CHAPTER 28

Liftings

Werner Strauss

Mathematisches Institut A, Universität Stuttgart, Postfach 80 11 40, D-70511 Stuttgart, Germany
E-mail: strauss@mathematik.uni-stuttgart.de

Nikolaos D. Macheras

Department of Statistics, University of Piraeus, 80 Karaoli and Dimitriou str., 185 34 Piraeus, Greece
E-mail: macheras@unipi.gr

Kazimierz Musiał

Institute of Mathematics, Wrocław University, Pl. Grundwaldzki 2/4, 50-384 Wrocław, Poland
E-mail: musial@math.uni.wroc.pl

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Introduction

The classical monograph of A. and C. Ionescu Tulcea (1969a) provides a systematic exposition of almost all results about liftings known at that time, in particular the basic existence theorems of J. von Neumann and D. Maharam were given with a new and more direct proof. The significance and power of the existence of liftings was illustrated there by important applications to mathematical analysis, e.g., to the point realization for automorphisms of spaces of measurable functions, to disintegration of measures, representation of integral operators which is equivalent to the differentiation of vector valued measures, and to the separable modification of stochastic processes. By a proper use of liftings they succeeded to put many classical results in their final form. Practically at the same time Kölzow (1968) proved the equivalence of the existence of liftings, Vitali differentiation, and Dunford Pettis theorem for locally convex spaces.

Nevertheless many interesting problems were left and formed the starting point for further developments in this field, such as the existence proof for densities on arbitrary finite measure spaces of Graf and von Weizsäcker (1975), the negative solution of the so-called 'strong lifting problem' by Losert (1979), the discussion of the existence of (strong) Borel liftings by Mokobodzki (1975), Fremlin (1977) and that of the non-existence of translation invariant Borel liftings for Haar measures by Johnson (1980), Talagrand (1982), Kupka and Prikry (1983), Losert (1983), and Burke (1993), the notion of lifting compactness studied by Bellow (1980) as well as Edgar and Talagrand (1980), existence results for strong Baire liftings by Grekas and Gryllakis (1991), the application of forcing methods for the non-existence of certain types of liftings (an example which demonstrates that lifting theory provides challenging problems for other areas of mathematics) by Shelah (1983), Burke and Just (1991) and Burke and Shelah (1992), and the discussion of permanence results mainly in products of probability spaces starting with a paper of Talagrand (1989) and subsequently developed by Burke (1995), Fremlin (200?), and the authors to mention probably the most important ones.

Because of the large number of contributions, in this article we can only give an overview of these developments with short indications of methods and proofs. Concerning the abounding number of applications we had to restrict ourselves drastically to either the most spectacular ones or to the most recent ones, giving only references for all the others.

As far as we could single out, we have tried to incorporate any paper dealing with liftings in the list of references at the end of this article. Sometimes we found it difficult to give full credit to authors, since many results in that field are circulated unpublished, but on the other hand they have become folklore.

1. Terminology

For a measure space (Ω, Σ, μ) we denote by Σ/μ its *measure algebra* (a Boolean algebra under its canonical Boolean operations), and $r : \Sigma \to \Sigma/\mu$ is the *canonical map* (a Boolean homomorphism). We assume throughout that μ is nontrivial, i.e., $\mu(\Omega) > 0$.

$\mathbb{K}$ stands for one of the fields $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers, $\overline{\mathbb{R}}$ for the extended real line, and $\mathbb{N} = \{1, 2, 3, \ldots\}$ stands for the set of natural numbers.

$\mathcal{L}^0_{\mathbb{K}}(\mu)$ is the space of all $\mathbb{K}$-valued Σ-measurable maps on Ω. $\mathcal{L}^0_{\mathbb{K}}(\mu)$ is a $\mathbb{K}$-algebra under *pointwise* addition and multiplication together with multiplication by scalars from $\mathbb{K}$. $\mathcal{L}^0(\mu) := \mathcal{L}^0_{\mathbb{R}}(\mu)$ and $\mathcal{L}^0_{\overline{\mathbb{R}}}(\mu)$, the space of all $\overline{\mathbb{R}}$-valued Σ-measurable functions are lattices under *pointwise* order. The subspace $\mathcal{L}^\infty_{\mathbb{K}}(\mu)$ of $\mathcal{L}^0_{\mathbb{K}}(\mu)$ consisting of all strictly bounded $f \in \mathcal{L}^0_{\mathbb{K}}(\mu)$ (i.e., $\|f\| := \sup_{\omega\in\Omega} |f(\omega)| < \infty$), is a Banach algebra and $\mathcal{L}^\infty(\mu) := \mathcal{L}^\infty_{\mathbb{R}}(\mu)$ is a Banach lattice.

A set $N \in \Sigma$ with $\mu(N) = 0$ is called a *μ-null set* and the σ-ideal of all μ-null sets is denoted by Σ_0 and Σ_f is the ideal of all sets $A \in \Sigma$ with $\mu(A) < \infty$. For $A, B \in \Sigma$ we write $A = B$ a.e. (μ), or only $A \equiv B$ for short if there arise no doubts about the measure, if $A \triangle B$, the symmetric difference of A and B, is a μ-null set, and we write $f = g$ a.e. (μ), or $f \equiv g$ for short, if $\{f \neq g\} \in \Sigma_0$ for $f, g \in \mathcal{L}^0_{\mathbb{K}}(\mu) \cup \mathcal{L}^0_{\overline{\mathbb{R}}}(\mu)$. The equivalence class of all functions in $\mathcal{L}^0_{\mathbb{K}}(\mu) \cup \mathcal{L}^0_{\overline{\mathbb{R}}}(\mu)$ or of all sets in Σ, that are μ-a.e. equal to f or to A will be denoted by $f^\bullet$ or by $A^\bullet$, respectively. Equivalent functions or sets will never be identified.

The (Carathéodory) completion of (Ω, Σ, μ) is written $(\Omega, \widehat{\Sigma}, \widehat{\mu})$. The σ-algebra generated by a family $\mathcal{L}$ of subsets of Ω is denoted by $\sigma(\mathcal{L})$.

A measure space is *locally determined* if $\Sigma = \{A \subseteq \Omega\colon\ A \cap B \in \Sigma$ for $A \in \Sigma_f\}$ and (Ω, Σ, μ) (or just μ) is *semi-finite*, i.e., $\mu(A) = \sup\{\mu(B)\colon\ A \supseteq B \in \Sigma_f\}$ for any $A \in \Sigma$. A measure space (Ω, Σ, μ) is *localizable* if it is semi-finite and, for any $\mathcal{E} \subseteq \Sigma$ there exists an $H \in \Sigma$ such that (i) $E \setminus H \in \Sigma_0$ for any $E \in \mathcal{E}$, (ii) if $G \in \Sigma$ and $E \setminus G \in \Sigma_0$ for every $E \in \mathcal{E}$ then $H \setminus G \in \Sigma_0$. It will be convenient to call such a set H *an essential supremum* of $\mathcal{E}$ in Σ (see Fremlin (1980, A6B)).

If (Θ, T, ν) is a measure space and $f : \Omega \to \Theta$ is a measurable function such that $\nu(B) = \mu(f^{-1}(B))$ for all $B \in T$ then ν is called the range of μ via f and we write $\nu = f(\mu)$.

A *pretopological measure space* is a quadruple $(\Omega, \mathcal{T}, \Sigma, \mu)$ such that $(\Omega, \mathcal{T})$ is a topological space and (Ω, Σ, μ) is a measure space. This notion is introduced only in order to avoid repetitions of the form $(\Omega, \mathcal{T}, \Sigma, \mu)$ such that $(\Omega, \mathcal{T})$ is a topological space and (Ω, Σ, μ) is a measure space.

A *topological measure space* is a quadruple $(\Omega, \mathcal{T}, \Sigma, \mu)$ such that $(\Omega, \mathcal{T})$ is a Hausdorff topological space and (Ω, Σ, μ) is a measure space with $\mathcal{T} \subseteq \Sigma$. Let $(\Omega, \mathcal{T}, \Sigma, \mu)$ be a topological measure space. The measure μ is *τ-additive* if for any increasing family $\langle G_i \rangle_{i\in I}$ of open subsets of Ω we have

$$\mu\Bigl(\bigcup_{i\in I} G_i\Bigr) = \sup_{i\in I} \mu(G_i).$$

For any Hausdorff topological space $(\Omega, \mathcal{T})$ we denote by $\mathcal{B}(\Omega)$ its Borel σ-algebra, i.e., the σ-algebra generated by $\mathcal{T}$. The Baire σ-algebra of $(\Omega, \mathcal{T})$, i.e., the σ-algebra generated by the system of all cozero subsets of Ω, is written $\mathcal{B}_0(\Omega)$. We call $(\Omega, \mathcal{T}, \mathcal{B}, \mu)$ a *Baire* or *Borel measure space*, respectively if $(\Omega, \mathcal{B}, \mu)$ is the completion of a finite measure space $(\Omega, \mathcal{A}, \nu)$, where $\mathcal{A} = \mathcal{B}_0(\Omega)$ or $\mathcal{A} = \mathcal{B}(\Omega)$, respectively. Let $(\Omega, \mathcal{T})$ be a Hausdorff topological space. A measure on $\mathcal{B}_0(\Omega)$ is called *completion regular*, if for any $B \in \mathcal{B}(\Omega)$ there exist $A_1, A_2 \in \mathcal{B}_0(\Omega)$ such that $A_1 \subseteq B \subseteq A_2$ and $\mu(A_2 \setminus A_1) = 0$. A measure μ on

$\mathcal{B}(\Omega)$ is completion regular if its restriction to $\mathcal{B}_0(\Omega)$ is completion regular. A topological measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$ is called a *category measure space* if Σ is the system of all sets with the Baire property with respect to $\mathcal{T}$ and Σ_0 is equal to the system of all sets of the 1st category in Ω. We use the notion of a *quasi-Radon measure space* in the sense of Fremlin (1974, 72A).

For an arbitrary probability space (Ω, Σ, μ) we call $(R, \mathcal{T}, \mathcal{R}, \nu)$ its associated *hyperstonian space* if R is the Stone space of the measure algebra of (Ω, Σ, μ), $\mathcal{T}$ the topology generated by $\{s(a)\colon a \in \Sigma/\mu\}$, where $s(a) \subseteq R$ is the corresponding closed–open set of a according to the Stone duality, $\mathcal{R}$ denotes the σ-algebra of all subsets of R with the Baire property (namely those sets $A \subseteq R$ such that $A \triangle U$ is a first category set for some open subset U of R), and $\nu = \tilde{\mu} \circ \pi : \mathcal{R} \to \mathbb{R}$ where $\pi : \mathcal{R} \to \Sigma/\mu$ is the canonical epimorphism and $\tilde{\mu} : \Sigma/\mu \to \mathbb{R}$ is unambiguously defined by $\tilde{\mu}(a) := \mu(A)$ if $a = A^\bullet$ for $A \in \Sigma$.

Throughout we assume the validity of the Axiom of Choice.

2. Existence of liftings and densities

A *lifting* for a given measure space (Ω, Σ, μ) is a Boolean homomorphism $\rho : \Sigma \to \Sigma$ with the additional properties

$$\rho(A) \equiv A \tag{L1}$$

and

$$\rho(A) = \rho(B) \quad \text{if } A \equiv B, \tag{L2}$$

i.e., besides (L1) and (L2) ρ satisfies more explicitly the equations

$$\rho(\emptyset) = \emptyset \quad \text{and} \quad \rho(\Omega) = \Omega, \tag{B1}$$

$$\rho(A \cap B) = \rho(A) \cap \rho(B), \tag{B2}$$

and

$$\rho(A \cup B) = \rho(A) \cup \rho(B), \tag{B3}$$

if $A, B \in \Sigma$. It follows

$$\rho(A^c) = [\rho(A)]^c \quad \text{for } A \in \Sigma. \tag{B4}$$

Conversely (B2) and (B4) imply (B3). We denote by $\Lambda(\mu)$ the space of all liftings for (Ω, Σ, μ).

There is another way of looking at liftings. For $\rho \in \Lambda(\mu)$ we can define unambiguously a Boolean homomorphism $\rho^\bullet : \Sigma/\mu \to \Sigma$ by means of

$$\rho^\bullet(A^\bullet) := \rho(A) \quad \text{if } A \in \Sigma$$

with the property $r \circ \rho^\bullet = \mathrm{id}_{\Sigma/\mu}$, the identical map of Σ/μ. For this reason it is perhaps more appropriate to call $\rho^\bullet$ lifting (J. von Neumann's original definition), since this reveals its algebraic character more precisely. But in applications the first given definition seems to be more in common use. It is also clear that the measure μ enters only through the σ-ideal Σ_0 of its null sets into the lifting. Therefore measures on Σ producing the same null set ideal produce the same liftings, i.e., liftings depend in fact on the triple $(\Omega, \Sigma, \Sigma_0)$, where Σ_0 is an ideal in Σ and there are generalizations of the notion of lifting along these lines already in the paper of von Neumann and Stone (1935).

In 1931 A. Haar raised the problem of the existence of a lifting for the Lebesgue measure space on $\mathbb{R}^d$, $d \in \mathbb{N}$. J. von Neumann gave a positive solution based on the classical *Lebesgue density*. Clearly the same problem can be raised for arbitrary measure spaces and there too, as we will see below, densities provide a useful step in the construction of liftings. A *lower density* for the measure space (Ω, Σ, μ) is a map $\delta : \Sigma \to \Sigma$ with the properties (L1), (L2), (B1), and (B2) and for the notion of *upper density* we have to replace (B2) by (B3) therein. If we define $\delta^c(A) := (\delta(A^c))^c$ for arbitrary maps $\delta : \Sigma \to \Sigma$ then the operation $\delta \to \delta^c$ is a bijection from the space $\vartheta(\mu)$ of all lower densities onto the space $Y(\mu)$ of all upper densities obeying the law $(\delta^c)^c = \delta$ for all $\delta \in \vartheta(\mu) \cup Y(\mu)$, where $\delta^c = \delta$ if and only if $\delta \in \Lambda(\mu)$. For this reason we consider only lower densities and call them "densities" for short.

The best known example of a (lower) density (being no lifting) is the *Lebesgue density* D, defined by means of

$$D(A) := \left\{ x \in \mathbb{R}^d \colon \lim_{\delta \searrow 0} \frac{\mu(A \cap B_\delta(x))}{\mu(B_\delta(x))} = 1 \right\}$$

for Lebesgue measurable sets $A \subseteq \mathbb{R}^d$, μ the Lebesgue measure on $\mathbb{R}^d$, and $B_\delta(x)$ the ball of center x and radius $\delta > 0$. Lebesgue's celebrated density theorem is just (L1) while the other axioms of a lower density follow more or less by technicalities (see, e.g., Oxtoby (1971, Theorems 3.20 and 3.21)). Starting from the Lebesgue density J. von Neumann (1931) constructed liftings for the Lebesgue measure on $\mathbb{R}^d$ by a process which has been generalized (see Graf and von Weizsäcker (1976), Traynor (1974)) to arbitrary densities on measure spaces (Ω, Σ, μ) and, at the same time, was made more transparent in the following way. For $\delta \in \vartheta(\mu)$ and $\omega \in \Omega$ define a filterbase

$$\mathcal{B}(\omega) := \{ A \in \Sigma \colon \omega \in \delta(A) \}$$

and apply the axiom of choice to find an ultrafilter $\mathcal{U}(\omega)$ finer than $\mathcal{B}(\omega)$. Then put

$$\rho(A) := \{ \omega \in \Omega \colon A \in \mathcal{U}(\omega) \} \quad \text{if } A \in \Sigma.$$

It follows $\delta(A) \subseteq \rho(A) \subseteq \delta^c(A)$ for $A \in \Sigma$ and this implies for *complete* measure spaces (Ω, Σ, μ) that $\rho(A) \in \Sigma$ and ρ satisfies (L1), while all other properties of a lifting are immediate by construction.

THEOREM 2.1. *If the measure space (Ω, Σ, μ) is complete, then for any $\delta \in \vartheta(\mu)$ there exists a $\rho \in \Lambda(\mu)$ such that $\delta(A) \subseteq \rho(A) \subseteq \delta^c(A)$ for all $A \in \Sigma$.*

By an application of Theorem 2.1 to the Lebesgue density the first existence result of von Neumann (1931) is now immediate.

COROLLARY 2.2. *There exists a lifting for the Lebesgue measure space on* $\mathbb{R}^d$, $d \in \mathbb{N}$.

The construction above leading to Theorem 2.1 makes it obvious that the liftings of the corollary rely on a more or less arbitrary choice of an ultrafilter in a non-constructive way with the consequence that any trace of uniqueness or "naturalness" is hopelessly lost. Though starting from the, in a certain sense "natural" Lebesgue density, we cannot single out some sort of "canonical" lifting. Taking any lifting ρ for the Lebesgue measure on $\mathbb{R}$ obtained by this process, we cannot without further information answer such a simple question as, e.g., $0 \in \rho(]-\infty, 0])$ or $0 \in \rho(]0, \infty[)$. On the other hand it can be easily seen that the axiom of choice is necessary for producing a lifting for the Lebesgue measure space (see, e.g., Burke (1993)). For general measure spaces not even a "natural" density is at hand as it was for the Lebesgue measure space. But for finite (even incomplete) measure spaces (Ω, Σ, μ) with $\mu(\Omega) > 0$ a density can be constructed by transfinite induction using the following two extension lemmata for densities. Clearly it is sufficient to consider probability spaces for simplicity.

LEMMA 2.3. *Let* (Ω, Σ, μ) *be a probability space,* η *a* σ*-subalgebra of* Σ *with* $\Sigma_0 \subseteq \eta$, $A \in \Sigma$, *and denote by* $\overline{\eta}$ *the* σ*-subalgebra of* Σ *generated by* $\eta \cup \{A\}$. *Then for every* $\delta \in \vartheta(\mu \mid \eta)$ *there exists a* $\overline{\delta} \in \vartheta(\mu \mid \overline{\eta})$ *extending* δ. *For* $\delta \in \Lambda(\mu \mid \eta)$ *may be chosen* $\overline{\delta} \in \Lambda(\mu \mid \overline{\eta})$.

First note $\overline{\eta} = \{(D \cap A) \cup (E \cap A^c)\colon D, E \in \eta\}$ and choose elements $B, C \in \eta$ such that

$$B = \operatorname{ess\,inf}\{D \in \eta\colon A \subseteq D\} \quad \text{and} \quad C = \operatorname{ess\,inf}\{D \in \eta\colon A^c \subseteq D\}.$$

Then

$$\begin{aligned}&\overline{\delta}\big((D \cap A) \cup (E \cap A^c)\big)\\ &\quad := \big[A \cap \delta\big((D \cap B) \cup (E \cap B^c)\big)\big] \cup \big[A^c \cap \delta\big((E \cap C) \cup (D \cap C^c)\big)\big],\end{aligned}$$

if $D, E \in \eta$ is a solution given by Graf and von Weizsäcker (1976).

LEMMA 2.4. *Let* (Ω, Σ, μ) *be a probability space,* $\langle \eta_n \rangle_{n \in \mathbb{N}}$ *an increasing sequence of* σ*-subalgebras of* Σ *with* $\Sigma_0 \subseteq \eta_1$, *and let* η_∞ *be the* σ*-subalgebra of* Σ *generated by* $\bigcup_{n \in \mathbb{N}} \eta_n$. *For each* $n \in \mathbb{N}$ *let be given* $\delta_n \in \vartheta(\mu \mid \eta_n)$ *with* $\delta_n \mid \eta_m = \delta_m$ *for all* $m \leqslant n$. *Then exists a* $\delta_\infty \in \vartheta(\mu \mid \eta_\infty)$ *satisfying* $\delta_\infty \mid \eta_n = \delta_n$ *for all* $n \in \mathbb{N}$.

If $E_\eta(f)$ denotes a version of the conditional expectation of $f \in \mathcal{L}^\infty(\mu)$ with respect to the σ-subalgebra η of Σ we may define (following Graf and von Weizsäcker (1976))

$$\delta_\infty(A) = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \delta_m\big(\{E_{\eta_m}(\chi_A) \geqslant 1 - 1/k\}\big) \quad \text{if } A \in \eta_\infty.$$

From Doob's martingale convergence theorem (see, e.g., Fremlin (1989, Chapter 22, 1.6)) follows $\delta_\infty(A) = A$ a.e. (μ).

A measure space (Ω, Σ, μ) is called *strictly localizable* if there exists a family $\langle A_i \rangle_{i\in I}$ in Σ of pairwise disjoint sets $A_i \in \Sigma$ with $\mu(A_i) < \infty$ for all $i \in I$ such that

$$\bigcup_{i\in I} A_i = \Omega,\ \Sigma = \{A \subseteq \Omega\colon\ A \cap A_i \in \Sigma \text{ for all } i \in I\},$$

and

$$\mu(A) = \sum_{i\in I} \mu(A \cap A_i) \quad \text{for } A \in \Sigma.$$

Such a family $\langle A_i \rangle_{i\in I}$ is called a *decomposition* of Ω. We will derive existence theorems from the following strictly more general extension theorem.

THEOREM 2.5. *Let be given a strictly localizable measure space (Ω, Σ, μ), a σ-subalgebra η of Σ with $\Sigma_0 \subseteq \eta$, and a density $\delta \in \vartheta(\mu \mid \eta)$. Then there exists a density $\overline{\delta} \in \vartheta(\mu)$ with $\overline{\delta} \mid \eta = \delta$.*

This and the next theorem are in fact theorems about probability spaces since the generalization to strictly localizable spaces is obvious and purely technical. We give a short indication of the proof for a probability space (Ω, Σ, μ) for later reference. Any proof uses induction in one form or another. We can apply induction taking the following steps.

