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A CONVERGENCE THEOREM FOR THE BIRKHOFF INTEGRAL

MAREK BALCERZAK, KAZIMIERZ MUSIAŁ

Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: We propose an essential improvement of a convergence theorem for the Birkhoff integral. We also obtain the respective version of this result for the convergence associated with an ideal on $\mathbb{N}$.

Keywords: convergence theorems for integrals, Pettis integral, Birkhoff integral.

1. Introduction

Several kinds of integrals for Banach space valued functions are known. For each of them, convergence theorems are always important because of their possible applications. Recently, we have obtained a new Vitali-type convergence theorem for the Pettis integral [2] using the notion of scalar equi-convergence in measure for a sequence of Banach space valued functions. We will use it in the main result of this paper. Our pourpose is to improve a convergence theorem for the Birkhoff integral due to Rodríguez [24]. Besides this, we provide the respective example witnessing that our improvement is essential, and we formulate a counterpart of the theorem for the convergence associated with an ideal on $\mathbb{N} := \{1, 2, \dots\}$.

Through the paper, (Ω, Σ, μ) stands for a complete probability space. A family $\mathcal{F}$ of real-valued Lebesgue integrable functions on Ω is said to be *uniformly integrable* if $\sup\{\int_\Omega |f| d\mu \colon f \in \mathcal{F}\} < \infty$ and for every $\varepsilon > 0$ there is $\delta > 0$ such that $\int_A |f| d\mu < \varepsilon$ for all $f \in \mathcal{F}$ and $A \in \Sigma$ with $\mu(A) < \delta$. Throughout, X is a real Banach space with its dual X^* and $B(X) := \{x \in X \colon ||x|| \leqslant 1\}$. For $f \colon \Omega \to X$ we denote $\mathcal{Z}_f := \{x^* f \colon \|x^*\| \leqslant 1\}$.

We refer the reader to [6] and [18] for basic terminology from the theory of integral for vector-valued functions. A scalarly measurable function f is called *Pettis integrable* if $x^* f \in L_1(\Omega, \mu)$ for all $x^* \in X^*$, and for each $E \in \Sigma$ there

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is $\nu_f(E) \in X$ such that $x^*\nu_f(E) = \int_E x^* f d\mu$ for all $x^* \in X^*$. Then $\nu_f(E)$ is called the *Pettis integral* of f over E with respect to μ. The Pettis integral is more general than the Bochner integral, usually treated as a counterpart of the Lebesgue integral for X-valued strongly measurable functions.

The space of X-valued Pettis μ-integrable functions can be endowed with a norm defined by $\|f\|_P := \sup_{\|x^*\|\leqslant 1} \int_\Omega |x^* f| \, d\mu$. It is known that in general this space is not complete. An equivalent norm can be defined by $|||f||| = \sup_{E\in\Sigma} \left\|\int_E f \, d\mu\right\|$. It follows from this fact that the convergence in the Pettis norm coincides with the uniform convergence of the integrals on the σ-algebra Σ. For a survey on the Pettis integral, see [18] (cf. also [17]).

A sequence (f_n) of X-valued scalarly measurable functions is called *scalarly convergent in measure* to a scalarly measurable function $f\colon \Omega \to X$ if for each $x^* \in X^*$ the sequence $(x^* f_n)$ is convergent in measure to $x^* f$. The following Vitali-type theorem for Pettis integral is due to Musiał [16, Theorem 1] (see also [17, Theorem 8.1] and [18, Theorem 5.2]).

Theorem 1. *[16] Let f_n, $n \in \mathbb{N}$, be Pettis integrable functions from Ω to X such that $\bigcup_{n\in\mathbb{N}} Z_{f_n}$ is uniformly integrable and (f_n) is scalarly convergent in measure to f. Then f is Pettis integrable and $\int_E f_n \to \int_E f$ weakly for each $E \in \Sigma$.*

In [2] we introduced a stronger notion called a scalar equi-convergence in measure. Namely, we say that a sequence of scalarly measurable functions $f_n\colon \Omega \to X$, $n \in \mathbb{N}$, is *scalarly equi-convergent in measure* to a scalarly measurable function $f\colon \Omega \to X$ if for every $\delta > 0$ we have

$$\lim_n \sup_{\|x^*\|\leqslant 1} \mu\{t \in \Omega\colon |x^* f_n(t) - x^* f(t)| > \delta\} = 0.$$

Note that, if a sequence of scalarly integrable functions $f_n\colon \Omega \to X$, $n \in \mathbb{N}$, is convergent to a scalarly integrable function $f\colon \Omega \to X$ in the Pettis norm, then it is scalarly equi-convergent in measure to $f\colon \Omega \to X$.

