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A decomposition theorem for Banach space valued fuzzy Henstock integral ☆

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Abstract

We establish the following decomposition theorem for fuzzy mappings with values in a Banach space: a fuzzy mapping is fuzzy Henstock integrable if and only if it can be represented as a sum of a fuzzy McShane integrable fuzzy mapping and of a fuzzy Henstock integrable fuzzy mapping generated by a Henstock integrable function.

Keywords: Fuzzy Henstock integral; Fuzzy McShane integral; Henstock–Kurzweil and McShane equi-integrability

1. Introduction

In this paper we continue the investigation of the Henstock integral started in [2–4] for set-valued functions and in [1] in case of fuzzy number valued functions, but we consider now a more general setting of fuzzy mappings on Banach spaces.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and let $E^n = \{u : \mathbb{R}^n \to [0, 1] : u$ fulfills (1)–(4) of Definition 2.1 with $X = \mathbb{R}^n\}$. It has been proven in [1] that a fuzzy-number valued function $\widetilde{\Gamma} : [a, b] \to E^n$ is fuzzy Henstock integrable if and only if $\widetilde{\Gamma}$ can be represented as $\widetilde{\Gamma}(t) = \widetilde{G}(t) + \widetilde{f}(t)$, where $\widetilde{G} : [a, b] \to E^n$ is fuzzy McShane integrable and $\widetilde{f}$ is a fuzzy Henstock integrable fuzzy number valued function generated by a Henstock integrable selection of $\widetilde{\Gamma}$.

In the current paper we consider the fuzzy Henstock and McShane integrals for functions taking values in the fuzzy number space $\mathcal{F}_c(X)$ (see Definition 2.1) in place of E^n. In Section 3 we give a characterization of the fuzzy-number mappings which are fuzzy Henstock or McShane integrable by means of the equi-integrability of the support functions (Proposition 3.3). The main result of this paper, a decomposition theorem generalizing that of [1], is in Section 4 (Theorem 4.2):

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A fuzzy mapping $\widetilde{\Gamma}:[a,b]\to\mathcal{F}_c(X)$ is fuzzy Henstock integrable if and only if $\widetilde{\Gamma}$ can be represented as $\widetilde{\Gamma}(t)=\widetilde{G}(t)+\widetilde{f}(t)$, $t\in[a,b]$, where $\widetilde{G}:[a,b]\to\mathcal{F}_c(X)$ is fuzzy McShane integrable and $\widetilde{f}$ is a fuzzy Henstock integrable fuzzy number valued function generated by a Henstock integrable selection of $\widetilde{\Gamma}$.

The idea of the proof is similar to that from [1]. Differences are caused by topological differences between $\mathbb{R}^n$ and an infinite dimensional Banach space X. First of all by the fact that the closed unit ball in X^* is never norm compact, if X is infinite dimensional. I have tried to avoid unnecessary repetitions from [1] but still the main body of the paper is very similar. The essential tool to prove the decomposition theorem is [1, Theorem 4.2] that provides sufficient conditions guaranteeing the McShane equi-integrability of a family of nonnegative real valued Henstock–Kurzweil equi-integrable functions. The second important result applied here is [5, Theorem 3.3], repeated here as Theorem 2.6, necessary in case of non-separable Banach spaces. If the Banach space X under consideration is separable, one may apply [3, Theorem 2] instead.

2. Basic facts

Let X be an arbitrary Banach space endowed with the norm $\|\cdot\|$. We denote by $B(X)$ its closed unit ball and by $\sigma(X^*,X)$ or w^* the weak* topology of X^*. $ck(X)$ is the family of all nonempty compact convex subsets of X endowed with the *Hausdorff distance*

$$d_H(A,B):=\max\Big\{\sup_{x\in A}\inf_{y\in B}\|x-y\|,\ \sup_{y\in B}\inf_{x\in A}\|x-y\|\Big\},$$

and the operations

$$A+B:=\{x+y:x\in A,\ y\in B\},\qquad kA:=\{kx:x\in A\}.$$

The space $ck(X)$ endowed with the Hausdorff distance is a complete metric space. For every $A\in ck(X)$ the *support function* of A is denoted by $s(\cdot,A)$ and defined by $s(x,A)=\sup\{\langle x,y\rangle:y\in A\}$, for each $x\in X$. Clearly the map $x\longmapsto s(x,A)$ is sublinear on X and $-s(-x,A)=\inf\{\langle x,y\rangle:y\in A\}$, for each $x\in X$.