(A) Choose the smallest cardinal $\mathbf{d}$ with the property that there exists a collection $\mathcal{M} \subseteq \Sigma$ of cardinality $\mathbf{d}$ such that the σ-algebra generated by $\eta \cup \mathcal{M}$ is dense in Σ for the pseudometric generated by μ. Let $\mathcal{M} = \langle M_\alpha \rangle_{\alpha<\kappa}$ be indexed by ordinals less than κ, where κ is the first ordinal of cardinality $\mathbf{d}$. For $\alpha \leqslant \kappa$ denote by η_α the σ-subalgebra of Σ generated by $\eta \cup \{M_\beta\colon\ \beta < \alpha\}$, where we may assume $M_\alpha \notin \eta_\alpha$ for $\alpha < \kappa$. Inductively we can construct a family $\langle \delta_\alpha \rangle_{\alpha\leqslant\kappa}$ of densities $\delta_\alpha \in \vartheta(\mu \mid \eta_\alpha)$ with $\delta_\beta \mid \eta_\alpha = \delta_\alpha$ for $\alpha \leqslant \beta \leqslant \kappa$.

(B) The induction starts with $\delta_0 = \delta$.

(C) The step from α to $\alpha + 1$ is covered by Lemma 2.3.

(D) For a limit ordinal $\alpha \leqslant \kappa$ of an uncountable cofinality put $\delta_\alpha = \bigcup_{\beta<\alpha} \delta_\beta$.

(E) For a limit ordinal $\alpha \leqslant \kappa$ of countable cofinality apply Lemma 2.4.

(F) Finally put $\overline{\delta} := \delta_\kappa$.

Densities constructed in this way will become important later on in Section 6. We therefore turn their proof in a definition and call any density δ for a probability space (Ω, Σ, μ) *admissible* if it is constructed inductively taking the steps (A) to (F) above starting from the σ-subalgebra $\eta = \sigma(\Sigma_0)$; $A\vartheta(\mu)$ denotes the family of all admissible densities for μ. Clearly $A\vartheta(\mu) \neq \emptyset$.

The last theorem and Theorem 2.1 imply the next result.

THEOREM 2.6. *Let be given a strictly localizable complete measure space (Ω, Σ, μ), a σ-subalgebra η of Σ with $\Sigma_0 \subseteq \eta$, and a lifting $\rho \in \Lambda(\mu \mid \eta)$. Then there exists a lifting $\overline{\rho} \in \Lambda(\mu)$ with $\overline{\rho} \mid \eta = \rho$.*

We call any $\rho \in \Lambda(\mu)$ *admissibly generated* if there exists a density $\delta \in A\vartheta(\mu)$ such that $\delta(A) \subseteq \rho(A)$ for all $A \in \Sigma$; $AG\Lambda(\mu)$ is the (for complete probability spaces clearly non-empty) class of all admissibly generated liftings for μ.

If in the last two theorems η is the σ-subalgebra of Σ generated by Σ_0 with the obvious lifting on it, we get the following existence theorems of Graf and von Weizsäcker (1976) for densities and of von Neumann (1931) and Maharam (1958) for liftings, respectively.

DENSITY THEOREM. *For every nontrivial strictly localizable measure space there exists a density.*

LIFTING THEOREM. *For every nontrivial strictly localizable complete measure space there exists a lifting.*

Radon measure spaces are strictly localizable since they have a *concassage* (see, e.g., Schwartz (1973)) and hence the lifting theorem applies.

COROLLARY. *Each nontrivial Radon measure space has a lifting.*

Around 1942 J. von Neumann gave an oral proof to S. Kakutani and D. Maharam for the lifting theorem, but "the proof was unfortunately forgotten beyond hope of reconstruction" according to Maharam (1958). In 1958 D. Maharam gave a different proof based on her structure theorem for measure algebras, reducing the general case to the product space $\{0, 1\}^{\kappa}$, where κ is any infinite ordinal, hence only needing a special instance of the martingale convergence theorem. The proof indicated above using general martingale convergence theorem in connection with induction was given by A. and C. Ionescu Tulcea (1961) and is today, in one form or another, standard in literature. It is an open problem whether (even for probability spaces) the lifting theorem remains true without the assumption of completeness. The most interesting case is when measure is defined on the σ-algebra of Borel sets. A lifting $\rho \in \Lambda(\mu)$ is called a *Borel lifting* for a topological measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$ if $\rho(A) \in \mathcal{B}(\Omega)$ for all $A \in \Sigma$, and a similar definition applies for *Baire liftings*. According to Shelah (1983) it is consistent with ZFC that there exists no Borel lifting for Lebesgue measure on $[0, 1]$. On the other hand already von Neumann and Stone (1935) proved the existence of a lifting for the Borel measure space on $[0, 1]$ under the assumption of the *continuum hypothesis*. This has been later generalized to the following result of Mokobodzki (1975) and Fremlin (1977).

THEOREM 2.7 (CH). *Subject to the continuum hypothesis any σ-finite measure space with a measure algebra of cardinality less or equal to ω_2 has a lifting.*

The assumption of the strict localizability implies that the basic measure space is locally determined. Within the class of all *complete locally determined (c.l.d. for short) measure spaces*, the strict localizability is in fact a necessary condition for the existence of a lifting. To see it we need only to consider for a measure space weaker types of decompositions, so called *decompositions* (ND) and (D), respectively, i.e., families $\langle A_i \rangle_{i \in I}$ in Σ_f such that $\mu(A) = \sum_{i \in I} \mu(A \cap A_i)$ for all $A \in \Sigma_f$ as well as $\mu(A_i \cap A_j) = 0$ and $A_i \cap A_j = \emptyset$, respectively for $i \neq j$ in I (see Ellis and Snow (1963)). Due to an application of the axiom of choice decompositions (ND) exist for any measure space and a lifting converts a decomposition (ND) into a decomposition (D). If $\langle A_i \rangle_{i \in I}$ is such a decomposition (D) it follows $\sup_{i \in I} A_i^\bullet = \Omega^\bullet$ and by Fremlin (1978, Theorem 2), the measure space (Ω, Σ, μ) is c.l.d. It is obvious that this argument remains true for a much weaker type of "lifting", the so-called *orthogonal lifting*, i.e., for a map φ from Σ into itself with the properties (L1), (L2), as well as (O) $\varphi(A) \cap \varphi(B) = \emptyset$ for all $A, B \in \Sigma$ with $A \cap B = \emptyset$. Another such weaker notion has been considered by Kölzow (1968), the *monotonous lifting*, i.e., a map λ from Σ into itself with the properties (L1), (L2), and (M) $\lambda(A) \subseteq \lambda(B)$ if $A \subseteq B$ for $A, B \in \Sigma$. Any density is a monotonous lifting, and from the existence of a monotonous lifting λ follows that of an orthogonal lifting φ by taking $\varphi(A) := \lambda(A) \cap \lambda^c(A)$ if $A \in \Sigma$, see Gapaillard (1973) and Strauss (1971). Bichteler (1972) considers *pre-densities*, i.e., maps λ from Σ into itself with the properties (L1), (L2), (B1), and $\lambda(A_1) \cap \cdots \cap \lambda(A_k) = \emptyset$ if $A_1 \cap \cdots \cap A_k = \emptyset$ for $A_1, \ldots, A_k \in \Sigma$ and shows that the existence of a pre-density implies the existence of a density in complete measure spaces. For this reason we get the somewhat surprising result that within the class of all c.l.d. measure spaces the existence of a lifting is equivalent to the existence of considerably weaker types of set functions.

THEOREM 2.8. *For a c.l.d. measure space (Ω, Σ, μ) the following conditions are equivalent.*

(i) *There exists a lifting for (Ω, Σ, μ).*
(ii) *There exists a density for (Ω, Σ, μ).*
(iii) *There exists a pre-density for (Ω, Σ, μ).*
(iv) *There exists a monotonous lifting for (Ω, Σ, μ).*
(v) *There exists an orthogonal lifting for (Ω, Σ, μ).*
(vi) *The measure space (Ω, Σ, μ) is strictly localizable.*

Any measure space has a c.l.d. version (see, e.g., Fremlin (1978)) with the same measure algebra if the measure space is localizable. At this point we should mention that localizable c.l.d. measure spaces are precisely spaces with the Radon–Nikodým property, equivalently the Riesz property (see Segal (1951)) and that by Fremlin (1978) there exist c.l.d. localizable, non strictly localizable measure spaces, i.e., within the class of all c.l.d. measure spaces the class of measure spaces with lifting is strictly smaller than the class of all measure spaces with the Radon–Nikodým property. According to Halmos (1950, Section 31 (9), p. 131) there exist measure spaces without decomposition (D). For such spaces even an orthogonal lifting cannot exist.

3. Liftings for functions

Given a measure space (Ω, Σ, μ) a *(function-)lifting* for $\mathcal{L}^\infty_{\mathbb{K}}(\mu)$ is a $\mathbb{K}$-algebra homomorphism $\varphi: \mathcal{L}^\infty_{\mathbb{K}}(\mu) \to \mathcal{L}^\infty_{\mathbb{K}}(\mu)$ (i.e., a $\mathbb{K}$-linear, multiplicative map) with the additional properties

$$\varphi(f) \equiv f, \tag{11}$$

$$\varphi(f) = \varphi(g) \quad \text{if } f \equiv g \text{ for } f, g \in \mathcal{L}^\infty_{\mathbb{K}}(\mu), \tag{12}$$

and

$$\varphi(1) = 1. \tag{n}$$

We denote by $\Lambda^{\infty}_{\mathbb{K}}(\mu)$ the space of all liftings for $\mathcal{L}^{\infty}_{\mathbb{K}}(\mu)$. Any lifting φ for $\mathcal{L}^{\infty}(\mu)$ is a lattice homomorphism, i.e., $\varphi(|f|) = |\varphi(f)|$ and $\varphi(f^{\pm}) = \varphi(f)^{\pm}$ for $f \in \mathcal{L}^{\infty}(\mu)$ (see A. and C. Ionescu Tulcea (1969a)). The original question of A. Haar was about the existence of a lifting φ for $\mathcal{L}^{\infty}_{\mathbb{K}}(\mu)$, μ the Lebesgue measure on $\mathbb{R}^d$, $d \in \mathbb{N}$ satisfying $\varphi(\bar{f}) = \overline{\varphi(f)}$ for $f \in \mathcal{L}^{\infty}_{\mathbb{K}}(\mu)$. This problem can be easily reduced to the existence of liftings for sets in the following way. Given $\rho \in \Lambda(\mu)$ put for any simple function $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ with $\alpha_i \in \mathbb{K}$, $A_i \in \Sigma$ $(i = 1, \dots, n)$,

$$\rho^{\infty}(f) := \sum_{i=1}^{n} \alpha_i \chi_{\rho(A_i)}.$$

This defines unambiguously a $\mathbb{K}$-algebra endomorphism of the algebra $S_{\mathbb{K}}(\Sigma)$ of all $\mathbb{K}$-simple functions over Σ of norm $\leqslant 1$ having a unique extension to an algebra endomorphism of $\mathcal{L}^{\infty}_{\mathbb{K}}(\mu)$ with $\rho^{\infty}(\bar{f}) = \overline{\rho^{\infty}(f)}$, since $S_{\mathbb{K}}(\Sigma)$ is dense in $\mathcal{L}^{\infty}_{\mathbb{K}}(\mu)$ by the Lebesgue ladder theorem. Conversely for any $\varphi \in \Lambda^{\infty}_{\mathbb{K}}(\mu)$ with $\varphi(\bar{f}) = \overline{\varphi(f)}$ for $f \in \mathcal{L}^{\infty}_{\mathbb{K}}(\mu)$ we can define a $\rho \in \Lambda(\mu)$ by means of

$$\rho(A) := \{\varphi(\chi_A) = 1\} \quad \text{for } A \in \Sigma.$$

PROPOSITION 3.1. *The map $\rho \in \Lambda(\mu) \to \rho^{\infty} \in \Lambda^{\infty}_{\mathbb{K}}(\mu)$ is a bijection which is completely determined by the equation $\rho^{\infty}(\chi_A) = \chi_{\rho(A)}$ for $A \in \Sigma$.*

A different proof of Proposition 3.1 was given by von Neumann (1931) in his first paper on liftings based on the formula

$$\rho^0(f)(\omega) := \inf\{r \in \mathbb{Q} \colon \omega \in \rho(\{f < r\})\}$$
$$\text{if } \rho : \Sigma \to \Sigma,\ f \in \mathcal{L}^0_{\mathbb{R}}(\mu),\ \text{and } \omega \in \Omega.$$

Then ρ^0 and ρ^{∞} can be identified on $\mathcal{L}^{\infty}(\mu)$ since $\rho^0|\mathcal{L}^{\infty}(\mu) \in \Lambda^{\infty}(\mu)$ and $\rho^0(\chi_A) = \chi_{\rho(A)}$. As a corollary we get the next result.

THEOREM 3.2. *For any c.l.d. measure space (Ω, Σ, μ) the following conditions are all equivalent.*

(i) *The measure space (Ω, Σ, μ) is strictly localizable.*
(ii) *The measure space (Ω, Σ, μ) has a lifting.*
(iii) *There exists a lifting φ for $\mathcal{L}^{\infty}_{\mathbb{K}}(\mu)$ such that $\varphi(\bar{f}) = \overline{\varphi(f)}$ for all $f \in \mathcal{L}^{\infty}_{\mathbb{K}}(\mu)$.*

J. von Neumann noticed already in his first paper from 1931 that for the Lebesgue measure on $\mathbb{R}$ the last theorem no longer holds true if the algebra $\mathcal{L}^{\infty}_{\mathbb{K}}(\mu)$ is replaced by the algebra $\mathcal{L}^0_{\mathbb{K}}(\mu)$.

As for liftings of sets there exist weakenings of the notion of lifting for functions which arise naturally in de Possel derivation as well as in Fourier analysis in case of one dimensional Lebesgue measure. Such a type is the *linear lifting* ψ for $\mathcal{L}^\infty(\mu)$, i.e., a map $\psi:\mathcal{L}^\infty(\mu)\to\mathcal{L}^\infty(\mu)$ which is a positive linear map with the additional properties (l1), (l2) and (n) (hence $\psi(\alpha)=\alpha$ for all $\alpha\in\mathbb{R}$). Clearly $\Lambda^\infty(\mu)\subseteq\mathcal{G}(\mu)$, where we write $\mathcal{G}(\mu)$ for the class of all linear liftings and indeed $\Lambda^\infty(\mu):=\Lambda^\infty_{\mathbb{R}}(\mu)$. Note that a linear lifting is already a lifting if it is a lattice homomorphism of $\mathcal{L}^\infty(\mu)$ into itself. In fact for $f:=\chi_A-\chi_{A^c}$, if $A\in\Sigma$ follows $\psi(f^+)\wedge\psi(f^-)=0$. Since $\psi(f^+)=1-\psi(f^-)$, the inequality $0<\psi(f^-)<1$ implies $0<\psi(f^+)<1$, a contradiction. Hence $\psi(f)\in\{0,1\}^\Omega$.

By a classical result of Lebesgue (1910) we have

$$f(x)=\lim_{k\to\infty}\frac{\int_{B_{r_k}(x)}f\,d\mu}{\mu(B_{r_k}(x))}\quad\text{for a.a. } x\in\mathbb{R}^d$$

for any sequence $0<r_k\to0$ $(k\to\infty)$, and any Lebesgue integrable f, if μ is the Lebesgue measure on $\mathbb{R}^d$, $d\in\mathbb{N}$. For $r_k:=1/k$, $x\in\mathbb{R}^d$ and $f\in\mathcal{L}^\infty(\mu)$ writing

$$u_k(f,x)=\frac{\int_{B_{r_k}(x)}f\,d\mu}{\mu(B_{r_k}(x))}$$

we get

$$\left|u_k(f,x)\right|\leqslant\|f\|_\infty\quad\text{for all } k\in\mathbb{N}\text{ and all } x\in\mathbb{R}^d.$$

Therefore, if we choose a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, then there exists

$$\psi_1(f)(x):=\lim_{k\in\mathcal{U}}u_k(f,x)\quad\text{for all } f\in\mathcal{L}^\infty(\mu)\text{ and } x\in\mathbb{R}^d$$

and $\psi_1\in\mathcal{G}(\mu)$, since by Lebesgue's result $\psi_1(f)=f$ a.e. (μ), hence $\psi_1\in\mathcal{L}^\infty(\mu)$ by completeness of μ while all the other properties of a linear lifting follow for ψ_1 by definition. This result can be generalized to more general measure spaces with suitable derivation bases. For $d=1$, μ the Lebesgue measure on $]-\pi,\pi]$ similar results are obtained by Cesaró and Abel summability for the Fourier series of a Lebesgue integrable function f on $]-\pi,\pi]$. In fact, if

$$\sigma_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)K_n(x-t)\,d\mu(t)$$

is the nth *Cesaró-mean*,

$$K_n(x)=\frac{1}{n}\left(\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right)^2$$

the *Fejér kernel*, and

$$f_r(\theta) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)P_r(\theta - t)\,d\mu(t), \quad 0 \leqslant r < 1,\ -\pi < \theta \leqslant \pi$$

is the harmonic of f, where $P_r(\theta) := (1-r^2)/(1-2r\cos\theta + r^2)$ denotes the *Poisson kernel* of f, then $\lim_{n\to\infty}\sigma_n(x) = f(x)$ by a theorem of Lebesgue (1905) (see Zygmund (1968, 3.9, p. 90)) and $\lim_{r\to 1} f_r(x) = f(x)$ for a.a. $x \in]-\pi,\pi]$ by Fatou (1906) (see Hoffmann (1965)). If $f_n := f_{1-1/n}$ then again $|\sigma_n(x)|,\ |f_n(x)| \leqslant \|f\|_\infty$ for $n \in \mathbb{N}$, $x \in]-\pi,\pi]$ for all $f \in \mathcal{L}^\infty(\mu)$ and if the ultrafilter $\mathcal{U}$ is chosen as above, then by means of

$$\psi_2(f)(x) := \lim_{n\in\mathcal{U}} \sigma_n(x), \quad \psi_3(f)(x) := \lim_{n\in\mathcal{U}} f_n(x), \quad x \in]-\pi,\pi],$$

can be defined $\psi_2, \psi_3 \in \mathcal{G}(\mu)$.

There are two procedures (both due to A. and C. Ionescu Tulcea (1969a)) for converting a linear lifting $\psi \in \mathcal{G}(\mu)$ into a lifting in $\Lambda^\infty(\mu)$ provided the basic measure space is complete. First note that by means of

$$\underline{\psi}(A) := \{\psi(\chi_A) = 1\}, \quad \overline{\psi}(A) := \{\psi(\chi_A) > 0\} \quad \text{for } A \in \Sigma$$

we can define $\underline{\psi} \in \vartheta(\mu)$, $\overline{\psi} \in Y(\mu)$ with $\overline{\psi} = (\underline{\psi})^c$ (see (1969a, p. 36)), so a solution is given by choosing a $\rho \in \Lambda(\mu)$ with $\underline{\psi}(A) \subseteq \rho(A) \subseteq \overline{\psi}(A)$ for $A \in \Sigma$, according to Theorem 2.1. Then apply Proposition 3.1 of this section. But according to A. and C. Ionescu Tulcea (1969a, Chapter III, Section 2, Theorem 1), there is no need for resorting to liftings for sets due to the following result paralleling Theorem 2.1.

THEOREM 3.3. *Let be given a complete measure space* (Ω, Σ, μ). *For any* $\psi \in \mathcal{G}(\mu)$ *the set* $\mathcal{G}_\psi := \{\upsilon \in \mathcal{G}(\mu)\colon\ \chi_{\underline{\psi}(A)} \leqslant \upsilon(\chi_A) \leqslant \chi_{\overline{\psi}(A)},\ A \in \Sigma\}$ *is a non-empty, convex and compact subset of the locally convex space* $\mathbb{R}^{\mathcal{L}^\infty(\mu)\times\Omega}$. *An element* $\rho \in \mathcal{G}_\psi$ *is extremal in* $\mathcal{G}_\psi$ *if and only if* ρ *is a lifting for* $\mathcal{L}^\infty(\mu)$.

The existence of an extremal element in $\mathcal{G}_\psi$ follows now from the Krein–Milman theorem.

In general measure spaces there are no "natural" linear liftings at hand, a situation very similar to that for densities. But again there is an inductive construction which parallels that one for densities in Section 2.

LEMMA 3.4. *If* (Ω, Σ, μ), η, A *and* $\overline{\eta}$ *are as in Lemma* 2.3 *of Section* 2 *then for each* $\psi \in \mathcal{G}(\mu \mid \eta)$ *there exists a* $\overline{\psi} \in \mathcal{G}(\mu \mid \overline{\eta})$ *extending* ψ. *For* $\psi \in \Lambda^\infty(\mu \mid \eta)$ *may be chosen* $\overline{\psi} \in \Lambda^\infty(\mu \mid \overline{\eta})$.

First note that $\mathcal{L}^\infty(\mu \mid \overline{\eta}) = \{f\chi_A + g\chi_{A^c}\colon\ f, g \in \mathcal{L}^\infty(\mu \mid \eta)\}$, and if $B, C \in \eta$ are defined as in Lemma 2.3 of Section 2 then put

$$\overline{\psi}(f\chi_A + g\chi_{A^c}) := \psi(f\chi_B + g\chi_{B^c})\chi_A + \psi(f\chi_{C^c} + g\chi_C)\chi_{A^c}$$

for $f, g \in \mathcal{L}^\infty(\mu \mid \eta)$, a formula completely analogous to the corresponding formula for densities in Lemma 2.3 of Section 2 (see Graf and von Weizsäcker (1976, p. 156)).

