INTEGRATION OF MULTIFUNCTIONS

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1 Preliminaries.

Throughout (Ω, Σ, μ) is a complete probability space, Σ_μ^+ is the collection of all sets of positive measure and X is a Banach space with its dual X^*. The closed unit ball of X is denoted by $B(X)$. $cwk(X)$ denotes the family of all nonempty convex weakly compact subsets of X and $ck(X)$ is the collection of all nonempty convex and compact subsets of X. $cb(X)$ is the collection of all nonempty closed bounded and convex subsets of X and $c(X)$ denotes the collection of all nonempty closed convex subsets of X. For every $C \in c(X)$ the *support function of* C is denoted by $s(\cdot, C)$ and defined on X^* by

$$s(x^*, C) = \sup\{\langle x^*, x\rangle : \ x \in C\} \qquad \text{for each } x^* \in X^* .$$

$\tau(X^*, X)$ denotes the topology of uniform convergence on elements of $cwk(X)$ and the weak*-topology of X^* will be denoted by $\sigma(X^*, X)$. If Y is a subspace of X, then $Y^\perp := \{x^* \in X^* : \forall\, y \in Y\ x^*(y) = 0\}$ is the annihilator of Y in X^*.
A map $\Gamma\colon \Omega \to c(X)$ is called a *multifunction*.
A function $f\colon \Omega \to X$ is called a *selection of* Γ if $f(\omega) \in \Gamma(\omega)$, for every $\omega \in \Omega$. If $A \subset X$ is nonempty, then we write $|A|\colon = \sup\{\|x\|\colon x \in A\}$.
A map $M\colon \Sigma \to c(X)$ is called a *multimeasure* if $s(x^*, M(\cdot))$ is a measure, for every $x^* \in X^*$. If M is a point map, then we talk about measure.

2 Aumann integral

Definition 2.1 Let X be a separable Banach space. A multifunction $\Gamma : \Omega \to c(X)$ is said to be measurable if for every open $U \subset X$, we have

$$\{w \in \Omega : \Gamma(\omega) \cap U \neq \emptyset\} \in \Sigma.$$

A measurable $\Gamma; \Omega \to c(X)$ is said to be Aumann integrable, if $\mathcal{S}_\Gamma$, the set of all Bochner integrable selections of Γ is nonempty.

The Aumann integral (Aumann (1965)) of Γ over $E \in \Sigma$ is defined by

$$(A) \int_E \Gamma \, d\mu := \left\{ \int_E f \, d\mu : f \in \mathcal{S}_\Gamma \right\}.$$

Example 2.2 Let X be a separable Banach space. Let $f\colon \Omega \to X$ be a Bochner integrable function and let $r\colon \Omega \to (0, \infty)$ be an integrable function.
Define $\Gamma\colon \Omega \to cb(X)$ by

$$\Gamma(\omega) \colon = B(f(\omega), r(\omega)) \, ,$$

where $B(x, r)$ is the closed ball with its center in x and of radius r. One can easily check that Γ is measurable.
Then, Γ is Aumann integrable in $cb(X)$ and

$$\forall \; E \in \Sigma \;\; (A) \int_E \Gamma \, d\mu = B \left(\int_E f \, d\mu, \int_E r \, d\mu \right).$$

3 Pettis integral

Definition 3.1 A multifunction Γ is said to be *scalarly measurable*, if for every $x^* \in X^*$, the map $s(x^*, \Gamma(\cdot))$ is measurable. A multifunction $\Gamma\colon \Omega \to c(X)$ is *scalarly integrable* if $s(x^*, \Gamma)$ is integrable for every $x^* \in X^*$.
We associate with each scalarly integrable $\Gamma\colon \Omega \to c(X)$ a sublinear operator $\boldsymbol{T_\Gamma : X^* \to L_1(\mu)}$, defined by $T_\Gamma(x^*) := s(x^*, \Gamma)$.

Definition 3.2 A scalarly integrable multifunction $\Gamma\colon \Omega \to c(X)$ is *Pettis integrable* in $c(X)$ [$cb(X)$, $ck(X)$, $cwk(X)$] if for each $A \in \Sigma$ there exists a set $M_\Gamma(A) \in c(X)$ [$cb(X)$, $ck(X)$, $cwk(X)$, respectively] such that

$$s(x^*, M_\Gamma(A)) = \int_A s(x^*, \Gamma) \, d\mu \tag{1}$$

for every $x^* \in X^*$. We set $(P) \int_A \Gamma \, d\mu := M_\Gamma(A)$ and call $M_\Gamma(A)$ the *Pettis integral of* Γ *over* A. It follows from (2) that M_Γ is a μ-continuous multimeasure. □

If Γ is an X-valued function, then we have a Pettis integrable function.

Definition 3.3 Let V be a topological space and $s : V \to \mathbb{R}$ be a function. s is said to be *lower semicontinuous*, if for each $\alpha \in \mathbb{R}$ the set $\{v \in V : s(v) \leq \alpha\}$ is closed in V.