According to Hörmander's equality (cf. [6], p. 9), for A and B non empty members of $ck(X)$ we have the equality

$$d_H(A,B)=\sup_{x\in B(X^*)}\big|s(x,A)-s(x,B)\big|.$$

Definition 2.1. The *generalized fuzzy number space* $\mathcal{F}_c(X)$ is defined as the set

$$\mathcal{F}_c(X)=\big\{u:X\to[0,1]:u \text{ satisfies conditions (1)–(4) below}\big\}:$$

(1) u is a normal fuzzy set, i.e. there exists $x_0\in X$, such that $u(x_0)=1$;
(2) u is quasiconcave, i.e. $u(tx+(1-t)y)\geqslant\min\{u(x),u(y)\}$ for any $x,y\in X$, $t\in[0,1]$;
(3) u is upper semi-continuous;
(4) $\operatorname{supp}u=\overline{\{x\in X:u(x)>0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Each $u\in\mathcal{F}_c(X)$ is called a generalized fuzzy number on X. For $r\in(0,1]$ let $[u]^r=\{x\in X:u(x)\geqslant r\}$ and $[u]^0=\overline{\bigcup_{s\in(0,1]}[u]^s}$. If $u\in\mathcal{F}_c(X)$ and $r\in[0,1]$, then $[u]^r\in ck(X)$.

In the sequel we will use the following representation theorem (cf. [7]).

Theorem 2.2. *If $u\in\mathcal{F}_c(X)$, then*

(i) $[u]^r\in ck(X)$, *for all* $r\in[0,1]$;
(ii) $[u]^{r_2}\subset[u]^{r_1}$, *for* $0\leqslant r_1\leqslant r_2\leqslant 1$;
(iii) if (r_k) is a nondecreasing sequence converging to $r>0$, then

$$[u]^r=\bigcap_{k\geqslant 1}[u]^{r_k}.$$

Conversely, if $\{A_r : r \in [0,1]\}$ *is a family of subsets of* X *satisfying (i)–(iii), then there exists a unique* $u \in \mathcal{F}_c(X)$ *such that* $[u]^r = A_r$ *for* $r \in (0,1]$ *and* $[u]^0 = \overline{\bigcup_{0<r\leqslant 1}[u]^r} \subset A_0$.

Define $D : \mathcal{F}_c(X) \times \mathcal{F}_c(X) \to \mathbb{R}^+ \cup \{0\}$ by the equation

$$D(u,v) = \sup_{r\in[0,1]} d_H\big([u]^r, [v]^r\big).$$

$(\mathcal{F}_c(X), D)$ is a metric space.

For $u, v \in \mathcal{F}_c(X)$ and $k \in \mathbb{R}$ the addition $u+v$ and the scalar multiplication ku are defined respectively by

$$[u+v]^r := [u]^r + [v]^r \quad \text{and} \quad [ku]^r := k[u]^r \quad \text{for every } r \in [0,1].$$