LEMMA 3.5. *If (Ω, Σ, μ) is a complete probability space, $\langle \eta_n \rangle_{n\in\mathbb{N}}$ and η_∞ are as in Lemma* 2.4 *of Section 2 then for all $\psi_n \in \mathcal{G}(\mu \mid \eta_n)$ with $\psi_n \mid \eta_m = \psi_m$ if $m \leqslant n \in \mathbb{N}$ there exists a $\psi_\infty \in \mathcal{G}(\mu \mid \eta_\infty)$ satisfying $\psi_\infty \mid \eta_n = \psi_n$ for $n \in \mathbb{N}$.*

As in the above examples choose again a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and put

$$\psi_\infty(f)(\omega) := \lim_{n\in\mathcal{U}} \psi_n\big(E_{\eta_n}(f)\big)(\omega) \quad \text{for } f \in \mathcal{L}^\infty(\mu \mid \eta_\infty),\ \omega \in \Omega.$$

(Here $\lim_{n\in\mathcal{U}}$ could be replaced by any Banach limit, see Dunford and Schwartz (1958, Chapter II, 4.22–23).) The existence of the limit is guaranteed by $|\psi_n(E_{\eta_n}(f))(\omega)| \leqslant \|f\|_\infty$ for $n \in \mathbb{N}$ and $\omega \in \Omega$. Now Doob's martingale convergence theorem implies $\psi_\infty(f) = f$ a.e. $(\mu \mid \eta_\infty)$. Since (Ω, Σ, μ) is assumed to be *complete* this implies $\psi_\infty(f) \in \mathcal{L}^\infty(\mu \mid \eta_\infty)$. All other properties of a linear lifting are immediate by the definition of ψ_∞ as a limit. Using Lemmata 3.4 and 3.5 for linear liftings in the same way as for densities in Section 2 in an inductive proof taking exactly the same steps (A) to (F) exhibited after Theorem 2.5 we get extension theorems for linear liftings and in the same way as for densities the (non-empty) class $A\mathcal{G}(\mu)$ of *admissible linear liftings*. But note that we need completeness of the basic measure space.

THEOREM 3.6. *If (Ω, Σ, μ) is a strictly localizable, complete measure space and η a σ-subalgebra of Σ with $\Sigma_0 \subseteq \eta$ then for any $\psi \in \mathcal{G}(\mu \mid \eta)$ there exists a $\overline{\psi} \in \mathcal{G}(\mu)$ with $\overline{\psi} \mid \eta = \psi$.*

If we take here η as the σ-algebra generated by Σ_0 then the existence of a linear lifting follows. It should be however noted that according to Burke and Shelah (1992) it is consistent with ZFC, the Zermelo–Fraenkel set theory including the axiom of choice, that $\mathcal{L}^\infty(\mu)$ admits no linear lifting for many non-complete probability spaces including Borel measure space on $[0, 1]$. For this reason there is no need for stating an existence result for linear liftings on strictly localizable complete measure spaces, since there the better result of the existence of liftings is available. In the context of extensions naturally arises the problem of mapping of liftings and densities.

THEOREM 3.7. *Let (Ω, Σ, μ) and (Θ, T, ν) be measure spaces together with a measurable map $f\colon \Omega \to \Theta$ such that $\nu = f(\mu)$ is the image measure of μ under f. If (Ω, Σ, μ) is strictly localizable then for every $\zeta \in \vartheta(\nu)$ there exists a $\delta \in \vartheta(\mu)$ such that*

$$\delta\big(f^{-1}(B)\big) = f^{-1}\big(\zeta(B)\big) \tag{C}$$

for all $B \in T$. If in addition (Ω, Σ, μ) is complete then for given $\zeta \in \Lambda(\nu)$ we may choose $\delta \in \Lambda(\mu)$ satisfying equation (C) *and in that case we have*

$$\delta^{\infty}(h \circ f) = \zeta^{\infty}(h) \circ f \tag{C^∞}$$

for all $h \in \mathcal{L}^{\infty}(\nu)$.

In fact, if η is the σ-subalgebra of Σ generated by Σ_0 and $\{f^{-1}(B)\colon\ B \in T\}$, a density δ_0 for $\mu \mid \eta$ is defined by means of $\delta_0(A) := f^{-1}(\zeta(B))$ if $A \in \eta$ and $A \equiv f^{-1}(B)$ for some $B \in T$. By Theorem 2.5 we can extend δ_0 to a density δ on Σ. If ζ is a lifting then δ_0 is too and we can apply Theorem 2.6 to extend it to a lifting on Σ because (Ω, Σ, μ) is assumed to be complete.

Since we have applied extension theorems for getting δ from ζ it it obvious that δ is not uniquely determined by ζ via the equation (C). We call any $\delta \in \vartheta(\mu)$ satisfying the equation (C) an *inverse density* of ζ and write $f^{-1}(\zeta)$ for the class of all inverse densities for ζ. On the other hand for a *surjective* map f any $\zeta \in \vartheta(\nu)$ satisfying (C) is uniquely determined by $\delta \in \vartheta(\mu)$ since then $\zeta(B) = f(\delta(f^{-1}(B)))$ for $B \in T$. For this reason we call ζ the *direct density* of δ and write $\zeta = f(\delta)$. In the same way we can define the *inverse lifting* and the *direct lifting*. But note that for given $\rho \in \Lambda(\mu)$ in general no direct lifting exists. Clearly $\bigcup_{\zeta \in \vartheta(\nu)} f^{-1}(\zeta)$ is the class of all densities for μ having a direct density, and similarly for liftings, but no inner characterization for the elements in this class is known and as yet very poor partial results can be given, e.g., if $T := \{B \subseteq \Theta\colon\ f^{-1}(B)\}$ and f is injective then for any $\rho \in \Lambda(\mu)$ there exists a direct lifting, see Macheras and Strauss (1992, Lemma 2.2). We refer to Kupka (1983) for his 'projection' Theorem 2.7 which gives a positive result for the 'projection' of strong liftings in the presence of a disintegration. Another way of projecting from products onto its factors is discussed by Macheras and Strauss (2000).

For linear liftings similar results as for liftings can be obtained by an application of the extension Theorem 3.6.

THEOREM 3.8. *If (Ω, Σ, μ) is a complete, strictly localizable measure space and the map f is surjective then for each $\varphi \in \mathcal{G}(\nu)$ exists a $\psi \in \mathcal{G}(\mu)$ such that*

$$\psi(h \circ f) = \big(\varphi(h)\big) \circ f \tag{C^∞}$$

for all $h \in \mathcal{L}^{\infty}(\nu)$. Again ψ can be chosen as a lifting if φ is a lifting.

Starting from the last theorem the *inverse and direct linear lifting* can be defined in analogy to the inverse and direct lifting. Note that the equation (C^∞) implies $\underline{\psi} \in f^{-1}(\underline{\varphi})$ and $\overline{\psi} \in f^{-1}(\overline{\varphi})$ if $\underline{\psi}$ is the lower density defined by $\underline{\psi}(A) := \{\psi(\chi_A) = 1\}$ and $\overline{\psi}$ is the upper density defined by the formula $\overline{\psi}(A) := \{\psi(\chi_A) > 0\}$ for $A \in \Sigma$.

The above indicated construction of linear liftings by means of de Possel derivation has some sort of converse. As a preparation we need the following result in which (i), (ii), (iii) were given by Maharam (1958), Theorem 4 for lifted sets. It is remarkable in itself since in general uncountable unions of measurable sets are no longer measurable, but it has interesting consequences in addition.

THEOREM 3.9. *Let be given a c.l.d. measure space* (Ω, Σ, μ) *with a density* $\delta \in \vartheta(\mu)$. *Then for every non-empty collection* $\mathcal{B} \subseteq \Sigma$ *with* $B \subseteq \delta(B)$ *for* $B \in \mathcal{B}$ *we have*

(i) $\bigcup \mathcal{B} \in \Sigma$;
(ii) $\bigcup \mathcal{B} \subseteq \bigcup_{B \in \mathcal{B}} \delta(B) \subseteq \delta(\cup \mathcal{B})$;
(iii) $\bigvee_{B \in \mathcal{B}} B^{\bullet} = (\cup \mathcal{B})^{\bullet}$ ($\bigvee$ *denotes the upper bound of* $\mathcal{B}$ *in* Σ/μ);
(iv) $\Omega_0 := \bigcup_{A \in \Sigma_f} \delta(A) \in \Sigma$ *and* $\Omega_0 = \Omega$ *a.e.* (μ);
(v) $\sup_{B \in \mathcal{B}} \mu(B) = \mu(\bigcup \mathcal{B})$ *if* $\mathcal{B}$ *is directed upwards*;
(vi) (Ω, Σ, μ) *is localizable.*

There is a corresponding version for liftings of functions given by A. and C. Ionescu Tulcea (1969a). J. Gapaillard asserts condition (i) of the last theorem for monotonous liftings with a proof being convincing at least for finite measure spaces. As a first consequence of Theorem 3.9 we get *Vitali derivation bases* (see Kölzow (1968, Section 12) for definition) in c.l.d. measure spaces with lifting by means of the next result of Kölzow (1968).

THEOREM 3.10. *For given c.l.d. measure space* (Ω, Σ, μ) *with a lifting* $\rho \in \Lambda(\mu)$ *put* $R := \bigcup_{A \in \Sigma_f} \rho(A)$ *and define* $g_\rho(\omega) := \{A \in \Sigma_f\colon \omega \in A \subseteq \rho(A)\}$ *and* $a_\rho(\omega) := \{g\colon g$ *is a cofinal subset of* $g_\rho(\omega)\}$ *for* $\omega \in R$. *Then* $\langle a_\rho(\omega)\rangle_{\omega \in R}$ *is a strong Vitali derivation basis.*

If conversely a *weak* Vitali derivation basis $a = \langle a(\omega)\rangle_{\omega \in R}$, $R^c \in \Sigma_0$ is given in a c.l.d. measure space (Ω, Σ, μ) we define the *lower de Possel derivative* with respect to a

$$\underline{D}_a(f)(\omega) = \inf_{g \in a(\omega)} \liminf_{A \in g} \frac{\int_A f \, d\mu}{\mu(A)}$$

for all locally integrable functions, i.e., all Σ-measurable functions such that $f\chi_A$ is μ-integrable for all $A \in \Sigma_f$, and $\omega \in \Omega$. Put $D_a(A) := \{\underline{D}_a(\chi_A) = 1\}$ for $A \in \Sigma$. Then D_a is a density for μ which can be converted into a lifting by Theorem 2.1, i.e., we have the converse of Theorem 3.10 which is also due to Kölzow (1968), see also C. Ionescu Tulcea (1971) and Sion (1973).

COROLLARY 3.11. *For each c.l.d. measure space* (Ω, Σ, μ) *with* $\mu(\Omega) > 0$ *there exists a lifting if and only if the measure space has a weak (strong) Vitali derivation basis.*

In the situation of Theorem 3.10 the upper and lower *de Possel derivatives* with respect to a_ρ are given by

$$\overline{D}(f)(\omega) = \limsup_{A \in a_\rho(\omega)} \frac{\int_A f \, d\mu}{\mu(A)}, \qquad \underline{D}(f)(\omega) = \liminf_{A \in a_\rho(\omega)} \frac{\int_A f \, d\mu}{\mu(A)}$$

for all locally integrable functions, i.e., for all Σ-measurable functions such that $f\chi_A$ is μ-integrable for all $A \in \Sigma_f$ and all $\omega \in R := \bigcup_{A \in \Sigma_f} \rho(A)$. If in addition $R = \Omega$ then they satisfy $\overline{D}(f) = \underline{D}(f) = \rho(f)$ for all $f \in \mathcal{L}^\infty(\mu)$, i.e., the lifting appears as a de Possel derivation. The assumption $\bigcup_{A\in\Sigma_f}\rho(A)=\Omega$ is trivially satisfied for every probability space and can be achieved for any σ-finite measure space.

A *linear lifting* ρ for $\mathcal{L}^p(\mu)$ is a positive linear map from $\mathcal{L}^p(\mu)$ into $\mathcal{L}^p(\mu)$ satisfying the basic properties (l1) and (l2) of liftings, where $\mathcal{L}^p(\mu)$ is the Banach space of all finitely *real-valued functions* $f\in\mathcal{L}^0(\mu)$ with $\|f\|_p := \int|f|\,d\mu<\infty$, for $1\leqslant p<\infty$. A. and C. Ionescu Tulcea (1969a, Chapter IV, Section 4, Theorem 6) noticed that there can't exist a linear lifting for $\mathcal{L}^p(\mu)$, $1\leqslant p<\infty$, if there exists a non-negligible measurable set A which is *diffuse*, i.e., whose class $A^\bullet\in\Sigma/\mu$ does not contain any atom (a similar conclusion holds true in case $p=0$ under an obvious definition of the linear lifting for $\mathcal{L}^0(\mu)$).

With a similar proof one can see that under the same assumption there can't exist a lifting $\rho\in\Lambda(\mu)$ satisfying the additional condition $\rho(\bigcap_{n=1}^\infty A_n)=\bigcap_{n=1}^\infty\rho(A_n)$ if $A_n\in\Sigma$, $n\in\mathbb{N}$, in fact only the properties (L1) and (L2) of a lifting are needed for this conclusion.

All these results are on the basis of ZFC, the Zermelo–Fraenkel set theory including the axiom of choice. In Solovay's model of Zermelo–Fraenkel set theory (where the axiom of choice fails) the above mentioned proof for the non-existence of a linear lifting for $\mathcal{L}^p(\mu)$ in case $1\leqslant p<\infty$ carries over to the case $p=\infty$ since $(L^\infty(\mu))'=L^1(\mu)$ (a result of Christensen (1974)), hence $\mathcal{G}(\mu)=\emptyset$ for Lebesgue measure μ, saying again that some sort of non-constructive tool like the axiom of choice is needed for an existence proof of a lifting even for Lebesgue measure (see Graf and von Weizsäcker (1976)).

4. Liftings on topological spaces

Throughout this section $(\Omega,\mathcal{T},\Sigma,\mu)$ is a pretopological measure space with $(\Omega,\mathcal{T})$ being Hausdorff. Theorem 3.9 gives raise to two classical "lifting topologies" of A. and C. Ionescu Tulcea (1969a), which according to A. and C. Ionescu Tulcea (1964b, p. 445) are partially tracing back to J.C. Oxtoby and which convert lifted functions into functions continuous with respect to the lifting topology. For given measure space (Ω,Σ,μ) and $\delta\in\vartheta(\mu)$ the collection $\mathcal{D}_\delta:=\{\delta(A)\colon A\in\Sigma\}$ is a basis for a topology t_δ on Ω. We don't assert $t_\delta\subseteq\Sigma$ at the moment, but it follows from Theorem 3.9 that $t_\delta\cap\Sigma\subseteq\tau_\delta:=\{A\in\Sigma\colon A\subseteq\delta(A)\}$.

The next theorem gives a collection of the basic properties of the *density* and *lifting topologies* t_δ, τ_δ, t_ρ, and τ_ρ, taken from A. and C. Ionescu Tulcea (1969a) and A. Ionescu Tulcea (1967a). Before we state it, we need a definition. If $(\Omega,\mathcal{T},\Sigma,\mu)$ is a pretopological measure space, a density $\delta\in\vartheta(\mu)$ is called $\mathcal{T}$*-strong*, if $G\subseteq\delta(G)$ for all $G\in\mathcal{T}\cap\Sigma$, and topologies $\mathcal{T}$ satisfying such a condition are called *compatible* with δ by A. and C. Ionescu Tulcea (1969a). It is well known that the Lebesgue density D on the Lebesgue measure space on $\mathbb{R}^d$ is $\mathcal{E}^d$-strong for the euclidean topology $\mathcal{E}^d$ on $\mathbb{R}^d$, and so is any lifting ρ for the Lebesgue measure satisfying $D(A)\subseteq\rho(A)$ for all $A\in\Sigma$ (such liftings exist by Theorem 2.1).

THEOREM 4.1. *For given c.l.d. measure spaces (Ω,Σ,μ) with $\delta\in\vartheta(\mu)$ and $\rho\in\Lambda(\mu)$ we have the following results.*

(i) $t_\delta\subseteq\tau_\delta\subseteq\Sigma$.

(ii) *$\tau_\delta \cap \Sigma_0 = \emptyset$ and for all $A \in \Sigma$ there exists a $G \in \tau_\delta$ with $A \triangle G \in \Sigma_0$.*
(iii) *A subset $K \subseteq \Omega$ is of the 1st category with respect to the topology τ_δ if and only if K is closed and nowhere dense or equivalently if $K \in \Sigma_0$.*
(iv) *The topologies t_ρ and τ_ρ are extremally disconnected and $t_\rho \subseteq \tau_\rho$.*
(v) *The topology t_ρ (hence τ_ρ) is Hausdorff if the set $\{\rho(A)\colon A \in \Sigma\}$ separates the points of Ω.*
(vi) *$C_{\overline{\mathbb{R}}}(\Omega, t_\rho) = C_{\overline{\mathbb{R}}}(\Omega, \tau_\rho) = \{\rho^0(f)\colon f \in \overline{\mathcal{L}}^0(\mu)\}$ and $\rho^0(f)$ is the unique continuous function with respect to t_ρ (respectively, τ_ρ) in $f^\bullet$.*
(vii) *$C_b(\Omega, t_\rho) = C_b(\Omega, \tau_\rho) = \{\rho^\infty(f)\colon f \in \mathcal{L}^\infty(\mu)\}$.*
(viii) *If t_ρ is Hausdorff then the Stone space of the measure algebra Σ/Σ_0 is the Stone Cech compactification of (Ω, t_ρ).*

If $\mathcal{T} \in \{t_\rho, \tau_\rho\}$ we can add the following conditions:

(ix) *We have $cl_{\mathcal{T}}(A) = \rho(A)$ for all $A \in \mathcal{T}$ and $\Sigma = \mathcal{B}(\Omega, \tau_\rho)$.*
(x) *ρ is the unique $\mathcal{T}$-strong lifting for the topological measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$.*
(xi) *The topological measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$ is a quasi-Radon measure space.*

Here (i) follows from Theorem 3.9 and (ii) and (iii) are from A. Ionescu Tulcea (1967a). A proof for (iv) to (viii) is available in A. and C. Ionescu Tulcea (1969a, Chapter V, Section 3), as well as the equation $C_{\overline{\mathbb{R}}}(\Omega, t_\rho) = C_{\overline{\mathbb{R}}}(\Omega, \tau_\rho)$ in (v).

If δ is the Lebesgue density (see Section 2) of the Lebesgue measure space (Ω, Σ, μ) on $\mathbb{R}$ and ρ is a lifting with $\delta(A) \subseteq \rho(A)$ for $A \in \Sigma$ then τ_ρ is completely regular and $t_\rho = \tau_\rho$ by A. and C. Ionescu Tulcea (1969a, Chapter V, Section 4, Theorem 2). But there exists a $\rho_0 \in \Lambda(\mu)$ such that τ_{ρ_0} is finer than the euclidean topology on $\mathbb{R}$ and $t_{\rho_0} \neq \tau_{\rho_0}$. For any $\lambda \in \Lambda(\mu)$ with τ_λ finer than the euclidean topology on $\mathbb{R}$ the topology τ_λ is not normal and every τ_λ-compact $K \subseteq \mathbb{R}$ is finite by A. Ionescu Tulcea (1967a).

It is now easy to see that generally the topological measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$ from (vii) will not be a Radon measure space. Take for instance as (Ω, Σ, μ) the Lebesgue measure space with $\mathcal{T} := \tau_\rho$ for a lifting $\rho \in \Lambda(\mu)$ for which τ_ρ is finer than the euclidean topology $\mathcal{E}_d$ on $\mathbb{R}$. Since by the above remark any τ_ρ-compact subset K of $\mathbb{R}$ is finite the inner regularity $\mu(A) = \sup\{\mu(K)\colon K \subseteq A, K\ \tau_\rho\text{-compact}\}$ with respect to τ_ρ-compact subsets K of $\mathbb{R}$ can't be true if $\mu(A) > 0$.

If (Ω, Σ, μ) and (Θ, T, ν) are measure spaces, $\delta \in \vartheta(\mu)$, and $\zeta \in \vartheta(\nu)$ then a Σ–T-measurable map $f : \Omega \to \Theta$ is τ_δ–τ_ζ-continuous if and only if $f^{-1}(\zeta(B)) \subseteq \delta(f^{-1}(\zeta(B)))$ for all $B \in T$. If moreover $\delta \in \Lambda(\mu)$, and $\zeta \in \Lambda(\nu)$ these conditions are equivalent with $f^{-1}(\zeta(B)) = \delta(f^{-1}(B))$ for all $B \in T$ as well as with $\zeta(h) \circ f = \delta(h \circ f)$ for all $h \in \mathcal{L}^\infty(\mu)$.

PROPOSITION 4.2. *If $(\Omega, \mathcal{T}, \Sigma, \mu)$ and $(\Theta, \mathcal{S}, T, \nu)$ are topological measure spaces, the map f is a $\mathcal{T}$–$\mathcal{S}$-continuous surjection, and $f(\delta)$ exists for $\delta \in \vartheta(\mu)$, then $f(\delta)$ is $\mathcal{T}$-strong provided δ is $\mathcal{T}$-strong.*

But note that under the assumptions of Proposition 4.2 for a strong lifting ζ for ν an inverse lifting $\delta \in f^{-1}(\zeta)$ needs not to be strong, see the example before Theorem 6.20.

