THE COMPLETENESS IN SPACES 
OF BOUNDED PETTIS INTEGRABLE FUNCTIONS 
AND IN SPACES OF BOUNDED FUNCTIONS 
SATISFYING THE LAW OF LARGE NUMBERS

Abstract. It has been proven already by Pettis [5] that the space $P(\mu, X)$ of Pettis
integrable functions may be non-complete when endowed with the semivariation norm
of the integrals. Then Thomas [9] proved that the space is almost always non-complete.
In view of the Open Mapping Theorem in such a case no complete equivalent norm
can be defined on $P(\mu, X)$. The question is now whether there are interesting linear
subsets of $P(\mu, X)$ where a complete norm does exist. In this paper we consider two
such subspaces: the space $P_{\infty}(\mu, X)$ of scalarly bounded Pettis integrable functions and
the space $LLN_{\infty}(\mu, X)$ of scalarly bounded functions satisfying the strong law of large
numbers. We prove that in several cases these spaces are complete.

Introduction
Throughout the paper $(\Omega, \Sigma, \mu)$ stands for a complete probability space, $\rho$
is a fixed lifting on $L_{\infty}(\mu)$, $X$ is a Banach space and $B(X)$ is the closed
unit ball in $X$. Given $X$ we set
$$\tilde{B} := \{x^{**} \in B(X^{**}) :$$
$$x^{**} \text{ is a weak}\star\text{-cluster point of a countable subset of } B(X)\}.$$ 

$\lambda$ is the Lebesgue measure on the unit interval $[0,1]$ and $\mathcal{L}$ denotes the
 corresponding $\sigma$-algebra of Lebesgue measurable sets.

We say that Axiom L (cf. [1]) is satisfied if $[0,1]$ cannot be covered by less
then the continuum closed sets of the Lebesgue measure zero. It is known (cf.
[1]) that Axiom L is a consequence of Martin's Axiom.

1991 Mathematics Subject Classification: Primary:46G10; Secondary:28B05, 28A51
Key words and phrases: Pettis integral, law of large numbers, lifting.
A function \( f : \Omega \to X \) is said to be Pettis integrable with respect to \( \mu \) if it is weakly measurable and for each \( E \in \Sigma \) there exists \( \nu_f(E) \in X \) satisfying for each functional \( x^* \in X^* \) the equality \( x^* \nu_f(E) = \int_E x^* f \, d\mu \). It is known (cf. [3]) that the measure \( \nu_f : \Sigma \to X \) is of \( \sigma \)-finite variation. Identifying weakly equivalent Pettis integrable functions we get a linear space which we denote by \( P(\mu, X) \). It is well known that the space can be normed by setting

\[
\|f\|_{P_1} := \sup_{\|x^*\| \leq 1} \int_{\Omega} |x^* f| \, d\mu.
\]

We denote by \( P_\infty(\mu, X) \) the linear space

\[
\{f \in P(\mu, X) : \|f\|_{P_\infty} := \sup_{\|x^*\| \leq 1} \|x^* f\|_\infty < \infty\},
\]

where \( \|x^* f\|_\infty \) is the \( L_\infty(\mu) \)-norm of \( x^* f \). One can easily check that \( \| \cdot \|_{P_\infty} \) is a norm. Then, let \( P_\infty(\mu, X) := \{f \in P_\infty(\mu, X) : \nu_f(\Sigma) \text{ is norm relatively compact}\} \).

If \( f : \Omega \to X^* \) is a weak*-measurable and weak*-bounded function (i.e. there exists \( M > 0 \) such that for each \( x \in X \) the inequality \( |xf| \leq M \|x\| \) holds \( \mu \)-a.e.), and \( \rho : L_\infty(\mu) \to L_\infty(\mu) \) is a lifting, then \( \rho_0(f) : \Omega \to X^* \) is the unique function (see [2]) satisfying for each \( x \in X \) the equality

\[
(x, \rho_0(f)) = \rho((x, f)).
\]

It is a consequence of Theorem III.3.3 from [2], that

\[
\|\rho_0(f)\| := \sup\{\|\rho((x, f))\| : \|x\| \leq 1\}
\]

is a measurable function.

