphi \in \vartheta\big(\boldsymbol{\mathcal{M}}(\boldsymbol{X} \times \boldsymbol{Y})\big) :& \\ \forall\, \boldsymbol{E} \in \mathfrak{B}_c(\boldsymbol{X} \times \boldsymbol{Y})\ \boldsymbol{\xi_1}(\boldsymbol{E}) \subseteq \varphi(\boldsymbol{E})& \\ \&\ \forall\, \boldsymbol{x} \in \boldsymbol{X}\, \forall \boldsymbol{E} \in \mathfrak{B}_c(\boldsymbol{X} \times \boldsymbol{Y})& \\ [\varphi(\boldsymbol{E})]_x \subseteq \boldsymbol{\sigma}\big([\varphi(\boldsymbol{E})]_x\big)& \\ \&\ \forall\, \boldsymbol{C} \in \mathfrak{B}_c(\boldsymbol{X})\, \forall \boldsymbol{x} \in \boldsymbol{X}& \\ [\varphi(\boldsymbol{C} \times \boldsymbol{Y})]_x \in \{\emptyset, \boldsymbol{Y}\}\}.&\end{aligned}$$

We order $\boldsymbol{\Phi}$ by inclusion and take a maximal element. □

Corollary 1 . *Let Y be a separable metric space without isolated points. If ρ, σ and π_1 are liftings satisfying Theorem 8 (with $X = P(\mathbb{N})$), then for each $y \in Y$*
there exists

$$\boldsymbol{E} \in \mathfrak{B}_c(\boldsymbol{P}(\mathbb{N}) \times \boldsymbol{Y})$$

such that

$$[\boldsymbol{\pi_1}(\boldsymbol{E})]^y \notin \mathfrak{B}_c(\boldsymbol{P}(\mathbb{N})).$$

□

It follows from the above corollary, that Theorem 8 cannot be in general improved.
Thus, in general if $\rho \in \Lambda\big(\mathcal{M}(X)\big)$ and $\sigma \in \Lambda\big(\mathcal{M}(Y)\big)$ are arbitrary, then there is no
$\pi \in \Lambda\big(\mathcal{M}(X \times Y)\big)$ such that

(α) $\boldsymbol{\pi} \in \boldsymbol{\rho} \otimes \boldsymbol{\sigma}$;

(β) $[\boldsymbol{\pi}(\boldsymbol{E})]_x = \boldsymbol{\sigma}\Big([\boldsymbol{\pi}(\boldsymbol{E})]_x\Big)$
for all $\boldsymbol{x} \in \boldsymbol{X}$ and all $\boldsymbol{E} \in \mathfrak{B}_c(\boldsymbol{X} \times \boldsymbol{Y})$;

(δ) $[\boldsymbol{\pi}(\boldsymbol{E})]^y = \boldsymbol{\rho}\Big([\boldsymbol{\pi}(\boldsymbol{E})]^y\Big)$

for all $\boldsymbol{y} \in \boldsymbol{Y}$ and all $\boldsymbol{E} \in \mathfrak{B}_c(\boldsymbol{X} \times \boldsymbol{Y})$.

References

[1] . M. Burke, N.D.Macheras, K. Musiał and W.Strauss, - Category product densities and liftings, Topology and its Applications 153(2006), 1164-1191.

[2] . M. Burke, N.D.Macheras, K. Musiał and W.Strauss, - Various products of category densities and liftings, Topology and its Appl. 156(2009), 1253-1270.

[3] Ionescu Tulcea, A. and C. (1969) - *Topics in the theory of lifting* (Springer-Verlag,Berlin-Heidelberg-New York).

[4] Macheras, N. D., Musiał, K. and Strauss, W. - On products of admissible liftings, J. for Analysis and Appl. 18(1999), 651-667.

[5] K. Musiał, W.Strauss and N.D.Macheras, Product liftings and densities with lifting invariant and density invariant sections, Fundamenta Math. 166 (2000), 281-303.

[6] W.Strauss, N.D.Macheras and K. Musiał. Liftings, Handbook of Measure Theory II, chapter 28, 1131-1184, North Holland. 2002.

[7] Talagrand, M. (1988) - On liftings and regularization of stochastic processes, *Probability Theory and Related Fields* 78, 127-134.