DIFFERENTIATION OF AN ADDITIVE INTERVAL MEASURE WITH VALUES IN A CONJUGATE BANACH SPACE

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Abstract: We present a complete characterization of finitely additive interval measures with values in conjugate Banach spaces which can be represented as Henstock-Kurzweil-Gelfand integrals. If the range space has the weak Radon-Nikodym property (WRNP), then we precisely describe when these integrals are in fact Henstock-Kurzweil-Pettis integrals.

Keywords: Kurzweil-Henstock integral, Pettis integral, variational measure.

1. Notations and preliminaries

Let \([0,1]\) be the unit interval of the real line equipped with the usual topology and the Lebesgue measure \(\lambda\). We denote by \(\mathcal{I}\) the family of all nontrivial closed subintervals of \([0,1]\), by \(\mathcal{L}\) the family of all Lebesgue measurable subsets of \([0,1]\) and by \(\mathcal{L}^+\) the family of all Lebesgue measurable subsets of \([0,1]\) of positive measure.

If \(E \subseteq \mathcal{L}\), then its Lebesgue measure is denoted by \(|E|\) or \(\lambda(E)\). Throughout \(X\) is a Banach space with its dual \(X^*\). The closed unit ball of \(X\) is denoted by \(B(X)\).

A mapping \(\nu: \mathcal{L} \to X\) is said to be an \(X\)-valued measure if \(\nu\) is countably additive in the norm topology of \(X\). If \(\mu\) is a positive measure on \(\mathcal{L}\) or an \(X\)-valued measure, then by \(\mu \ll \lambda\) we mean that \(|E| = 0\) implies \(\mu(E) = 0\). We say then that \(\mu\) is \(\lambda\)-continuous. The variation of an \(X\)-valued measure \(\nu\) is denoted by \(|\nu|\).

\(\tau(X^*,X)\) is the Mackey topology on \(X^*\) and \(\tau_c(X^*,X)\) is the topology of uniform convergence on compact subsets of \(X\). It is known (cf. [12]) that \(\tau_c(X^*,X)\) coincides on \(B(X^*)\) with the weak*i-*topology \(\sigma(X^*,X)\).

A partition in \([0,1]\) is a finite collection of pairs \(\mathcal{P} = \{(I_1,t_1), \ldots, (I_p,t_p)\}\), where \(I_1, \ldots, I_p\) are non-overlapping subintervals of \([0,1]\) and \(t_i \in I_i\), for all \(i \leq p\).

Given a subset \(E\) of \([0,1]\), we say that the partition \(\mathcal{P}\) is anchored on \(E\) if \(t_i \in E\)
for each $i = 1, \ldots, p$. If $\bigcup_{i=1}^{p} I_i = [0, 1]$ we say that $\mathcal{P}$ is a partition of $[0, 1]$. A gauge on $E \subset [0, 1]$ is a positive function on $E$. For a given gauge $\delta$, we say that a partition $\{(I_i, t_i), \ldots, (I_p, t_p)\}$ is $\delta$-fine if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1, \ldots, p$.

Given two real numbers $a, b$, we denote by the symbol $< a, b >$ the interval $[\min\{a, b\}, \max\{a, b\}]$.

**Definition 1.1.** A function $f : [0, 1] \to \mathbb{R}$ is said to be Henstock-Kurzweil integrable, or simply HK-integrable, on $[0, 1]$ if there exists $w \in \mathbb{R}$ with the following property: for every $\varepsilon > 0$ there exists a gauge $\delta$ on $[0, 1]$ such that

$$\left| \sum_{i=1}^{p} f(t_i)|I_i| - w \right| < \varepsilon,$$

for each $\delta$-fine partition $\mathcal{P} = \{(I_i, t_i), \ldots, (I_p, t_p)\}$ of $[0, 1]$.

We set $(HK) \int_{0}^{1} f d\lambda := w$. By $HK[0, 1]$ is denoted the set of all HK-integrable functions $f : [0, 1] \to \mathbb{R}$.

It is well known that if $f \in HK[0, 1]$ then $f$ is HK-integrable on each $I \in \mathcal{I}$. We call the additive interval function $F(I) := (HK) \int_{I} f d\lambda$ the HK-primitive of $f$.

**Definition 1.2.** A function $f : [0, 1] \to X$ is said to be scalarly Henstock-Kurzweil integrable if, for each $x^* \in X^*$, the function $x^* f$ is Henstock-Kurzweil integrable. A scalarly Henstock-Kurzweil integrable function $f$ is said to be Henstock-Kurzweil-Pettis integrable (or simply HKP-integrable) if for each $I \in \mathcal{I}$ there exists $w_I \in X$ such that

$$\langle x^*, w_I \rangle = \int_{I} \langle x^*, f(t) \rangle dt,$$

for every $x^* \in X^*$.

We call $w_I$ the Henstock-Kurzweil-Pettis integral of $f$ over $I$ and we write $(HKP) \int_{0}^{1} f(t) dt := w_I$.