Let $[a,b]$ be a bounded closed interval of the real line equipped by the Lebesgue measure λ. We denote by $\mathcal{L}$ and by $\mathcal{I}$ the families of all Lebesgue measurable subsets of $[a,b]$ and of all closed subintervals of $[a,b]$, respectively. If $I \in \mathcal{I}$, then $|I|$ denotes its length. A *partition in* $[a,b]$ is a collection of pairs $\mathcal{P} = \{(I_i, t_i) : i = 1, \dots, p\}$, where I_i, are non-overlapping subintervals of $[a,b]$ and t_i are points of $[a,b]$, $i = 1, \dots, p$. If $\bigcup_{i=1}^p I_i = [a,b]$ we say that $\mathcal{P}$ is a *partition of* $[a,b]$. If $t_i \in I_i$, $i = 1, \dots, p$, we say that $\mathcal{P}$ is a *Perron partition of* $[a,b]$. A *gauge* on $[a,b]$ is a positive function on $[a,b]$. For a given gauge δ on $[a,b]$, we say that a partition $\mathcal{P} = \{(I_i, t_i) : i = 1, \dots, p\}$ is δ*-fine* if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1, \dots, p$.

Given $f : [a,b] \to X$ and a partition $\mathcal{P} = \{(I_i, t_i) : i = 1, \dots, p\}$ in $[a,b]$ we set

$$\sigma(f, \mathcal{P}) = \sum_{i=1}^{p} |I_i| f(t_i).$$

Let us recall the definitions of McShane and Henstock integral for X-valued functions.

Definition 2.3. A function $g : [a,b] \to X$ is said to be *McShane* (resp. *Henstock*) *integrable on* $[a,b]$ if there exists a vector $w \in X$ with the following property: for every $\epsilon > 0$ there exists a gauge δ on $[a,b]$ such that

$$\big\|\sigma(g, \mathcal{P}) - w\big\| < \varepsilon$$

for each δ-fine partition (resp. Perron partition) $\mathcal{P}$ of $[a,b]$. We set $(MS)\int_a^b g(t)\,dt := w$ (resp. $(H)\int_a^b g(t)\,dt := w$).

In case of $X = \mathbb{R}$, g is said to be *Henstock–Kurzweil* integrable. We denote by $\mathcal{MS}[a,b]$ (resp. $\mathcal{HK}[a,b]$) the set of all real valued McShane (resp. Henstock–Kurzweil) integrable functions on $[a,b]$.

A set-valued function $\varGamma : [a,b] \to ck(X)$ is said to be *scalarly measurable* if for every $x^* \in X^*$, the map $s(x^*, \varGamma(\cdot))$ is measurable. A set-valued function $\varGamma : [a,b] \to ck(X)$ is said to be *scalarly Lebesgue* (resp. *scalarly Henstock–Kurzweil*) *integrable on* $[a,b]$ if for each $x^* \in X^*$ the real function $s(x^*, \varGamma(t))$ is integrable (resp. Henstock–Kurzweil integrable) on $[a,b]$.

A function $f : [a,b] \to X$ is called a *selection* of a set-valued function $\varGamma : [a,b] \to ck(X)$ if, for every $t \in [a,b]$, one has $f(t) \in \varGamma(t)$. By $\mathcal{S}_H(\varGamma)$ we denote the family of all scalarly measurable selections of $\varGamma$ that are Henstock integrable.

Definition 2.4. (See [8].) A set-valued function $\varGamma : [a,b] \to ck(X)$ is said to be *Pettis integrable* in $ck(X)$ if $\varGamma$ is scalarly Lebesgue integrable on $[a,b]$ and for each $A \in \mathcal{L}$ there exists a set $W_A \in ck(X)$ such that for each $x^* \in X^*$, we have

$$s\big(x^*, W_A\big) = (L)\int_A s\big(x^*, \varGamma(t)\big)\,dt,$$

where (L) stands for Lebesgue. Then we set $(P)\int_A \varGamma(t)\,dt := W_A$, for each $A \in \mathcal{L}$. One can find in [8] examples of $ck(X)$-valued multifunctions that are Pettis integrable in the family of closed convex subsets of X but not in $ck(X)$.

