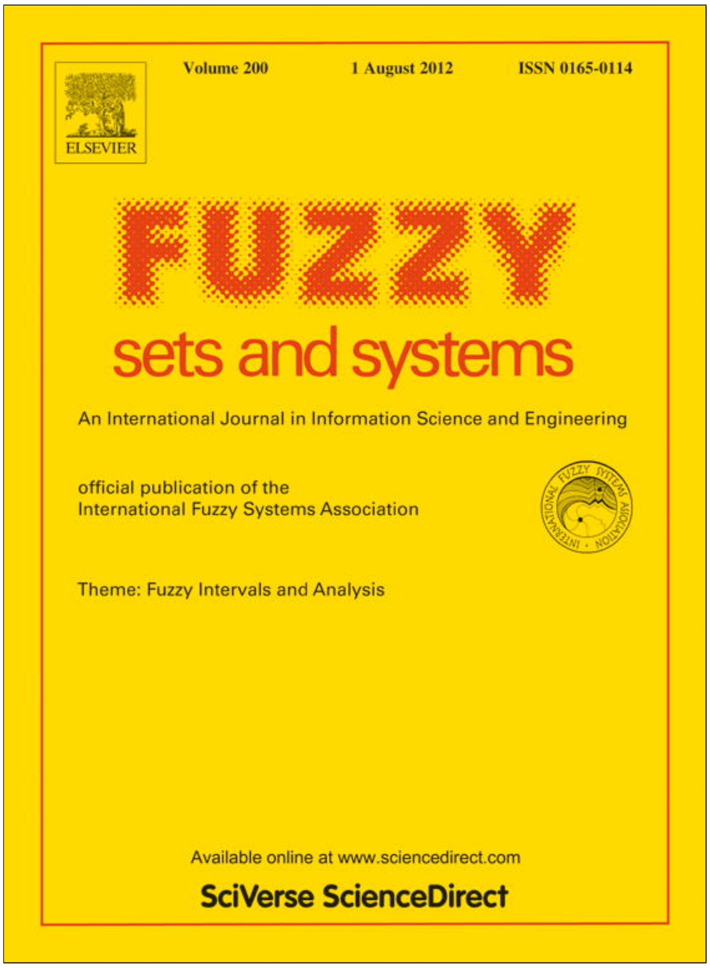
Volume 200
1 August 2012
ISSN 0165-0114
ELSEVIER
FUZZY
sets and systems
An International Journal in Information Science and Engineering
official publication of the
International Fuzzy Systems Association
Theme: Fuzzy Intervals and Analysis
Available online at www.sciencedirect.com
SciVerse ScienceDirect

Available online at www.sciencedirect.com
SciVerse ScienceDirect
Fuzzy Sets and Systems 200 (2012) 36–47
FUZZY sets and systems
www.elsevier.com/locate/fss

A decomposition theorem for the fuzzy Henstock integral ☆

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Received 17 January 2011; received in revised form 10 December 2011; accepted 12 December 2011
Available online 21 December 2011

Abstract

We study the fuzzy Henstock and the fuzzy McShane integrals for fuzzy-number valued functions. The main purpose of this paper is to establish the following decomposition theorem: a fuzzy-number valued function is fuzzy Henstock integrable if and only if it can be represented as a sum of a fuzzy McShane integrable fuzzy-number valued function and of a fuzzy Henstock integrable fuzzy number valued function generated by a Henstock integrable function.

Keywords: Fuzzy Henstock integral; Fuzzy McShane integral; Henstock–Kurzweil and McShane equi-integrability

1. Introduction

The Henstock integral introduced, independently, in 1957 by Kurzweil [11] and, in 1963 by Henstock [8], by a simple modification of Riemann's method turned out to be more general than that of Lebesgue. It is a powerful tool which integrates highly oscillating functions that the Lebesgue integral fails to integrate.

In this paper we continue the study of the Henstock integral for set valued functions started in [2–4] and we consider the more general setting of fuzzy-number valued functions. In case of the fuzzy-number space E^1, the fuzzy Henstock integral has been introduced and studied by Wu and Gong in [17,18]. It is an extension of the integrals introduced in [12,10]. Here we consider the fuzzy Henstock and the fuzzy McShane integrals for functions taking values in the fuzzy number space E^n. In Section 3 we give a characterization of the fuzzy-number valued functions which are fuzzy Henstock or McShane integrable by means of the equi-integrability of the support functions (see Proposition 3.5). As an application of this characterization we prove that the family of all fuzzy Henstock (resp. McShane) integrable fuzzy-number valued functions is properly enclosed in that of all weakly fuzzy Henstock (resp. McShane) integrable fuzzy-number valued functions. The main result of this paper is in Section 4 (Theorem 4.1):

A fuzzy-number valued function $\widetilde{\Gamma}\colon [a,b] \to E^n$ is fuzzy Henstock integrable if and only if $\widetilde{\Gamma}$ can be represented as $\widetilde{\Gamma}(t) = \widetilde{G}(t) + \widetilde{f}(t)$, where $\widetilde{G}\colon [a,b] \to E^n$ is fuzzy McShane integrable and $\widetilde{f}$ is a fuzzy Henstock integrable fuzzy number valued function generated by a Henstock integrable selection of $\widetilde{\Gamma}$.

☆ The authors were partially supported by the Grant 2008EEZ4N7 of MURST (Prin 2008) and by the Grant N N201 416139 of MNiSW of Poland.
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doi:10.1016/j.fss.2011.12.006

This result is a generalization of a similar decomposition for set valued functions (see [3,4]), but the technique used here is different. The essential tool to prove it is Theorem 4.2 that provides sufficient conditions guaranteeing the McShane equi-integrability of a family of nonnegative real valued Henstock–Kurzweil equi-integrable functions. Theorem 4.2 gives also new contributions to the theory of integration of real valued functions.

2. Basic facts

Let R^n be the n-dimensional Euclidean space endowed with the Euclidean norm $\|\cdot\|$. We denote by S^{n-1} its closed unit ball and by $ck(R^n)$ the family of all nonempty compact convex subsets of R^n endowed with the *Hausdorff distance*

$$d_H(A,B):=\max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|,\ \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\},$$

and the operations

$$A+B:=\{x+y:\ x\in A, y\in B\},\quad kA:=\{kx:\ x\in A\}.$$

The space $ck(R^n)$ endowed with the Hausdorff distance is a complete metric space. For every $A\in ck(R^n)$ the *support function* of A is denoted by $s(\cdot,A)$ and defined by $s(x,A)=\sup\{\langle x,y\rangle:\ y\in A\}$, for each $x\in R^n$. Clearly the map $x\longmapsto s(x,A)$ is sublinear on R^n and $-s(-x,A)=\inf\{\langle x,y\rangle:\ y\in A\}$, for each $x\in R^n$.

