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THE COMPLETENESS IN SPACES OF BOUNDED PETTIS INTEGRABLE FUNCTIONS AND IN SPACES OF BOUNDED FUNCTIONS SATISFYING THE LAW OF LARGE NUMBERS

Dedicated to Prof. Kazimierz Urbanik
on the occasion of his 70^{th} birthday

Abstract. It has been proven already by Pettis [5] that the space $P(\mu, X)$ of Pettis integrable functions may be non-complete when endowed with the semivariation norm of the integrals. Then Thomas [9] proved that the space is almost always non-complete. In view of the Open Mapping Theorem in such a case no complete equivalent norm can be defined on $P(\mu, X)$. The question is now whether there are interesting linear subsets of $P(\mu, X)$ where a complete norm does exist. In this paper we consider two such subspaces: the space $P_\infty(\mu, X)$ of scalarly bounded Pettis integrable functions and the space $LLN_\infty(\mu, X)$ of scalarly bounded functions satisfying the strong law of large numbers. We prove that in several cases these spaces are complete.

Introduction

Throughout the paper (Ω, Σ, μ) stands for a complete probability space, ρ is a fixed lifting on $L_\infty(\mu)$, X is a Banach space and $B(X)$ is the closed unit ball in X. Given X we set

$$\widetilde{B} := \{x^{**} \in B(X^{**}) : \\ x^{**} \text{ is a weak}^*\text{-cluster point of a countable subset of } B(X)\}.$$

λ is the Lebesgue measure on the unit interval $[0,1]$ and $\mathcal{L}$ denotes the corresponding σ-algebra of Lebesgue measurable sets.

We say that *Axiom L* (cf. [1]) is satisfied if $[0,1]$ *cannot be covered by less then the continuum closed sets of the Lebesgue measure zero.* It is known (cf. [1]) that Axiom L is a consequence of Martin's Axiom.

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A function $f : \Omega \to X$ is said to be Pettis integrable with respect to μ if it is weakly measurable and for each $E \in \Sigma$ there exists $\nu_f(E) \in X$ satisfying for each functional $x^* \in X^*$ the equality $x^*\nu_f(E) = \int_E x^* f \, d\mu$. It is known (cf. [3]) that the measure $\nu_f : \Sigma \to X$ is of $\sigma-$finite variation. Identifying weakly equivalent Pettis integrable functions we get a linear space which we denote by $P(\mu, X)$. It is well known that the space can be normed by setting

$$\|f\|_{P_1} := \sup_{\|x^*\| \le 1} \int_\Omega |x^* f| \, d\mu \, .$$

We denote by $P_\infty(\mu, X)$ the linear space

$$\{f \in P(\mu, X) : \|f\|_{P_\infty} := \sup_{\|x^*\| \le 1} \|x^* f\|_\infty < \infty\} ,$$

where $\|x^* f\|_\infty$ is the $L_\infty(\mu)$-norm of $x^* f$. One can easily check that $\|\cdot\|_{P_\infty}$ is a norm. Then, let $P^c_\infty(\mu, X) := \{f \in P_\infty(\mu, X) : \nu_f(\Sigma)$ is norm relatively compact$\}$.

If $f : \Omega \to X^*$ is a weak*-measurable and weak*-bounded function (i.e. there exists $M > 0$ such that for each $x \in X$ the inequality $|xf| \le M\|x\|$ holds μ-a.e.), and $\rho : L_\infty(\mu) \to \mathcal{L}_\infty(\mu)$ is a lifting, then $\rho_0(f) : \Omega \to X^*$ is the unique function (see [2]) satisfying for each $x \in X$ the equality

$$\langle x, \rho_0(f) \rangle = \rho(\langle x, f \rangle) \, .$$

It is a consequence of Theorem III.3.3 from [2], that

$$\|\rho_0(f)\| := \sup\{|\rho(\langle x, f \rangle)| : \|x\| \le 1\}$$

is a measurable function.

Following [8] we are going to introduce now the space $LLN(\mu, X)$ of $X-$valued functions satisfying the law of large numbers. It is defined in the following way:

$$LLN(\mu, X) = \Big\{ f : \Omega \to X : \\ \exists a_f \in X \lim_{n\to\infty} \Big\| a_f - \frac{1}{n} \sum_{i=1}^n f(\omega_i) \Big\| = 0 \quad \text{for} \quad \mu^\infty\text{-a.e. } (\omega_i) \in \Omega^\infty \Big\}$$

where μ^∞ is the countable direct product of μ on Ω^∞ – the countable product of Ω.

The space $LLN(\mu, X)$ will be considered with the Glivenko-Cantelli seminorm, defined in [8] for an arbitrary function $f : \Omega \to X$ by the formula

$$\|f\|_{GC} = \limsup_n \int^* g_n \, d\mu^\infty \, ,$$

where

$$g_n(\omega) = \sup_{\|x^*\| \le 1} \frac{1}{n} \sum_{i \le n} |x^*(f(\omega_i))| \,.$$

According to [8], the GC-seminorm and the Pettis seminorm are equivalent on $LLN(\mu, X)$. In particular functions in $LLN(\mu, X)$ that are weakly equivalent are not distinguishable by the GC-norm. This permits us to identify weakly equivalent elements of $LLN(\mu, X)$ and investigate the quotient space.