We denote by $HKP([0, 1], X)$ the set of all $X$-valued Henstock-Kurzweil-Pettis integrable functions on $[0, 1]$ (functions that are scalarly equivalent are identified).

**Definition 1.3.** A function $f : [0, 1] \to X^*$ is said to be $w^*$-scalarly Henstock-Kurzweil integrable if, for each $x \in X$, the function $x^* f$ is Henstock-Kurzweil integrable. A $w^*$-scalarly Henstock-Kurzweil integrable function $f : [0, 1] \to X^*$ is said to be Henstock-Kurzweil-Gelfand integrable (or simply HKG-integrable) if, for each interval $I \in \mathcal{I}$, there exists a vector $\Psi(I) \in X^*$ such that for every $x \in X$

$$\langle x, \Psi(I) \rangle = (HKG) \int_{I} \langle x, f(t) \rangle dt.$$

We call $\Psi(I)$ the Henstock-Kurzweil-Gelfand integral of $f$ over $I$ and we write $(HKG) \int_{0}^{1} f(t) dt := \Psi(I)$. $\Psi$ is called the HKG-primitive of $f$. 
Following the proof of [9, Theorem 3] (with suitable changes), it is easy to see that a function $f: [0,1] \to X^*$ is HKG-integrable if and only if $f$ is $w^*$-scalarly Henstock-Kurzweil integrable.

Throughout, we identify a function $\Psi: [0,1] \to X$ (resp. $\Psi: [0,1] \to X^*$) with the additive interval function $\Psi: I \to X$ (resp. $\Psi: I \to X^*$) defined by $\Psi(I) = \Psi(b) - \Psi(a)$, if $I = [a, b]$. And conversely, with each $\Psi: I \to X$, (resp. $\Psi: I \to X^*$) we associate $\Psi: [0,1] \to X$ (resp. $\Psi: [0,1] \to X^*$) by setting $\Psi(t) = \Psi([0,t])$.

**Definition 1.4.** A function $f: [0,1] \to X$ is said to be *scalarly measurable* (scalarly integrable) if, for each $x^* \in X^*$, the function $x^*f$ is Lebesgue measurable (integrable). A scalarly integrable function $f: [0,1] \to X$ is said to be *Pettis integrable* if, for each set $A \in \mathcal{L}$ there exists a vector $\nu_f(A) \in X$ such that for every $x^* \in X^*$

$$\langle x^*, \nu_f(A) \rangle = \int_A \langle x^*, f(t) \rangle \, dt.$$ 

We call $\nu_f(A)$ the *Pettis integral of $f$ over $A$* and we write $(P)\int_A f(t) \, dt := \nu_f(A)$. It is known (see [15]) that $\nu_f: \mathcal{L} \to X$ is a measure of $\sigma$-finite variation.

**Definition 1.5.** A function $f: [0,1] \to X^*$ is said to be *$w^*$-scalarly measurable* (resp. *$w^*$-scalarly integrable*) if, for each $x \in X$, the function $xf$ is Lebesgue measurable (resp. integrable). It is well known that each $w^*$-scalarly integrable function $f: [0,1] \to X^*$ is *Gelfand integrable*, that is, for each set $A \in \mathcal{L}$, there exists a vector $\nu(A) \in X^*$ such that

$$\langle x, \nu(A) \rangle = \int_A \langle x, f(t) \rangle \, dt,$$

for every $x \in X$.

We call the set function $\nu: \mathcal{L} \to X^*$ the *Gelfand integral of $f$ on $[0,1]$* and we write $(G)\int_A f(t) \, dt := \nu(A)$.

**Definition 1.6.** A function $f: [0,1] \to X^*$ is said to be *weak*-scalarly bounded on $E$ if

$$\exists M > 0 \forall x \in B(X) \ |\langle x, f \rangle| \leq M \quad \text{a.e. on } E.$$ 

A function $f: [0,1] \to X$ is said to be *scalarly bounded on $E$*, if it is weak*-scalarly bounded, when considered as an $X^\ast\ast$-valued function.

**Definition 1.7.** Let $\Phi: [0,1] \to X$ be a function. If there is a function $\Phi'_p : [0,1] \to X$ such that for each $x^* \in X^*$

$$\lim_{h \to 0} \frac{x^* \langle \Phi \lt t, t + h \rangle}{|h|} = x^* \langle \Phi'_p(t) \rangle,$$

for almost all $t \in [0,1]$ (the exceptional sets depend on $x^*$), then $\Phi$ is said to be *pseudo-differentiable on $[0,1]$*, with pseudo-derivative $\Phi'_p$ (see [16], p. 300).
Let \( \Phi : [0, 1] \to X^* \) be a function. If there is a function \( \Phi' : [0, 1] \to X^* \) such that for each \( x \in X \)
\[
\lim_{h \to 0} \frac{x(\Phi(t) + h) - x(\Phi(t))}{|h|} = x(\Phi'(t)),
\]
for almost all \( t \in [0, 1] \) (the exceptional sets depend on \( x \)), then \( \Phi \) is said to be \( \omega^* \)-pseudo-differentiable on \([0, 1]\), with \( \omega^* \)-pseudo-derivative \( \Phi' \).