If $f\colon \Omega \to X$ is a scalarly measurable function, we can define (cf. [2]) the following translation invariant F-norm

$$|f|_\mu := \inf\left\{\lambda > 0\colon \sup_{\|x^*\|\leqslant 1} \mu\{|x^* f| \geqslant \lambda\} \leqslant \lambda\right\}.$$

The convergence in this F-norm is equivalent to scalar equi-convergence in measure.

Scalar equi-convergence in measure can be compared with other kinds of convergence as follows.

Lemma 2. *[2] Let $f_n\colon \Omega \to X$, $n \in \mathbb{N}$, and $f\colon \Omega \to X$ be scalarly measurable functions. We then have (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (D) where*

(A) *(f_n) is μ-a.e. convergent in the norm topology of X to f;*
(B) $\forall\, \delta > 0$ $\lim_n \mu_*\{\|f_n - f\| > \delta\} = 0$ *(μ_* is inner measure induced by μ);*
(C) *(f_n) is scalarly equi-convergent in measure to f;*
(D) *(f_n) is scalarly convergent in measure to f.*

It was observed in [2] that no implication stated in Lemma 2 is reversible.

A new Vitali-type convergence theorem obtained in [2] is the following. It improves [24, Theorem 2.8] and [19, Corollary 5.3].

Theorem 3. *[2] Let functions $f_n : \Omega \to X$, $n \in \mathbb{N}$, be Pettis integrable and let $f : \Omega \to X$ be scalarly measurable. The following conditions are equivalent:*

- (a) *(f_n) is scalarly equi-convergent in measure to f and $\bigcup_n \mathcal{Z}_{f_n}$ is uniformly integrable;*
- (b) *f is Pettis integrable and $\lim_n \|f_n - f\|_P = 0$.*

In particular, (a) *implies that $\lim_n \|\int_E f_n\,d\mu - \int_E f\,d\mu\| = 0$ uniformly with respect to $E \in \Sigma$.*

2. Results

In the recent years, a number of works have been devoted to the Birkhoff integral [4] located between the Bochner and the Pettis integrals (see [5], [22], [23], [24], [3]). A function $f : \Omega \to X$ is called *Birkhoff integrable* with integral $x = \int_\Omega f\,d\mu \in X$ if for every $\varepsilon > 0$ there is a countable partition (A_m) of Ω with $A_m \in \Sigma$ such that, for any choice of points $t_m \in A_m$, the series $\sum_m f(t_m)\mu(A_m)$ converges unconditionally in X and $\|\sum_m f(t_m)\mu(A_m) - x\| \leqslant \varepsilon$. Cascales and Rodríguez [5] discovered that $f : \Omega \to X$ is Birkhoff integrable if and only if $\mathcal{Z}_f$ is uniformly integrable and has the Bourgain property. (A family $\mathcal{H} \subseteq \mathbb{R}^\Omega$ is said to have the *Bourgain property* if for every $\varepsilon > 0$ and every $A \in \Sigma$ with $\mu(A) > 0$ there are $A_1, \ldots, A_n \in \Sigma$, $A_i \subseteq A$ with $\mu(A_i) > 0$ and $\min\limits_{1 \leqslant i \leqslant n} \operatorname{osc}(h|A_i) \leqslant \varepsilon$ for each $h \in \mathcal{H}$.)