According to Hörmander's equality (cf. [9, p. 9]), for A and B nonempty members of $ck(R^n)$ we have the equality

$$d_H(A,B)=\sup_{x\in S^{n-1}}|s(x,A)-s(x,B)|. \tag{1}$$

Definition 2.1. The *n-dimensional fuzzy number space* E^n is defined as the set

$$E^n=\{u: R^n\to[0,1]:\ u \text{ satisfies conditions (1)–(4) below}\}:$$

(1) u is a normal fuzzy set, i.e. there exists $x_0\in R^n$, such that $u(x_0)=1$;
(2) u is a convex fuzzy set, i.e. $u(tx+(1-t)y)\geq\min\{u(x),u(y)\}$ for any $x,y\in R^n$, $t\in[0,1]$;
(3) u is upper semi-continuous;
(4) supp $u=\overline{\{x\in R^n: u(x)>0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

For $r\in(0,1]$ let $[u]^r=\{x\in R^n:\ u(x)\geq r\}$ and $[u]^0=\overline{\bigcup_{s\in(0,1]}[u]^s}$. If $u\in E^n$ and $r\in[0,1]$, then $[u]^r\in ck(R^n)$.

In the sequel we will use the following representation theorem (see [1,19]).

Theorem 2.2. *If $u\in E^n$, then*

(i) $[u]^r\in ck(R^n)$, *for all* $r\in[0,1]$;
(ii) $[u]^{r_2}\subset[u]^{r_1}$, *for* $0\leq r_1\leq r_2\leq 1$;
(iii) *if* (r_k) *is a nondecreasing sequence converging to* $r>0$, *then*

$$[u]^r=\bigcap_{k\geq 1}[u]^{r_k}.$$

Conversely, if $\{A_r:\ r\in[0,1]\}$ *is a family of subsets of* R^n *satisfying* (i)–(iii), *then there exists a unique* $u\in E^n$ *such that* $[u]^r=A_r$ *for* $r\in(0,1]$ *and* $[u]^0=\overline{\bigcup_{0<r\leq 1}[u]^r}\subset A_0$.

Define $D: E^n\times E^n\to R^+\cup\{0\}$ by the equation

$$D(u,v)=\sup_{r\in[0,1]}d_H([u]^r,[v]^r).$$

(E^n,D) is a complete metric space (see [1,19]).

For $u,\ v \in E^n$ and $k \in R$ the addition and the scalar multiplication are defined respectively by

$$[u+v]^r := [u]^r + [v]^r \quad \text{and} \quad [ku]^r := k[u]^r.$$

Let $[a,b]$ be a bounded closed interval of the real line equipped by the Lebesgue measure λ. We denote by $\mathcal{L}$ and by $\mathcal{I}$ the families of all Lebesgue measurable subsets of $[a,b]$ and of all closed subintervals of $[a,b]$, respectively. If $I \in \mathcal{I}$, then $|I|$ denotes its length. A *partition in* $[a,b]$ is a collection of pairs $\mathcal{P} = \{(I_i, t_i) :\ i = 1, \ldots, p\}$, where I_i, are non-overlapping subintervals of $[a,b]$ and t_i are points of $[a,b]$, $i = 1, \ldots, p$. If $\bigcup_{i=1}^p I_i = [a,b]$ we say that $\mathcal{P}$ is a *partition of* $[a,b]$. If $t_i \in I_i$, $i = 1, \ldots, p$, we say that $\mathcal{P}$ is a *Perron partition of* $[a,b]$. A *gauge* on $[a,b]$ is a positive function on $[a,b]$. For a given gauge δ on $[a,b]$, we say that a partition $\mathcal{P} = \{(I_i, t_i) :\ i = 1, \ldots, p\}$ is δ*-fine* if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1, \ldots, p$.

Given $f: [a,b] \to R^n$ and a partition $\mathcal{P} = \{(I_i, t_i) :\ \ i = 1, \ldots, p\}$ in $[a,b]$ we set

$$\sigma(f, \mathcal{P}) = \sum_{i=1}^{p} |I_i| f(t_i).$$

Let us recall the definitions of McShane and Henstock integral for R^n-valued functions.

Definition 2.3. A function $g: [a,b] \to R^n$ is said to be *McShane* (resp. *Henstock*) *integrable on* $[a,b]$ if there exists a vector $w \in R^n$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that

$$\|\sigma(g, \mathcal{P}) - w\| < \varepsilon$$

for each δ-fine partition (resp. Perron partition) $\mathcal{P}$ of $[a,b]$. We set $(Mc)\int_a^b g(t)\,dt := w$ (resp. $(H)\int_a^b g(t)\,dt := w$).

In case $n = 1$, g is said to be *Henstock–Kurzweil* integrable. We denote by $\mathcal{M}c[a,b]$ (resp. $\mathcal{HK}[a,b]$) the set of all real valued McShane (resp. Henstock–Kurzweil) integrable functions on $[a,b]$.

Remark 2.4. We recall that McShane, Pettis and Bochner integrability coincide for functions taking values in a finite dimensional space.

A set-valued function $\Gamma: [a,b] \to ck(R^n)$ is said to be *Effros measurable* (or for short *measurable*) if for each open subset O of R^n, the set $\{t \in [a,b] : \Gamma(t) \cap O \neq \emptyset\}$ is a measurable set. Γ is said to be *scalarly measurable* if for every $x \in R^n$, the map $s(x, \Gamma(\cdot))$ is measurable. It is however well known that in case of $ck(R^n)$-valued multifunctions the scalar measurability yields the measurability. A set-valued function $\Gamma: [a,b] \to ck(R^n)$ is said to be *scalarly* (resp. *scalarly Henstock–Kurzweil*) *integrable on* $[a,b]$ if for each $x \in R^n$ the real function $s(x, \Gamma(t))$ is integrable (resp. Henstock–Kurzweil integrable) on $[a,b]$.

A function $f: [a,b] \to R^n$ is called a *selection* of a set-valued function $\Gamma: [a,b] \to ck(R^n)$ if, for every $t \in [a,b]$, one has $f(t) \in \Gamma(t)$. By $\mathcal{S}(\Gamma)$ (resp. $\mathcal{S}_H(\Gamma)$) we denote the family of all measurable selections of Γ that are integrable (resp. Henstock integrable).