2. Variational measures

**Definition 2.1.** Given an additive interval function \( \Phi : \mathcal{I} \to X \), a gauge \( \delta \) and a set \( E \subset [0, 1] \) we define
\[
Var(\Phi, \delta, E) = \sup \left\{ \sum_{i=1}^{p} |\Phi(I_i)| : \{I_i, t_i : i = 1, \ldots, p\} \text{ \( \delta \)-fine} \right\}
\]
if \( E \neq \emptyset \) and \( Var(\Phi, \delta, \emptyset) = 0 \). Then we set
\[
V_\Phi(E) = \inf \{Var(\Phi, \delta, E) : \delta \text{ is a gauge on } E\}
\]
if \( E \neq \emptyset \) and \( V_\Phi(\emptyset) = 0 \).

We call \( V_\Phi \) the variational measure generated by \( \Phi \). \( V_\Phi \) is known to be a metric outer measure in \([0, 1]\) (see [17]). In particular, \( V_\Phi \) restricted to Borel subsets of \([0, 1]\) is a measure. We say that \( V_\Phi \) is absolutely continuous with respect to \( \lambda \) (we write then \( V_\Phi \ll \lambda \)), if \( \lambda(E) = 0 \) yields \( V_\Phi(E) = 0 \), for all \( E \in \mathcal{L} \). Notice that if \( V_\Phi \ll \lambda \), then given \( \varepsilon > 0 \) and \( \emptyset \neq E \in \mathcal{L} \) with \( |E| = 0 \), there exists a gauge \( \delta \) such that \( Var(\Phi, \delta', E) < \varepsilon \), for every \( \delta' \leq \delta \).

If \( \Phi \) is continuous, then \( V_\Phi(I) \leq |\Phi(I)| \) for every \( I \in \mathcal{I} \), where
\[
|\Phi(I)| = \sup \left\{ \sum_{i=1}^{p} |\Phi(I_i)| : I_i \text{ are non-overlapping subintervals of } I \right\}.
\]

We would like to remark that if \( \Phi \) is discontinuous the inequality \( V_\Phi(I) \leq |\Phi(I)| \) may fail. As an example consider \( \Phi \) on \([0, 1]\) defined in the following way: \( \Phi(t) = 1 \) for \( t \in [0, 1/2) \), \( \Phi(t) = 0 \) for \( t \in [1/2, 1] \). \( \Phi \) is not continuous, and \( V_\Phi([1/2, 1]) = 1 > |\Phi([1/2, 1])| = 0 \).

Moreover we say that a variational measure \( V_\Phi \) is \( \sigma \)-finite if there is a sequence of (pairwise disjoint) sets \( F_n \) covering \([0, 1]\) and such that \( V_\Phi(F_n) < \infty \), for every \( n \in \mathbb{N} \).

By a result of Thomson (see [17, Theorem 3.15]) it follows that the sets \( F_n \) in the previous definition can be taken from \( \mathcal{L} \).

We recall that a function \( \Phi : [0, 1] \to X \) is said to be \( BV^* \) on a set \( E \subset [0, 1] \) if
\[
sup \sum_{i=1}^{n} \omega(\Phi(J_i)) < +\infty,
\]
where the supremum is taken over all finite collections \( \{J_1, \ldots, J_n\} \) of non overlapping intervals in \( \mathcal{I} \) with end-points in \( E \), and the symbol \( \omega(\Phi(J)) \) stands for \( \sup \{||\Phi(u) - \Phi(z)|| : u, z \in J\} \). The function \( \Phi \) is said to be \( BVG^* \) on \([0, 1]\) if \( [0, 1] = \bigcup_{n} E_n \) and \( \Phi \) is \( BV^* \) on each \( E_n \).
Differentiation of an additive interval measure with values in a conjugate Banach space

In the following we will use the following results proved in [2].

**Proposition 2.2.** Let \( \Phi : \mathcal{I} \to X \) be an additive interval function.

1. If \( V_\Phi \ll \lambda \), then \( \Phi \) is continuous on \([0, 1]\) and \( V_\Phi \) is \( \sigma \)-finite.
2. \( V_\Phi \) is \( \sigma \)-finite if and only if \( \Phi \) is \( BVG^* \) on \([0, 1]\).

In case of a separable Banach space \( X \) and \( \Phi \) being an HKP-integral we are able to describe the variational measure \( V_\Phi \) more precisely. Our result generalizes a well known fact for real valued functions.