Given $\Gamma : [a,b] \to ck(X)$ and a partition $\mathcal{P} = \{(I_i, t_i) : i = 1, \ldots, p\}$ in $[a,b]$ we set

$$\sigma(\Gamma, \mathcal{P}) = \sum_{i=1}^{p} |I_i| \Gamma(t_i).$$

Definition 2.5. A set-valued function $\Gamma : [a,b] \to ck(X)$ is said to be *Henstock* (resp. *McShane*) *integrable* on $[a,b]$ if there exists a nonempty bounded, closed and convex set $W \subset X$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that for each δ-fine Perron partition (resp. partition) $\mathcal{P} = \{(I_i, t_i) : i = 1, \ldots, p\}$ of $[a,b]$, we have

$$d_H\big(W, \sigma(\Gamma, \mathcal{P})\big) < \varepsilon.$$

Notice that since d_H is a complete metric on $ck(X)$, the set W is necessarily compact.

The following theorem has been proven in [5, Theorem 3.3] (if X is separable, then the same result, but with a different proof, is contained in [3, Theorem 2]).

Theorem 2.6. *Let $\Gamma : [a,b] \to ck(X)$ be a scalarly Henstock–Kurzweil integrable set-valued function. Then the following conditions are equivalent:*

(i) Γ is Henstock integrable;
(ii) $\mathcal{S}_H(\Gamma) \neq \emptyset$ and for every $f \in \mathcal{S}_H(\Gamma)$ the multifunction $G : [a,b] \to ck(X)$ defined by $\Gamma(t) = G(t) + f(t)$ is McShane integrable;
(iii) there exists $f \in \mathcal{S}_H(\Gamma)$ such that the multifunction $G : [a,b] \to ck(X)$ defined by $\Gamma(t) = G(t) + f(t)$ is McShane integrable.

3. Weakly fuzzy Henstock and fuzzy Henstock integral

Each mapping $\widetilde{\Gamma} : [a,b] \to \mathcal{F}_c(X)$ is called a fuzzy mapping on X. For each $r \in [0,1]$ we set $\widetilde{\Gamma}_r(t) = [\widetilde{\Gamma}(t)]^r$.

A fuzzy mapping $\widetilde{\Gamma} : [a,b] \to \mathcal{F}_c(X)$ is said to be *scalarly Lebesgue* (resp. *scalarly Henstock–Kurzweil*) *integrable* on $[a,b]$ if for all $r \in [0,1]$ the set-valued function $\widetilde{\Gamma}_r : [a,b] \to ck(X)$ is scalarly Lebesgue (resp. scalarly Henstock–Kurzweil) integrable.

Definition 3.1. A fuzzy mapping $\widetilde{\Gamma} : [a,b] \to \mathcal{F}_c(X)$ is said to be *weakly fuzzy Henstock* (or *weakly fuzzy Pettis* or *weakly fuzzy McShane*) integrable in $\mathcal{F}_c(X)$ if for every $r \in [0,1]$ the set-valued function $\widetilde{\Gamma}_r(t)$ is Henstock (or Pettis or McShane) integrable in $ck(X)$ and there exists a generalized fuzzy number $\widetilde{A} \in \mathcal{F}_c(X)$ such that for any $r \in [0,1]$ and for any $x^* \in X^*$ we have

$$s\big(x^*, [\widetilde{A}]^r\big) = (HK) \int_a^b s\big(x^*, \widetilde{\Gamma}_r(t)\big)\, dt,$$

(or

$$s\big(x^*, [\widetilde{A}]^r\big) = (L) \int_a^b s\big(x^*, \widetilde{\Gamma}_r(t)\big)\, dt,$$

respectively).

Definition 3.2. (See [9].) A fuzzy mapping $\widetilde{\Gamma} : [a,b] \to \mathcal{F}_c(X)$ is said to be *fuzzy Henstock* (resp. *fuzzy McShane*) integrable on $[a,b]$ if there exists a fuzzy number $\tilde{A} \in \mathcal{F}_c(X)$ such that for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that for every δ-fine Perron partition (resp. partition) $\mathcal{P}$ of $[a,b]$, we have

$$D\big(\tilde{A}, \sigma(\widetilde{\Gamma}, \mathcal{P})\big) < \varepsilon,$$

where $\sigma(\widetilde{\Gamma}, \mathcal{P}) = \sum_{i=1}^{p} |I_i| \widetilde{\Gamma}(t_i)$. We write $(FH) \int_a^b \widetilde{\Gamma}(t)\, dt := \tilde{A}$ (resp. $(FMS) \int_a^b \widetilde{\Gamma}(t)\, dt := \tilde{A}$).