Following [8] we are going to introduce now the space \( LLN(\mu, X) \) of \( X \)-valued functions satisfying the law of large numbers. It is defined in the following way:

\[
LLN(\mu, X) = \left\{ f : \Omega \to X : \exists a_f \in X \lim_{n \to \infty} \left\| a_f - \frac{1}{n} \sum_{i=1}^{n} f(\omega_i) \right\| = 0 \text{ for } \mu^\infty \text{-a.e. } (\omega_i) \in \Omega^\infty \right\}
\]

where \( \mu^\infty \) is the countable direct product of \( \mu \) on \( \Omega^\infty \) — the countable product of \( \Omega \).

The space \( LLN(\mu, X) \) will be considered with the Glivenko-Cantelli seminorm, defined in [8] for an arbitrary function \( f : \Omega \to X \) by the formula

\[
\|f\|_{GC} = \limsup_n \sup_{\|x\| \leq 1} g_n \, d\mu^\infty,
\]

where

\[
g_n := \max \left\{ \left| f(\omega) - \frac{1}{n} \sum_{i=1}^{n} f(\omega_i) \right| : \omega \in \Omega^\infty \right\}.
\]
Completeness in spaces of Pettis integrable functions

where

$$g_n(\omega) = \sup_{\|x^*\| \leq 1} \frac{1}{n} \sum_{i \leq n} |x^* (f(\omega_i))| .$$

According to [8], the GC-seminorm and the Pettis seminorm are equivalent on $LLN(\mu, X)$. In particular functions in $LLN(\mu, X)$ that are weakly equivalent are not distinguishable by the GC-norm. This permits us to identify weakly equivalent elements of $LLN(\mu, X)$ and investigate the quotient space.

Identifying weakly equivalent functions—we denote by $LLN_\infty(\mu, X)$ the linear space

$$\left\{ f \in LLN(\mu, X) : \|f\|_{P_\infty} := \sup_{\|x^*\| \leq 1} \|x^* f\|_\infty < \infty \right\} .$$

1. Completeness of $P_\infty(\mu, X)$

In many cases $P_\infty(\mu, X) = L_\infty(\mu, X)$ (in the sense of isomorphic isometry). It is so in the case of measure compact spaces (so in particular for separable or weakly compactly generated $X$) and for $X$ possessing RNP. In general however the above equality is false. In spite of this the space $P_\infty(\mu, X)$ is often complete.

**Proposition 1.** If $X$ has the WRNP, then $P_\infty(\mu, X)$ is complete.

**Proof.** Let $(f_n)_{n \in \mathbb{N}} \subset P_\infty(\mu, X)$ be a Cauchy sequence. It is clear that for each $E \in \Sigma$ the sequence $(\nu f_n(E))_{n \in \mathbb{N}}$ is Cauchy in $X$ and so it is convergent to an $X$-valued measure $\nu$. A simple calculation shows that there is $M > 0$ such that $\|\nu(E)\| \leq M \mu(E)$ for each $E \in \Sigma$. According to the assumptions there is a scalarly bounded Pettis integrable density $f$ of $\nu$ with respect to $\mu$. Since $(f_n)_{n \in \mathbb{N}}$ is Cauchy in $P_\infty(\mu, X)$ it is also Cauchy in $P(\mu, X)$. It follows that for each $x^* \in X^*$

$$\lim_{n \to \infty} \int |x^* f_n - x^* f| d\mu = 0 .$$

Consequently, for each $x^* \in X^*$ the sequence $(x^* f_n)$ is convergent in measure to $x^* f$. Since at the same time the sequence $(x^* f_n)$ is Cauchy in $L_\infty(\mu)$, it is convergent to $x^* f$ in $L_\infty(\mu)$. Together with the Cauchy condition in $P_\infty(\mu, X)$, this yields the convergence

$$\lim_{n \to \infty} \|f_n - f\|_{P_\infty} = 0 .$$

The WRNP however is not necessary for the completeness of $P_\infty(\mu, X)$. Assuming Axiom L we can get some results without the assumption of the WRNP. We are going to begin with the following two simple facts.
LEMMA 2. If $f \in P(\mu, X^*)$ then for each $x^{**} \in B(X^{**})$ there exists $x_0^{**} \in \tilde{B}$ such that $x^{**} f = x_0^{**} f \mu$–a.e.