**Proposition 2.3.** Assume that \( X \) is a separable Banach space, \( \Phi : \mathcal{I} \to X \) is additive and

\[
\Phi(I) = (HKP) \int_I f(t) \, dt.
\]

If \( V_\Phi \ll \lambda \), then

\[
V_\Phi(E) = \int_E ||f|| \, dt, \quad \text{for every } E \in \mathcal{L}.
\]

**Proof.** By Proposition 2.2, \( V_\Phi \) is \( \sigma \)-finite and so \( \Phi \) is a \( BVG^* \) function. Moreover, by [13, Theorem 9], for each measurable set \( E \), we have

\[
V_\Phi(E) = \int_E |\Phi'(t)| \, dt
\]

where the symbol \( |\Phi'(t)| \) denotes the upper absolute derivative of \( \Phi \) in \( t \), that is

\[
|\Phi'(t)| = \limsup_{h \to 0} \frac{||\Phi(t+h) - \Phi(t)||}{|h|}.
\]

Let us observe that since \( \Phi \) is the HKP-primitive of \( f \), then \( f \) is a pseudo-derivative of \( \Phi \). Now, since \( X \) is separable, then by a result in an unpublished paper of Gordon [11] (see also [13]), \( \Phi \) is differentiable a.e. on \([0, 1]\) with derivative \( f \). So

\[
|\Phi'(t)| = ||f|| \text{ a.e. on } [0, 1] \text{ and this completes the proof.}
\]

**Question 2.4.** Do we have always \( V_\Phi(E) = \int_E ||f(t)|| \, dt \) or \( V_\Phi(E) \ll \lambda \) for every \( E \in \mathcal{L} \), if the function \( ||f|| \) is measurable?

Besides the above variational measure we define the following two outer measures, introduced for technical reasons only:

\[
W_\Phi^x(E) = \sup_{x^* \in H(X^*)} V_{x^* \Phi}(E), \quad \text{if } \Phi : \mathcal{I} \to X
\]

and

\[
W_\Phi^x(E) = \sup_{x \in B(X)} V_{x \Phi}(E), \quad \text{if } \Phi : \mathcal{I} \to X^*.
\]
In general, the two outer measures are not metric and not all Borel subsets of \([0,1]\) are measurable with respect to them.

Let us observe that if \(\Phi : I \to X^*\) is an additive interval function, then by the definitions of variational measures, we have:

\[
W^\phi_{\Phi}(E) \leq W^\phi(E) \leq V^\phi(E)
\]

for every \(E \subseteq [0,1]\). In fact, for every \(I \in \mathcal{I}\), \(x^* \in B(X^{**})\) and \(x \in B(X)\), we have: \(|x^*\Phi(I)| \leq ||\Phi(I)||\) and \(|x\Phi(I)| \leq ||\Phi(I)||\). So \(V^\phi_{\Phi}(E) \leq V^\phi(E),\)

\[V^\phi_{\Phi}(E) \leq V^\phi(E)\]

and inequalities (1) follow.

**Definition 2.5.** Let \(V\) be one of the above introduced outer measures and let \(AV := \{V^\phi(E) : |E| > 0\}\) be the average range of \(V\). We say that \(AV\) is locally bounded if there are sets \(E_n \in \mathcal{E}\) such that \(|\bigcup_n E_n| = 1\) and \(V(E_n \cap E) \leq n|E_n \cap E|\), for every \(n \in \mathbb{N}\) and \(E \in \mathcal{E}\).

**Proposition 2.6.** Let \(\Phi : I \to X\). If \(V^\phi \ll \lambda\), then \(AV^\phi\) is locally bounded.

**Proof.** By Proposition 2.2 we have that \(V^\phi\) is \(\sigma\)-finite. Since \(V^\phi|_\mathcal{E}\) is a measure, applying the Radon-Nikodym Theorem, we conclude that \(AV^\phi\) is locally bounded.

**Remark 2.7.** Assume that

\[
\Phi(I) = (HKP) \int_I f(t) \, dt.
\]

In general \(V^\phi\) is neither \(\sigma\)-finite nor absolutely continuous. In fact, if \(V^\phi\) is \(\sigma\)-finite, then by Proposition 2.2, \(\Phi\) is a \(BVG_*\) function. So, if \(X\) has the RNP, then \(\Phi\) is a.e. differentiable (see [1, Theorem 3.6]). But by a result in [5] we know that in each infinite dimensional Banach space (in particular in a conjugate space with the RNP) there exist strongly measurable Pettis (and then Henstock-Kurzweil-Pettis) integrable functions whose Pettis integrals are nowhere differentiable. Each such function is HKP-integrable and induces a non-\(\sigma\)-finite variational measure \(V^\phi\).

In the general case the following characterization holds.

**Proposition 2.8.** A function \(\Phi : [0,1] \to X\) is an HKP-primitive (of a function \(f\)) if and only if \(W^\phi_{\Phi} \ll \lambda\) and \(\Phi\) is pseudo-differentiable (with pseudo-derivative \(f\)).

**Proof.** The proof follows at once from the characterization of the primitives of real valued \(HK\)-integrable functions (see [3]).

### 3. Henstock-Kurzweil-Gelfand integral

The following result gives a full description of \(X^*\)-valued additive interval measures that can be represented as an HKG-integral.