Identifying weakly equivalent functions—we denote by $LLN_\infty(\mu, X)$ the linear space

$$\{f \in LLN(\mu, X) : \|f\|_{P_\infty} := \sup_{\|x^*\| \le 1} \|x^* f\|_\infty < \infty\} \,.$$

1. Completeness of $P_\infty(\mu, X)$

In many cases $P_\infty(\mu, X) = L_\infty(\mu, X)$ (in the sense of isomorphic isometry). It is so in the case of measure compact spaces (so in particular for separable or weakly compactly generated X) and for X possessing RNP. In general however the above equality is false. In spite of this the space $P_\infty(\mu, X)$ is often complete.

PROPOSITION 1. *If X has the WRNP, then $P_\infty(\mu, X)$ is complete.*

Proof. Let $\langle f_n \rangle_{n \in \mathbf{N}} \subset P_\infty(\mu, X)$ be a Cauchy sequence. It is clear that for each $E \in \Sigma$ the sequence $\langle \nu_{f_n}(E) \rangle_{n \in \mathbf{N}}$ is Cauchy in X and so it is convergent to an X−valued measure ν. A simple calculation shows that there is $M > 0$ such that $\|\nu(E)\| \le M\mu(E)$ for each $E \in \Sigma$. According to the assumptions there is a scalarly bounded Pettis integrable density f of ν with respect to μ. Since $\langle f_n \rangle_{n \in \mathbf{N}}$ is Cauchy in $P_\infty(\mu, X)$ it is also Cauchy in $P(\mu, X)$. It follows that for each $x^* \in X^*$

$$\lim_{n \to \infty} \int_\Omega |x^* f_n - x^* f| \, d\mu = 0 \,.$$

Consequently, for each $x^* \in X^*$ the sequence $\langle x^* f_n \rangle$ is convergent in measure to $x^* f$. Since at the same time the sequence $\langle x^* f_n \rangle$ is Cauchy in $L_\infty(\mu)$, it is convergent to $x^* f$ in $L_\infty(\mu)$. Together with the Cauchy condition in $P_\infty(\mu, X)$, this yields the convergence

$$\lim_{n \to \infty} \|f_n - f\|_{P_\infty} = 0 \,. \blacksquare$$

The WRNP however is not necessary for the completeness of $P_\infty(\mu, X)$. Assuming Axiom L we can get some results without the assumption of the WRNP. We are going to begin with the following two simple facts.

LEMMA 2. *If $f \in P(\mu, X^*)$ then for each $x^{**} \in B(X^{**})$ there exists $x_0^{**} \in \widetilde{B}$ such that $x^{**}f = x_0^{**}f$ μ–a.e.*

Proof. Take x^{**} with $\|x^{**}\| = 1$ and let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net of functionals from the unit ball of X weak*–converging to x^{**}. Since $f \in P(\mu, X^*)$ the operator $T : X^{**} \longrightarrow L_1(\mu)$ given by $T(x^{**}) = x^{**}f$ is weak*–weakly continuous (cf [3]) and so $x_\alpha f \longrightarrow x^{**}f$ weakly in $L_1(\mu)$. By a theorem of Mazur, one can find $y_n \in conv\{x_\alpha : \alpha \in A\}$ such that $\lim y_n f = x^{**}f$ μ – a.e.. Now, if x_0^{**} is an arbitrary weak*–cluster point of $\{y_n : n \in \mathbf{N}\}$, then it satisfies the required equality. ∎

LEMMA 3. *If $f \in P_\infty(\mu, X^*)$, then*

$$\sup_{\|x^{**}\| \leq 1} \|x^{**}f\|_\infty = \sup_{\|x\| \leq 1} \|xf\|_\infty .$$

Proof. Let $a := \sup_{\|x\| \leq 1} \|xf\|_\infty$ and $b := \sup_{\|x^{**}\| \leq 1} \|x^{**}f\|_\infty$. According to Lemma 2 for each $x^{**} \in B(X^{**})$ there exists $\tilde{x} \in \widetilde{B}$ satisfying the equality $x^{**}f = \tilde{x}f$ μ–a.e.. Consequently,

$$b = \sup\{\|\tilde{x}f\|_\infty : \tilde{x} \in \widetilde{B}\} .$$

Given $\tilde{x} \in \widetilde{B}$ with $\|\tilde{x}\| = 1$, let $\langle x_\alpha \rangle_{\alpha \in A}$ be a countable net from the unit ball of X that is weak*– convergent to $\tilde{x}$. By the assumption $|x_\alpha f| \leq a$ μ–a.e. for each α. Since the net consists of countably many different elements, we have also $|\tilde{x}f| \leq a$ μ–a.e. Thus $\|\tilde{x}f\|_\infty \leq a$ for each $\tilde{x}$ and so $b \leq a$. This completes the proof. ∎