**Proof.** Take $x^{**}$ with $\|x^{**}\| = 1$ and let $(x_\alpha)_{\alpha \in A}$ be a net of functionals from the unit ball of $X$ weak*–converging to $x^{**}$. Since $f \in P(\mu, X^*)$ the operator $T : X^{**} \to L_1(\mu)$ given by $T(x^{**}) = x^{**} f$ is weak*–weakly continuous (cf [3]) and so $x_\alpha f \to x^{**} f$ weakly in $L_1(\mu)$. By a theorem of Mazur, one can find $y_n \in \text{conv}\{x_\alpha : \alpha \in A\}$ such that $\lim y_n f = x^{**} f \mu$–a.e. Now, if $x_0^{**}$ is an arbitrary weak*–cluster point of $\{y_n : n \in \mathbb{N}\}$, then it satisfies the required equality. 

LEMMA 3. If $f \in P_\infty(\mu, X^*)$, then

$$\sup_{\|x^{**}\| \leq 1} \|x^{**} f\|_\infty = \sup_{\|x\| \leq 1} \|x f\|_\infty.$$ 

**Proof.** Let $a := \sup_{\|x^{**}\| \leq 1} \|x f\|_\infty$ and $b := \sup_{\|x^{**}\| \leq 1} \|x^{**} f\|_\infty$. According to Lemma 2 for each $x^{**} \in B(X^{**})$ there exists $\tilde{x} \in \tilde{B}$ satisfying the equality $x^{**} f = \tilde{x} f \mu$–a.e. Consequently,

$$b = \sup \{\|\tilde{x} f\|_\infty : \tilde{x} \in \tilde{B}\}.$$

Given $\tilde{x} \in \tilde{B}$ with $\|\tilde{x}\| = 1$, let $(x_\alpha)_{\alpha \in A}$ be a countable net from the unit ball of $X$ that is weak*–convergent to $\tilde{x}$. By the assumption $|x_\alpha f| \leq a \mu$–a.e. for each $\alpha$. Since the net consists of countably many different elements, we have also $|\tilde{x} f| \leq a \mu$–a.e. Thus $\|\tilde{x} f\|_\infty \leq a$ for each $\tilde{x}$ and so $b \leq a$. This completes the proof.

**Theorem 4.** (Axiom L) If $\mu$ is perfect then $P_\infty(\mu, X^*)$ is complete.

**Proof.** Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $P_\infty(\mu, X^*)$. Applying Lemma 3 we see that for each $m, n \in \mathbb{N}$

$$\sup_{\|x^{**}\| \leq 1} \|x^{**} f_n - x^{**} f_m\|_\infty = \sup_{\|x\| \leq 1} \|x f_n - x f_m\|_\infty.$$

Now,

$$\sup_{\|x\| \leq 1} \|x f_n - x f_m\|_\infty = \sup_{\|x\| \leq 1} \|x \rho_0(f_n) - x \rho_0(f_m)\|_\infty$$

$$= \sup_{\|x\| \leq 1} \sup_{\omega} |x \rho_0(f_n)(\omega) - x \rho_0(f_m)(\omega)|$$

$$= \sup_{\omega} \|\rho_0(f_n)(\omega) - \rho_0(f_m)(\omega)\|.$$

Consequently, the sequence $(\rho_0(f_n))$ is uniformly convergent to a function $h : \Omega \to X^*$ such that $h = \rho_0(h)$. Since for each $x^{**} \in \tilde{B}$ the functions $\rho_0(f_n)$ are measurable, according to [7], Theorem 6-2-1 (where the Axiom L is used), the functions $\rho_0(f_n)$ are in $P_\infty(\mu, X^*)$ and so $h$ is weakly measurable. Then, it is a consequence of the Lebesgue Convergence Theorem (see [3]) that $h \in P_\infty(\mu, X^*)$. 