By means the notion of equi-integrability it is possible to characterize the fuzzy Henstock and the fuzzy McShane integrability. We recall that a family $\{g_\alpha : \alpha \in \mathbb{A}\}$ of real valued functions in $\mathcal{HK}[a,b]$ (resp. $\mathcal{MS}[a,b]$) is said to be *Henstock–Kurzweil* (resp. *McShane*) *equi-integrable on* $[a,b]$ whenever for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\sup\left\{\left|\sigma(g_\alpha,\mathcal{P}) - (HK)\int_a^b g_\alpha(t)\,dt\right| : \alpha \in \mathbb{A}\right\} < \varepsilon \quad \left(\text{resp. } \sup\left\{\left|\sigma(g_\alpha,\mathcal{P}) - (L)\int_a^b g_\alpha(t)\,dt\right| : \alpha \in \mathbb{A}\right\} < \varepsilon\right)$$

for each δ-fine Perron partition (resp. partition) $\mathcal{P}$ of $[a,b]$.

Proposition 3.3. *Let $\widetilde{\Gamma} : [a,b] \to \mathcal{F}_c(X)$ be a scalarly Henstock–Kurzweil (resp. scalarly Lebesgue) integrable fuzzy mapping. Then the following are equivalent:*

(j) $\widetilde{\Gamma}$ is fuzzy Henstock (resp. McShane) integrable on $[a,b]$;
(jj) the collection $\{s(x^*, \widetilde{\Gamma}_r(\cdot)) : x^* \in B(X^*)$ and $0 \leqslant r \leqslant 1\}$ is Henstock–Kurzweil (resp. McShane) equi-integrable.

The proof of the above proposition is similar to that of [1, Proposition 3.5].

It follows from the definitions that each fuzzy Henstock (resp. McShane) integrable function is also weakly fuzzy Henstock (resp. McShane) integrable. It has been shown in [1, Example 3.6] that even in case of finite dimensional Banach space the family of all weakly fuzzy Henstock (resp. McShane) integrable fuzzy-number valued functions is larger than the family of all fuzzy Henstock (resp. McShane) integrable fuzzy-number valued functions.

4. A decomposition of the fuzzy Henstock integral

Before proving the main result we need yet the following fact that is a very special case of a general theorem proved in [8, Theorem 3.3].

Proposition 4.1. *Let $G : [a,b] \to ck(X)$ be a multifunction that is Pettis integrable in $ck(X)$ and whose support functions are non negative. Then the set*

$$\mathbb{S} = \{s(x^*, G(\cdot)) : x^* \in B(X^*)\}$$

is totally bounded in $L^1[a,b]$.

Proof. Let $M_G(E)$ be the Pettis integral of G on the set $E \in \mathcal{L}$. Moreover, let $\{x_n^* : n \in \mathbb{N}\} \subset B(X^*)$ be an arbitrary sequence and let $\{x_\alpha^*\}_{\alpha\in\mathbb{A}}$ be a subnet of $\{x_n^* : n \in \mathbb{N}\}$ that is weak*-converging to a functional $x_0^* \in B(X^*)$. Since the set $M_G[a,b]$ is norm compact, the net is uniformly convergent to x_0^* on $M_G[a,b]$. It follows that there exists a subsequence $\{x_{n_k}^* : k \in \mathbb{N}\}$ that is also uniformly convergent to x_0^* on $M_G[a,b]$. We have then

$$\lim_k (L)\int_a^b s(x_{n_k}^* - x_0^*, G(t))\,dt = \lim_k s(x_{n_k}^* - x_0^*, M_G[a,b]) = 0.$$

Since the support functions are non-negative and subadditive the sequence $\{s(x_{n_k}^*, G)\}_k$ is convergent in $L_1(\mu)$ to $s(x_0^*, G)$. Consequently, $\mathbb{S}$ is totally bounded in $L^1[a,b]$.