Several convergence theorems for the Birkhoff integral were discussed in [22], [23], [24] and [3]. Rodríguez showed in [22], [23] that the classical Lebesgue dominated convergence theorem need not hold for the Birkhoff integral. Following [3], we say that a family $\{f_n : n \in \mathbb{N}\} \subseteq X^\Omega$ is *Birkhoff equi-integrable* if for every $\varepsilon > 0$ there is a countable Σ-partition (A_m) of Ω such that for any choice of points $t_m \in A_m$ we have:

- for each $\delta > 0$ there is $k \in \mathbb{N}$ such that $\|\sum_{m \in M} f_n(t_m)\mu(A_m)\| \leqslant \delta$ for every finite set $M \subseteq \mathbb{N}$ disjoint from $\{1, \ldots, k\}$ and all $n \in \mathbb{N}$ (in particular, each series $\sum_m f_n(t_m)\mu(A_m)$, $n \in \mathbb{N}$, converges unconditionally in X);
- $\|\sum_m f_n(t_m)\mu(A_m) - \int_\Omega f_n d\mu\| \leqslant \varepsilon$ for all $n \in \mathbb{N}$.

Note that each member of a Birkhoff equi-integrable family is Birkhoff integrable and every infinite subset of a Birkhoff equi-integrable family is Birkhoff equi-integrable.

The following result was first proved in [3] for norm convergence, and shown again in a different way in [24] where also weak convergence was considered.

Theorem 4 ([24], [3]). *Let $f : \Omega \to X$ and $f_n : \Omega \to X$, $n \in \mathbb{N}$, where $\{f_n : n \in \mathbb{N}\}$ is Birkhoff equi-integrable. If (f_n) is convergent pointwise in norm (weakly) to f then f is Birkhoff integrable and $\int_E f_n d\mu \to \int_E f\,d\mu$ in norm (weakly) for every $E \in \Sigma$.*

We propose the following improvement of this theorem:

Theorem 5. *Let (f_n) be a pointwise bounded sequence of Birkhoff equi-integrable functions $f_n\colon \Omega \to X$, $n \in \mathbb{N}$, which is scalarly convergent in measure to a function $f\colon \Omega \to X$. If $\mathcal{Z}_f$ is contained in the pointwise closure of $\bigcup_n \mathcal{Z}_{f_n}$, then f is Birkhoff integrable and*

$$\lim_n \int_\Omega |x^* f_n - x^* f|\, d\mu = 0 \qquad \textit{for every } x^* \in X^*. \tag{1}$$

Moreover, if the sequence (f_n) is scalarly equi-convergent in measure to f, then

$$\lim_n \|f_n - f\|_P = 0. \tag{2}$$

In particular,

$$\lim_n \left\| \int_E f_n\, d\mu - \int_E f\, d\mu \right\| = 0 \qquad \textit{uniformly with respect to } E \in \Sigma.$$

Proof. According to [24, Proposition 2.11] the set $\bigcup_n \mathcal{Z}_{f_n}$ is uniformly integrable and has the Bourgain property. By the assumption, $\mathcal{Z}_f$ is contained in the pointwise closure of $\bigcup_n \mathcal{Z}_{f_n}$, and we know that the Bourgain property is preserved by taking pointwise closures (cf. [21, Theorem 11]). Consequently, $\mathcal{Z}_f$ has the Bourgain property. Applying the assumed convergence in measure, we obtain the uniform integrability of the set $\mathcal{Z}_f$. Then applying the Cascales-Rodríguez theorem [5], we get the Birkhoff integrability of f. Condition (1) follows from Theorem 1, and condition (2) is a consequence of Theorem 3. ∎

It is obvious that the above result generalizes Theorem 4 in the case of weak convergence. However, to show that this improvement is essential, we need an example.

Example 1. Let (Ω, Σ, μ) be a non-atomic probability space such that there exists a sequence $(E_n)_n$ of elements of Σ generating an algebra that is μ-dense in Σ. If X is separable, $l_1 \not\subseteq X$ and X^* is non-separable, then there exists a Pettis integrable bounded function $f\colon \Omega \to X^*$ that is not weak* equivalent to any strongly measurable X^*-valued function (see [14]). Without loss of generality, we may assume that for a lifting ρ on $L_\infty(\mu)$ the function f satisfies for every $x \in X$ the equality $xf = \rho(xf)$. It follows then from [17, Corollary 12.1] that the set $\{xf\colon \|x\| \leqslant 1\}$ and then $\mathcal{Z}_f$ have the Bourgain property. In virtue of [5], f is Birkhoff integrable.