THEOREM 4. *(Axiom L) If μ is perfect then $P_\infty(\mu, X^*)$ is complete.*

Proof. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $P_\infty(\mu, X^*)$. Applying Lemma 3 we see that for each $m, n \in \mathbb{N}$

$$\sup_{\|x^{**}\| \leq 1} \|x^{**}f_n - x^{**}f_m\|_\infty = \sup_{\|x\| \leq 1} \|xf_n - xf_m\|_\infty .$$

Now,

$$\begin{aligned} \sup_{\|x\| \leq 1} \|xf_n - xf_m\|_\infty &= \sup_{\|x\| \leq 1} \|x\rho_0(f_n) - x\rho_0(f_m)\|_\infty \\ &= \sup_{\|x\| \leq 1} \sup_{\omega} |x\rho_0(f_n)(\omega) - x\rho_0(f_m)(\omega)| \\ &= \sup_{\omega} \|\rho_0(f_n)(\omega) - \rho_0(f_m)(\omega)\| . \end{aligned} \tag{1}$$

Consequently, the sequence $(\rho_0(f_n))$ is uniformly convergent to a function $h : \Omega \longrightarrow X^*$ such that $h = \rho_0(h)$. Since for each $x^{**} \in \widetilde{B}$ the functions $\rho_0(f_n)$ are measurable, according to [7], Theorem 6-2-1 (where the Axiom L is used), the functions $\rho_0(f_n)$ are in $P_\infty(\mu, X^*)$ and so h is weakly measurable. Then, it is a consequence of the Lebesgue Convergence Theorem (see [3]) that $h \in P_\infty(\mu, X^*)$.

Thus, using (1), with h rather then f_m, we get

$$\lim_n \|f_n - h\|_{P_\infty} = \lim_n \sup_{\|x\|\le 1} \|xf_n - xh\|_\infty$$
$$= \lim_n \sup_\omega \|\rho_0(f_n)(\omega) - \rho_0(h)(\omega)\| = 0\,.$$

This proves the completeness of $P_\infty(\mu, X^*)$. ■

Notice that according to a result of Stegall [1], if μ is perfect then $P_\infty(\mu, X^*) = P^c_\infty(\mu, X^*)$.

The above proof makes it obvious that in fact the following more general result holds true:

THEOREM 5. *Let μ and X be arbitrary. If for each countable family $\mathcal{F} \subset P_\infty(\mu, X^*)$ there exists a lifting ρ such that $\rho_0(f)$ is μ−Pettis-integrable for each $f \in \mathcal{F}$, then $P_\infty(\mu, X^*)$ and $P^c_\infty(\mu, X^*)$ are complete.*

The question of whether a lifting of $f \in P_\infty(\mu, X^*)$ is Pettis integrable was implicitely posed in [7]. Rybakov [6] undertook an attempt to solve the problem, but his approach turned out to be wrong (see Math. Reviews 98h #20007).

COROLLARY 6. *If X is separable, then for each μ the spaces $P^c_\infty(\mu, X^*)$ and $P_\infty(\mu, X^*)$ are complete.*

2. Completeness of $LLN_\infty(\mu, X)$

It has been proven in [4] that if X is infinite dimensional and μ is not purely atomic, then $LLN(\mu, X)$ is non-complete. In the case of $LLN_\infty(\mu, X^*)$ the completeness problem is solved affirmatively.

THEOREM 7. *The space $LLN_\infty(\mu, X^*)$ is complete.*

Proof. Let ρ be a consistent lifting on $L_\infty(\mu)$ and let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $LLN_\infty(\mu, X^*)$. As in the proof of Theorem 4 we get the equality

$$\sup_{\|x^{**}\|\le 1} \|x^{**}f_n - x^{**}f_m\|_\infty = \sup_\omega \|\rho_0(f_n)(\omega) - \rho_0(f_m)(\omega)\|\,.$$

It follows that the sequence $\langle\rho_0(f_n)\rangle$ is uniformly Cauchy in the norm topology of X^*. Let $h : \Omega \longrightarrow X^*$ be the pointwise limit of the sequence $\langle\rho_0(f_n)\rangle$. The uniform convergence yields the equality $h = \rho_0(h)$. Moreover, since each f_n is properly measurable and ρ is consistent, the function $\rho_0(f_n)$ is also properly measurable. Clearly it is also pointwise bounded by the function $\|\rho_0(f_n)\| \in L_\infty(\mu)$. Consequently, it follows from [8], Theorem 26, that $\rho_0(f_n) \in LLN_\infty(\mu, X^*)$. The uniform convergence of the sequence $\langle\rho_0(f_n)\rangle$ yields $h \in LLN(\mu, X^*)$ and the convergence of $\langle\rho_0(f_n)\rangle$ to h in $LLN_\infty(\mu, X^*)$.

This proves the completeness of $LLN_\infty(\mu, X^*)$. ■

Considering each X- valued function as an X^{**}- valued function we get the following result in case of an arbitrary Banach space X:

THEOREM 8. *The completion of the space $LLN_\infty(\mu, X)$ is a subspace of $LLN_\infty(\mu, X^{**})$. If Axiom L is satisfied and μ is perfect then the completion of $P_\infty(\mu, X)$ is a subspace of $P_\infty(\mu, X^{**})$.*

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