Definition 2.5. A measurable set-valued function $\Gamma: [a,b] \to ck(R^n)$ is said to be *Aumann integrable on* $[a,b]$ if $\mathcal{S}(\Gamma) \neq \emptyset$. Then we define

$$(A)\int_a^b \Gamma(t)\,dt := \left\{ \int_a^b f(t)\,dt : f \in \mathcal{S}(\Gamma) \right\}.$$

Definition 2.6 (*See Amri and Hess [5]*). A set-valued function $\Gamma: [a,b] \to ck(R^n)$ is said to be *Pettis integrable* on $[a,b]$ if Γ is scalarly integrable on $[a,b]$ and for each $A \in \mathcal{L}$ there exists a set $W_A \in ck(R^n)$ such that for each $x \in R^n$, we have

$$s(x, W_A) = \int_A s(x, \Gamma(t))\,dt.$$

Then we set $W_A = (P)\int_A \Gamma(t)\,dt$, for each $A \in \mathcal{L}$.

Given $\Gamma\colon [a,b] \to ck(R^n)$ and a partition $\mathcal{P} = \{(I_i, t_i):\quad i = 1, \ldots, p\}$ in $[a,b]$ we set

$$\sigma(\Gamma, \mathcal{P}) = \sum_{i=1}^{p} |I_i|\Gamma(t_i).$$

Definition 2.7. A set-valued function $\Gamma\colon [a,b] \to ck(R^n)$ is said to be *Henstock* (resp. *McShane*) *integrable* on $[a,b]$ if there exists a nonempty set $W \in ck(R^n)$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that for each δ-fine Perron partition (resp. partition) $\mathcal{P}$ of $[a,b]$, we have

$$d_H(W, \sigma(\Gamma, \mathcal{P})) < \varepsilon.$$

Remark 2.8. We recall that Pettis and McShane integrals coincide for set-valued functions taking values in $ck(R^n)$, with the same value for the integrals (see [3, Proposition 2]). Then, taking into account Remark 2.4, by [5, Theorems 3.7 and 5.4] it follows that for scalarly integrable set-valued functions taking values in a finite dimensional space the Pettis, the McShane and the Aumann integrability coincide (with the same value for the integrals).

The following theorem has been proven in [3] (with R^n replaced by an arbitrary separable Banach space and the McShane integral in place of the Aumann integral).

Theorem 2.9. *Let $\Gamma\colon [a,b] \to ck(R^n)$ be a scalarly Henstock–Kurzweil integrable set valued function. Then the following conditions are equivalent*:

(i) *Γ is Henstock integrable*;
(ii) *for every $f \in \mathcal{S}_H(\Gamma)$ the multifunction $G\colon [a,b] \to ck(R^n)$ defined by $\Gamma(t) = G(t) + f(t)$ is Aumann integrable*;
(iii) *there exists $f \in \mathcal{S}_H(\Gamma)$ such that the multifunction $G\colon [a,b] \to ck(R^n)$ defined by $\Gamma(t) = G(t) + f(t)$ is Aumann integrable*;
(iv) *every measurable selection of Γ is Henstock integrable*.

3. Weakly fuzzy Henstock and fuzzy Henstock integral

We recall that a fuzzy-number valued function $\widetilde{\Gamma}\colon [a,b] \to E^n$ is said to be *strongly measurable* if for every $r \in [0,1]$ the set valued function $[\widetilde{\Gamma}]^r\colon [a,b] \to ck(R^n)$ is graph measurable (see [9, p. 141]) . Since the range space R^n is finite dimensional this is equivalent to the measurability of all support functions $s(x, [\widetilde{\Gamma(\cdot)}]^r)$, $x \in S^{n-1}$.

From now on we set

$$\widetilde{\Gamma}_r(t) = [\widetilde{\Gamma}(t)]^r.$$

A fuzzy-number-valued function $\widetilde{\Gamma}\colon [a,b] \to E^n$ is said to be *scalarly* (resp. *scalarly Henstock–Kurzweil*) *integrable* on $[a,b]$ if for all $r \in [0,1]$ the set-valued function $\widetilde{\Gamma}_r\colon [a,b] \to ck(R^n)$ is scalarly (resp. scalarly Henstock–Kurzweil) integrable.

Definition 3.1. A fuzzy-number valued function $\widetilde{\Gamma}\colon [a,b] \to E^n$ is said to be *weakly fuzzy Henstock* or *weakly fuzzy Pettis* or *weakly fuzzy McShane* integrable on $[a,b]$ if for every $r \in [0,1]$ the set-valued function $\widetilde{\Gamma}_r$ is Henstock or Pettis or McShane integrable on $[a,b]$ and there exists a fuzzy number $\widetilde{A} \in E^n$ such that for any $r \in [0,1]$ and for any $x \in R^n$ we have

$$s(x, [\widetilde{A}]^r) = (HK)\int_a^b s(x, \widetilde{\Gamma}_r(t))\, dt$$

or

$$s(x, [\widetilde{A}]^r) = \int_a^b s(x, \widetilde{\Gamma}_r(t))\, dt,$$

respectively.

Definition 3.2. A fuzzy-number valued function $\widetilde{\Gamma}\colon [a,b] \to E^n$ is said to be *fuzzy Aumann integrable* on $[a,b]$ if there exists a fuzzy number $\widetilde{A} \in E^n$ such that for every $r \in [0,1]$ the set-valued function $\widetilde{\Gamma}_r$ is Aumann integrable on $[a,b]$ and $[\widetilde{A}]^r = (A)\int_a^b \widetilde{\Gamma}_r(t)\,dt$. We write $(FA)\int_a^b \widetilde{\Gamma}(t)\,dt := \widetilde{A}$.

Remark 3.3. Since Pettis, McShane and Aumann integrals coincide for set-valued functions taking values in a finite dimensional space, then also the fuzzy Aumann, the weakly fuzzy Pettis and the weakly fuzzy McShane integrals coincide.

Definition 3.4 (*See Wu and Gong [18]*). A fuzzy-number-valued function $\widetilde{\Gamma}\colon [a,b] \to E^n$ is said to be *fuzzy Henstock* (resp. *fuzzy McShane*) integrable on $[a,b]$ if there exists a fuzzy number $\tilde{A} \in E^n$ such that for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that for every δ-fine Perron partition (resp. partition) $\mathcal{P}$ of $[a,b]$, we have

$$D(\tilde{A}, \sigma(\Gamma, \mathcal{P})) < \varepsilon.$$

We write $(FH)\int_a^b \widetilde{\Gamma}(t)\,dt := \tilde{A}$ (resp. $(FMc)\int_a^b \widetilde{\Gamma}(t)\,dt := \tilde{A}$).