For each $n \in \mathbb{N}$ let π_n be the partition generated by the sets $E_1, \ldots, E_n$. For each $n \in \mathbb{N}$ let

$$f_n := \sum_{E \in \pi_n} \frac{(P)\int_E f\, d\mu}{\mu(E)} \qquad \text{with the convention } 0/0 = 0\,.$$

One can easily check that $\{(f_n, \sigma(\pi_n))\colon n \in \mathbb{N}\}$ is a bounded martingale; in particular, for each $x^{**} \in X^{**}$, the sequence $\{((x^{**} f_n), \sigma(\pi_n))\colon n \in \mathbb{N}\}$ is a real valued uniformly integrable martingale. Moreover, $\mathbb{E}(x^{**}f|\sigma(\pi_n)) = x^{**}f_n$ μ–a.e. for every $n \in \mathbb{N}$. Hence, if $\widetilde{\Sigma} = \sigma(\{E_n : n \in \mathbb{N}\}$, then $\lim_n x^{**}f_n = \mathbb{E}((x^{**}f)|\widetilde{\Sigma}) = x^{**}f$ in $L_1(\mu|\widetilde{\Sigma})$ and μ–a.e.

If $\{x_k : k \in \mathbb{N}\}$ is a norm dense in $B(X)$ then, one can extract $N \in \Sigma$ of measure zero such that for each k and each $t \notin N$ we have $\lim_n x_k f_n(t) = x_k f(t)$. Since

$$\sup_n \sup_{t \in \Omega} \max\{\|f_n(t)\|, \|f(t)\|\} < \infty,$$

it follows that $\lim_n xf_n(t) = xf(t)$ for every $x \in X$ and every $t \notin N$.

Set now, for each $n \in \mathbb{N}$, $g_n := f_n\chi_{N^c}$ and $g := f\chi_{N^c}$. It is obvious that g is Birkhoff integrable and $g_n \to g$ pointwise in the weak* topology. Since g is not scalarly equivalent to any strongly measurable function, no subsequence of $(g_n)_n$ can converge a.e. weakly to g.

By the same reason, g cannot be a pointwise weak limit of any sequence of strongly measurable functions.

But X is separable, and so, due to Rosenthal's subsequence theorem, if $x^{**} \in B(X^{**})$, then there is a subsequence $(y_k)_k$ in $B(X)$ satisfying the equality $\lim_k y_k = x^{**}$ in the weak* topology of X^{**}. It follows that $x^{**}g$ is in the pointwise closure of $\{xg : \|x\| \leqslant 1\}$. But each xg is in the pointwise closure of the set $\{xg_n : n \in \mathbb{N}\}$. This proves that $\mathcal{Z}_g$ is contained in the pointwise closure of $\bigcup_n \mathcal{Z}_{g_n}$. ∎

Recently, extensive studies have been developed in various applications of a generalized kind of convergence associated with an ideal (or, equivalently, with a filter) of subsets of $\mathbb{N}$. (cf. [13, 20, 9, 10, 11, 8, 7, 1, 12]). If $\mathcal{I}$ is an ideal of subsets of $\mathbb{N}$, we say (cf. [13], [20]) that a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is *$\mathcal{I}$-convergent* to $x \in \mathbb{R}$ if for every $\varepsilon > 0$ we have $\{n \in \mathbb{N} : |x_n - x| > \varepsilon\} \in \mathcal{I}$. We then write $\mathcal{I}\text{-}\lim_n x_n = x$. Note that the usual convergence implies $\mathcal{I}$-convergence, while the converse is not true in general. If functions $f : \Omega \to \mathbb{R}$ and $f_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, are measurable, we say (cf. [1]) that (f_n) is *$\mathcal{I}$-convergent in measure* to f whenever

$$\mathcal{I}\text{-}\lim_n \mu(\{t \in \Omega : |f_n(t) - f(t)| > \delta\}) = 0$$

for every $\delta > 0$.

The following Vitali-type theorem was proved in [2].