**Theorem 3.1.** An additive function \(\Phi : I \to X^*\) is an HKG-primitive if and only if \(W^\phi_{\Phi} \ll \lambda\) and \(AW^\phi_{\Phi}\) is locally bounded.
Proof. Assume first that \( f : [0, 1] \to X^* \) is HKG-integrable and let \( \Phi(I) = (HKG) \int_I f(t) \, dt \) for every \( I \in \mathcal{I} \). Since \( xf \in HK[0, 1] \) for every \( x \in X \), we have \( V_{x, \Phi} \ll \lambda \), and so also \( W_{x, \Phi} \ll \lambda \). Moreover, according [14, Corollary 3.1] there are pairwise disjoint sets \( E_n \in \mathcal{L} \) such that \( \bigcup_n E_n = [0, 1] \) and \( |x f \chi_{E_n}| \leq n \) a.e., for each \( x \in B(X) \) (the exceptional sets depend on \( x \)). It follows that every \( f \chi_{E_n} \) is Gelfand integrable.

According to [4] and [6] we have also

\[
V_{x, \Phi}(E \cap E_n) = \int_{E \cap E_n} |x f(t)| \, dt \leq n|E \cap E_n|\|x\|
\]

for every \( E \in \mathcal{L} \) and \( n \in \mathbb{N} \). Hence \( W_{x, \Phi}(E \cap E_n) \leq n|E \cap E_n| \) and consequently \( AW_{x, \Phi} \) is locally bounded.

Assume now that \( W_{x, \Phi} \ll \lambda \) and \( AW_{x, \Phi} \) is locally bounded. Then \( V_{x, \Phi} \ll \lambda \) for every \( x \in X \). According to [3], for every \( x \in B(X) \), let \( f_x \in HK[0, 1] \) be such that

\[
\langle x, \Phi(I) \rangle = (HK) \int_I f_x(t) \, dt \quad \text{for every } I \in \mathcal{I}.
\]

Let \( \rho \) be a lifting on \( L_\infty[0, 1] \). Since \( AW_{x, \Phi} \) is locally bounded, there are pairwise disjoint sets \( E_n = \rho(E_n) \in \mathcal{L} \) such that \( |\bigcup_n E_n| = 1 \) and

\[
W_{x, \Phi}(E_n \cap E) \leq n|E_n \cap E|, \quad \text{for every } n \in \mathbb{N} \text{ and } E \in \mathcal{L}.
\]  

(2)

According to [4] and [6], then

\[
V_{x, \Phi}(E) = \int_E |f_x(t)| \, dt \quad \text{for every } E \in \mathcal{L} \text{ and } x \in X.
\]  

(3)

In particular (3) holds true for measurable \( E \subseteq E_n \). It follows from (2) and (3) that for every \( n \in \mathbb{N} \) and \( x \in B(X) \) we have \( |f_x \chi_{E_n}| \leq n\chi_{E_n} \) a.e. In particular

\[
|\rho(f_x)(t)\chi_{E_n}(t)| = \rho(|f_x|)(t)\chi_{E_n}(t) \leq n \quad \text{for every } t \in [0, 1], x \in B(X) \text{ and } n \in \mathbb{N}.
\]

Define now a function \( f : [0, 1] \to X^* \) by setting for each \( x \in X \)

\[
\langle x, f(t) \rangle = \begin{cases} 
\rho(f_x)(t)\chi_{E_n}(t) & \text{if } t \in E_n \\
0 & \text{if } t \notin \bigcup_n E_n
\end{cases}
\]

For each \( t \in E_n \) the function \( x \mapsto \langle x, f(t) \rangle \) is linear and \( |\langle x, f(t) \rangle| \leq n\|x\| \). If \( t \notin \bigcup_n E_n \), then \( f(t) = 0 \). It follows that \( f(t) \in X^* \), for every \( t \).

Since \( \langle x, f \rangle \overset{\text{a.e.}}{=} f_x \in HK[0, 1] \), we get the representation

\[
\langle x, \Phi(I) \rangle = (HK) \int_I \langle x, f(t) \rangle \, dt \quad \text{for every } I \in \mathcal{I}.
\]  

(4)

of \( \Phi \) as an HKG-integral of \( f \).
It follows from the construction of $f$ that it is $w^*$-scalarly bounded, hence Gelfand integrable on every $E_n$. It is a consequence of lifting measurability properties that $\|f\|$ is measurable on every $E_n$, and so on $[0,1]$.

If $X^*$ has the WRNP, then according to [14, Proposition 12.3] and [14, Corollary 3.1], $f$ is Pettis integrable and scalarly bounded on each $E_n$. Thus, we can formulate the following consequence of the proof of Theorem 3.1:

**Corollary 3.2.** Assume that $\Phi : I \to X^*$ is an HKG-primitive. Then there exists a function $f : [0,1] \to X^*$ such that $f$ is a weak* pseudo-derivative of $\Phi$ and there exists a sequence of pairwise disjoint sets $E_n \in \mathcal{L}$ such that $\bigcup_n E_n = [0,1]$, $f$ is weak* scalarly bounded and Gelfand integrable on every $E_n$, $n \in \mathbb{N}$, $\mathcal{W}_\mathcal{X}(E_n) < \infty$ and $\|f\|$ is measurable.