If $X = \mathbb{R}^n$, we may simply apply the norm compactness of the unit ball of $\mathbb{R}^n$ (as it has been done in [1]) but in case of infinite dimensional X the unit ball is never norm compact. That is one of the essential differences between this paper and [1]. □

The following decomposition theorem is the main result of the paper.

Theorem 4.2. *Let $\widetilde{\Gamma}:[a,b]\to\mathcal{F}_c(X)$ be a fuzzy mapping. Then the following conditions are equivalent:*

(A) $\widetilde{\Gamma}$ is fuzzy Henstock integrable;
(B) $\mathcal{S}_H(\widetilde{\Gamma}_1)\neq\emptyset$ and for every Henstock integrable function $f\in\mathcal{S}_H(\widetilde{\Gamma}_1)$ the fuzzy mapping $\widetilde{G}:[a,b]\to\mathcal{F}_c(X)$ defined by $\widetilde{\Gamma}(t)=\widetilde{G}(t)+\widetilde{f}(t)$ (where $\widetilde{f}(t)=\chi_{\{f(t)\}}$) is fuzzy McShane integrable on $[a,b]$.
(C) There exists a Henstock integrable function $f\in\mathcal{S}_H(\widetilde{\Gamma}_1)$ such that the fuzzy mapping $\widetilde{G}:[a,b]\to\mathcal{F}_c(X)$ defined by $\widetilde{\Gamma}(t)=\widetilde{G}(t)+\widetilde{f}(t)$ is fuzzy McShane integrable on $[a,b]$.

If (B) or (C) are fulfilled, then

$$(FH)\int_a^b\widetilde{\Gamma}(t)\,dt=(FMS)\int_a^b\widetilde{G}(t)\,dt+(H)\int_a^b f(t)\,dt; \tag{1}$$

Proof of Theorem 4.2. $(A)\Rightarrow(B)$. We assume that $\widetilde{\Gamma}$ is fuzzy Henstock integrable. Then for each $r\in[0,1]$ the set function $\widetilde{\Gamma}_r$ is Henstock integrable. So, according to Theorem 2.6, $\mathcal{S}_H(\widetilde{\Gamma}_1)\neq\emptyset$. Let us fix $f\in\mathcal{S}_H(\widetilde{\Gamma}_1)$ and define a fuzzy mapping $\widetilde{f}:[a,b]\to\mathcal{F}_c(X)$ as follows: $\widetilde{f}(t)=\chi_{\{f(t)\}}$, for each $t\in[a,b]$. Then define $\widetilde{G}:[a,b]\to\mathcal{F}_c(X)$ by setting $\widetilde{G}(t):=\widetilde{\Gamma}(t)-\widetilde{f}(t)$. To prove that $\widetilde{G}(t)$ is fuzzy McShane integrable on $[a,b]$, by Proposition 3.3 it is enough to show that the collection

$$\mathbb{B}:=\big\{s\big(x^*,\widetilde{G}_r(\cdot)\big):x^*\in B\big(X^*\big)\text{ and }0\leqslant r\leqslant 1\big\}$$

is McShane equi-integrable. To this end we are going to prove that $\mathbb{B}$ fulfills the hypotheses of [1, Theorem 4.2]. Since $\widetilde{\Gamma}$ is fuzzy Henstock integrable, it follows from Proposition 3.3 that the family of functions

$$\big\{s\big(x^*,\widetilde{\Gamma}_r(\cdot)\big):x^*\in B\big(X^*\big)\text{ and }0\leqslant r\leqslant 1\big\}$$

is Henstock–Kurzweil equi-integrable. Moreover, for each $r\in[0,1]$ the set-function $\widetilde{\Gamma}_r(t)$ is Henstock integrable and

$$\widetilde{\Gamma}_r(t)=\widetilde{G}_r(t)+f(t). \tag{2}$$

Hence, for $r\in[0,1]$ and $x^*\in X$,

$$s\big(x^*,\widetilde{G}_r(t)\big)=s\big(x^*,\widetilde{\Gamma}_r(t)\big)-\big\langle x^*,f(t)\big\rangle.$$