Theorem 6 ([2]). *Let (f_n) be a uniformly (Lebesgue) integrable sequence of functions $f_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, $\mathcal{I}$-convergent in measure to a measurable function $f : \Omega \to \mathbb{R}$. Then f is integrable and $\mathcal{I}\text{-}\lim_n \int_\Omega |f_n - f| = 0$.*

For our pourposes we need two else definitions. Let $f_n : \Omega \to X$, $n \in \mathbb{N}$ and $f_n : \Omega \to X$ be scalarly measurable functions. We say that (f_n) is *$\mathcal{I}$-scalarly convergent in measure to* f if (x^*f_n) is $\mathcal{I}$-convergent in measure to x^*f for each $x^* \in X^*$. We say (cf. [2]) that the sequence (f_n) is *$\mathcal{I}$-scalarly equi-convergent in measure* to f if for every $\delta > 0$ we have

$$\mathcal{I}\text{-}\lim_n \sup_{\|x^*\| \leqslant 1} \mu(\{t \in \Omega : |x^*f_n(t) - x^*f(t)| > \delta\}) = 0.$$

The following result is an $\mathcal{I}$-version of Theorem 5.

Theorem 7. *Assume that (f_n) is a pointwise bounded Birkhoff equi-integrable sequence of functions $f_n\colon \Omega \to X$, $n \in \mathbb{N}$, which is $\mathcal{I}$-scalarly convergent in measure to a scalarly measurable function $f\colon \Omega \to X$. If $\mathcal{Z}_f$ is contained in the pointwise closure of $\bigcup_{n\in\mathbb{N}} \mathcal{Z}_{f_n}$, then f is Birkhoff integrable and*

$$\mathcal{I}\text{-}\lim_n \int_\Omega |x^* f_n - x^* f|\, d\mu = 0 \qquad \text{for every } x^* \in X^*. \tag{3}$$

Moreover, if (f_n) is $\mathcal{I}$-scalarly equi-convergent in measure to f, then $\mathcal{I}\text{-}\lim_n \|f_n - f\|_P = 0$. In particular,

$$\mathcal{I}\text{-}\lim_n \left\| \int_E f_n\, d\mu - \int_E f\, d\mu \right\| = 0 \qquad \text{uniformly with respect to } E \in \Sigma. \tag{4}$$

Proof. To show that f is Birkhoff integrable we proceed as in the proof of Theorem 5 (the uniform integrability of $\mathcal{Z}_f$ follows from the assumed convergence of (f_n), the uniform integrability of $\bigcup_n \mathcal{Z}_{f_n}$, and Theorem 6). Condition (3) follows from Theorem 6 applied to the sequence $(x^* f)$ for every fixed $x^* \in X^*$.

Assume that (f_n) is $\mathcal{I}$-scalarly equi-convergent in measure to f. Then we modify simply the argument used in the final part of the proof of Theorem 3 (cf. [2]). Fix $\varepsilon > 0$ and pick $\delta > 0$ such that $\int_A |x^* f_n - x^* f| d\mu < \varepsilon$ for all $n \in \mathbb{N}$, $\|x^*\| \leqslant 1$ and $A \in \Sigma$ with $\mu(A) < \delta$. By the assumption of the $\mathcal{I}$-scalar equi-convergence, pick $E \in \mathcal{I}$ such that

$$\sup_{\|x^*\|\leqslant 1} \mu(\{t \in \Omega\colon |x^* f_n(t) - x^* f(t)| > \varepsilon\}) < \delta \qquad \text{for all } n \in \mathbb{N} \setminus E.$$

Then for all $n \in \mathbb{N} \setminus E$ we have

$$\begin{aligned} \|f_n - f\|_P &= \sup_{\|x^*\|\leqslant 1} \int_\Omega |x^* f_n - x^* f| d\mu \\ &\leqslant \sup_{\|x^*\|\leqslant 1} \int_{\{|x^* f_n - x^* f| > \varepsilon\}} |x^* f_n - x^* f| d\mu \\ &\quad + \sup_{\|x^*\|\leqslant 1} \int_{\{|x^* f_n - x^* f| \leqslant \varepsilon\}} |x^* f_n - x^* f| d\mu < 2\varepsilon. \end{aligned}$$

This yields $\mathcal{I}\text{-}\lim_n \|f_n - f\|_P = 0$ and consequently, condition (4) holds. ∎

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Addresses: Marek Balcerzak: Institute of Mathematics, Łódź University of Technology, ul. Wólczańska 215, 93-005 Łódź, Poland;
Kazimierz Musiał: Institute of Mathematics, University of Wrocław, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland.

E-mail: marek.balcerzak@p.lodz.pl, musial@math.uni.wroc.pl

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