As we state below, the last theorem gives in fact a topological characterization for c.l.d. measure spaces having a lifting and there are again characterizations by weaker types of "liftings" in an abounding number, hence we can mention only the most spectacular ones. We call a map φ from $\mathcal{L}^\infty(\mu)$ into itself satisfying the properties (11), (12) a *monotonous lifting* if in addition $\varphi(f) \leqslant \varphi(g)$ for $f, g \in \mathcal{L}^\infty(\mu)$ and $f \leqslant g$, it is called a *bounded linear lifting* if φ is a linear map with $\|\varphi\| := \sup\{\|\varphi(f)\|/\|f\|_\infty\colon 0 < \|f\|_\infty < \infty\} < \infty$, where $\|f\|$ is the strict supremum and $\|f\|_\infty$ the essential supremum of a function $f \in \mathcal{L}^\infty(\mu)$, it is called a *function lower* respectively *upper density* if $\varphi(f \wedge g) = \varphi(f) \wedge \varphi(g)$ and $\varphi(f \vee g) = \varphi(f) \vee \varphi(g)$, respectively for $f, g \in \mathcal{L}^\infty(\mu)$.

THEOREM 4.3. *For given c.l.d. measure spaces (Ω, Σ, μ) the following conditions are all equivalent with the existence of a lifting for $\mathcal{L}^\infty(\mu)$.*

(i) *There exists a topology $\mathcal{T} \subseteq \Sigma$ such that the topological measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$ is a category measure space.*
(ii) *There exists a topology $\mathcal{T} \subseteq \Sigma$ such that $\mathcal{T} \cap \Sigma_0 = \emptyset$ and a set $K \subseteq \Omega$ is of first category if and only if it is closed and nowhere dense in Ω.*
(iii) *There exists a topology $\mathcal{T} \subseteq \Sigma$ such that* $\operatorname{card}(f^\bullet \cap C_b(\Omega)) = 1$ *for any $f \in \mathcal{L}^\infty(\mu)$, where $C_b(\Omega)$ denotes the space of all $\mathcal{T}$-continuous, bounded real-valued functions on Ω.*
(iv) *There exists a monotonous lifting for $\mathcal{L}^\infty(\mu)$.*
(v) *There exists a function lower respectively upper density for $\mathcal{L}^\infty(\mu)$.*
(vi) *There exists a linear lifting for $\mathcal{L}^\infty(\mu)$.*
(vii) *There exists a bounded linear lifting φ for $\mathcal{L}^\infty(\mu)$ of norm $\|\varphi\| < 3$.*

For (i) and (ii) compare Graf (1973), for the sophisticated equivalence proof for (vii) see Erben (1983), where an example is given that the bound 3 cannot be improved.

We call $\delta \in \vartheta(\mu)$ (and $\psi \in \mathcal{G}(\mu)$) *almost $\mathcal{T}$-strong*, if there exists a set $N \in \Sigma_0$ such that for all $G \in \mathcal{T} \cap \Sigma$ follows $G \setminus N \subseteq \delta(G)$ (respectively $\psi(f) \mid N^c = f \mid N^c$ for all $f \in C_b(\Omega) \cap \mathcal{L}^\infty(\mu)$). δ and ψ are called $\mathcal{T}$-strong in case $N = \emptyset$. It is obvious that $\delta \in \vartheta(\mu)$ is $\mathcal{T}$-strong if and only if $\mathcal{T} \subseteq \tau_\delta$.

If $\mathcal{E}_d$ is the euclidean topology of $\mathbb{R}^d$ then it is well known that the Lebesgue density D and any lifting ρ for the Lebesgue measure with $D(A) \subseteq \rho(A)$ for $A \in \Sigma$ are $\mathcal{E}_d$-strong, and so are the linear liftings ψ_1, ψ_2 from Section 3 obtained by L. Fejér's theorem (see Hoffmann (1965, pages 20 and 33)). Any hyperstonian space has a uniquely determined strong lifting, which is given by choosing the unique continuous function from each equivalence class of $L^\infty(\mu)$. For any complete measure space (Ω, Σ, μ) any $\rho \in \Lambda(\mu)$ is t_ρ-strong as well as τ_ρ-strong by the definition of t_ρ and τ_ρ.

PROPOSITION 4.4. *If the topology $\mathcal{T}$ is $T_{3\frac{1}{2}}$ then for each $\rho \in \Lambda(\mu)$ the following conditions are equivalent.*

(i) *ρ is almost $\mathcal{T}$-strong.*
(ii) *ρ^∞ is almost $\mathcal{T}$-strong.*
(iii) *There exists a $N \in \Sigma_0$ such that $\rho(F) \subseteq F \cup N$ for all closed $F \in \Sigma$.*
(iv) *There exists a $N \in \Sigma_0$ such that $\rho^0(f) \mid N^c = f \mid N^c$ for all $f \in \overline{C}(\Omega) \cap \mathcal{L}^0(\mu)$, where $\overline{C}(\Omega)$ denotes the space of all continuous functions from Ω into $\overline{\mathbb{R}}$.*
(v) *There exists a $N \in \Sigma_0$ such that $\rho^\infty(f) \mid N^c = f \mid N^c$ for all $f \in C_b(\Omega) \cap \mathcal{L}^\infty(\mu)$, where $C_b(\Omega)$ denotes the space of all bounded continuous functions from Ω into $\mathbb{R}$.*

Here we can choose the same set $N \in \Sigma_0$ in (i) *to* (iv), *where in particular $N = \emptyset$ for $\mathcal{T}$-strong ρ and remember that ρ, ρ^{∞}, and ρ^0 are in biunique correspondence by means of the equations $\rho^{\infty}(\chi_A) = \rho^0(\chi_A) = \chi_{\rho(A)}$ for $A \in \Sigma$.*

The implication (i) $\Rightarrow$ (iv) is quickly achieved by observing $\rho^0 = \rho_0$ if $\rho_0(f) = \sup\{r \in \mathbb{Q}\colon \omega \in \rho(\{f > r\})\}$ since then $\rho_0(f) \mid N^c \geqslant f \mid N^c$ for $f \in \overline{C}(\Omega)$ and $\rho_0(-f) = -\rho_0(f)$ hence $\rho_0(f) \mid N^c = f \mid N^c$ if if ρ is almost $\mathcal{T}$-strong with universal set $N \in \Sigma_0$. The implication (iv) $\Rightarrow$ (ii) and the equivalence of (i) and (ii) are obvious.

Moreover (ii) $\Rightarrow$ (i) works for almost $\mathcal{T}$-strong $\psi \in \mathcal{G}(\mu)$, since for a $T_{3\frac{1}{2}}$ topology $\mathcal{T}$ follows $\chi_G = \sup\{h \in C_b(\Omega)\colon h \leqslant \chi_G\}$ if $G \in \mathcal{T}$ hence $\chi_{G\setminus N} \leqslant \psi(\chi_G) \mid N^c$ therefore $G \setminus N \subseteq \underline{\psi}(G)$ for $G \in \Sigma \cap \mathcal{T}$ if $\underline{\psi}$ is defined by $\underline{\psi}(A) = \{\psi(\chi_A) = 1\}$ for $A \in \Sigma$, where for $\rho \in \Lambda(\mu)$ follows $\underline{\rho} = \rho$. So we have in addition the following result.

PROPOSITION 4.5. *If the topology $\mathcal{T}$ is $T_{3\frac{1}{2}}$ and the measure space (Ω, Σ, μ) is complete then the existence of a $\mathcal{T}$-almost strong density for μ is equivalent with the existence of a $\mathcal{T}$-almost strong (linear) lifting. Here we may replace "almost strong" by "strong".*

If a topological measure space admits a $\mathcal{T}$-strong density its measure has to be of full support, i.e., $\mathrm{supp}(\mu) = \Omega$, since then for all $G \in \mathcal{T}$ follows $G = \emptyset$ from $G \in \Sigma_0$. The notion of the almost strong lifting allows us to cover in full generality the cases with $\mathrm{supp}(\mu) \neq \Omega$, for which the notion of strong lifting is inappropriate.

PROPOSITION 4.6. *If (Ω, Σ, μ) is a complete measure space with $\mathrm{supp}(\mu) = \Omega$, then from the existence of a $\mathcal{T}$-almost strong lifting for μ follows the existence of a $\mathcal{T}$-strong lifting.*

This is easily achieved for $\rho \in \Lambda(\mu)$ with $G \setminus N \subseteq \rho(G)$ for $G \in \mathcal{T} \cap \Sigma$, $N \in \Sigma_0$ by choosing an ultrafilter $\mathcal{U}(\omega)$ finer than the filter basis $\{G \in \mathcal{T}\colon \omega \in G\}$ for $\omega \in N$ and putting

$$\overline{\rho}(A) := \big(\rho(A) \cap N^c\big) \cup \big\{\omega \in N\colon A \in \mathcal{U}(\omega)\big\} \quad \text{for } A \in \Sigma.$$

The space $(\Omega, \mathcal{T}, \Sigma, \mu)$ (or just μ) has the *almost strong lifting property* (*ASLP* for short), if there exists an almost $\mathcal{T}$-strong lifting for μ. Since for general spaces $(\Omega, \mathcal{T}, \Sigma, \mu)$ there is no "natural" candidate for an almost strong lifting, one possible issue is to check whether arbitrary liftings are almost strong. This leads to the stronger notion of *universal strong lifting property* (*USLP* for short), which says that $\Lambda(\mu) \neq \emptyset$ and every $\rho \in \Lambda(\mu)$ is almost $\mathcal{T}$-strong. Results sufficient for applications (polish spaces) rely on the next result which is a generalization of a result from Maher (1978), see also Fremlin (200?), and Macheras and Strauss (1996a). Before we state it we need a modification of the purely topological notion of network (see Gruenhage (1984)). A family $\mathcal{F} \subseteq \Sigma$ is called a *measurable network* for the pretopological measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$ if for each $G \in \mathcal{T}$ there exists a subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that $G = \cup\mathcal{G}$; we denote by $mnw(\mu)$ the least cardinal of a measurable network for $(\Omega, \mathcal{T}, \Sigma, \mu)$.

LEMMA 4.7. *If $(\Omega, \mathcal{T}, \Sigma, \mu)$ is a pretopological measure space with a measurable network $\mathcal{F}$ then $\rho \in \Lambda(\mu)$ is almost $\mathcal{T}$-strong if there exists $N \in \Sigma_0$ such that $G \setminus N \subseteq \rho(G)$ for all $G \in \mathcal{F}$.*

The last lemma remains true if we replace liftings by monotonous liftings.

It follows that a complete topological measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$ has the *USLP* if there exists a cardinal $\aleph$ with $mnw(\mu) < \aleph$ and if for any family $\mathcal{A} \subseteq \Sigma$ with $\text{card}(\mathcal{A}) < \aleph$ and $\mu(A) = 0$ for $A \in \mathcal{A}$ follows $\cup\mathcal{A} \in \Sigma$ and $\mu(\cup\mathcal{A}) = 0$. Indeed for given lifting $\rho \in \Lambda(\mu)$ apply Lemma 4.7 for $N := \bigcup_{F \in \mathcal{F}}(F \setminus \rho(F))$ if $\mathcal{F}$ is a measurable network of cardinality $mnw(\mu)$. For $\aleph = \aleph_1$ this gives the next result of Fremlin (200?, 453F).

COROLLARY 4.8. *A complete strictly localizable topological measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$ with a countable measurable network has the USLP.*

In particular a complete strictly localizable topological measure spaces $(\Omega, \mathcal{T}, \Sigma, \mu)$ possesses the *USLP* if their topology $\mathcal{T}$ is second countable. This is an improvement of a result of Graf (1975), see Maher (1978). In particular, polish as well as locally compact metrizable measure spaces have the *USLP* (see A. and C. Ionescu Tulcea (1969a, Chapter VII, Theorem 8)). The next result is from Macheras, Musiał and Strauss (200?a).

PROPOSITION 4.9. *Let $(\Omega, \mathcal{T}, \Sigma, \mu)$ and $(\Theta, \mathcal{S}, T, \nu)$ be Borel probability spaces and $f : \Omega \to \Theta$ a measure preserving map. Suppose that ν admits a strong lifting τ which has an almost strong inverse image lifting in $\Lambda(\mu)$. Then there exists a strong inverse lifting $\rho \in \Lambda(\mu)$ of τ under f if and only if $f^{-1}(\tau(B)) \cap G \neq \emptyset$ implies $\mu(f^{-1}(B) \cap G) > 0$ for all $B \in \mathcal{S}$ and all $G \in \mathcal{T}$.*

To sketch the proof of the above proposition, let $\varphi \in \Lambda(\mu)$ be an almost strong inverse lifting of τ under f. Then there exists a null set $N \in \Sigma_0$ such that

$$G \subseteq \varphi(G) \cup N \quad \text{for each } G \in \mathcal{T}.$$

For each $\omega \in N$, let

$$\mathcal{A}(\omega) := \{A\colon A \in f^{-1}(T),\ \omega \in \varphi(A)\},$$
$$\mathcal{E}(\omega) := \{f^{-1}(G)\colon G \in \mathcal{S},\ f(\omega) \in G\},$$

and

$$\mathcal{H}(\omega) := \{A \cap G\colon A \in \mathcal{A}(\omega),\ G \in \mathcal{E}(\omega)\}$$

and $\mathcal{F}(\omega) \subseteq \Sigma$ be the filter defined by

$$\mathcal{F}(\omega) := \{E \in \Sigma\colon \exists F \in \mathcal{H}(\omega) \text{ with } F \subseteq E \text{ a.e. } (\mu)\}.$$

Define a density $\tilde{\rho} \in \vartheta(\mu)$ by means of

$$\tilde{\rho}(E) := \left[\varphi(E) \cap N^c\right] \cup \rho^*(E) \quad \text{for each } E \in \Sigma,$$

where $\rho^*(E) := \{\omega \in N\colon E \in \mathcal{F}(\omega)\}$.

It can be shown that $\tilde{\rho} \in \vartheta(\mu) \cap f^{-1}(\tau)$ and that $\tilde{\rho}$ is strong. By Theorem 2.1 there exists a $\rho \in \Lambda(\mu)$ such that $\tilde{\rho}(E) \subseteq \rho(E)$ for each $E \in \Sigma$. It follows that ρ is a strong inverse lifting of τ under f.

A measure μ on a totally ordered space $(\Omega, \leqslant)$ is a measure defined on the Borel σ-algebra $\mathcal{B}(\Omega)$ generated by the order topology, Σ is the completion of $\mathcal{B}(\Omega)$ under μ, where we denote the extension of μ to Σ again by μ. Then the quadruple $(\Omega, \leqslant, \Sigma, \mu)$ is called a *totally ordered measure space*. For such spaces Sapounakis (1983) proved the existence of strong liftings.

THEOREM 4.10. *Let $(\Omega, \leqslant, \Sigma, \mu)$ be a totally ordered measure space such that its measure is of full support. Then there exists a strong lifting for μ.*

For Baire and Borel measure spaces necessary conditions for the *ASLP* are given by Babiker and Strauss (1980a) as well as by K.P. Dalgas.

PROPOSITION 4.11. *If $(\Omega, \mathcal{T}, \mathcal{B}, \mu)$ is a Baire measure space with finite measure and a completely regular topology $\mathcal{T}$ then the ASLP implies that the measure μ is τ-additive and completion regular. For any topological measure space $(\Omega, \mathcal{T}, \mathcal{B}, \mu)$ with the ASLP the measure μ is necessarily τ-additive.*

It follows, e.g., that the Wiener measure restricted to the completion of the Baire σ-algebra on $\overline{\mathbb{R}}^{[0,1]}$ does not have the *ASLP*. But the Wiener measure considered on the completion of the Borel σ-algebra has the *USLP* (see, e.g., Macheras and Strauss (1996b)).

Clearly the last proposition raises the problem whether the necessary conditions for the *ASLP* given there are sufficient? By Babiker and Knowles (1978), there exists a Baire measure space $(\Omega, \mathcal{T}, \mathcal{B}, \mu)$ with compact Ω, finite and non-atomic μ of full support, that is τ-additive but not completion regular. This space is an example of a compact Baire measure space without the *ASLP*.

Fremlin (1979) has given an example of a Radon measure space on a compact set with completion regular μ of full support but without the *ASLP*, improving a result of Losert (1979) which lacked completion regularity. The completion regularity is of interest here because in important known cases where the *ASLP* holds true, the completion regularity is also fulfilled. Mokobodzki (1975) and Fremlin (1977) gave a converse for Proposition 4.11. It was noticed by Dalgas that this result is the main step towards a result on strong Borel liftings improving the result of Mokobodzki and Fremlin. In the subsequent theorem of Dalgas (199?) $\mathbf{c}$ stands for the cardinal of the set of reals. Different approaches have been given by Musiał (1973) and Lloyd (1974).

THEOREM 4.12 (CH). *Let be given a topological measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$ with finite measure μ and $\mathcal{T}$ possessing a basis of the cardinality less or equal to $\mathbf{c}$. Then,*

$(\Omega, \mathcal{T}, \Sigma, \mu)$ has a strong Borel lifting if and only if $\Sigma \subseteq \mathcal{B}_\mu$, $\operatorname{supp}(\mu) = \Omega$, and μ is τ-additive ($\mathcal{B}_\mu$ is the completion of $\mathcal{B}(\Omega)$ with respect to $\mu|\mathcal{B}(\Omega)$). The theorem remains true for σ-finite μ if μ is moderated or all finite Borel measures on $(\Omega, \mathcal{T}, \Sigma, \mu)$ are τ-additive.

The proof of the last result relies on the following theorem of Fremlin (1977).

THEOREM 4.13. *Let be given a topological measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$ with $\Sigma \subseteq \mathcal{B}_\mu$ satisfying the following conditions.*

(i) *For any $A \in \Sigma$ and any $\mathcal{G} \subseteq \mathcal{T}$ with $\mu(A \cap G) = 0$ for all $G \in \mathcal{G}$ follows $\mu(A \cap \bigcup\mathcal{G}) = 0$.*
(ii) *If κ is the cardinal of the measure algebra Σ/μ then the union of fewer than κ sets of measure zero is measurable and of measure zero.*
(iii) $\operatorname{supp}(\mu) = \Omega$.

Then there exists a strong Borel lifting for $(\Omega, \mathcal{T}, \Sigma, \mu)$.

Burke (1993a, Proposition 3.6) gives an elementary proof that the existence of a Borel lifting for the Borel–Lebesgue measure space on $[0, 1]$ implies already the existence of a strong Borel lifting for this measure space. It follows from the example of Losert (1979) that the Theorem 4.12 is no longer valid without the cardinality restriction. Dalgas gives an example that for σ-finite measures spaces additional hypotheses must be imposed for a corresponding characterization. Mauldin (1978) gives related results, following the original results of von Neumann (1931). The first one states that subject to the continuum hypothesis, any Borel measure space $(\Omega, \mathcal{T}, \mathcal{B}, \mu)$ has a Borel lifting if $(\Omega, \mathcal{B}(\Omega))$ is Borel isomorphic to a universally measurable subset of the unit interval $[0, 1]$ of the reals, thus generalizing a result of A. and C. Ionescu Tulcea (1969a, p. 182). The second one applies Martin's axiom to construct liftings ranging in the **c**-*algebra* $\mathcal{B}_{\mathbf{c}}(\Omega)$ of the topological space $(\Omega, \mathcal{T})$, i.e., in the the smallest algebra containing the Borel σ-algebra $\mathcal{B}(\Omega)$ and being closed under unions of less than **c** sets. If the continuum hypothesis does not hold true the **c**-algebra may be much larger than the Borel σ-algebra. This result is one of the very few ones in lifting theory applying Martin's axiom. For more results on Borel liftings we refer to Section 5 on translation invariant liftings.

Assuming the continuum hypothesis, first existence results for strong Baire liftings have been given by Losert (1980), see also Talagrand (1978a), for a restricted class of measures, i.e., for any product of less than or equal to $\aleph_2$ Radon probability measures of full support, each on a compact metric space. Grekas and Gryllakis (1992) improved this result as well as a result of their own from 1991 by the following theorem.

THEOREM 4.14 (CH). *Subject to the continuum hypothesis every product of less than or equal to $\aleph_2$ many completion regular probability measures, each supported on a product of less than or equal to $\aleph_1$ many compact metric spaces admits a strong Baire lifting.*

The proof is based on a result for measures on product spaces satisfying a certain condition which reduces in case of compact metric factor spaces to completion regularity. The next theorem from Babiker and Strauss (1980a) gives necessary and sufficient conditions for the *USLP*.

THEOREM 4.15. *Let $(\Omega, \mathcal{T})$ be a locally metrizable space. Then a Borel measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$ with a finite measure μ has the USLP if and only if μ is τ-additive, and a Baire measure space $(\Omega, \mathcal{T}, \mathcal{B}, \mu)$ with finite measure μ and completely regular topology $\mathcal{T}$ has the USLP if and only if μ is τ-additive and completion regular.*

This theorem implies that under the mild set theoretic assumption of the non-existence of measurable cardinals every metrizable space with finite Borel or Baire measure has the *USLP*.

COROLLARY 4.16. *Suppose that every closed discrete subspace of the metric space $(\Omega, \mathcal{T})$ has non-measurable cardinality. Then every Baire (respectively Borel) measure space $(\Omega, \mathcal{T}, \mathcal{B}, \mu)$ with a finite measure μ has the USLP.*

For existence of strong liftings on products and projective limits, respectively we refer to the Sections 6.2 and 6.3 below.

The interest in strong liftings comes from the following theorem of A. and C. Ionescu Tulcea (1969a) on the existence of strict disintegrations (see the same book for terminology).

THEOREM 4.17. *A finite Radon measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$ over a compact set Ω with* $\operatorname{supp}(\mu) = \Omega$ *has the ASLP if and only if for each Radon measure space $(X, \mathcal{S}, \mathcal{A}, \nu)$ over a compact set X and each continuous surjection $p : \Omega \to X$ with $\nu = p(\mu)$ there exists a strict disintegration of ν with respect to p.*

The notion of a strong lifting has been generalized to the so-called $\mathcal{H}$-lifting, for which we refer to Levin (1975) and for existence to Babiker and Strauss (1980a, 1980c). Bichteler (1970, 1971) shows that the problem of the existence of strong liftings for Radon measures on locally compact Hausdorff spaces can be reduced to the problem of the existence of strong liftings for Radon measures on products of unit intervals. Bichteler (1972) shows that the set of all signed Radon measures μ on a locally compact Hausdorff space X such that $|\mu|$ admits an almost strong lifting is a band in the Dedekind complete lattice of all Radon measures on X, see also Bichteler (1973) as well as C. Ionescu Tulcea and Maher (1971).