Applying Theorem 2.6 to each set-function $\widetilde{\Gamma}_r$, we obtain McShane integrability of each set function $\widetilde{G}_r(t)$. Since the function f is Henstock integrable, $\mathbb{B}$ is Henstock–Kurzweil equi-integrable. Since all support functions of $\widetilde{G}_r(t)$ are non negative it follows that if $0\leqslant r_1\leqslant r_2\leqslant 1$, then $\widetilde{G}_{r_2}(t)\subset\widetilde{G}_{r_1}(t)\subset\widetilde{G}_0(t)$, and

$$0\leqslant s\big(x^*,\widetilde{G}_{r_2}(t)\big)\leqslant s\big(x^*,\widetilde{G}_{r_1}(t)\big)\leqslant s\big(x^*,\widetilde{G}_0(t)\big), \tag{3}$$

for every $x^*\in B(X^*)$.

Thus, the family $\mathbb{B}$ is pointwise bounded. We shall prove yet that $\mathbb{B}$ is also totally bounded in $L^1[a,b]$.

Claim. *If $g_r(x^*):=\int_a^b s(x^*,\widetilde{G}_r(t))\,dt$, for each $x^*\in B(X^*)$ and $r\in[0,1]$, then for each r the function g_r is weak*-continuous and the family $\{g_r:r\in[0,1]\}$ is norm relatively compact in $C(B(X^*),\sigma(X^*,X))$, the space of real functions on $B(X^*)$, continuous with respect to the weak* topology.*

Proof. Since each function $\widetilde{G}_r$ is Pettis integrable in $ck(X)$, the functions g_r are weak* continuous (see [8, Theorem 1.4]). Moreover, it follows from (3) that

$$0\leqslant g_r\big(x^*\big)\leqslant g_0\big(x^*\big)\quad\text{for every }0<r\leqslant 1\text{ and }x^*\in X^*$$

and so, if $x^*,y^*\in B(X^*)$, then

$$g_r\big(x^*\big)-g_r\big(y^*\big)\leqslant g_r\big(x^*-y^*\big)\leqslant g_0\big(x^*-y^*\big)$$

and further

$$\left|g_r(x^*) - g_r(y^*)\right| \leqslant g_0(x^* - y^*) + g_0(y^* - x^*).$$

It follows that the collection $\{g_r : 0 \leqslant r \leqslant 1\}$ is equicontinuous on $(B(X^*), \sigma(X^*, X))$, because g_0 (being weak*-continuous on the weak*-compact set $B(X^*)$) is uniformly continuous on $(B(X^*), \sigma(X^*, X))$. Moreover, since $0 \leqslant g_r(x^*) \leqslant g_0(x^*)$ for each $r \in [0,1]$ and each $x^* \in B(X^*)$, it follows from Ascoli's theorem that the family $\{g_r : r \in [0,1]\}$ is norm relatively compact in $C(B(X^*), \sigma(X^*, X))$. □

It follows from the Claim that the family $\{g_r : r \in [0,1]\}$ is totally bounded in $C(B(X^*), \sigma(X^*, X))$. That is, given $\varepsilon > 0$, there exist reals $r_1, \ldots, r_m \in [0,1]$ such that

$$\forall r \in [0,1]\ \exists i \leqslant m : \|g_r - g_{r_i}\|_{C(B(X^*),w^*)} < \varepsilon/2.$$