Thus, using (1), with \( h \) rather than \( \mu \), we get
\[
\lim_{n} \|f_n - h\|_{P^\infty_{00}} = \lim_{n} \sup_{\|x\|_{L} \leq 1} \|xf_n - xh\|_{\infty} = \lim_{n} \sup_{\omega} \|\rho_0(f_n)(\omega) - \rho_0(h)(\omega)\|_{X^*} = 0.
\]
This proves the completeness of \( P^\infty_{00}(\mu, X^*) \). □

Notice that according to a result of Stegall [1], if \( \mu \) is perfect then \( P^\infty_{00}(\mu, X^*) = P^\infty_{00}(\mu, X^*) \).

The above proof makes it obvious that in fact the following more general result holds true:

**Theorem 5.** Let \( \mu \) and \( X \) be arbitrary. If for each countable family \( \mathcal{F} \subset P^\infty_{00}(\mu, X^*) \) there exists a lifting \( \rho \) such that \( \rho(f) \) is \( \mu \)-Pettis-integrable for each \( f \in \mathcal{F} \), then \( P^\infty_{00}(\mu, X^*) \) and \( P^\infty_{00}(\mu, X^*) \) are complete.

The question of whether a lifting of \( f \in P^\infty(\mu, X^*) \) is Pettis integrable was implicitly posed in [7]. Rybakov [6] undertook an attempt to solve the problem, but his approach turned out to be wrong (see Math. Reviews 98h #20007).

**Corollary 6.** If \( X \) is separable, then for each \( \mu \) the spaces \( P^\infty_{00}(\mu, X^*) \) and \( P^\infty(\mu, X^*) \) are complete.

2. **Completeness of \( LLN^\infty(\mu, X) \)**

It has been proven in [4] that if \( X \) is infinite dimensional and \( \mu \) is not purely atomic, then \( LLN(\mu, X) \) is non-complete. In the case of \( LLN^\infty(\mu, X^*) \) the completeness problem is solved affirmatively.

**Theorem 7.** The space \( LLN^\infty(\mu, X^*) \) is complete.

**Proof.** Let \( \rho \) be a consistent lifting on \( L^\infty(\mu) \) and let \( (f_n)_{n=1}^{\infty} \) be a Cauchy sequence in \( LLN^\infty(\mu, X^*) \). As in the proof of Theorem 4 we get the equality
\[
\sup_{\|x^*\|_{L} \leq 1} \|x^* f_n - x^* f_m\|_{\infty} = \sup_{\omega} \|\rho_0(f_n)(\omega) - \rho_0(f_m)(\omega)\|.
\]
It follows that the sequence \( \{\rho_0(f_n)\} \) is uniformly Cauchy in the norm topology of \( X^* \). Let \( h : \Omega \rightarrow X^* \) be the pointwise limit of the sequence \( \{\rho_0(f_n)\} \). The uniform convergence yields the equality \( h = \rho_0(h) \). Moreover, since each \( f_n \) is properly measurable and \( \rho \) is consistent, the function \( \rho_0(f_n) \) is also properly measurable. Clearly it is also pointwise bounded by the function \( \|\rho_0(f_n)\| \in L^\infty(\mu) \). Consequently, it follows from [8], Theorem 26, that \( \rho_0(f_n) \in LLN^\infty(\mu, X^*) \). The uniform convergence of the sequence \( \{\rho_0(f_n)\} \) yields \( h \in LLN(\mu, X^*) \) and the convergence of \( \{\rho_0(f_n)\} \) to \( h \) in \( LLN^\infty(\mu, X^*) \).

This proves the completeness of \( LLN^\infty(\mu, X^*) \). □
Considering each $X$-valued function as an $X^{**}$-valued function we get the following result in case of an arbitrary Banach space $X$:

**THEOREM 8.** The completion of the space $LLN_{\infty}(\mu, X)$ is a subspace of $LLN_{\infty}(\mu, X^{**})$. If Axiom $L$ is satisfied and $\mu$ is perfect then the completion of $P_{\infty}(\mu, X)$ is a subspace of $P_{\infty}(\mu, X^{**})$.

**References**