For weakenings of the notion of strong lifting such as the so-called *almost strong predensity*, the *idempotent lifting*, and the *almost τ-continuous lifting*, respectively we refer to Bichteler (1972), Georgiou (1974), Grekas (1989), and Rinkewitz (1997), respectively.

5. Liftings on topological groups

Throughout this section a *topological group X carrying Haar measures* together with its σ-algebra Σ of *Haar measurable sets* (the common domain of all its left and right Haar measures) and a left Haar measure μ on Σ are given. Then (X, Σ, μ) is called a *Haar measure space* over X by Fremlin (200?). (The case of a right Haar measure can be reduced to that of the left one.) The map $x \in X \to x^{-1} \in X$ is the inversion operation. For $s \in X$,

$A \subseteq X$, and $f : X \to \mathbb{K}$ we consider $sA := \{sx\colon\ x \in A\}$, the *left s-translate* of A as well as $\gamma(s)f$, the *left s-translate* of f defined by means of

$$\bigl(\gamma(s)f\bigr)(x) := f\bigl(s^{-1}x\bigr) \quad \text{for all } x \in X.$$

A density $\delta \in \vartheta(\mu)$ and a linear lifting $\psi \in \mathcal{G}(\mu)$, respectively is called *left-translation invariant* if

$$\delta(sA) = s\delta(A) \quad \text{for all } A \in \Sigma \text{ and all } s \in X$$

and

$$\psi\bigl(\gamma(s)f\bigr) = \gamma(s)\bigl(\psi(f)\bigr) \quad \text{for all } f \in \mathcal{L}^{\infty}(\mu) \text{ and all } s \in X,$$

respectively. If again (see Section 3) ρ^{∞} and ρ^{0} are the liftings for functions in $\mathcal{L}^{\infty}_{\mathbb{K}}(\mu)$ and $\mathcal{L}^{0}_{\overline{\mathbb{R}}}(\mu)$ uniquely generated by a lifting $\rho \in \Lambda(\mu)$, then ρ is left-translation invariant if and only if ρ^{∞} and ρ^{0} are such.

In the above definition we have fixed explicitly a Haar measure μ on X for easier reference. But the definition of a left-translation invariant density and (linear) lifting is completely independent of this choice, since all left (and right) Haar measures produce the same domain Σ and the same null sets Σ_0. For this reason we can speak of Haar densities and Haar (linear) liftings. In Abelian groups we simply speak on translation invariant densities and (linear) liftings. For the next proposition compare A. and C. Ionescu Tulcea (1967, Section 3, Proposition 1) and Fremlin (200?, 448C).

PROPOSITION 5.1. *Any left-translation invariant linear lifting is strong. The same holds true for translation invariant densities.*

The Lebesgue density D defined in Section 2 for the Lebesgue measure space on $\mathbb{R}^d$, $d \in \mathbb{N}$, is translation invariant and similarly the examples ψ_2, ψ_3 quoted in Section 3 provide translation invariant linear liftings on the circle group. A. and C. Ionescu Tulcea (1967, p. 90), show that in contrast to the uniqueness of the Haar measure, Haar densities are by no means uniquely determined even in case of the Lebesgue measure on the real line. By the next two theorems (which are the analogues of the Theorems 2.1 and 3.3, respectively) translation invariant densities and linear liftings can be converted into translation invariant liftings.

THEOREM 5.2. *For any left-translation invariant density δ for μ there exists a left-translation invariant lifting ρ for μ with $\delta(A) \subseteq \rho(A)$ for all $A \in \Sigma$.*

In fact by Theorem 2.1 we may first choose a $\tau \in \Lambda(\mu)$ with $\delta(A) \subseteq \tau(A)$ for all $A \in \Sigma$ and then take

$$\rho(A) := \bigl\{x \in X\colon\ e \in \tau\bigl(x^{-1}A\bigr)\bigr\} \quad \text{for all } A \in \Sigma,$$

where e denotes the identity of X.

THEOREM 5.3. *For any left-translation invariant linear lifting ψ for μ there exists an extremal element $\rho \in \mathcal{G}_\psi$ (see Theorem 3.3 for the definition) which is a left-translation invariant lifting for μ.*

Compare A. and C. Ionescu Tulcea (1967, Section 6, Corollary 3) for the last theorem.

The construction of left-translation invariant densities and linear liftings goes through the induction steps (A) to (E) lined out for Theorem 2.5. But these steps become now more complicated since the construction has to run through translation invariant σ-subalgebras. In particular, in the step from an ordinal α to its successor ordinal $\alpha + 1$ the extra difficulties are a consequence of the fact that translation invariant σ-subalgebra for $\alpha + 1$ is much larger than that one generated by the σ-subalgebra for α and the new element which enters. A. and C. Ionescu Tulcea (1967) had overcome these difficulties by exploiting special structural properties of locally compact groups, in particular of Lie groups, getting the next fundamental result.

THEOREM 5.4. *For each Haar measure space (X, Σ, μ) on a locally compact group X there exists a left-translation invariant lifting.*

The last step from a locally compact group to a group X carrying Haar measures which we are going to state now has been done by Fremlin (200?) using the fact that for a Haar measure space (X, Σ, μ) on X there exists a Haar measure space (Z, T, ν) on a locally compact group Z and a continuous map $f : X \to Z$ such that $\nu = f(\mu)$ and for all $E \in \Sigma$ there exists a set $F \in T$ with $f^{-1}(F) \subseteq E$ and $E \setminus f^{-1}(F) \in \Sigma_0$.

COROLLARY 5.5. *For any topological group carrying a Haar measure there exists a left-translation invariant lifting adequate for all its Haar measures.*

For the special group $X = \{0, 1\}^I$ for a non-empty index set I with its "usual Haar measure" the proof of the existence of a Haar density from Fremlin (200?, 345C), can be modified to give a proof of the existence of an admissible translation invariant density and a translation invariant admissible linear lifting being in particular a strong admissible density and strong linear lifting, respectively.

By Theorem 5.2 there exists an admissibly generated translation invariant lifting, hence by Proposition 5.1 an admissibly generated strong lifting.

Concerning translation invariant Borel liftings for Haar measure spaces Johnson (1980) proved that in ZFC no translation invariant lifting for the Haar measure on the circle group $\mathbb{R}/\mathbb{Z}$ can be a Borel lifting. This result was extended by Talagrand (1982) to non discrete compact Abelian groups. While R.A. Johnson used results from topological dynamics on the circle group M. Talagrand's proof is simpler since it is based on the well known construction of the Cantor set. Generalizing Johnson's procedure Kupka and Prikry (1983) succeeded to extend M. Talagrand's result to non-discrete (possibly non-Abelian) locally compact groups replacing also Borel liftings by more general liftings ρ with Baire property, i.e., $\rho(A)$ is a set with Baire property for any $A \in \Sigma$ (note that every Borel set has this property). M. Talagrand's simpler method was developed for the non-Abelian case first by Losert (1983) and subsequently by Burke (1993) who succeeded in the next stated theorem to improve the result of J. Kupka and K. Prikry.

THEOREM 5.6. *In each non-discrete locally compact group X there exists a Borel set E such that for an arbitrary left-translation invariant lifting ρ for the Haar measure space (X, Σ, μ), the set $\rho(E)$ is not universally measurable and does not have the Baire property.*

Here a set $A \subseteq X$ is called *universally measurable* if A is μ-measurable for every Radon measure μ on X.

Fremlin (1989) gives a list of open problems.

QUESTIONS 5.7. (1) Is there a cardinal κ such that the Haar measure on $\{0, 1\}^\kappa$ has no Baire lifting?

This problem is connected with the next unsolved one about the existence of product liftings (see Section 6 for the definition) in incomplete probability spaces.

(2) Do there exist product liftings for the Borel measure space on $[0, 1]^2$?

(3) Does $\{0, 1\}^{\aleph_3}$ possess a Borel lifting?

By Mokobodzki's Theorem 2.7, subject to CH, the spaces $\{0, 1\}^{\aleph_i}$ for $i = 0, 1, 2$ have Baire liftings.

According to Burke and Shelah (1992) it is consistently true with ZFC that there is no Borel lifting on $\{0, 1\}^\kappa$ for any κ.

(4) Is the existence of a Borel lifting for $\{0, 1\}^\omega$ consistent with $2^{\aleph_0} > \aleph_3$?

The problem of existence of translation invariant liftings has been generalized to the problem of the existence of $\mathcal{G}$*-invariant (linear) liftings* for a given set $\mathcal{G}$ of bi-measurable maps $s : \Omega \to \Omega$ over a measure space (Ω, Σ, μ) by A. and C. Ionescu Tulcea (1969a, p. 182) and Maher (1974, p. 69). Besides the positive solution for the group of left translations in topological groups carrying Haar measures from above (see Corollary 5.5) another positive solution was given by A. Ionescu Tulcea (1965) for the existence of a $\mathcal{G}$-invariant linear lifting if (Ω, Σ, μ) is strictly localizable and $\mathcal{G}$ is a countable amenable group. On the other hand von Weizsäcker (1977) proved on the basis of a lemma on automorphisms for complete Boolean algebras criteria for the non-existence of $\mathcal{G}$-invariant liftings. In particular he gives a maximality argument for the set of left translations in the situation of Theorem 5.4, more precisely he showed in von Weizsäcker (1976), Corollary A.3. on the basis of his fixed-point result the next theorem.

THEOREM 5.8. *If X is a connected locally compact group with left Haar measure then for every set G of continuous bi-measurable and null-set preserving bijections on X which is strictly larger than the set of all left translations there exists no lifting commuting with G.*

6. Permanence of liftings

6.1. *Liftings in products*

We denote by $(\prod_{i \in I} \Omega_i, \bigotimes_{i \in I} \Sigma_i, \bigotimes_{i \in I} \mu_i)$ or by $\bigotimes_{i \in I}(\Omega_i, \Sigma_i, \mu_i)$ or by $(\Omega_I, \Sigma_I, \mu_I)$ the product probability space of the probability spaces $(\Omega_i, \Sigma_i, \mu_i)$ $(i \in I)$ and $(\prod_{i \in I} \Omega_i, \widehat{\bigotimes}_{i \in I} \Sigma_i, \widehat{\bigotimes}_{i \in I} \mu_i)$ or $\widehat{\bigotimes}_{i \in I}(\Omega_i, \Sigma_i, \mu_i)$ denotes its (Carathéodory) completion. For each

$\emptyset \neq J \subseteq I$ we denote by $(\Omega_J, \Sigma_J, \mu_J)$ the product measure space $\bigotimes_{i\in J}(\Omega_i, \Sigma_i, \mu_i)$. For any $\emptyset \neq J \subseteq I$ the canonical projection of Ω_I onto Ω_J is denoted by p_J, where $p_i := p_{\{i\}}$ if $i \in I$ and the σ-algebra $p_J^{-1}(\Sigma_J) \subseteq \Sigma_I$ is written Σ_J^*. For a probability space (Θ, T, ν) and a non-empty set I we write (Θ^I, T^I, ν^I) for the product probability space $\bigotimes_{i\in J}(\Omega_i, \Sigma_i, \mu_i)$ with all its factors $(\Omega_i, \Sigma_i, \mu_i)$ equal to (Θ, T, ν) for $i \in I$.

If f is a function defined on $\prod_{i\in I}\Omega_i$ and $(\omega_{i_1}, \ldots, \omega_{i_n}) \in \prod_{k=1}^n \Omega_{i_k}$ are fixed, then $f_{(\omega_{i_1},\ldots,\omega_{i_n})}$ is the function on $\prod_{i\in I\setminus\{i_1,\ldots,i_n\}}\Omega_i$ obtained from f by fixing $(\omega_{i_1}, \ldots, \omega_{i_n})$. In a similar way the sets $E_{(\omega_{i_1},\ldots,\omega_{i_n})}$ being sections of a set $E \subset \prod_{i\in I}\Omega_i$ are defined. In case of the product of two spaces, we shall be using the notation f_ω, f^θ and E_ω, E^θ rather.

During the last fifteen years a good deal of research in lifting theory concentrated on the problem of the existence of liftings compatible with the product structure of probability spaces leading to different types of compatibility of increasing complication, starting from Talagrand (1989).

Throughout what follows let be given a family $\langle(\Omega_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ of probability spaces with index set $I \neq \emptyset$ and a probability space (Ω, Σ, μ) with $\Omega = \Omega_I$, $\Sigma \supseteq \Sigma_I$, $\mu \mid \Sigma_I = \mu_I$. If $\emptyset \neq J \subseteq I$, then $\mathcal{L}_J^* := \{f \circ p_J\colon\ f \in \mathcal{L}^\infty(\mu_J)\}$ is the set of functions, determined by coordinates in J. Moreover,

$$\overset{*}{\bigotimes_{i\in J}} A_i := p_J^{-1}\Big(\prod_{i\in J} A_i\Big) \quad \text{for } A_i \in \Sigma_i,\ i \in J$$

and

$$\overset{*}{\bigotimes_{i\in J}} f_i := \Big(\prod_{i\in J} f_i\Big) \circ p_J \quad \text{for } f_i \in \mathcal{L}^\infty(\mu_i),\ i \in J$$

are the *cylinder sets* and *functions*, respectively. We call $\delta \in \vartheta(\mu)$ a *product-density* for μ if for all $i \in I$ exists $\delta_i \in \vartheta(\mu_i)$ such that

$$\delta\Big(\overset{*}{\bigotimes_{i\in F}} A_i\Big) = \overset{*}{\bigotimes_{i\in F}} \delta_i(A_i) \quad \text{for all } A_i \in \Sigma_i,\ i \in F \subseteq I,\ F \text{ finite.} \tag{P}$$

The δ_i, $i \in I$ are uniquely determined by δ via (P) but conversely δ is only uniquely determined on the cylinder sets by the family $\langle\delta_i\rangle_{i\in I}$ via (P). We therefore write $\delta \in \bigotimes_{i\in I}\delta_i$ and call δ_i the ith *marginal* of δ. $\vartheta_p(\mu)$ is the class of all product-densities, and $\Lambda_p(\mu) := \vartheta_p(\mu) \cap \Lambda(\mu)$ is the class of all product-liftings for μ. Clearly the marginals of any $\rho \in \Lambda_p(\mu)$ are in $\Lambda(\mu_i)$ for $i \in I$. In the same way we can define the class $\mathcal{G}_p(\mu)$ of all *product linear liftings* and the class $\Lambda_p^\infty(\mu) := \mathcal{G}_p(\mu) \cap \Lambda^\infty(\mu)$ of all *product (function-) liftings*, where indeed $\rho \in \Lambda_p(\mu)$ if and only if $\rho^\infty \in \Lambda_p^\infty(\mu)$ and in that case the marginals satisfy the equation $(\rho_i)^\infty = (\rho^\infty)_i$ if $i \in I$. The *product-lifting* was first investigated in A. and C. Ionescu Tulcea (1969a, Chapter VIII). Then Talagrand (1982) introduced a *consistent lifting*. A lifting ρ for a complete probability space (Θ, T, ν) is said to be *consistent* if for each $n \in \mathbb{N}$ there exists a product lifting $\rho^n \in \Lambda(\widehat{\bigotimes}^n \nu)$ with all its ith marginals equal to ρ for $i = 1, \ldots, n$. *Consistent densities* can be defined in the same way. The existence problem for consistent liftings has been definitely solved by Talagrand (1989).

THEOREM 6.1. *For each complete probability space (Θ, T, ν) there exists a consistent lifting $\rho \in \Lambda(\nu)$.*

It is however not true that each lifting is consistent. Talagrand (1988) assuming (CH) constructed such a Borel lifting ρ on $[0, 1]$ with respect to Lebesgue measure that its product ρ^2 defined on Borel rectangles by $\rho^2(A \times B) = \rho(A) \times \rho(B)$ cannot be extended to a lifting on Lebesgue measurable subsets of the unit square (in particular it cannot be also extended to a Borel lifting).

QUESTION 6.2 (*cf. Burke* (1993)). Can one produce in ZFC a lifting for Lebesgue measure on $[0, 1]$ such that ρ^2 does not extend to a lifting on $[0, 1]^2$?

QUESTION 6.3 (*cf. Burke* (1993)). Is it a theorem of ZFC that there is no Borel lifting ρ on $[0, 1]$ such that ρ^2 extends to a Borel lifting on $[0, 1]^2$?

QUESTION 6.4 (*A.H. Stone*). Is there consistently a Borel lifting on $[0, 1]$ of bounded Borel class?

The existence of a product lifting in case of a product of arbitrary complete probability spaces was solved by Macheras and Strauss (1996d). For an improvement of this result see Theorem 6.11 below.

$\delta \in \vartheta(\mu)$ and $\psi \in \mathcal{G}(\mu)$ will be called *respecting coordinates* for μ if $\delta(\Sigma_J^*) \subseteq \Sigma_J^*$ and $\psi(\mathcal{L}_J^*) \subseteq \mathcal{L}_J^*$ for all $\emptyset \neq J \subseteq I$. The notion traces back to works of Burke (1995) and Fremlin (200?). For finite index set I all the above concepts make sense for σ-finite measure spaces $(\Omega_i, \Sigma_i, \mu_i)$, $i \in I$, instead of probability spaces, since in that case there exists a unique product measure. But again it is only a matter of technique to carry over the results. So we are going to consider probability spaces only. The first essential result concerning liftings respecting coordinates is due to Burke (1995). It has been obtained by an application of a theorem of Erdős–Rado.

THEOREM 6.5. *For any finite family $\langle(\Omega_i, \Sigma_i, \mu_i)\rangle_{i=1}^n$ of complete probability spaces there exists a lifting for $\widehat{\bigotimes}_{i=1}^n \mu_i$ respecting coordinates.*

Fremlin (200?, 345G), attempting to prove an infinite version of Theorem 6.5 took first into account densities respecting coordinates. Applying D. Maharam's theorem on the structure of measure algebras (cf. Fremlin (1989, 3.9)) Fremlin (200?) has got the following result.

THEOREM 6.6. *For any family $\langle(\Omega_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ of complete probability spaces there exists a separately additive density δ for $\widehat{\mu}_I$ respecting coordinates.*

Separate additivity above (D.H. Fremlin called it the $(*)$ property) means that

$$\delta(A \cup B) = \delta(A) \cup \delta(B) \quad \text{for any } A \in \Sigma_J^* \text{ and } B \in \Sigma_K^* \tag{$*$}$$

for all disjoint $J, K \subseteq I$.

It seems that the proof procedure applied above is restricted to complete probability spaces and gives no information about the marginals. Probably without the completeness assumptions one cannot obtain the separate additivity above. The admissible densities have convenient properties from the product point of view and for this reason they are the natural candidates for marginals in the existence results below. The next result of Macheras, Musiał and Strauss (1999) gives not only a solution of the existence problem for densities respecting coordinates and arbitrary probability spaces but at the same time it produces a unifying approach to all former existence results for existence of product densities and liftings as well as to those respecting coordinates. The conclusion of this result is on one side weaker since it can not be shown that the density which is respecting coordinates is separately additive in addition but on the other hand its proof is quite elementary (in particular D. Maharam's theorem is not applied there) and the product probability needs no completion.

THEOREM 6.7. *Let $\langle(\Omega_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces. If $i_0 \in I$ is fixed, then for each $\delta_{i_0} \in \vartheta(\mu_{i_0})$ and for arbitrary $\delta_i \in A\vartheta(\mu_i)$ with $i \in I \setminus \{i_0\}$ there exists a $\delta \in \vartheta(\mu)$ such that δ respects coordinates and $\delta \in \bigotimes_{i\in I} \delta_i$. In particular the theorem holds true if all the densities δ_i, $i \in I$, are identical and admissible. As a consequence, it follows that each admissible density is a consistent density.*

Since $A\vartheta(\mu_i) \neq \emptyset$ if $i \in I$ the above result is a generalization of Macheras and Strauss (1995). The proof of Theorem 6.7 is based in principle on the inductive steps exhibited before Theorem 2.6, but this time the induction is more complicated (see Macheras, Musiał and Strauss (1999) for details). Applying Theorem 6.6 Fremlin (200?) proved the following nice result.

THEOREM 6.8. *For any family $\langle(\Omega_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ of complete Maharam homogeneous probability spaces there exists a lifting for $\widehat{\bigotimes}_{i\in I} \mu_i$ respecting coordinates.*

The proof of Theorem 6.8 has been reduced to the next Theorem 6.9 of Fremlin (200?) which is of an even more special nature by using the fact that the measure algebra of any Maharam homogeneous probability space is isomorphic to some $\{0,1\}^{I_i}$ with its usual measure ν_i for $i \in I$.

THEOREM 6.9. *For any set I any translation invariant lifting for the usual measure μ on $\{0,1\}^I$ respects coordinates.*

Then Macheras, Musiał and Strauss (200?c) applying Theorem 6.22 to the conclusion of Theorem 6.8 were able to generalize Theorem 6.8 to the following form:

THEOREM 6.10. *For an arbitrary family $\langle(\Omega_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ of complete Maharam homogeneous probability spaces and arbitrary finite collection $\langle(\Theta_j, T_j, \nu_j)\rangle_{j\in J}$ of complete probability spaces there exists a lifting for $(\bigotimes_{i\in I}\mu_i)\mathbin{\widehat{\otimes}}(\bigotimes_{j\in J}\nu_j)$ respecting coordinates.*

In case when $I=\emptyset$ one gets Burke's theorem with a completely new proof. It is as yet an open problem whether for arbitrary families of complete probability spaces there exist liftings respecting coordinates. Only partial results are known. The next result seem to be the most far reaching for general complete probability spaces at the moment. (See Macheras, Musiał and Strauss (1999) as well as Fremlin (200?) for this result.)