If $X^*$ has the WRNP, then $f$ and the sets $E_n \ n \in \mathbb{N}$ can be taken in such a way that $f$ is Pettis integrable and scalarly bounded on each $E_n$.

If $V_\Phi \ll \lambda$, then by Proposition 2.6, $AV_\Phi$ is locally bounded. Consequently, in view of (1), $\mathcal{W}_\mathcal{X}(E_n) < \infty$ and so on $[0,1]$.

**Proposition 3.3.** Let $\Phi : I \to X^*$ be additive and such that $V_\Phi \ll \lambda$. Then $\Phi$ is an HKG-primitive.

4. Henstock-Kurzweil-Pettis integral

We begin with the following characterization of Pettis integrability that holds true in case of an arbitrary perfect measure in place of the Lebesgue one.

**Proposition 4.1.** For a scalarly integrable function $f : [0,1] \to X$ the following conditions are equivalent:

(i) $f$ is Pettis integrable;

(ii) the mapping $X^* \ni x^* \mapsto x^* f \in L_1[0,1]$ is $\tau_c(X^*,X)$-norm continuous;

(iii) the mapping $X^* \ni x^* \mapsto x^* f \in L_1[0,1]$ is $\tau(X^*,X)$-norm continuous.

**Proof.** (i) $\Rightarrow$ (ii) Since $f$ is Pettis integrable, the functional $x^* \mapsto \int_E(x^*, f(t)) \ dt$ is, for each $E \in \mathcal{L}$, weak*-continuous (cf. [14]). Due to Stegall’s result [8], the set $\nu_f(\mathcal{L})$ is norm relatively compact. Hence, if $x^*_0 \xrightarrow{\tau(X^*,X)} x^*_0$, then $x^*_0 \xrightarrow{\tau} x^*_0$ uniformly on $\nu_f(\mathcal{L})$. It follows that $\lim_n \frac{1}{n} \int_0^1 |x^*_0 f(t) - x^*_0 f(t)| \ dt = 0$.

(ii) $\Rightarrow$ (iii) The proof is almost the same.

(iii) $\Rightarrow$ (i) If $x^*_0 \xrightarrow{\tau(X^*,X)} x^*_0$, then $\int_E(x^*_0 f(t)) \ dt \xrightarrow{\tau} \int_E(x^*_0 f(t)) \ dt$ for each $E \in \mathcal{L}$. Thus, the functional $x^* \mapsto \int_E(x^*, f(t)) \ dt$ is, for each $E \in \mathcal{L}$, weak*-continuous. Consequently, $f$ is Pettis integrable (see [14]).

(ii) $\Rightarrow$ (i) The proof is the same, but now we assume that $B(X^*) \ni x^* \xrightarrow{\tau(X^*,X)} x^*_0$. We obtain now the weak* continuity of the functionals $x^* \mapsto \int_E(x^*, f(t)) \ dt$ on $B(X^*)$, but due to the Banach-Dieudonné Theorem (see [12, p. 154]) this yields its weak* continuity. Consequently, $f$ is Pettis integrable (see [14]).
In order to obtain a complete characterization of the HKP-primitive of functions taking values in a dual space with the WRNP, we need some preliminary results.

**Proposition 4.2.** Assume that \( \Phi : I \to X \) is of the form

\[
\Phi(I) = (HKP) \int_I f(t) \, dt,
\]

for each \( I \in \mathcal{I} \).

Then, for each \( I \in \mathcal{I} \), the mapping \( x^* \to \int_I (x^*, f(t)) \, dt \) is weak*-continuous. Moreover, there exists a partition \([0, 1] = \bigcup_k H_k\) such that, for every \( k \in \mathbb{N} \), \( f \) is Pettis integrable and scalarly bounded on \( H_k \), \( \text{AW}^\phi_x(H_k) < \infty \) and the functional \( x^* \to \tau_x (x^*, X) \) is \( \tau_e(X^*, X) \)-continuous.

**Proof.** The first continuity fact has been proven in [7]. Exactly as in the proof of Theorem 3.1 one can obtain a sequence of pairwise disjoint sets \( E_n \in \mathcal{L} \) such that \( \text{AW}^\phi_x(E_n) < \infty \), for each \( n \in \mathbb{N} \). It follows also from [7, Corollary 1] that there exists a decomposition \([0, 1] = \bigcup_k F_k\) into sets of positive measure such that \( f \) is Pettis integrable and scalarly bounded on each \( F_k \). Denote by \( \{H_k : k \in \mathbb{N}\} \) the collection of all intersections \( E_n \cap F_m \) of positive measure. Then, by Proposition 4.1, for each \( k \), the function \( x^* \to x^* f|_{H_k} \) is \( \tau_x(X^*, X) \)-norm continuous as a map from \( X^* \) to \( L_1(\lambda|_{H_k}) \), because \( f \) is Pettis integrable on \( H_k \). Consequently, if \( x^*_\alpha \to x_0^* \) in \( X^* \), then according to [4] and [6] we have

\[
\lim_{\alpha} \text{V}_x^{x^*} (x_0^*) = \lim_{\alpha} \int_{H_k} |x^*_\alpha f(t) - x^*_0 f(t)| \, dt = 0.
\]

**Lemma 4.3 (see [1, Lemma 3.3]).** Let \( Y \) be a Banach space and let \( \nu : \mathcal{L} \to Y \) be a \( \lambda \)-continuous measure of finite variation. If \( \Phi : I \to X \) is defined by \( \Phi(I) := \nu(I) \), for all \( I \in \mathcal{I} \), then \( \Phi \) is finite, \( \text{V}_x \ll \lambda \) and \( \text{V}_x(E) \ll |\nu(E)| \), whenever \( E \in \mathcal{L} \).