By means the notion of equi-integrability it is possible to characterize the fuzzy Henstock and the fuzzy McShane integrability. We recall that a family $\{g_\alpha\}$ of real valued functions in $\mathcal{HK}[a,b]$ (resp. $\mathcal{M}c[a,b]$) is said to be *Henstock–Kurzweil* (resp. *McShane*) *equi-integrable on* $[a,b]$ whenever for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\sup_\alpha \left| \sigma(g_\alpha, \mathcal{P}) - (HK)\int_a^b g_\alpha(t)\,dt \right| < \varepsilon,$$

$$\left(\text{resp. } \sup_\alpha \left| \sigma(g_\alpha, \mathcal{P}) - \int_a^b g_\alpha(t)\,dt \right| < \varepsilon \right)$$

for each δ-fine Perron partition (resp. partition) $\mathcal{P}$ of $[a,b]$.

Proposition 3.5. *Let $\widetilde{\Gamma}\colon [a,b] \to E^n$ be a Henstock–Kurzweil scalarly (resp. scalarly) integrable fuzzy-number-valued function. Then the following are equivalent:*

(j) *$\widetilde{\Gamma}$ is fuzzy Henstock (resp. McShane) integrable on $[a,b]$;*
(jj) *the collection $\{s(x, \widetilde{\Gamma}_r(\cdot)) : x \in S^{n-1}$ and $0 \le r \le 1\}$ is Henstock–Kurzweil (resp. McShane) equi-integrable.*

Proof. (j) $\Rightarrow$ (jj). According to Hörmander's equality and the definition of metric D in E^n we have

$$D\left(\tilde{A}, \sum_{i=1}^p |I_i|\,\widetilde{\Gamma}(t_i)\right) = \sup_{r\in[a,b]} \sup_{x \in S^{n-1}} \left| s(x, [\tilde{A}]^r) - \sum_{i=1}^p s(x, \widetilde{\Gamma}_r(t_i))\,|I_i| \right|. \tag{2}$$

So the implication holds true.

(jj) $\Rightarrow$ (j). Let us fix $r \in [0,1]$. Since the collection $\{s(x, \widetilde{\Gamma}_r(\cdot)) : x \in S^{n-1}\}$ is Henstock–Kurzweil (resp. McShane) equi-integrable, by [3, Proposition 1] there exists $A_r \in ck(R^n)$ such that for each $x \in S^{n-1}$

$$s(x, A_r) = (HK)\int_a^b s(x, \widetilde{\Gamma}_r(t))\,dt, \tag{3}$$

$$\left(\text{resp. } s(x, A_r) = \int_a^b s(x, \widetilde{\Gamma}_r(t))\,dt \right). \tag{4}$$

Now we are going to prove that the family $\{A_r\colon r \in [0,1]\}$ satisfies properties (i)–(iii) of Theorem 2.2. Since $A_r \in ck(R^n)$ it remains to prove only (ii) and (iii). Let $0 \le r_1 \le r_2 \le 1$. By Theorem 2.2 we have $\widetilde{\Gamma}_{r_2}(t) \subset \widetilde{\Gamma}_{r_1}(t)$, for each $t \in [a,b]$. Therefore

$$s(x, A_{r_2}) \le s(x, A_{r_1}),$$

for each $x \in R^n$. Then, as a consequence of the separation theorem for convex sets, we also infer the inclusion $A_{r_2} \subset A_{r_1}$ and property (ii) is satisfied. If (r_k) is a nondecreasing sequence converging to $r > 0$, then for each $t \in [a, b]$ we have

$$\widetilde{\Gamma}_r(t) = \bigcap_{k \ge 1} \widetilde{\Gamma}_{r_k}(t).$$

Consequently (see [16, Proposition 1]) *

$$s(x, \widetilde{\Gamma}_r(t)) = \lim_k s(x, \widetilde{\Gamma}_{r_k}(t)),$$

for each $t \in [a, b]$ and $x \in R^n$.

By hypothesis, for each $x \in R^n$, the sequence of real valued functions $(s(x, \widetilde{\Gamma}_{r_k}(\cdot)))$ is Henstock–Kurzweil (resp. McShane) equi-integrable. So we have (see [15])

$$s(x, A_r) = (HK)\int_a^b s(x, \widetilde{\Gamma}_r(t))\,dt = \lim_k (HK)\int_a^b s(x, \widetilde{\Gamma}_{r_k}(t))\,dt = \lim_k s(x, A_{r_k}) = s\left(x, \bigcap_{k \ge 1} A_{r_k}\right),$$

$$\left[\text{resp. } s(x, A_r) = \int_a^b s(x, \widetilde{\Gamma}_r(t))\,dt = \lim_k \int_a^b s(x, \widetilde{\Gamma}_{r_k}(t))\,dt = \lim_k s(x, A_{r_k}) = s\left(x, \bigcap_{k \ge 1} A_{r_k}\right)\right].$$

Since the above equalities hold for each $x \in R^n$, we obtain $A_r = \bigcap_{k \ge 1} A_{r_k}$ and property (iii) is satisfied. Therefore according to Theorem 2.2 there exists a unique $u \in E^n$ such that $[u]^r = A_r$ for $r \in (0, 1]$ and $[u]^0 = \overline{\bigcup_{s \in (0,1]} [u]^s} \subset A_0$. * Taking into account (3) (resp. (4)) and the definition of the distance D we get the fuzzy Henstock (resp. McShane) integrability of $\widetilde{\Gamma}$ on $[a, b]$ with the fuzzy Henstock (resp. McShane) integral equal to u. □

As a direct consequence of Proposition 3.5 we have the following characterization of the fuzzy Henstock and fuzzy McShane integrability:

Corollary 3.6. *A fuzzy-number-valued function $\widetilde{\Gamma}: [a, b] \to E^n$ is fuzzy Henstock (resp. fuzzy McShane) integrable on $[a, b]$ if and only if it is weakly fuzzy Henstock (resp. weakly fuzzy McShane) integrable on $[a, b]$ and the collection $\{s(x, \widetilde{\Gamma}_r(\cdot)) : x \in S^{n-1}$ and $0 \le r \le 1\}$ is Henstock–Kurzweil (resp. McShane) equi-integrable.*

Now we use Proposition 3.5 to show that the family of all weakly fuzzy Henstock (resp. McShane) integrable functions is wider than the family of all fuzzy Henstock (resp. McShane) integrable fuzzy-number-valued functions. At first it may look strange since we are in $\mathbb{R}^n$ and the Henstock (resp. McShane) integral of $ck(\mathbb{R}^n)$-valued multifunctions defined with the help of support functions coincides with that defined with the help of the Hausdorff distance. In particular, for each $0 \le r \le 1$ the family $\{s(x, \widetilde{\Gamma}_r(\cdot)) : x \in S^{n-1}\}$ is Henstock–Kurzweil (resp. McShane) equi-integrable. But it is known that an infinite union of equi-integrable families may be not equi-integrable. Thus, the fuzzy approach may change the situation. In fact, in the example below we show even more. We prove that there exists a weakly fuzzy McShane integrable fuzzy-number-valued function on [0,1] that is not fuzzy Henstock integrable (then also not fuzzy McShane integrable).