THEOREM 6.11. *Let κ be an ordinal and $\langle(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)\rangle_{\alpha<\kappa}$ be a family of complete probability spaces. Then for each $\rho_0\in\Lambda(\mu_0)$ and each collection $\{\rho_\alpha\in AG\Lambda(\mu_\alpha)\colon 0<\alpha<\kappa\}$ there exists a lifting $\pi\in\bigotimes_{\alpha<\kappa}\rho_\alpha$ respecting coordinates of each initial segment of κ.*

Considering linear liftings a general existence theorem was given by Macheras, Musiał and Strauss (2000). The proof is based on the extension Lemmata 3.4 and 3.5.

THEOREM 6.12. *$\langle(\Omega_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of complete probability spaces with product probability space (Ω, Σ, μ). If $i_0\in I$ is fixed, then for each $\tau_{i_0}\in\mathcal{G}(\mu_{i_0})$ and for arbitrary $\tau_i\in A\mathcal{G}(\mu_i)$ with $i\in I\setminus\{i_0\}$ there exists a $\varphi\in\mathcal{G}(\widehat{\mu})$ such that φ respects coordinates and $\varphi\in\bigotimes_{i\in I}\tau_i$.*

The above result gives also an alternative proof of Theorem 6.6.

It seems however that the extreme point method of Theorem 3.3 does not work while trying to convert a linear lifting respecting coordinates found in Theorem 6.12 into a lifting respecting coordinates.

6.2. *Strong liftings in products*

Though weaker than liftings respecting coordinates, product liftings are useful for transporting strong liftings from the factors onto the product as well as for attacking a problem posed by Kupka (1983), whether the product of two topological (especially Radon) probability spaces has the *ASLP* if the factors have this property. In particular it is natural to ask, whether the product of hyperstonian spaces has the *ASLP*. It is known from Macheras and Strauss (1996c) that the only possible candidate for a strong lifting on the ordinary completed product $(X\times X, \mathcal{T}\times\mathcal{T}, H\mathbin{\widehat{\otimes}}H, \nu\mathbin{\widehat{\otimes}}\nu)$ of a hyperstonian probability space $(X, \mathcal{T}, H, \nu)$ with itself is necessarily a product lifting of the canonical strong liftings in the factors. If $(X, \mathcal{T}, H, \nu)$ is the hyperstonian space of, e.g., the Lebesgue probability space on $[0,1]$, it follows from the next theorem that there exists no strong lifting for $\nu\mathbin{\widehat{\otimes}}\nu$ since $\mathcal{T}\times\mathcal{T}\not\subseteq H\mathbin{\widehat{\otimes}}H$, the latter a result of Fremlin (1976). But let us remark that subject to the continuum-hypothesis the Radon product of $(X, \mathcal{T}, H, \nu)$ with itself has the *ASLP* by Theorem 4.12 (see Macheras and Strauss (1996c, Section 3, Remark 5) for details). It remains an open problem, whether the Radon product of two Radon probability spaces with the *ASLP* has the *ASLP*, in particular it is unknown, whether the Radon product of two hyperstonian measure spaces has the *ASLP*.

Let us remark that the existence result for product liftings, Theorem 6.11, can be stated in an equivalent form in terms of products of lifting topologies which implies Theorem 6.13, a permanence result for strong liftings on products. For the proofs of the next three theorems see Macheras and Strauss (1996d) as well as (1996c). If $\langle(\Omega_i, \mathcal{T}_i)\rangle_{i\in I}$ is a family of topological spaces we write $\prod_{i\in I} \mathcal{T}_i$ for the *product topology* on $\prod_{i\in I} \Omega_i$.

THEOREM 6.13. *Let be given a family $\langle(\Omega_i, \mathcal{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ (I a non-empty set) of complete topological probability spaces, $\rho_i \in \Lambda(\mu_i)$ for each $i \in I$, let (Ω, Σ, μ) be a complete probability space such that $\Omega = \prod_{i\in I} \Omega_i$, $\widehat{\otimes}\Sigma_i \subseteq \Sigma$, $\mu|\widehat{\bigotimes}_{i\in I}\Sigma_i = \widehat{\bigotimes}_{i\in I}\mu_i$, and let $\pi \in \Lambda(\mu)$ satisfy $\pi \in \bigotimes_{i\in I}\rho_i$. Then we have that $\prod_{i\in I} \mathcal{T}_i \subseteq \Sigma$ and π is strong with respect to $\prod_{i\in I} \mathcal{T}_i$ if and only if all ρ_i are strong for $i \in I$.*

If $(X, \mathcal{T}, H, \nu)$ is again the hyperstonian space of the Lebesgue probability space on $[0, 1]$ then it follows from Theorem 6.13 the surprising fact that the unique strong lifting σ for ν is neither consistent nor admissible and any consistent lifting ρ for ν is not strong. In particular, ν has the *ASLP* but not the *USLP* and the lifting topologies t_σ and τ_σ satisfy $t_\sigma \times t_\sigma, \tau_\sigma \times \tau_\sigma \not\subseteq H \widehat{\otimes} H$.

Under a mild set-theoretic assumption, i.e., assuming the non-existence of measurable cardinals, a purely topological result of P.C. Curtis, M. Hendricksen, and J.R. Isbell (see Gillman (1960, p. 53)) says that a product of two topological spaces is extremally disconnected if and only if one factor is extremally disconnected and the other one is discrete. This implies the following result whose assumptions are satisfied, e.g., by the Lebesgue measure space on [0, 1]. It answers the question, whether the product of lifting topologies is a lifting topology to the negative.

THEOREM 6.14. *If we assume the non-existence of measurable cardinals then for two complete probability spaces $(\Omega_i, \Sigma_i, \mu_i)$ and $\rho_i \in \Lambda(\mu_i)$ with non-discrete lifting topologies t_{ρ_i}, τ_{ρ_i} $(i = 1, 2)$ the product topologies $t_{\rho_1} \times t_{\rho_2}$, $\tau_{\rho_1} \times \tau_{\rho_2}$ are not extremally disconnected. If $(\Omega_1 \times \Omega_2, \Sigma, \mu)$ is a complete probability space such that $\Sigma_1 \widehat{\otimes} \Sigma_2 \subseteq \Sigma$, $\mu|\Sigma_1 \widehat{\otimes} \Sigma_2 = \mu_1 \widehat{\otimes} \mu_2$, $t_{\rho_1} \times t_{\rho_2}$, $\tau_{\rho_1} \times \tau_{\rho_2} \subseteq \Sigma$, then in particular we have*

$$t_{\rho_1} \times t_{\rho_2} \neq t_\pi, \tau_\pi \quad \text{and} \quad \tau_{\rho_1} \times \tau_{\rho_2} \neq t_\pi, \tau_\pi$$

for each $\pi \in \Lambda(\mu)$, and if π is strong with respect to $t_{\rho_1} \times t_{\rho_2}$ respectively $\tau_{\rho_1} \times \tau_{\rho_2}$ (for example if $\pi \in \rho_1 \otimes \rho_2$) then

$$t_{\rho_1} \times t_{\rho_2} \subset t_\pi \quad \text{and} \quad \tau_{\rho_1} \times \tau_{\rho_2} \subset \tau_\pi, \quad \text{but not equality.}$$

The next result is an analogue of Theorem 6.11 for strong liftings.

THEOREM 6.15. *Let κ be an ordinal, $\langle(\Omega_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha)\rangle_{\alpha<\kappa}$ any family of complete topological probability spaces with completed product space (Ω, Σ, μ) and product topology $\mathcal{T}$. Suppose that all measures μ_α $(\alpha < \kappa)$ have full support and that μ_0 has* *the ASLP while all other measures μ_α for $0 < \alpha < \kappa$ have the USLP. Then for any strong lifting $\rho_0 \in \Lambda(\mu_0)$ there exist strong liftings $\rho_\alpha \in \Lambda(\mu_\alpha)$ for $0 < \alpha < \kappa$ and a strong lifting $\pi \in \Lambda(\mu)$ such that $\pi \in \bigotimes_{\alpha<\kappa} \rho_\alpha$, $\mathcal{T} \subseteq \Sigma$ and*

$$\pi\Big(A \times \prod_{\alpha \leqslant \gamma < \kappa} \Omega_\gamma\Big) = \Big(\bigotimes_{\xi<\alpha} \rho_\xi\Big)(A) \times \prod_{\alpha \leqslant \gamma < \kappa} \Omega_\gamma$$

for all $A \in \widehat{\bigotimes}_{\xi<\alpha} \Sigma_\xi$ and all $\alpha < \kappa$, that is π is a strong lifting respecting individual coordinates and initial segments of coordinates.

In particular, if $\Sigma_\alpha = \widehat{\mathcal{B}}(\Omega_\alpha)$ for all $\alpha < \kappa$ then $\Sigma = \widehat{\mathcal{B}}(\Omega)$ and if all the measures μ_α $(\alpha < \kappa)$ are completion regular then μ is also completion regular.

The last theorem together with Corollary 4.8 imply the next classical result of Maharam (1958) and Kakutani (1943).

COROLLARY 6.16. *For every ordinal κ there exists a strong lifting for the usual measure μ on $\{0, 1\}^\kappa$.*

Another consequence of the last theorem together with Corollary 4.8 is the next one given by Fremlin (200?, 453H).

COROLLARY 6.17. *Let $\langle(\Omega_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha)\rangle_{\alpha\in I}$ be a family of complete topological probability spaces with completed product measure space (Ω, Σ, μ) and product topology $\mathcal{T}$. Suppose that all measures μ_α $(\alpha \in I)$ have full support and that every $\mathcal{T}_\alpha$ has a countable measurable network. Then μ is a τ-additive topological measure and has a $\mathcal{T}$-strong lifting.*

6.3. *Liftings on projective limits*

Throughout what follows $\langle\Omega_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I\rangle$ denotes a projective system of probability spaces which will be assumed complete if liftings are involved. If $I = \kappa$, where κ is an infinite cardinal, we say that the projective system $\langle\Omega_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, \kappa\rangle$ of probability spaces is *continuous*, if for every limit ordinal $\xi < \kappa$ we have that $(\Omega_\xi, \Sigma_\xi, \mu_\xi, \langle f_\alpha\rangle_{\alpha<\xi})$ is the projective limit of the projective system $\langle\Omega_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, \xi\rangle$. For projective systems of topological probability spaces a similar definition can be posed.

As for products and inductive limits suitable notions of compatibility of projective systems and liftings or densities are crucial. A family $\langle\rho_\alpha\rangle_{\alpha\in I}$ of densities $\rho_\alpha \in \vartheta(\mu_\alpha)$ is called *self-consistent*, if

$$\rho_\beta\big(f_{\alpha\beta}^{-1}(A)\big) = f_{\alpha\beta}^{-1}\big(\rho_\alpha(A)\big) \tag{P}$$

for all $A \in \Sigma_\alpha$ and all $\alpha, \beta \in I$ with $\alpha \leqslant \beta$. For liftings $\rho_\alpha \in \Lambda(\mu_\alpha)$, $\alpha \in I$ this is equivalent to the τ_{ρ_β}–τ_{ρ_α}-continuity of the maps $f_{\alpha\beta}$ as well as with the equations

$$\rho_\beta^\infty(h \circ f_{\alpha\beta}) = \big(\rho_\alpha^\infty(h)\big) \circ f_{\alpha\beta} \tag{P$^\infty$}$$

for all $h \in \mathcal{L}^{\infty}(\mu_\alpha)$ and all $\alpha, \beta \in I$ with $\alpha \leqslant \beta$ by Section 3. The latter equation can be taken as a definition of a *self-consistent family of linear liftings* for the above projective system. If $(\Omega, \Sigma, \mu, \langle f_\alpha \rangle_{\alpha \in I})$ is the projective limit of the above projective system a density $\rho \in \Lambda(\mu)$ is called a *projective limit* of the self-consistent family $\langle \rho_\alpha \rangle_{\alpha \in I}$ if

$$\rho\big(f_\alpha^{-1}(A)\big) = f_\alpha^{-1}\big(\rho_\alpha(A)\big)$$

for all $A \in \Sigma$ and all $\alpha \in I$. Again for liftings (where we take the completed projective limit) this is equivalent to the τ_{ρ_α}–τ_ρ-continuity of the maps f_α as well as with the validity of the equations

$$\rho^{\infty}(h \circ f_\alpha) = \big(\rho_\alpha^{\infty}(h)\big) \circ f_\alpha$$

for all $h \in \mathcal{L}^{\infty}(\mu_\alpha)$ and all $\alpha \in I$ and this equation can be taken as the definition of the projective limit of linear liftings. In symbols we write $\rho \in \operatorname{proj}_{\alpha \in I} \lim \rho_\alpha$ for densities and (linear) liftings. We assume throughout that all canonical projections f_α of a projective limit $(\Omega, \Sigma, \mu, \langle f_\alpha \rangle_{\alpha \in I})$ of probability spaces are surjections.

Suppose we have a projective system $(\Omega, \Sigma, \mu, \langle f_\alpha \rangle_{\alpha < \kappa})$. In general, it is not obvious whether there exists at all a self-consistent family of densities or liftings associated with the system. By Macheras, Musiał and Strauss (200?a) there is an answer to the positive in the next result.

THEOREM 6.18. *Let κ be an infinite ordinal and $\langle \Omega_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, \kappa \rangle$ a continuous projective system of probability spaces with projective limit $(\Omega, \Sigma, \mu, \langle f_\alpha \rangle_{\alpha < \kappa})$. Then there exist a self-consistent family $\langle \delta_\alpha \rangle_{\alpha < \kappa}$ of densities $\delta_\alpha \in \vartheta(\mu_\alpha)$ and a density $\delta \in \vartheta(\mu)$ such that δ is a projective limit of the system $\langle \delta_\alpha \rangle_{\alpha < \kappa}$.*

If all probability spaces of the projective system are complete and the projective limit is taken completed, then we may in the above replace the word "density" by "linear lifting" and "lifting", respectively, throughout.

We will give examples below showing the existence of self-consistent families over non well ordered index set without projective limit. Necessary and sufficient conditions for the existence of projective limit liftings are given by Macheras and Strauss (1994).

THEOREM 6.19. *Let be given a projective system $\langle \Omega_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I \rangle$ of complete probability spaces with completed projective limit $(\Omega, \Sigma, \mu, \langle f_\alpha \rangle_{\alpha \in I})$, a self-consistent family $\langle \rho_\alpha \rangle_{\alpha \in I}$ of strong liftings $\rho_\alpha \in \Lambda(\mu_\alpha)$ and a lifting $\rho \in \Lambda(\mu)$. Then the following conditions are all equivalent:*

(i) *the projective limit topology τ of $\langle \tau_{\rho_\alpha} \rangle_{\alpha \in I}$ is contained in Σ and ρ is strong w.r.t. τ;*
(ii) *the projective limit topology $\mathcal{T}$ of $\langle t_{\rho_\alpha} \rangle_{\alpha \in I}$ is contained in Σ and ρ is strong w.r.t. $\mathcal{T}$;*
(iii) *ρ is a projective limit of $\langle \rho_\alpha \rangle_{\alpha \in I}$;*
(iv) *$\tau \subseteq \tau_\rho$;*
(v) *$\mathcal{T} \subseteq t_\rho$.*

For projective systems of (complete) topological probability spaces we may ask for the existence of self-consistent families of strong liftings and densities and of strong projective liftings. In general self-consistent families of strong liftings do not have any projective limit lifting as shown in the next example.

Let $X := \{0,1\}^I$, where I is an index set with $\operatorname{card}(I) = \aleph_2$, and let μ be the probability measure on $\widehat{\mathcal{B}}_0(X)$ constructed by Fremlin (1979). Denote by $\mathcal{F}(I)$ the system of all finite subsets of I. For any $\alpha \in \mathcal{F}(I)$ put $X_\alpha := \{0,1\}^\alpha$ and $\mu_\alpha := f_\alpha(\mu)$, where f_α is the canonical projection from X onto X_α, and denote by $\mathcal{T}_\alpha$ the discrete topology on X_α. For any $\alpha, \beta \in \mathcal{F}(I)$ with $\alpha \leqslant \beta$ denote by $f_{\alpha\beta}$ the canonical projection from X_β onto X_α. If by $\mathcal{T}$ is denoted the projective limit topology of the family $\langle \mathcal{T}_\alpha \rangle_{\alpha\in\mathcal{F}(I)}$, then $(X, \widehat{\mathcal{B}}_0(X), \mu, \langle f_\alpha \rangle_{\alpha\in\mathcal{F}(I)})$ is the projective limit of the system $\langle X_\alpha, \widehat{\mathcal{B}}_0(X_\alpha), \mu_\alpha, f_{\alpha\beta}, \mathcal{F}(I)\rangle$. For any $\alpha \in \mathcal{F}(I)$ the map ρ_α from $\widehat{\mathcal{B}}_0(X_\alpha)$ onto itself defined by

$$\rho_\alpha(A) = A \quad \text{for any } A \in \widehat{\mathcal{B}}_0(X_\alpha)$$

is a strong lifting in $\Lambda(\mu_\alpha)$ and the family $\langle \rho_\alpha \rangle_{\alpha\in\mathcal{F}(I)}$ is self-consistent. Since μ does not admit a strong lifting, in particular there cannot exist any lifting $\rho \in \Lambda(\mu)$, which is a projective limit of $\langle \rho_\alpha \rangle_{\alpha\in\mathcal{F}(I)}$, because in such a case we should have $G = \rho(G)$ for any element G of the family

$$\mathcal{G} := \{f_\alpha^{-1}(G_\alpha) \colon G_\alpha \in \mathcal{T}_\alpha, \alpha \in \mathcal{F}(I)\}.$$

But since $\mathcal{G}$ is a basis for the topology $\mathcal{T}$, it follows from Lemma 4.7 that ρ should be strong; this yields a contradiction. The same example shows also that *an inverse lifting of a strong one under a continuous and measure preserving map is not in general strong*. In fact, for each $\alpha \in \mathcal{F}(I)$ there exists by Theorem 3.7 a $\tilde{\rho}_\alpha \in \Lambda(\mu)$ which is an inverse lifting of ρ_α under f_α. But according to Fremlin (1979) $\tilde{\rho}_\alpha$ cannot be strong. But for well ordered index set there is always a positive solution due to Macheras, Musiał and Strauss (200?a).

THEOREM 6.20. *Let κ be an infinite ordinal and $\langle X_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, \kappa\rangle$ be a projective system of complete topological probability spaces. Suppose that $(X, \Sigma, \mu, \langle f_\alpha \rangle_{\alpha<\kappa})$ is the projective limit of the above projective system and $\langle \varphi_\alpha \rangle_{\alpha<\kappa}$ is a self-consistent family of strong densities $\varphi_\alpha \in \vartheta(\mu_\alpha)$. Then for the projective limit topology $\mathcal{T}$ of $\langle \mathcal{T}_\alpha \rangle_{\alpha<\kappa}$ follows $\mathcal{T} \subseteq \widehat{\Sigma}$ and there exists a strong density $\varphi \in \vartheta(\widehat{\mu})$ such that φ is a projective limit of the system $\langle \varphi_\alpha \rangle_{\alpha<\kappa}$.*

In particular, if for every $\alpha < \kappa$ we have $\Sigma_\alpha = \widehat{\mathcal{B}}(X_\alpha)$ then $\widehat{\Sigma} = \widehat{\mathcal{B}}(X)$ and, if in addition $\widehat{\Sigma} = \widehat{\mathcal{B}}_0(X)$ and all the measures μ_α are completion regular, then $\widehat{\mu}$ is also completion regular. The same is true if we replace "densities" by "liftings" throughout.

The last result raises the problem of the existence of a self-consistent family of strong liftings for projective systems of complete topological probability spaces. Even for well ordered index set the answer is to the negative in general as witnessed by the following example. Take Fremlin's Radon probability measure μ on $X := \{0,1\}^{\aleph_2}$ which has no strong lifting and is supported by X (cf. Fremlin (1979)). If κ is the smallest ordinal of cardinality $\aleph_2$, then $(X, \widehat{\mathcal{B}}(X), \mu, \langle f_\alpha \rangle_{\alpha<\kappa})$ is the projective limit of the projective system $\langle X_\alpha, \widehat{\mathcal{B}}(X_\alpha), \mu_\alpha, f_{\alpha\beta}, \kappa \rangle$ of probability spaces, where $X_\alpha := \{0,1\}^\alpha$, $f_{\alpha\beta}$ (respectively f_α) is the canonical projection from X_β onto X_α (respectively from X onto X_α), and μ_α is the image measure $f_\alpha(\mu)$ on $\widehat{\mathcal{B}}(X_\alpha)$ for $\alpha \leqslant \beta < \kappa$. Assume that there exists a self-consistent family $\langle \rho_\alpha \rangle_{\alpha<\kappa}$ of strong liftings $\rho_\alpha \in \Lambda(\mu_\alpha)$. Then by Theorem 6.20 there exists a strong lifting $\rho \in \Lambda(\mu)$; this yields a contradiction.

On the other hand we know two classes of spaces with projective systems admitting a self-consistent family of strong liftings even for arbitrary index set. These are the hyperstonian spaces with measure of full support and the extremally disconnected Baire spaces where each set of the first category is closed, endowed with a category measure (cf. Macheras and Strauss (1994, Remark 2.2(iii) (b) and (c)), which includes also the definitions of the above notions).