Proposition 3.4 *Let $\Gamma\colon \Omega \to c(X)$ be scalarly integrable. Then Γ is Pettis-integrable in $cb(X)$ if and only if the functional $x^* \longrightarrow \int_E s(x^*, \Gamma) \, d\mu$ is weak* lower semicontinuous for every $E \in \Sigma$.*

Theorem 3.5 *Let $\Gamma\colon \Omega \to c(X)$ be scalarly integrable. Then Γ is Pettis-integrable in $cwk(X)$ if and only if T_Γ is $\tau(X^*, X)$-weakly continuous.*

Theorem 3.6 *Let $\Gamma\colon \Omega \to c(X)$ be scalarly integrable. Then Γ is Pettis-integrable in $ck(X)$ if and only if T_Γ is $\sigma(X^*, X)$-weakly continuous on $B(X^*)$.*

Definition 3.7 We say that a space $Y \subset X$ determines a multifunction $\Gamma\colon \Omega \to c(X)$ if $s(x^*, \Gamma) = 0$ μ-a.e. for each $x^* \in Y^\perp$ (the exceptional sets depend on x^*). □

Theorem 3.8 *A scalarly integrable multifunction $\Gamma\colon \Omega \to cwk(X)$ is Pettis integrable in $cwk(X)$ if and only if it satisfies the following conditions*

(WC) $T_\Gamma\colon X^* \to L_1(\mu)$ is weakly compact;

(D) Γ is determined by a WCG space $Y \subseteq X$.

Theorem 3.9 *Let X be a Banach space not containing any isomorphic copy of c_0. If $\Gamma\colon \Omega \to c(X)$ is scalarly integrable and determined by a WCG space, then Γ is Pettis integrable in $cb(X)$. If $\Gamma\colon \Omega \to cwk(X)$ is scalarly integrable and determined by a WCG space, then Γ is Pettis integrable in $cwk(X)$.*

Theorem 3.10 *Let $\Gamma\colon \Omega \to cwk(X)$ be a scalarly integrable multifunction with weakly compact T_Γ. If each scalarly measurable selection of Γ is Pettis integrable, then Γ is Pettis integrable in $cwk(X)$.*

4 Henstock-Kurzweil-Pettis integral

Definition 4.1 A multifunction $\Gamma\colon [0,1] \to c(X)$ is *scalarly HK-integrable* if $s(x^*, \Gamma)$ is HK-integrable for every $x^* \in X^*$.
A scalarly HK-integrable multifunction $\Gamma\colon [0,1] \to c(X)$ is *Henstock-Kurzweil-Pettis integrable* in
$c(X)$ [$cb(X)$, $ck(X)$, $cwk(X)$], if for each interval $I \subset [0,1]$ there exists a set $M_\Gamma(I) \in c(X)$ [$cb(X)$, $ck(X)$, $cwk(X)$, respectively] such that

$$s(x^*, M_\Gamma(I)) = (HK) \int_I s(x^*, \Gamma)\, d\mu \tag{2}$$

for every $x^* \in X^*$.
If Γ is a function, then we have a HKP-integrable function.
We set $(HKP) \int_I \Gamma\, d\mu := M_\Gamma(I)$ and call $M_\Gamma(I)$ the *Henstock-Kurzweil-Pettis integral of Γ* over I. □

Let

$$(AHKP) \int_J \Gamma(t)\; dt := \overline{\left\{ (HKP) \int_J f(t) dt : f \in \mathcal{S}_{HKP}(\Gamma) \right\}}.$$

Theorem 4.2 *(*Di Piazza, Musiał(2009)) *Let $\Gamma : [0,1] \to cwk(X)$ be a scalarly measurable multifunction. Then the following conditions are equivalent:*

(i) *Γ is HKP-integrable in $cwk(X)$;*

(ii) *$\mathcal{S}_{HKP}(\Gamma) \neq \emptyset$ and for every $f \in \mathcal{S}_{HKP}(\Gamma)$ the multifunction $G : [0,1] \to cwk(X)$ defined by $\Gamma(t) = G(t) + f(t)$, is Pettis integrable in $cwk(X)$;*

(iii) *there exists $f \in \mathcal{S}_{HKP}(\Gamma)$ such that the multifunction $G : [0,1] \to cwk(X)$ defined by $\Gamma(t) = G(t) + f(t)$ is Pettis integrable in $cwk(X)$;*

(iv) *for each interval $I \in \mathcal{I}$, the set $(AKHP)\int_I \Gamma(t)\,dt$ belongs to $cwk(X)$ and*

$$s\left(x^*, (AHKP)\int_I \Gamma(t)\,dt\right) = (HK)\int_I s(x^*, \Gamma(t))\,dt$$

for all $x^* \in X^*$;

(v) *each scalarly measurable selector of Γ is HKP-integrable.*

Literatura

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