Example 3.7. It is enough to show that such a function exists for $n = 1$. Define $g_m = \chi_{[1-2^{-m},1]}$, $m = 1, 2, \ldots$ where χ_B denotes the characteristic function of the set B, and let $f_k = \sum_{m=1}^{k} g_m$, $k = 1, 2, \ldots$.

Remark that $f_k(t) \le f_{k+1}(t)$, for $t \in [0, 1]$, and set $\mathcal{O}_r(t) = [0, f_k(t)]$, $\mathcal{Q}_r = [0, 1 - 2^{-k}]$, for $(k+1)^{-1} < r \le k^{-1}$, $t \in [0, 1]$ and $k \in N$, $\mathcal{O}_0(t) = \overline{\bigcup_{r \in (0,1]} \mathcal{O}_r(t)}$, $\mathcal{Q}_0 = [0, 1]$. It is easy to check that $\mathcal{O}_r(t)$ and $\mathcal{Q}_r$ satisfy conditions (i)–(iii) of Theorem 2.2, for any $t \in [0, 1]$, then from Theorem 2.2 it is possible to define a function $\widetilde{\Gamma}: [0, 1] \to E^1$ and a fuzzy number $\widetilde{A}$ such that $\widetilde{\Gamma}_r(t) = \mathcal{O}_r(t)$ and $[\widetilde{A}]^r = \mathcal{Q}_r$ for all $0 < r \le 1$ and all $t \in [0, 1]$.

The fuzzy-number-valued function $\widetilde{\Gamma}$ is weakly fuzzy McShane (and then also weakly fuzzy Henstock) integrable. In fact for each $k \in N$ it is

$$\int_0^1 f_k(t)\,dt = \sum_{m=1}^{k} \int_0^1 g_m(t)\,dt = \sum_{m=1}^{k} \frac{1}{2^m} = 1 - \frac{1}{2^k},$$

and for each $x \in S^0 = [-1, 1]$ and each $(k+1)^{-1} < r \le k^{-1}$, $k = 1, 2, \ldots$, we have

$$s(x, [\widetilde{A}]^r) = \begin{cases} x(1-2^{-k}) & \text{if } 0 < x \le 1, \\ 0 & \text{if } -1 \le x \le 0, \end{cases}$$

$$s(x, \tilde{\Gamma}_r(t)) = \begin{cases} x f_k(t) & \text{if } 0 < x \le 1, \\ 0 & \text{if } -1 \le x \le 0, \end{cases}$$

and

$$\int_0^1 s(x, \tilde{\Gamma}_r(t))\,dt = \begin{cases} x \int_0^1 f_k(t)\,dt = x(1-2^{-k}) & \text{if } 0 < x \le 1, \\ 0 & \text{if } -1 \le x \le 0. \end{cases}$$

Now, by an application of Proposition 3.5, we are going to prove that $\widetilde{\Gamma}$ is not fuzzy McShane integrable. To this aim it is enough to show that the sequence (f_k) is not Henstock–Kurzweil equi-integrable.

Given a gauge δ on [0, 1], we define $\tilde{\delta}: [0, 1] \to R^+$ as follows:

$$\tilde{\delta}(t) = \begin{cases} \min\{\delta(t), t-(1-2^{-m+1}), 1-2^{-m}-t\} & \text{if } 1-2^{-m+1} < t < 1-2^{-m},\ m \in N, \\ \min\{\delta(t), 1-2^{-m}\} & \text{if } t = 1-2^{-m+1},\ m \in N, \\ \delta(1) & \text{if } t = 1. \end{cases}$$

Let $\mathcal{P} = \{(I_j, t_j)\}_{j=1}^p$ be a $\tilde{\delta}$-fine Perron partition of $[a, b]$. By the definition of $\tilde{\delta}$ it follows that $t_j = 1$ for some j. For simplicity we can assume that $j = p$. Let $q \in N$ be the first index such that $I_p \supset [1-2^{-q}, 1]$. Then, since $f_k(1) = k$, we have $f_k(t_p)|I_p| = k|I_p| \ge k2^{-q}$, for $k = 1, 2, \ldots$.

Moreover, since $f_k = m$ on $[1-2^{-m}, 1-2^{-(m+1)})$ for $k \ge m$, by the definition of $\tilde{\delta}$ it follows $\sum_{j=1}^{p-1} f_k(t_j)|I_j| \ge \sum_{m=1}^{q-2} m2^{-(m+1)}$, for $k \ge q$. Hence, for $k \ge q$,

$$\left|\sum_{j=1}^{p} f_k(t_j)|I_j| - \int_0^1 f_k(t)\,dt\right| \ge f_k(t_p)|I_p| + \sum_{j=1}^{p-1} f_i(t_j)|I_j| - \int_0^1 f_k(t)\,dt \ge \frac{k}{2^q} + \sum_{m=1}^{q-2} m2^{-(m+1)} - 1 + \frac{1}{2^k}.$$

Consequently, since q depends only on $\mathcal{P}$, we have

$$\lim_{k\to\infty}\left|\sum_{j=1}^{p} f_k(t_j)|I_j| - \int_0^1 f_k(t)\,dt\right| = \infty,$$

that gives the mentioned Henstock not equi-integrability of (f_k). □

4. A decomposition of the fuzzy Henstock integral

The following decomposition theorem is the main result of the paper.

Theorem 4.1. *Let $\widetilde{\Gamma}: [a, b] \to E^n$ be a fuzzy-number valued function on $[a, b]$. Then the following conditions are equivalent*:

(A) *$\widetilde{\Gamma}$ is fuzzy Henstock integrable.*

(B) *For every Henstock integrable function $f \in \mathcal{S}_H(\widetilde{\Gamma}_1)$ the fuzzy-number valued function $\widetilde{G}: [a, b] \to E^n$ defined by $\widetilde{\Gamma}(t) = \widetilde{G}(t) + \widetilde{f}(t)$ (where $\widetilde{f}(t) = \chi_{\{f(t)\}}$) is fuzzy McShane integrable on $[a, b]$ and*

$$\left[(FH)\int_a^b \widetilde{\Gamma}(t)\,dt\right]^r = \left[(FMc)\int_a^b \widetilde{G}(t)\,dt\right]^r + (H)\int_a^b f(t)\,dt, \tag{5}$$

for every $r \in [0, 1]$.