6.4. *Various Fubini products*

The most difficult type of lifting in products arises if we ask for lifted functions with measurable respectively lifted sections or even for Fubini-like formulas. There is still a comparatively satisfactory answer for densities in the next theorem. All results in this subsection are taken from the paper Musiał, Strauss and Macheras (2001) if not otherwise indicated.

THEOREM 6.21. *Let (Θ, T, ν) be an arbitrary probability space. If $\tau \in A\vartheta(\nu)$ then for each (Ω, Σ, μ) and each $\delta \in \vartheta(\mu)$ there exists $\varphi \in \vartheta(\mu \widehat{\otimes} \nu)$ with the following properties*:

(i) $\varphi \in \delta \otimes \tau$;

(ii) $[\varphi(E)]_\omega = \tau([\varphi(E)]_\omega)$ *for all* $\omega \in \Omega$ *and* $E \in \Sigma \widehat{\otimes} T$;

(iii) $[\varphi(E)]^\theta$ *is* $\widehat{\mu}$*-measurable for all* $\theta \in \Theta$ *and* $E \in \Sigma \widehat{\otimes} T$.

Admitting the completeness of both measure spaces one can find $\varphi \in \vartheta(\mu \widehat{\otimes} \nu)$ satisfying the properties (ii), (iii), *and the following two properties*:

(iv) $\varphi(A \times B) \subseteq \delta(A) \times \tau(B)$ *for all* $A \in \Sigma$ *and* $B \in T$;

(v) $\nu([\varphi(E)]_\omega \cup [\varphi(E^c)]_\omega) = 1$ *for all* $\omega \in \Omega$ *and* $E \in \Sigma \widehat{\otimes} T$.

The condition (iii) *cannot be improved essentially. More precisely, if $\delta \in \vartheta(\mu)$, $\tau \in A\vartheta(\nu)$ and $\varphi \in \vartheta(\mu \widehat{\otimes} \nu)$ are such that* (ii) *is satisfied and*

$$[\varphi(E)]^\theta = \delta([\varphi(E)]^\theta) \quad \textit{for all } \theta \in \Theta \textit{ and } E \in \Sigma \widehat{\otimes} T$$

then either μ or ν is purely atomic.

For liftings a corresponding result looks as follows:

THEOREM 6.22. *Let (Ω, Σ, μ) and (Θ, T, ν) be complete probability spaces. For each $\sigma \in AG\Lambda(\nu)$ and each $\rho \in \Lambda(\mu)$, there exists $\pi \in \Lambda(\mu \widehat{\otimes} \nu)$ such that the following conditions are satisfied*:

(a) $\pi \in \rho \otimes \sigma$;

(b) $[\pi(E)]_\omega = \sigma([\pi(E)]_\omega)$ *for all* $\omega \in \Omega$ *and* $E \in \Sigma \widehat{\otimes} T$.

Equivalently,

$$\pi^{\infty} \in \rho^{\infty} \otimes \sigma^{\infty} \quad \textit{and} \quad \left[\pi^{\infty}(f)\right]_{\omega} = \sigma^{\infty}\left(\left[\pi^{\infty}(f)\right]_{\omega}\right)$$

for each $f \in \mathcal{L}^{\infty}(\mu \widehat{\otimes} \nu)$ *and each* $\omega \in \Omega$.

The above result is in a sense the best possible, as it follows from the next theorem.

THEOREM 6.23. *Let* (Ω, Σ, μ) *be a complete non-atomic and perfect probability space and let* (Θ, T, ν) *be a complete non-atomic probability space. There exist no liftings* $\sigma \in \Lambda(\nu)$ *and* $\varphi \in \Lambda(\mu \widehat{\otimes} \nu)$ *satisfying the following two conditions:*

(j) *there exists* $\overline{\theta} \in \Theta$ *such that for each* $E \in \Sigma \widehat{\otimes} T$

$$\left[\varphi(E)\right]^{\overline{\theta}} \in \Sigma;$$

(jj) *for each* $E \in \Sigma \widehat{\otimes} T$ *there exists a set* $N_E \in \Sigma_0$ *such that*

$$\left[\varphi(E)\right]_{\omega} = \sigma\left(\left[\varphi(E)\right]_{\omega}\right) \quad \textit{for each } \omega \notin N_E.$$

In particular, no $\sigma \in \Lambda(\nu)$ *is an admissible density.*

So we are left with the following problem for given complete probability spaces (Ω, Σ, μ) and (Θ, T, ν).

QUESTION 6.24.

(i) Does there exist a lifting $\varphi^{\infty} \in \Lambda^{\infty}(\mu \widehat{\otimes} \nu)$ such that $[\varphi^{\infty}(f)]_{\omega} \in \mathcal{L}^{\infty}(\nu)$ and $[\varphi^{\infty}(f)]^{\theta} \in \mathcal{L}^{\infty}(\mu)$ for all $f \in \mathcal{L}^{\infty}(\mu \widehat{\otimes} \nu)$ and for all $(\omega, \theta) \in \Omega \times \Theta$?

(ii) Can we choose in addition $\varphi^{\infty} \in \Lambda_p^{\infty}(\mu \widehat{\otimes} \nu)$ in (i) for some marginal liftings?

In case both these complete probability spaces coincide with the Lebesgue probability space over [0, 1] subject to the continuum hypothesis we get an answer to the positive for Question 6.24(i) just by choosing a Borel lifting according to Theorem 4.12 for the 2-dimensional Lebesgue probability on $[0, 1]^2$. But what is the position without the continuum hypothesis?

In case of linear liftings Macheras, Musiał and Strauss (200?d) proved the following:

THEOREM 6.25. *Let* (Θ, T, ν) *be a complete separable probability space and* (Ω, Σ, μ) *a complete probability space. Then for each* $\rho \in \mathcal{G}(\mu)$ *and each* $\tau \in A\mathcal{G}(\nu)$ *there exists a* $\varphi \in \mathcal{G}(\mu \widehat{\otimes} \nu)$ *such that*

(i) $\varphi(g \otimes h) = \rho(g) \otimes \tau(h)$ *for all* $g \in \mathcal{L}^{\infty}(\mu)$ *and* $h \in \mathcal{L}^{\infty}(\nu)$;

(ii) *for each* $f \in \mathcal{L}^{\infty}(\mu \widehat{\otimes} \nu)$ *and each* $\omega \in \Omega$

$$\left[\varphi(f)\right]_{\omega} = \tau\left(\left[\varphi(f)\right]_{\omega}\right).$$

It remains an open question whether the separability assumption is necessary. However again there is no hope for a linear product lifting with all sections being lifting invariant. The subsequent theorem of Macheras, Musiał and Strauss (200?d) is a direct consequence of the appropriate part of Theorem 6.21.

THEOREM 6.26. *Let (Θ, T, ν) be a complete probability space and (Ω, Σ, μ) be an arbitrary complete probability space. Assume that there are $\rho \in \mathcal{G}(\mu)$, $\sigma \in \mathcal{G}(\nu)$ and $\varphi \in \mathcal{G}(\mu \widehat{\otimes} \nu)$ such that the inequalities*

$$[\varphi(f)]_\omega(\theta) = \sigma([\varphi(f)]_\omega)(\theta) \quad \textit{and} \quad [\varphi(f)]^\theta(\omega) = \rho([\varphi(f)]^\theta)(\omega)$$

hold true for all (ω, θ) and all $f \in \mathcal{L}^\infty(\mu \widehat{\otimes} \nu)$. Then at least one of the measures is purely atomic.

QUESTION 6.27. Does there exist $\varphi \in \mathcal{G}(\mu \widehat{\otimes} \nu)$, $\rho \in \mathcal{G}(\mu)$ and $\sigma \in \mathcal{G}(\nu)$ such that $\varphi \in \rho \otimes \sigma$, $[\varphi^\infty(f)]_\omega \in \mathcal{L}^\infty(\nu)$ and $[\varphi^\infty(f)]^\theta \in \mathcal{L}^\infty(\mu)$ for all $f \in \mathcal{L}^\infty(\mu \widehat{\otimes} \nu)$ and for all $(\omega, \theta) \in \Omega \times \Theta$?

QUESTION 6.28. Does there exist a $\varphi \in \mathcal{G}(\mu \widehat{\otimes} \nu)$ and a $\tau \in \mathcal{G}(\nu)$ such that for all $f \in \mathcal{L}^\infty(\mu \widehat{\otimes} \nu)$ exists $N_f \in \Sigma_0$ with $\tau([\varphi(f)]_\omega) = [\varphi(f)]_\omega \in \mathcal{L}^\infty(\nu)$ for all $\omega \notin N_f$ and $[\varphi(f)]^\theta \in \mathcal{L}^\infty(\mu)$ for all $\theta \in \Theta$?

For general complete probability spaces these problems are open as far as we know. The next two results are related to the topic of this section since they also concern some measurability properties of liftings from a different point of view. They are due to Christensen (1974).

THEOREM 6.29 (CH). *Let Σ be a separable σ-algebra on Ω (i.e., generated by countably many sets and containing all points) and let μ be a probability on Σ. Let $\mathcal{T}$ be the weak*-topology on the space $L^\infty(\mu)$. Then there exists $\rho \in \mathcal{G}(\mu)$ such that for each probability measure ν defined on the product σ-algebra $\mathcal{B}(L^\infty(\mu)) \otimes \Sigma$ the function $(f^\bullet, \omega) \to \rho(f)(\omega)$ is measurable with respect to the completion of ν. In particular each functional $f^\bullet \to \rho(f)(\omega)$ is universally measurable on $L^\infty(\mu)$.*

THEOREM 6.30. *We keep the notation of Theorem 6.29. Assume that μ is non-atomic and let $\mathcal{B}a(\mu)$ be the σ-algebra of sets possessing the Baire property with respect to the weak*-topology of $L^\infty(\mu)$. If $\rho \in \Lambda(\mu)$, then for almost every $\omega \in \Omega$ there exists a measure ν_ω on $\mathcal{B}a(\mu)$ such that the functional $f^\bullet \to \rho(f)(\omega)$ is not measurable with respect to the completion of ν_ω.*

6.5. *Applications of Fubini products to stochastic processes*

Theorem 6.22 has applications to functions of two variables and stochastic processes. All results in this subsection are taken from Musiał, Macheras and Strauss (2001). Let

(Ω, Σ, μ), (Θ, T, ν) be complete probability spaces and let $\{X_\theta\}_{\theta\in\Theta}$ be an arbitrary real-valued stochastic process on (Ω, Σ, μ). If $\{Y_\theta\}_{\theta\in\Theta}$ is another stochastic process then it is called to be *equivalent to* $\{X_\theta\}_{\theta\in\Theta}$ if for each $\theta \in \Theta$ we have $X_\theta = Y_\theta$ a.e. (μ) (the exceptional set may depend on θ) according to Talagrand (1987). $\{X_\theta\}_{\theta\in\Theta}$ is said to be measurable if the map $(\omega, \theta) \to X_\theta(\omega)$ is measurable with respect to the product σ-algebra $\Sigma \mathbin{\widehat{\otimes}} T$. $\{X_\theta\}_{\theta\in\Theta}$ is bounded if the family $\{X_\theta\colon \theta \in \Theta\}$ is bounded in $L^\infty(\mu)$. There are several papers concerning the existence of measurable (or separable) processes that are equivalent to a given process (cf. Cohn (1978), Talagrand (1987, 1988)). Sometimes a measurable process equivalent to a bounded $\{X_\theta\}_{\theta\in\Theta}$ can be defined by setting $Y_\theta = \rho(X_\theta)$, where $\rho \in \Lambda(\mu)$ and the initial process X or (Θ, T, ν) satisfy some additional conditions. In particular, the lifting ρ in Cohn (1978) is assumed to be strong and Ω is taken to be an interval. In general however, a strong lifting might not exist on an investigated topological measure space. With the help of Theorem 6.22 we get, – by a different method –, the following two results stated in Cohn (1978) under additional topological assumptions.

THEOREM 6.31. *For each $\sigma \in AG\Lambda(\nu)$ and each bounded measurable stochastic process $\{X_\theta\}_{\theta\in\Theta}$ on a space (Ω, Σ, μ) there is a collection of measurable functions $\{Y_\theta\}_{\theta\in\Theta}$ on (Ω, Σ, μ) satisfying the following conditions:*

(i) *$Y_.(\omega) = \sigma(Y_.(\omega))$ for each $\omega \in \Omega$.*

(ii) *There is $M_X \in T_0$ such that for every $\theta \notin M_X$ we have $X_\theta = Y_\theta$ a.e. (μ) and $\{Y_\theta\}_{\theta\notin M_X}$ is a measurable stochastic process on (Ω, Σ, μ).*

(iii) *There is $N_X \in \Sigma_0$ such that for every $\omega \notin N_X$*

$$X_.(\omega) = Y_.(\omega) \quad \text{a.e. } (\nu).$$

(iv) *If Θ is a separable metric space and $(X_\theta)_{\theta\notin M_X}$ is continuous in probability, then $(Y_\theta)_{\theta\notin M_X}$ is separable. Furthermore, every countable dense subset of $\Theta \setminus M_X$ is a separating set for $(Y_\theta)_{\theta\notin M_X}$.*

In the terminology of Cohn (1978) the process $\{Y_\theta\}_{\theta\in\Theta}$ is called *σ-canonical*. Another application of Theorem 6.22 is the next result.

THEOREM 6.32. *For each $\rho \in AG\Lambda(\mu)$ and each bounded measurable stochastic process $\{X_\theta\}_{\theta\in\Theta}$ on a space (Ω, Σ, μ) there is a measurable process $\{Y_\theta\}_{\theta\in\Theta}$ on (Ω, Σ, μ) that is equivalent to $\{X_\theta\}_{\theta\in\Theta}$ and satisfies the following conditions:*

(i) *$Y_\theta = \rho(Y_\theta)$ for each θ;*

(ii) *There exists a set $N_X \in \Sigma_0$ such that for each $\omega \notin N_X$ we have*

$$X_.(\omega) = Y_.(\omega) \quad \text{a.e. } (\nu).$$

Notice that in Theorem 6.32 we had to use properties of $\pi(X)$ in order to assure the plane measurability of Y, where π is given by Theorem 6.22. Direct defining of the process Y by setting $Y_\theta = \rho(X_\theta)$ for quite an arbitrary lifting might destroy the plane measurability properties of the process X (cf. Cohn (1978)).

One of the important problems in the theory of functions of two variables is pointing out of conditions guaranteeing the plane measurability of a separately measurable function. A notion of a stable set investigated by D.H. Fremlin and M. Talagrand (see Talagrand (1984)) turned out to be very fruitful in this field. In particular, Talagrand (1984, 10-2-1) proved that if Θ is a compact set, ν is a Radon measure on Θ, $f:\Omega\times\Theta\to\mathbb{R}$ is measurable as a function of the first variable and continuous as a function of the second variable, then f is measurable, provided the family $\{f^\theta\colon \theta\in\Theta\}$ is a stable set. Applying a result of Fremlin (1983) on stable sets together with Theorem 6.22, we get a similar result for functions that have lifting invariant sections. It may be considered as a strengthening of M. Talagrand's method from Talagrand (1988) of modification of a stochastic process with the help of consistent liftings.

THEOREM 6.33. *Let (Ω,Σ,μ), (Θ,T,ν) be complete probability spaces and, and $\{X_\theta\}_{\theta\in\Theta}$ be an arbitrary real-valued bounded stochastic process on (Ω,Σ,μ) with measurable paths (i.e., all functions X_ω are ν-measurable). If the set $\{X_\theta\colon \theta\in\Theta\}$ is stable then for each $\rho\in AG\Lambda(\mu)$ the process $\{Y_\theta\}_{\theta\in\Theta}$ given for each θ by $Y_\theta:=\rho(X_\theta)$ is $\Sigma\mathbin{\widehat{\otimes}}T$-measurable and equivalent to $\{X_\theta\}_{\theta\in\Theta}$.*

COROLLARY 6.34. *Let (Ω,Σ,μ), (Θ,T,ν) be complete probability spaces and let $f:\Omega\times\Theta\to\mathbb{R}$ be a separately measurable function such that the set $\{f^\theta\colon \theta\in\Theta\}$ is stable. If there exists $\rho\in AG\Lambda(\mu)$ such that for each $\theta\in\Theta$ the equality $\rho(f^\theta)=f^\theta$ holds true, then f is $\Sigma\mathbin{\widehat{\otimes}}T$-measurable.*

7. Liftings for abstract valued functions

Throughout let be given complete probability spaces (Ω,Σ,μ) and a completely regular Hausdorff space T. For any lifting $\rho\in\Lambda(\mu)$ a Baire-measurable function $f:\Omega\to T$ induces a Borel-measurable map $\rho_T(f)$ from Ω into βT, the Stone–Čech compactification of T, defined via the formula

$$h\circ\rho_T(f)=\rho(h\circ f)\quad \text{for } h\in C_b(T).$$

According to Bellow (1980) the map f is called *lifting compact* if for every $\rho\in\Lambda(\mu)$ we have $\rho_T(f)(\omega)\in T$ for μ-a.a. $\omega\in\Omega$. This implies $h\circ\rho_T(f)\equiv h\circ f$ a.e. (μ) for every $h\in C_b(T)$, where the null-set involved depends on $h\in C_b(T)$. If for lifting compact f the stronger equivalence $\rho_T(f)=f$ μ-a.e. holds true for any $\rho\in\Lambda(\mu)$, then f is called *strongly lifting compact* by Babiker et al. (1986), and a completely regular Hausdorff topological space T is (*strongly*) *lifting compact* by definition, if for any complete probability space (Ω,Σ,μ) every Baire-measurable map $f:\Omega\to T$ is (strongly) lifting compact. Lifting compactness of completely regular Hausdorff spaces is a property lying strictly in between strong measure compactness and measure compactness by Bellow (1980, 6.1) as well as Edgar and Talagrand (1980, Theorem 1). By Bellow (1980) lifting compactness has good stability properties.

THEOREM 7.1.
(i) *An arbitrary subspace of a compact metrizable space is lifting compact.*
(ii) *Every Baire set in a lifting compact space is lifting compact.*
(iii) *Any continuous bijection between completely regular spaces with Baire measurable inverse transforms a lifting compact space into a lifting compact one.*
(iv) *A countable product of lifting compact spaces is lifting compact.*

The results (ii) to (iv) are analogs of results proved by Moran (1969) for (strongly) measure compact spaces. A Banach space E under its weak topology is lifting compact if and only if every E-valued scalarly measurable function is scalarly equivalent to a Bochner measurable function by Bellow (1980, Section 6, Remark 2). Every subspace of a compact metric space is strongly lifting compact. Moreover, this holds true for every strongly lifting compact space. The next three theorems are taken from Babiker et al. (1986).

THEOREM 7.2.
(i) *An arbitrary subset of a strongly lifting compact space is strongly lifting compact.*
(ii) *Countable unions of strongly lifting compact Baire subsets of a completely regular Hausdorff space are strongly lifting compact.*
(iii) *A countable product of strongly lifting compact spaces is strongly lifting compact.*
(iv) *The image of a strongly lifting compact space under a continuous surjection with Baire measurable section is strongly lifting compact.*
(v) *A measure compact space is strongly lifting compact if every point has a strongly measure compact neighborhood.*

The next result states a relation between strong lifting compactness and the *USLP*. Its proof relies on Theorem 3.7.

THEOREM 7.3. *If the map $f:\Omega \to T$ is strongly lifting compact and ν is the image measure of μ under f on the σ-algebra $\mathcal{B} := \{B \subseteq T\colon f^{-1}(B) \in \Sigma\}$, then the topological measure space $(T, \mathcal{T}, \mathcal{B}, \nu)$ has the USLP, if $\mathcal{T}$ denotes the completely regular Hausdorff topology of T.*

From the last theorem together with the existence of a compact Radon probability space without strong lifting (see Section 4) follows that neither lifting compactness nor strong measure compactness imply strong lifting compactness, nor does strong lifting compactness imply strong measure compactness, as witnessed by the standard Lebesgue non-measurable subset of [0, 1]. For general metric spaces strong lifting compactness is equivalent with measure compactness, which means in that case that every closed discrete subspace has non-measurable cardinal.

It is an open problem, whether the converse of Theorem 7.3 is true, compare Macheras and Strauss (1992). The next result gives among other equivalent conditions an answer to the positive for $(E, \textit{weak})$, a metrizable locally convex spaces E under its weak topology. An essential tool for its proof is A. Tortrat's Theorem 8, from Tortrat (1975).

THEOREM 7.4. *For a metrizable locally convex space E the following conditions are all equivalent.*

(i) *$(E, weak)$ is strongly lifting compact.*
(ii) *Every Baire probability measure space based on $(E, weak)$ has the USLP.*
(iii) *Every Baire probability measure space based on $(E, weak)$ has the ASLP.*
(iv) *$(E, weak)$ is completion regular and measure compact.*
(v) *Every Baire probability measure μ on $(E, weak)$ is supported by a μ-measurable closed linear subspace of E which is separable with respect to the metric of E.*
(vi) *$(E, weak)$ is measure compact and every Borel subset of $(E, metric)$ is measurable with respect to any Baire probability measure on $(E, weak)$.*
(vii) *Every scalarly measurable function from a complete probability space into E agrees a.e. with a Bochner measurable function.*
(viii) *$(E, weak)$ is measure compact and $\{0\}$ is a Baire subset of E with respect to $(E, weak)$.*
(ix) *$(E, weak)$ is measure compact and there exists a sequence in E', the topological conjugate of E, which separates the points of E.*
(x) *$(E, weak)$ is measure compact and there is a continuous linear injection from $(E, metric)$ into $\mathbb{R}^{\mathbb{N}}$.*
(xi) *$(E, weak)$ is measure compact and submetrizable.*

If the locally convex space E is normable, we may add the following condition.