**Theorem 4.4.** Let \( X \) be a Banach space. Consider the following two properties of an additive interval function \( \Phi : I \to X \):

1. \( \text{W}^\phi_x \ll \lambda \) and there exists a decomposition \([0, 1] = \bigcup_k H_k\) of \([0, 1]\) into sets of positive measure such that for every \( k \in \mathbb{N} \) the function \( x^* \to \tau_x (x^*, X) \) is \( \tau_e(X^*, X) \)-continuous and \( \text{AW}^\phi_x(H_k) < \infty \).
2. There is an HKP-integrable function \( f : [0, 1] \to X \) such that

\[
\langle x^*, \Phi(I) \rangle = (HKP) \int_I (x^*, f(t)) \, dt \quad \text{for every } I \in \mathcal{I}.
\]

If (k) \( \Rightarrow \) (kk) for every additive \( \Phi : I \to X \), then \( X \) has the WRNP.

**Proof.** Let \( \nu : \mathcal{L} \to X \) be a \( \lambda \)-continuous measure of finite variation. Define \( \Phi : I \to X \) by \( \Phi(I) := \nu(I) \). It follows from Lemma 4.3 that \( \text{V}_x \ll \lambda \) and \( \text{V}_x \) is finite. So \( \Phi : I \to X \) is an additive interval measure such that \( \text{V}_{x^*\Phi} \ll \lambda \) for every \( x^* \in X^* \). Moreover, \( \text{V}_{x^*\Phi}(E) \ll |x^*\nu(E)| \), for every \( E \in \mathcal{L} \). Let \( \{x^*_\alpha\} \subset B(X^*) \) be a net of functionals that is \( \tau_e(X^*, X) \)-convergent to \( 0 \). Since \( \nu(\mathcal{L}) \) is a weakly
relatively compact subset of $X$, the net $(x_n^*, \nu)$ is uniformly convergent to zero on $\mathcal{L}$. Hence, $\lim_{\alpha} |x_n^*\nu|[0,1] = 0$. By the inequality $V_{x_n^*\phi}(E) \leq |x_n^*\nu|(E)$, for every $E \in \mathcal{L}$, we have also $\lim_{\alpha} V_{x_n^*\phi}[0,1] = 0$, what proves the weak*-continuity of the map $x^* \rightarrow V_{x^*\phi}[0,1]$.

We are going to prove yet the local boundedness of $W_{x^*}$. To do it notice that the classical Radon-Nikodým Theorem yields the existence of a decomposition $[0,1] = \bigcup_k H_k$ such that $|\nu|(E) \leq k|E|$, for every measurable $E \subset H_k$. It follows that

$$\frac{V_{x^*\phi}(E)}{|E|} \leq \frac{|x^*\nu|(E)}{|E|} \leq k$$

and hence $AW_{x^*}^\infty(H_k) < \infty$.

Thus, condition $(k)$ is satisfied. Hence, there is a Henstock-Kurzweil-Pettis integrable function $f : [0,1] \rightarrow X$ such that

$$\Phi(I) = (HK) \int_I f(t) \, dt, \quad \text{for every } I \in \mathcal{I}.$$  

Proceeding as in the proof of [2, Theorem 4.5] we see that $f$ is also Pettis integrable and $\nu$ is its indefinite Pettis integral.

Proposition 4.5. Let $X$ be an arbitrary Banach space and $\phi : I \rightarrow X$ be an additive interval function such that $W_{x^*}^\infty \ll \lambda$. Assume that there is a decomposition $[0,1] = \bigcup_k H_k$ into measurable sets of positive measure such that $V_{x^*\phi}(H_k) < \infty$ for every $k \in \mathbb{N}$ and every $x^* \in X^*$ and, for every $k \in \mathbb{N}$, the function $x^* \rightarrow V_{x^*\phi}(H_k)$ is sequentially weak*-continuous.

If $f : [0,1] \rightarrow X$ is a scalarly measurable function, then the set

$$K = \left\{ x^* \in X^* : x^* f \in HK[0,1] \text{ and } x^*\Phi(I) = (HK) \int_I (x^*, f(t)) \, dt, \forall I \in \mathcal{I} \right\}$$

is sequentially weak* closed.

If for every $k \in \mathbb{N}$, the function $x^* \rightarrow V_{x^*\phi}(H_k)$ is $\tau(X^*, X)$-continuous and $f$ is Pettis integrable on $H_k$, then $K$ is weak*-closed.