(C) *There exists a Henstock integrable function* $f \in \mathcal{S}_H([\widetilde{\Gamma}]^1)$ *such that the fuzzy-number valued function* $\widetilde{G}\colon [a,b] \to E^n$ *defined by* $\widetilde{\Gamma}(t) = \widetilde{G}(t) + \widetilde{f}(t)$ *is fuzzy McShane integrable on* $[a,b]$ *and*

$$\left[(FH)\int_a^b \widetilde{\Gamma}(t)\,dt\right]^r = \left[(FMc)\int_a^b \widetilde{G}(t)\,dt\right]^r + (H)\int_a^b f(t)\,dt, \tag{6}$$

for every $r \in [0,1]$.

As readers may easily observe it is quite easy to define $\widetilde{G}$ and $\widetilde{f}$. It is however not so simple to show that $\widetilde{G}$ is fuzzy McShane integrable. The proof of this fact is the most difficult part of our paper.

Before proving Theorem 4.1 we need some preliminary results. It is well known that if $f\colon [a,b] \to R$ is a nonnegative Henstock–Kurzweil integrable function, then f is McShane integrable. So one could expect that if $\mathbb{A}$ is a family of nonnegative Henstock–Kurzweil equi-integrable functions, then $\mathbb{A}$ is also McShane equi-integrable. At the moment we do not know if this is true, however under additional suitable conditions next theorem gives the expected McShane equi-integrability. The idea of our proof is taken from [6, Theorem 8].

Theorem 4.2. *Let* $S \neq \emptyset$ *be an arbitrary set and let* $\mathbb{A} = \{g_\alpha\colon [a,b] \to [0,\infty)\colon \alpha \in S\}$ *be a family of functions satisfying the following conditions*:

(a) $\mathbb{A}$ *is Henstock–Kurzweil equi-integrable.*
(b) $\mathbb{A}$ *is totally bounded in the* L^1 *norm.*
(c) $\mathbb{A}$ *is pointwise bounded.*

Then the family $\mathbb{A}$ *is also McShane equi-integrable.*

Proof. Given $\varepsilon > 0$ and $k \in N \cup \{0\}$ we set $\eta_k = 2^{-k}\varepsilon^2/(2+12(k+1))$. By (b) we can find $l_{k,0}, \ldots, l_{k,i(k)} \in \mathbb{A}$ such that $\mathbb{A} \subset \bigcup_{j=1}^{i(k)} B(l_{k,j}, \eta_k)$, where as usual $B(l,r)$ denotes the ball with center at l and radius r in the L^1 norm.

By (a) and by the remark that the functions $l_{k,0}, \ldots, l_{k,i(k)}$ are McShane integrable (since they are nonnegative and Henstock–Kurzweil integrable; see [7]) we can find a gauge δ_k in $[a,b]$ such that:

$$\left|\sigma(l_{k,j}, \mathcal{P}) - \int_a^b l_{k,j}\right| < \eta_k, \tag{7}$$

for every $j \le i(k)$ and every δ_k-fine partition $\mathcal{P}$ of $[a,b]$, and

$$\sup\left\{\left|\sigma(g_\alpha, \mathcal{P}) - (HK)\int_a^b g_\alpha\right| : \alpha \in \mathbb{A}\right\} < \eta_k, \tag{8}$$

for every δ_k-fine Perron partition $\mathcal{P}$ of $[a,b]$.

By (c) there exists $g\colon [a,b] \to [0,\infty)$ such that

$$0 \le g_\alpha(t) \le g(t), \tag{9}$$

for every $t \in [a,b]$ and every $g_\alpha \in \mathbb{A}$.

Now fix $g_\alpha \in \mathbb{A}$ and take $j_\alpha \le i(k)$ such that $\int_a^b |g_\alpha - l_{k,j_\alpha}| \le \eta_k$. Then

$$\left|\int_H (g_\alpha - l_{k,j_\alpha})\right| \le \eta_k, \tag{10}$$

for each $H \in \mathcal{L}$.

By (7) and [15, Lemma 3.5.6], if $\mathcal{S}$ is any partial δ_k-fine partition in $[a,b]$, we have

$$\left|\sigma(l_{k,j_\alpha}, \mathcal{S}) - \int_{\bigcup\{J:(J,t)\in\mathcal{S}\}} l_{k,j_\alpha}\right| \le \eta_k. \tag{11}$$

So, if $\mathcal{R}$ is any partial δ_k-fine Perron partition in $[a,b]$, by (8), (11), and by [15, Lemma 3.5.6], we get

$$\left|\sigma(g_\alpha,\mathcal{R})-\int_{\bigcup\{I:(I,t)\in\mathcal{R}\}} g_\alpha\right|\le \eta_k$$

and

$$\left|\sigma(g_\alpha-l_{k,j_\alpha},\mathcal{R})-\int_{\bigcup\{I:(I,t)\in\mathcal{R}\}} (g_\alpha-l_{k,j_\alpha})\right|\le 2\eta_k.$$

Consequently, by (10) we have

$$\sum_{(I,t)\in\mathcal{R}} |I|\,(g_\alpha-l_{k,j_\alpha})(t)=\sigma(g_\alpha-l_{k,j_\alpha},\mathcal{R})\le 3\eta_k. \tag{12}$$

Now set

$$V=\bigcup\{(t-\delta_k(t),t+\delta_k(t)):\ g_\alpha(t)-l_{k,j_\alpha}(t)\ge\varepsilon\}.$$

Then by [6, Lemma 6] applied to the function $g_\alpha-l_{k,j_\alpha}$ we have

$$\lambda([a,b]\cap V)\le 3\eta_k/\varepsilon.$$

Set

$$A_k=\{t\in[a,b]:\ k\le g(t)<k+1\}, \tag{13}$$

and define a new gauge δ on $[a,b]$ by setting $\delta(t)=\delta_k(t)$, whenever $t\in A_k$.