(xii) *$(E, weak)$ is measure compact and there is a continuous linear injection from $(E, norm)$ into $l^{\infty}(\mathbb{N})$.*

It can be seen from the last theorem that within the class of all metrizable locally convex spaces strongly lifting compact spaces can be characterized in a purely topological way. In Strauss (1992) the strong lifting compactness of conjugate Banach spaces under their *weak** topology has been discussed. In this class the equivalence of the condition (vii) in the last theorem with strong lifting compactness as well as with the conditions corresponding to conditions (ii) respectively (iii) of the last theorem breaks down. There are similar characterizations for strongly lifting compact functions in Babiker et al. (1986). The conjugate $M([0, 1])$ of $C_b([0, 1])$ is a non-separable Banach space which is submetrizable under its weak topology. The mild set-theoretic assumption that the continuum is measure compact implies that $(M([0, 1]), weak)$ is measure compact. Therefore by the last theorem $M([0, 1])$ is an example of a non-separable strongly lifting compact space.

In connection with the definition at the beginning of this section one should mention the following construction of a lifting of Banach space valued functions, which is commonly used in the context of differentiation of vector measures and integration of vector functions. If X is a Banach space and X' is the space of continuous functionals on X then for given function $f : \Omega \to X'$, satisfying $\langle x, f\rangle \in L^{\infty}(\mu)$, where by definition $\langle x, f\rangle(\omega) := (f(\omega))(x)$ for every $\omega \in \Omega$ and for every $x \in X$, we can for each $\rho \in \Lambda(\mu)$ define a function $\rho(f) : \Omega \to X'$ by setting $\langle x, \rho(f)(\omega)\rangle := \rho(\langle x, f\rangle)(\omega)$ where $x \in X$ for all $\omega \in \Omega$. Von Weizsäcker (1978) proved that $\rho(f)$ is measurable when X' is equipped with its *weak**-topology. This function is an essential tool in investigating several aspects of integration. More details and abandon references can be found in Musiał (2002).

In fact we have the following result equivalent to the existence of a lifting (compare A. and C. Ionescu Tulcea (1962, 1969a) and Kölzow (1968)):

THEOREM 7.5. *For given c.l.d. measure spaces* (Ω, Σ, μ) *the following conditions are all equivalent*:

(i) *There exists a lifting on* $\mathcal{L}^{\infty}(\mu)$;

(ii) *For any normed space* X *and any bounded linear operator* u *from* $L^1(\mu)$ *into* X' *(the topological dual space of* X*) there exists a map* f *from* Ω *into* X' *with* $\langle x, f\rangle \in \mathcal{L}^{\infty}(\mu)$ *for all* $x \in X$ *and* $\|f\| \leqslant \|u\|$ *such that* $\int \langle x, f\rangle g \, d\mu = \langle x, u(g^{\bullet})\rangle$ *for all* $x \in X$ *and all* $g \in \mathcal{L}^1(\mu)$.

8. Liftings and densities with respect to ideals of sets

The notion of measure-theoretical lifting and density is a particular case of more general notions of lifting and density with respect to an ideal of sets. One may call them respectively by $\mathcal{J}$-lifting and $\mathcal{J}$-density. Given a measurable space (Ω, Σ) and an ideal $\mathcal{J} \subset \Sigma$ one says that a Boolean homomorphism $\rho : \Sigma \to \Sigma$ is an $\mathcal{J}$-lifting if $\rho(A) \equiv A$ for all $A \in \Sigma$ and $\rho(A) = \rho(B)$ whenever $A \equiv B$ in the sense of $\mathcal{J}$. Similarly $\mathcal{J}$-density is introduced.

As in the case of measure spaces there is also an equivalent formulation for the Boolean algebra $\Sigma/\mathcal{J}$. In fact the original paper of von Neumann and Stone (1935) has been written in the language of ideals. In spite of this, the theory of $\mathcal{J}$-liftings is much less developed than the theory of measure liftings. Here are some known facts. Maharam (1977) noticed that the following result follows easily from Graf (1973):

THEOREM 8.1. *If* Ω *is a Baire space (i.e., no non-empty subset of* Ω *is of the first category in itself or, equivalently, in* Ω*), and* Σ *is the* σ*-algebra generated by Borel subsets and the* σ*-ideal* $\mathcal{C}$ *of sets of the first category, then there is a strong lifting on* $\Sigma/\mathcal{C}$.

For some time there has been an open question whether each Boolean algebra of the form $\Sigma/\mathcal{J}$ which additionally satisfies the countable chain condition has a lifting. Shelah (1998) answered it negatively.

THEOREM 8.2. *There is a cardinal* κ $(\kappa = 2^{\aleph_0++}$ *suffices) a* σ*-ideal* $\mathcal{J} \subset \mathcal{P}(\kappa)$ *and a* σ*-algebra* $\Sigma \subset \mathcal{P}(\kappa)$ *containing* $\mathcal{J}$ *and such that* $\Sigma/\mathcal{J}$ *satisfies the countable chain condition but* $\Sigma/\mathcal{J}$ *has no lifting.*

The next theorem of von Neumann and Stone (1935) is a classical tool for converting a density into a lifting in the situation of this section. Let $\aleph_\alpha$ be a fixed infinite cardinal. Let us remember that a lattice V is called *(conditionally)* κ*-complete* for some cardinal κ, if every subset W of V having cardinality $\leqslant \kappa$ (with lower and upper bounds in V) has an infimum and a supremum in V.

THEOREM 8.3. *Let* $\mathcal{J}$ *be an ideal in a Boolean algebra* $\mathcal{B}$ *which is conditionally* κ*-complete for all* $\kappa < \aleph_\alpha$. *If* $\operatorname{card}(\mathcal{B}/\mathcal{J}) \leqslant \aleph_\alpha$, *then for any* $\mathcal{J}$*-density* φ *there exists a* $\mathcal{J}$*-lifting* ρ *with* $\varphi(B) \subseteq \rho(B)$ *for all* $B \in \mathcal{B}$.

Von Weizsäcker (1976, Theorem B.3) shows that the condition card$(\mathcal{B}/\mathcal{J}) \leqslant \aleph_\alpha$ in the last theorem cannot be weakened to card$(\mathcal{B}/\mathcal{J}) \leqslant \aleph_{\alpha+1}$, even if one is interested only in the existence of linear liftings.

THEOREM 8.4. *Assume* $\sup \Gamma < \aleph_\alpha$ *for all families* Γ *of cardinals with* card$(\Gamma) < \aleph_\alpha$ *and* $\gamma < \aleph_\alpha$ *for all* $\gamma \in \Gamma$. *Then there exist set algebras* $\mathcal{B}_i$, *ideals* $\mathcal{J}_i$ *in* $\mathcal{B}_i$, $\mathcal{J}_i$*-densities* φ_i *for* $i = 1, 2$ *such that the following conditions hold true.*

(i) $\mathcal{J}_i$ *is* κ*-complete for all* $\kappa < \aleph_\alpha$ *for* $i = 1, 2$.

(ii) $\mathcal{B}_1/\mathcal{J}_1$ *and* $\mathcal{B}_2/\mathcal{J}_2$ *are isomorphic with* card$(\mathcal{B}_i/\mathcal{J}_i) = \aleph_{\alpha+1}$ *for* $i = 1, 2$.

(iii) *There exists no linear lifting* $\rho_1 : \mathcal{L}^\infty(\mathcal{B}_1) \to \mathcal{L}^\infty(\mathcal{B}_1)$ *satisfying* $\rho_1(\chi_B) \geqslant \chi_{\varphi_1(B)}$ *for all* $B \in \mathcal{B}_1$.

(iv) *There exists a linear lifting* $\rho_2 : \mathcal{L}^\infty(\mathcal{B}_2) \to \mathcal{L}^\infty(\mathcal{B}_2)$ *satisfying* $\rho_2(\chi_B) \geqslant \chi_{\varphi_2(B)}$ *for all* $B \in \mathcal{B}_2$, *but there is no* $\mathcal{J}_2$*-lifting* $\overline{\rho}_2$ *for* $\mathcal{B}_2$ *satisfying* $\varphi_1(B) \subseteq \overline{\rho}_2(B)$ *for all* $B \in \mathcal{B}_2$.

At last we formulate a few facts concerning liftings on the set of natural numbers.

It is well known that if $\mathcal{J}$ is the ideal of all finite subsets of $\mathbb{N}$, then there is no $\mathcal{J}$-lifting on $\mathcal{P}(\mathbb{N})$.

To formulate the next result denote by $\#A$ the cardinality of $A \subset \mathbb{N}$. Then, define a density measure $\mu : \mathcal{P}(\mathbb{N}) \to [0, 1]$ by setting for each $A \subseteq \mathbb{N}$

$$\mu(A) := \operatorname{Lim}_{n\to\infty} \frac{\#(A \cap \{1, 2, \ldots, n\})}{n},$$

where Lim is a Banach limit on l_∞.

With this concept, we have the following result of D. Maharam and P. Erdős (see Maharam (1976)):

THEOREM 8.5. *Let* μ *be a density measure on* $\mathbb{N}$ *and let* $\mathcal{J} := \{A \subset \mathbb{N}\colon\ \mu(A) = 0\}$. *Then there is no* $\mathcal{J}$*-lifting on* $\mathcal{P}(\mathbb{N})$.

9. Beyond $\mathcal{L}^\infty(\mu)$

As we have mentioned at the end of Section 3 it has been proved already by A. and C. Ionescu Tulcea (1969a) that there is no lifting on $\mathcal{L}^p(\mu)$ if $1 < p < \infty$ and μ is non-atomic. It turns out however that each lifting can be extended far beyond $\mathcal{L}^\infty(\mu)$, as the following result of Monakov-Rogozkin (1974) shows

THEOREM 9.1. *Given* $\rho \in \Lambda(\mu)$ *there exists an ideal* $\mathcal{K}(\rho)$ *in the lattice* $\mathcal{L}^0_R(\mu)$ *with the following properties*:

(i) $\mathcal{L}^\infty(\mu) \subset \mathcal{K}(\rho)$.

(ii) *There is a lifting* $\widetilde{\rho}$ *on* $\mathcal{K}(\rho)$ *which is an extension of* ρ.

(iii) $\mathcal{K}(\rho)$ *is the largest ideal in* $\mathcal{L}^0_R(\mu)$ *possessing the above two properties.*

(iv) *The extension of* ρ *onto* $\mathcal{K}(\rho)$ *is unique.*

More precisely, $f \in \mathcal{K}(\rho)$ if and only if $f/g \in \mathcal{L}^{\infty}(\mu)$ for a function $g \in \mathcal{L}_R^0(\mu)$ such that $g \geqslant 1$ and $\rho(1/g) = 1/g$. One can also say that $\mathcal{K}(\rho) = \pi(C(\Omega, \mathcal{T}))$ where π identifies functions μ-equivalent, $\mathcal{T}$ is one of the lifting topologies and $C(\Omega, \mathcal{T})$ is the space of $\mathcal{T}$-continuous real-valued functions.

Monakov-Rogozkin (1974) proves also that if $\rho \in \mathcal{G}(\mu) \setminus \Lambda(\mu)$ then a counterpart of $\mathcal{K}(\rho)$ for ρ does not exist.

10. Further applications

Lifting theory has so many applications in mathematical analysis that it is impossible to give a detailed account for all within this article since this would afford too much additional terminology. But we want to give references now for some of the applications which could not be mentioned as yet.

Jirina (1959) proved the existence of disintegrations for measures under separability assumptions. The most common procedure taken now is to eliminate separability by an application of liftings, see C. Ionescu Tulcea (1965b), A. and C. Ionescu Tulcea (1969a), Hoffmann-Jorgensen (1971), Pellaumail (1972), Chatterji (1973), Valadier (1973), Saint-Pierre (1975), Pachl (1978), Heller (1983), Babiker and Strauss (1982), Choksi and Duncan (unpublished), and Rinkewitz (1997). In A. and C. Ionescu Tulcea (1969a) and Babiker and Strauss (1982), the application of a lifting comes indirectly in via the Dunford Pettis theorem. Heller (1983) determined the class of all Baire measures which can be disintegrated with the help of liftings. Corresponding results are given by Rinkewitz (1997).

We are going to recall two particular applications of liftings to disintegration. Let (Θ, T) be a measurable space and let $p : \Omega \to \Theta$ be a measurable map (i.e., $p^{-1}(T) \subseteq \Sigma$). $P : \Sigma \times \Theta \to [0, 1]$ is a *regular conditional probability* if

(cp1) P_θ is a probability measure on Σ for all $\theta \in \Theta$;

(cp2) $\theta \to P_\theta(A)$ is $\nu := p(\mu)$-measurable for all $A \in \Sigma$;

(cp3) $\mu(A \cap p^{-1}(B)) = \int_B P_\theta(A)\, d\nu(\theta)$ for all $A \in \Sigma$ and all $B \in T$.

The next result was given by Hoffmann-Jørgensen (1971).

THEOREM 10.1. *Let Ω be a Hausdorff space, μ is a regular probability on $\mathcal{B}(\Omega)$ (i.e., μ is inner regular with respect to compact sets), (Θ, T) is a measurable space, $p : \Omega \to \Theta$ is a measurable function and $\nu = p(\mu)$. Then there exists a regular conditional probability $P : \mathcal{B}(\Omega) \times \Theta \to [0, 1]$ such that for every $\theta \in \Theta$ the measure P_θ is regular.*

PROOF (*Sketch*). We assume for the simplicity that Ω is compact. If $f \in C(\Omega)$ then we define a ν-continuous finite signed measure ν_f on T by setting for every $B \in T$

$$\mu_f(B) = \int_{p^{-1}(B)} f\, d\mu.$$

Let $p(\theta, f)$ be a Radon–Nikodým derivative of ν_f with respect to ν. One easily sees that $p(\theta, f)$ is bounded in $L^{\infty}(\nu)$ by $\|f\|_\infty$. Take an arbitrary $\rho \in \Lambda(\nu)$ and set for every

$f \in C(\Omega)$

$$\overline{p}(\cdot, f) := \rho(p(\cdot, f)).$$

It follows from the properties of ν_f that $\overline{p}(\theta, \cdot)$ is for each θ a positive continuous linear functional on $C(\Omega)$. Moreover, for every θ and $f \in C(\Omega)$ we have

$$\overline{p}(\theta, \chi_\Omega) = 1 \quad \text{and} \quad |\overline{p}(\theta, f)| \leqslant \|f\|_\infty.$$

Hence, for every θ there is a regular probability P_θ on $\mathcal{B}(\Omega)$ such that for all $f \in C(\Omega)$

$$\overline{p}(\theta, f) = \int_\Omega f \, dP_\theta.$$

The rest of the proof is a matter of calculations. □

The second application of liftings we are going to present here is an application to disintegration of compact measures. We follow the terminology of Pachl (1978).

DEFINITION 10.2. Let (Ω, Σ) be a measurable space and let (Θ, T, ν) be a probability space. Let κ be a probability on $\Sigma \otimes T$ such that $\kappa(\Omega \times B) = \nu(B)$ for every $B \in T$. Suppose that for every $\theta \in \Theta$ there is a σ-algebra Σ_θ on Ω and a probability P_θ on Σ_θ such that the following two conditions are satisfied:

(a) for each $A \in \Sigma$ there exists $N \in \mathcal{N}(\nu)$ such that $A \in \Sigma_\theta$ for all $\theta \in \Theta \setminus N$ and the function $\Theta \setminus N \ni \theta \to P_\theta(A)$ is $T|(\Theta \setminus N)$-measurable;

(b) if $A \in \Sigma$ and $B \in T$ then

$$\kappa(A \times B) = \int_B P_\theta(A) \, d\nu(\theta).$$

The collection $\{(\Sigma_\theta, P_\theta)\}_{\theta \in \Theta}$ is called a *ν-disintegration* of κ.

Here is the main result of Pachl (1978).

THEOREM 10.3. *Let (Ω, Σ, μ) and (Θ, T, ν) be probability spaces and let κ be a probability on $\Sigma \otimes T$ with marginals μ and ν. Assume that ν is complete and μ is approximated by an ω-compact lattice $\mathcal{K} \subset \Sigma$ which is closed under countable intersections. Then there is a ν-disintegration $\{(\Sigma_\theta, P_\theta)\}_{\theta \in \Theta}$ of κ such that $\Sigma_\theta \supset \mathcal{K}$ and $\mathcal{K}$ approximates P_θ for every θ.*

BASIC IDEA OF THE PROOF. Let ρ be a lifting on $L^\infty(\nu)$. By the Radon–Nikodým theorem, for each $A \in \Sigma$ there exists a T-measurable function h_A such that for every $B \in T$, we have

$$\kappa(A \times B) = \int_B h_A \, d\nu.$$

For each $\theta \in \Theta$ define a function β_θ on $\mathcal{K}$ by $\beta_\theta(K) := \rho(h_K(\theta))$. It needs some work to show that for each θ there exists a probability γ_θ on $\sigma(\mathcal{K})$ such that $\gamma_\theta \geqslant \beta_\theta$ on $\mathcal{K}$ and its completion P_θ gives a σ-algebra Σ_θ satisfying the requirements of the theorem (see Pachl (1978) for details). □

It should be mentioned here that using the above result J.K. Pachl proved that a restriction of a compact measure (in the sense of Marczewski (1953)) to a sub-σ-algebra is also a compact measure, thus answering a question of E. Marczewski.

For applications to the realization of homomorphisms see A. and C. Ionescu Tulcea (1969a, Chapter X), Graf (1980a), Babiker and Graf (1983), Vesterstrøm and Wils (1969), Fremlin (1989, 4.12 and 4.14), and Fremlin (200?).

Strongly connected with the above are applications of liftings to the existence of measurable selections and sections discussed by Edgar (1976), Talagrand (1978), Losert (1980), Graf (1980, 1982), and Kupka (1983). The results are too technical to be presented here so we mention only two of them, taken from Graf (1980a).

THEOREM 10.4. *Let $\Theta \neq \emptyset$ be a Hausdorff space. Let μ be a non-trivial complete strictly localizable measure which is locally determined and semifinite. Then, let $\Phi : \mathcal{B}(\Theta) \to \Sigma/\mu$ be a Boolean σ-homomorphism such that $\mu \circ \Phi$ is a Radon measure on Θ. Then there is a Σ–$\mathcal{B}(\Theta)$-measurable map $f : \Omega \to \Theta$ with $f^{-1}(B) \in \Phi(B)$ for all $B \in \mathcal{B}(\Theta)$, i.e., Φ is induced by f.*

PROOF (*Sketch*). If $\rho \in \Lambda(\mu)$ then $\rho \circ \Phi : \mathcal{B}(\Theta) \to \Sigma$ is a homomorphism. Let $\mathcal{K}(\Theta)$ be the collection of all compact subsets of Θ. Then, for each ω let $\mathcal{K}_\omega := \{K \in \mathcal{K}(\Theta) \colon \omega \in \rho \circ \Phi(K)\}$. One can easily see that $\cap\mathcal{K}_\omega \neq \emptyset$. In fact for almost all ω the set $\cap\mathcal{K}_\omega$ is a singleton. Consequently we can define $f : \Omega \to \Theta$ by taking $f(\omega) \in \cap\mathcal{K}_\omega$. f satisfies the conclusion of the theorem. □

The next result is related to the extension of measures. $\Theta \neq \emptyset$ is Hausdorff, (Ω, Σ, μ) is a finite measure space and $p : \Theta \to \Omega$ is a measurable map. Denote by $M(\mu, p)$ the collection of all positive measures ν on $\mathcal{B}(\Theta)$ such that $\mu = p(\nu)$.

THEOREM 10.5. *If $\nu \in M(\mu, p)$ is Radon, then ν is an extreme point of $M(\mu, p)$ if and only if there exists a $\widehat{\Sigma}$–$\mathcal{B}(\Theta)$ measurable weak section $f : \Omega \to \Theta$ for p with $\nu = f(\mu)$ (i.e., $\mu(A \triangle f^{-1}p^{-1}(A)) = 0$ for all $A \in \Sigma$).*

The separable modifications of stochastic processes and the domination of measures are treated by A. and C. Ionescu Tulcea (1969b; 1969a, Chapter VII, Section 7 as well as Chapter IX, Section 7). Here again the lifting eliminates restrictive hypotheses (metrizability, for example) needed previously. For separable measurable modifications of empirical processes and regularizations of stochastic processes see in addition Talagrand (1987, 1988), respectively. Schreiber et al. (1971), use liftings for the construction of probability measures corresponding to stochastic processes. This construction applies algebraic models for probability spaces introduced before by Dinculeanu and Foiaş (1968). In these models liftings apply as well.

A. and C. Ionescu Tulcea (cf. (1969a)) applied lifting in order to describe the space of functionals on the space of Bochner integrable functions. Further applications of that type can be found in Dinculeanu (1967) and several papers concerning vectorial integration. For applications of the consistent lifting to the Pettis integral see Talagrand (1984).

C. Ionescu Tulcea (1965b) applies the strong lifting property and linear liftings for the decomposition of measures into its ergodic parts and in (1965) he uses the translation invariant lifting (see Theorem 5.4) to solve a problem on almost stable sets in locally compact groups posed by A.B. Simon.

Maréchal (1969) gives a definition of a measurable field of Hilbert spaces over a Hausdorff space based on the almost strong lifting property and shows that isomorphisms between measurable fields can be induced by pointwise isomorphisms. She gives decompositions of operators on Hilbert space by means of liftings in Maréchal (1968, 1969).

Von Weizsäcker (1978) applies liftings to the regularization of functions with Radon image measures and arbitrary completely regular range. Applications to subdifferentials and convex functions are given by Levin (1975).

Graf (1995) gives an application of the lifting to self-similar measures. A similar procedure was taken by Schief in an unpublished note.

Another application of the lifting theory is the existence and uniqueness of preimages of a given measure (see, e.g., Edgar (1976) as well Graf (1980a, 1982)). Also Lipecki (1998) used liftings for the extension of measures.

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