But

$$\begin{aligned}\|g_r - g_{r_i}\|_{C(B(X^*),w^*)} &= \sup_{x^* \in B(X^*)} \left|(L)\int_a^b s\big(x^*, \widetilde{G}_r(t)\big)\,dt - (L)\int_a^b s\big(x^*, \widetilde{G}_{r_i}(t)\big)\,dt\right| \\ &= \sup_{x^* \in B(X^*)} \left|(L)\int_a^b \big[s\big(x^*, \widetilde{G}_r(t)\big) - s\big(x^*, \widetilde{G}_{r_i}(t)\big)\big]\,dt\right| \\ &= \sup_{x^* \in B(X^*)} (L)\int_a^b \big|s\big(x^*, \widetilde{G}_r(t)\big) - s\big(x^*, \widetilde{G}_{r_i}(t)\big)\big|\,dt,\end{aligned}$$

where the final equality follows from (3). Consequently, we have

$$(L)\int_a^b \big|s\big(x^*, \widetilde{G}_r(t)\big) - s\big(x^*, \widetilde{G}_{r_i}(t)\big)\big|\,dt < \varepsilon/2, \quad \text{for every } x^* \in B\big(X^*\big).$$

But from Proposition 4.1 we know that for each $i \leqslant m$ the family $\{s(x^*, \widetilde{G}_{r_i}) : x^* \in B(X^*)\}$ is totally bounded in $L_1[a,b]$. Hence, there are points $\{x^*_{i1}, \ldots, x^*_{ip_i}\} \subset B(X^*)$ such that if $x^* \in B(X^*)$ is arbitrary, then

$$(L)\int_a^b \big|s\big(x^*, \widetilde{G}_{r_i}(t)\big) - s\big(x^*_{ij}, \widetilde{G}_{r_i}(t)\big)\big|\,dt < \varepsilon/2, \quad \text{for a certain } j \leqslant p_i.$$

It follows that the set $\{s(x^*_{ij}, \widetilde{G}_{r_i}(\cdot)) : j \leqslant p_i, i \leqslant m\}$ is an ε-mesh of $\mathbb{B}$ in the norm of $L_1[a,b]$.

Thus, the collection $\mathbb{B}$ is McShane equi-integrable. Applying once again Proposition 3.3, we obtain the fuzzy McShane integrability of $\widetilde{G}$ on $[a,b]$.

The implication $(B) \Rightarrow (C)$ is obvious.

$(C) \Rightarrow (A)$. Let assume now that $\widetilde{\Gamma}(t) = \widetilde{G}(t) + \widetilde{f}(t)$, where $\widetilde{G}$ is a fuzzy mapping fuzzy McShane integrable on $[a,b]$ and f is a Henstock integrable function $f \in \mathcal{S}_H(\widetilde{\Gamma}_1)$. Then according to Proposition 3.3 we have that the collection

$$\mathbb{B} := \big\{s\big(x^*, \widetilde{G}_r(\cdot)\big) : x^* \in B\big(X^*\big) \text{ and } 0 \leqslant r \leqslant 1\big\}$$

is McShane equi-integrable. Therefore by the equality

$$s\big(x^*, \widetilde{\Gamma}_r(t)\big) = s\big(x^*, \widetilde{G}_r(t)\big) + \big\langle x^*, f(t)\big\rangle,$$

we infer that the collection

$$\big\{s\big(x^*, \widetilde{\Gamma}_r(\cdot)\big) : x^* \in B\big(X^*\big) \text{ and } 0 \leqslant r \leqslant 1\big\}$$

is Henstock–Kurzweil equi-integrable. Applying once again Proposition 3.3 we obtain the fuzzy Henstock integrability of $\widetilde{\Gamma}$.

Now, if (B) or (C) is satisfied, then it follows from (2) that

$$(H)\int_a^b \widetilde{\Gamma}_r(t)\,dt = (MS)\int_a^b \widetilde{G}_r(t)\,dt + (H)\int_a^b f(t)\,dt,$$

for every $r \in [0, 1]$. That immediately yields the equality (1). □

Remark 4.3. In case of a finite dimensional space $X = \mathbb{R}^n$ it has been proven in [1, Theorem 4.1, Claim 1] that each function g_r satisfied the Lipschitz condition, with a constant independent of $r \in [0, 1]$. In case of an infinite dimensional X such a result is not valid and so the proof of the Claim is different from that in [1].

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