Let $\mathcal{Q}=\{(J_i,t_i):\ i=1,\dots,p\}$ be a δ-fine partition of $[a,b]$ and set

$$T_k=\{i:\ i\le p,\ t_i\in A_k\},\quad H_k=\bigcup_{i\in T_k} J_i.$$

Since

$$\bigcup\{J_i:\ i\in T_k,\ g_\alpha(t_i)-l_{k,j_\alpha}(t_i)\ge\varepsilon\}\subset V,$$

we have

$$\sum_{\{i\in T_k:\ g_\alpha(t_i)-l_{k,j_\alpha}(t_i)\ge\varepsilon\}} |J_i|\le 3\eta_k/\varepsilon.$$

In a similar way we obtain

$$\sum_{\{i\in T_k:\ l_{k,j_\alpha}(t_i)-g_\alpha(t_i)\ge\varepsilon\}} |J_i|\le 3\eta_k/\varepsilon.$$

Moreover by (9) and (13) we have

$$|l_{k,j_\alpha}(t_i)-g_\alpha(t_i)|\le 2g(t_i)<2(k+1),$$

for every $t_i\in A_k$.

So

$$\sum_{i\in T_k} |J_i|\,|l_{k,j_\alpha}(t_i)-g_\alpha(t_i)|\le\varepsilon\lambda(H_k)+12\eta_k(k+1)/\varepsilon. \tag{14}$$

Hence by (10), (11) and (14) we obtain

$$\left|\int_{H_k} g_\alpha - \sum_{i\in T_k} |J_i|\, g_\alpha(t_i)\right| \le \left|\int_{H_k} (g_\alpha - l_{k,j_\alpha})\right| + \left|\int_{H_k} l_{k,j_\alpha} - \sum_{i\in T_k} |J_i|\, l_{k,j_\alpha}(t_i)\right| + \sum_{i\in T_k} |J_i|\, |l_{k,j_\alpha}(t_i) - g_\alpha(t_i)|$$

$$\le \eta_k + \eta_k + \varepsilon\lambda(H_k) + 12\eta_k(k+1)/\varepsilon < \varepsilon(2^{-k} + \lambda(H_k)).$$

Now remark that the sets A_k are pairwise disjoint and $\bigcup_{k=0}^{\infty} A_k = [a,b]$. Then only a finite number of sets H_k are nonempty and $\bigcup_{k=0}^{\infty} H_k = [a,b]$. Consequently, summing over k we get

$$\left|\int_a^b g_\alpha - \sigma(g_\alpha, \mathcal{Q})\right| < \varepsilon \sum_{k=0}^{\infty} (2^{-k} + \lambda(H_k)) = \varepsilon(2 + (b-a)).$$

Since this is true for any function g_α in $\mathbb{A}$ and for any δ-fine partition $\mathcal{Q}$ of $[a,b]$, the family $\mathbb{A}$ is McShane equi-integrable. □

We need yet the following fact that is a very special case of a general theorem proved in [13, Theorem 3.3].

Proposition 4.3. *Let $G: [a,b] \to ck(R^n)$ be a Pettis integrable multifunction whose support functions are nonnegative. Then the set*

$$\mathbb{S} = \{s(x, G(\cdot)) : x \in S^{n-1}\}$$

is totally bounded in $L^1[a,b]$.

Proof. Let $M_G(E)$ be the Pettis integral of G on the set $E \in \mathcal{L}$. Moreover, let $\{x_n : n \in \mathbb{N}\} \subset S^{n-1}$ be an arbitrary sequence and let $\{x_{n_k}\}_k$ be a subsequence converging to x_0. We have then

$$\lim_k \int_E s(x_{n_k} - x_0, G(t))\, dt = \lim_k s(x_{n_k} - x_0, M_G(E)) = 0 \quad \text{for every } E \in \mathcal{L},$$

and the convergence of the sequence $\{s(x_{n_k} - x_0, M_G(E))\}_k$ is uniform on $\mathcal{L}$, because $M_G(E) \subseteq M_G(\Omega)$, for every $E \in \mathcal{L}$. Thus, the sequence $\{s(x_{n_k}, G)\}_k$ is convergent in $L_1(\mu)$ to $s(x_0, G)$ (cf. [14, Proposition II.5.3]). □

Proof of Theorem 4.1. (A) ⇒ (B). Since $\widetilde{\Gamma}$ is fuzzy Henstock integrable, then for each $r \in [0,1]$ the set function $\widetilde{\Gamma}_r$ is Henstock integrable. So, according to Theorem 2.9, $\mathcal{S}_H(\widetilde{\Gamma}_1) \ne \emptyset$. Let us fix $f \in \mathcal{S}_H(\widetilde{\Gamma}_1)$ and define a fuzzy-number valued function $\widetilde{f}: [a,b] \to E^n$ as follows: $\widetilde{f}(t) = \chi_{\{f(t)\}}$, for each $t \in [a,b]$. Now define $\widetilde{G}: [a,b] \to E^n$ setting $\widetilde{G}(t) := \widetilde{\Gamma}(t) - \widetilde{f}(t)$. To prove that $\widetilde{G}(t)$ is fuzzy McShane integrable on $[a,b]$, by Proposition 3.5 it is enough to show that the collection

$$\mathbb{B} := \{s(x, \widetilde{G}_r(\cdot)) : x \in S^{n-1} \text{ and } 0 \le r \le 1\}$$

is McShane equi-integrable. To this end we are going to prove that $\mathbb{B}$ fulfils the hypotheses of Theorem 4.2. Since $\widetilde{\Gamma}$ is fuzzy Henstock integrable, it follows from Proposition 3.5 that the family of functions

$$\{s(x, \widetilde{\Gamma}_r(\cdot)) : x \in S^{n-1} \text{ and } 0 \le \mathrm{r} \le 1\}$$

is Henstock–Kurzweil equi-integrable. Moreover, for each $r \in [0,1]$ the set-function $\widetilde{\Gamma}_r$ is Henstock integrable and

$$\widetilde{\Gamma}_r(t) = \widetilde{G}_r(t) + f(t) \quad \text{for each } t \in [a,b]. \tag{15}$$

Then, for $r \in [0,1]$, $t \in [a,b]$ and $x \in R^n$, we have

$$s(x, \widetilde{G}_r(t)) = s(x, \widetilde{\Gamma}_r(t)) - \langle x, f(t)\rangle.$$

Now applying Theorem 2.9 to each set-function $\widetilde{\Gamma}_r$, we obtain that, for every $r \in [0,1]$, the set function $\widetilde{G}_r$ is Aumann and then Pettis integrable. Since the function f is Henstock integrable, we infer that the family $\mathbb{B}$ is Henstock–Kurzweil equi-integrable. We observe that all support functions of $\widetilde{G}_r(t)$ are nonnegative. Consequently, if $0 \le r_1 \le r_2 \le 1$, then $\widetilde{G}_{r_2}(t) \subset \widetilde{G}_{r_1}(t) \subset \widetilde{G}_0(t)$, and

$$0 \le s(x, \widetilde{G}_{r_2}(t)) \le s(x, \widetilde{G}_{r_1}(t)) \le s(x, \widetilde{G}_0(t)), \tag{16}$$

for every $x \in S^{n-1}$ and $t \in [a, b]$.