Proof. It is obvious that $K \neq \emptyset$ and $K$ is convex. Notice first that if $x^* \in K$, then $(x^*\Phi) = x^* f$ a.e. (see [10]). Let $\{x_n^*\} \subset K$ be such that $x_n^* \rightarrow x_0^*$ in the $w^*$-topology. We may assume, without loss of generality, that all $x_n^*$, $n = 0, 1, 2, ...$ belong to $B(X^*)$. By hypothesis $V_{x^*\phi} \ll \lambda$, and so there exists $g \in HK[0,1]$ such that $x_n^*\Phi(I) = (HK) \int_I g(t) \, dt$, for all $I \in \mathcal{I}$ (cf. [3]).

By the assumption and by [6, Corollary 3] we have, for each $k \in \mathbb{N}$,

$$\lim_{n} \int_{H_k} |x_n^*(t) - g(t)| \, dt = \lim_{n} V_{(x_n^*-x_0^*)\phi}(H_k) = 0.$$  

Hence, there is a subsequence $\{x_{k,n}^*\}_m \subset \{x_n^*\}$ with $\lim_{m} x_{k,n}^* f = g$, a.e. on $H_k$. It follows that $g = x_0^* f$ a.e. and so $x_0^* f \in HK[0,1]$. Moreover

$$\lim_{m} \int_{[0,1]} (x_{k,n}^*, f(t)) \, dt = \lim_{m} (x_{k,n}^*\Phi(I)) = (x_0^*, \Phi(I)) = \int_{[0,1]} (x_0^*, f(t)) \, dt.$$  

This yields $x_0^* \in K$ and so $K$ is weak* sequentially closed.
Assume now that $f$ is Pettis integrable on every $H_k$. We are going to prove that $K$ is weak*-closed. We know that for each $k \in \mathbb{N}$ the function $x^* \mapsto x^* \Phi(I)_{|H_k}$ is $\tau(X^*, X)$-norm continuous as a map from $X^*$ to $L_1(\lambda|H_k)$. Consequently, if $x^*_\alpha \tau(X^*, X) x^*_0$, then

$$
\lim_{\alpha} \int_{H_k} |x^*_\alpha f(t) - x^*_0 f(t)| dt = 0.
$$

By hypothesis $V(x^*_\alpha \Phi(I)) \ll \lambda$, and so there exists $g \in HK[0, 1]$ such that $x^*_0 \Phi(I) = (HK) \int g(t) dt$, for all $I \in \mathcal{I}$ and so $[0, \lambda$, Corollary 3] we have

$$
\lim_{\alpha} \int_{H_k} |x^*_\alpha f(t) - g(t)| dt = \lim_{\alpha} V(x^*_0 - x^*_0 \Phi(H_k)) = 0.
$$

It follows that $x^*_0 f = g \in HK[0, 1]$. Moreover

$$
\lim_{\alpha} \int_t \langle x^*_\alpha, f(t) \rangle dt = \lim_{\alpha} \langle x^*_\alpha, \Phi(I) \rangle = \langle x^*_0, \Phi(I) \rangle = \int_t \langle x^*_0, f(t) \rangle dt
$$

and so $x^*_0 \in K$. Thus, $K$ is $\tau(X^*, X)$-closed, and as it is convex, it is also weak*-closed.

Now we are ready to prove the main result of this section.

**Theorem 4.6.** Let $X$ be a Banach space such that $X^*$ has the WRNP and let $\Phi : \mathcal{I} \to X^*$ be an additive interval measure. Then the following two conditions are equivalent:

(j) $W^\phi \ll \lambda$ and there exists a decomposition $[0, 1] = \bigcup_k H_k$ of $[0, 1]$ into sets of positive measure such that for every $k \in \mathbb{N}$ the function $x^{**} \mapsto V_{x^{**}} \Phi(H_k)$ is weak*-continuous and $AW^\phi_\Phi(H_k) < \infty$.

(jj) There is an HKP-integrable function $f : [0, 1] \to X^*$ such that

$$
\langle x^{**}, \Phi(I) \rangle = (HK) \int_t \langle x^{**}, f(t) \rangle dt \quad \text{for every } I \in \mathcal{I}.
$$

Moreover, $f$ can be chosen in such a way that $\|f\|$ is a measurable function.

**Proof.** The implication $(jj) \Rightarrow (j)$ is a particular case of Proposition 4.2. In order to prove the implication $(j) \Rightarrow (jj)$, we may apply Theorem 3.1 to conclude that there exists a function $f : [0, 1] \to X^*$ that is HKG-integrable on $[0, 1]$ and Pettis integrable on each $H_k$, $k \in \mathbb{N}$. Proposition 4.5 yields the HKP-integrability of $f$ on $[0, 1]$.

**Remark 4.7.** According to Remark 2.7 each strongly measurable Pettis integrable (and hence also Henstok-Kurzweil-Pettis integrable) function with nowhere differentiable Pettis integral satisfies the conditions (j) and (jj) of Theorem 4.6 and has non-$\sigma$-finite variational measure $V_\phi$. 
References


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