So the family $\mathbb{B}$ is pointwise bounded. It remains to show that $\mathbb{B}$ is also totally bounded in $L^1[a, b]$.

Claim 1. *If $g_r(x) := \int_0^1 s(x, \widetilde{G}_r(t))\,dt$, for each $x \in S^{n-1}$ and $r \in [0, 1]$, then for each r the function g_r is continuous and the family $\{g_r : r \in [0, 1]\}$ is norm relatively compact in $C(S^{n-1})$, the space of real continuous functions on S^{n-1}.*

Proof. Given $x, y \in S^{n-1}$ and $r \in [0, 1]$, we have for $x \neq y$

$$\begin{aligned}
|g_r(x) - g_r(y)| &\le \int_a^b |s(x, \widetilde{G}_r(t)) - s(y, \widetilde{G}_r(t))|\,dt \le \int_a^b [s(x-y, \widetilde{G}_r(t)) + s(y-x, \widetilde{G}_r(t))]\,dt \\
&\le \|x-y\| \int_a^b \left[s\left(\frac{x-y}{\|x-y\|}, \widetilde{G}_r(t)\right) + s\left(\frac{y-x}{\|x-y\|}, \widetilde{G}_r(t)\right)\right] dt \\
&\le \|x-y\| \int_a^b \left[s\left(\frac{x-y}{\|x-y\|}, \widetilde{G}_0(t)\right) + s\left(\frac{y-x}{\|x-y\|}, \widetilde{G}_0(t)\right)\right] dt \\
&\le 2\|x-y\| \sup_{\|z\|\le 1} \int_a^b s(z, \widetilde{G}_0(t))\,dt.
\end{aligned}$$

But, since $\widetilde{G}_0$ is Pettis integrable, we have $\sup_{\|z\|\le 1} \int_a^b s(z, \widetilde{G}_0(t))\,dt < \infty$ (cf. [5, Theorem 5.5]). It follows that g_r satisfies the Lipschitz condition. Consequently the family $\{g_r : r \in [0, 1]\}$ is equicontinuous. Moreover, since $0 \le g_r(x) \le g_0(x)$ for each $r \in [0, 1]$ and each $x \in [a, b]$, from Ascoli's theorem follows that the family $\{g_r : r \in [0, 1]\}$ is norm relatively compact in $C(S^{n-1})$. □

Claim 2. *$\mathbb{B}$ is totally bounded in $L_1[a, b]$.*

Proof. Let us fix $\varepsilon > 0$. It follows from Claim 1 that the family $\{g_r : r \in [0, 1]\}$ is totally bounded in $C(S^{n-1})$. That is there exist reals $r_1, \ldots, r_m \in [0, 1]$ such that

$$\forall r \in [0, 1]\ \exists i \le m\ :\ \|g_r - g_{r_i}\|_{C(S^{n-1})} < \varepsilon/2.$$

But

$$\begin{aligned}
\|g_r - g_{r_i}\|_{C(S^{n-1})} &= \sup_{x \in S^{n-1}} \left| \int_a^b s(x, \widetilde{G}_r(t))\,dt - \int_a^b s(x, \widetilde{G}_{r_i}(t))\,dt \right| \\
&= \sup_{x \in S^{n-1}} \left| \int_a^b [s(x, \widetilde{G}_r(t)) - s(x, \widetilde{G}_{r_i}(t))]\,dt \right| = \sup_{x \in S^{n-1}} \int_a^b |s(x, \widetilde{G}_r(t)) - s(x, \widetilde{G}_{r_i}(t))|\,dt,
\end{aligned}$$

where the final equality follows from (16). Consequently, we have

$$\int_a^b |s(x, \widetilde{G}_r(t)) - s(x, \widetilde{G}_{r_i}(t))|\,dt < \varepsilon/2 \quad \text{for every } x \in S^{n-1}.$$

But from Proposition 4.3 we know that for each $i \le m$ the family $\{s(x, \widetilde{G}_{r_i}) : x \in S^{n-1}\}$ is totally bounded in $L_1[a, b]$. Hence, there are points $\{x_{1i}, \ldots, x_{p_i}\} \subset S^{n-1}$ such that if $x \in S^{n-1}$ is arbitrary, then

$$\int_a^b |s(x, \widetilde{G}_{r_i}(t)) - s(x_{ji}, \widetilde{G}_{r_i}(t))|\,dt < \varepsilon/2 \quad \text{for a certain } j \le p_i.$$

It follows that the set $\{s(x_{ji}, \widetilde{G}_{r_i}(\cdot)) : j \le p_i,\ i \le m\}$ is an ε-mesh of $\mathbb{B}$ in the norm of $L_1[a, b]$. □

Then the collection $\mathbb{B}$ is McShane equi-integrable and, applying once again Proposition 3.5, we get that $\widetilde{G}$ is fuzzy McShane integrable on $[a, b]$. Moreover equality (5) follows at once from equality (15).

The implication (B) $\Rightarrow$ (C) is obvious.

(C) $\Rightarrow$ (A). Let us assume now that $\widetilde{\Gamma}(t) = \widetilde{G}(t) + \widetilde{f}(t)$, where $\widetilde{G}$ is a fuzzy-number valued function, is fuzzy McShane integrable on $[a, b]$ and f is a Henstock integrable function $f \in \mathcal{S}_H([\widetilde{\Gamma}]^1)$. Then according to Proposition 3.5 we have that the collection

$$\mathbb{B}:=\{s(x, \widetilde{G}_r(\cdot)) : \ x \in S^{n-1} \text{ and } 0 \le r \le 1\}$$

is McShane equi-integrable. Therefore by the equality

$$s(x, \widetilde{\Gamma}_r(t)) = s(x, \widetilde{G}_r(t)) + \langle x, f(t)\rangle,$$

we infer that the collection

$$\{s(x, \widetilde{\Gamma}_r(\cdot)) : \ x \in S^{n-1} \text{ and } 0 \le r \le 1\}$$

is Henstock–Kurzweil equi-integrable. And applying once again Proposition 3.5 we obtain the fuzzy Henstock integrability of $\widetilde{\Gamma}$. □

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