CHAPTER 12

Pettis Integral

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Contents

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Introduction

Until 1968, there has been no significant progress in the theory of the Pettis integral since Pettis's (1938) original paper. The Pettis integral was known to be countably additive and absolutely continuous. Due to Pettis (1938) it was also known that the space of Pettis integrable functions defined on the unit interval, integrated with respect to the Lebesgue measure may be non-complete in the semivariation norm. There were also known some facts concerning Pettis integrability of strongly measurable functions and a few examples of non-strongly measurable but Pettis integrable functions but almost nothing more. Then, Rybakov (1968) proved that the Pettis integral is of σ-finite variation. In 1975, Thomas presented some negative aspects of Pettis integrability. He proved that the space of Pettis integrable functions with respect to a nonatomic measure in never complete if the range space is infinite dimensional and that for each infinite dimensional Banach space X there exists an X-valued function on $[0, 1]^2$ such that not almost all its section are Pettis integrable on $[0, 1]$. Thomas also suggested that probably one could find a Pettis integrable function f on $[0, 1]$ such that $\lim_{h\to 0} \|\frac{1}{h}\int_s^{s+h} f(t)\,dt\| = \infty$ on a set of positive Lebesgue measure.

Then started a series of papers developing the theory of Pettis integration.

Edgar (1979) investigated Banach spaces with the property that each scalarly integrable function with values in the space is Pettis integrable (so-called PIP) and proved several non-trivial facts about these spaces. He was also the first to consider the Lindolöf property (so-called (C) property) for closed convex sets in the context of Pettis integration.

Fremlin and Talagrand (1979) constructed an example of a Pettis integrable function with non relatively norm compact range of its integral, answering a question of Grothendieck formulated in the language of operators. The same paper contains a proof of Stegall theorem saying that on perfect measure spaces the range of a Pettis integral is always norm relatively compact.

Musiał (1979) (preprint version was published in 1976) called attention to the problem of the existence of Pettis integrable Radon–Nikodým derivatives of Banach space valued measures proving that there is a strong connection between conjugate spaces with all measures of finite variation possessing Pettis integrable densities and Banach spaces not containing any isomorphic copy of l_1.

I have tried to include all papers concerning Pettis integrability, but the text presents my own point of view on the subject. As a result some papers are only mentioned without details. There are some topics completely absent in this survey. In particular I consider only Banach spaces, in spite of several papers dealing with locally convex spaces, where the notion of the Pettis integral is quite natural. I also left out of account the approach via Stone transform due to D. Sentilles, for the benefit of liftings, which seem to be more measure theoretical. That approach however was fully exploited by Talagrand (1984). I skipped also some important geometric properties of Banach spaces possessing WRNP type properties (containing trees, extreme points), mainly because of limited amount of space. Totally overlooked is the theory of strongly measurable functions, which is discussed in several other papers and books (cf. Diestel and Uhl (1977). I also decided do not write anything about Fremlin's generalization of McShane integral (Fremlin (1995)), which is intermediate between Bochner and Pettis integral, in spite of thinking that the integral of Fremlin may turn out to be more suitable for several applications than the Pettis integral. Also the approaches of Phillips (1940), Birkhoff (1935), Dobrakov (1970a) and Erben and Grimeisen (1990) are beyond this paper. Most of the problems posed in the survey paper Musiał (1991) are still open, so I do not formulate new ones.

1. Preliminaries

Throughout (Ω, Σ, μ) is a *finite complete* measure space with $\mu \neq 0$. If not stated differently, the conjectural measure space will always be denoted by (Ω, Σ, μ). Σ_μ^+ is the collection of all measurable sets of positive measure, $\mathcal{N}(\mu)$ is the σ-ideal of μ-null sets and μ^* is the outer measure generated by μ. ρ will denote a lifting on (Ω, Σ, μ) (for the detailed information concerning liftings see Strauss, Macheras and Musiał (2002) in this Handbook). $\mathcal{L}_0(\mu)$ is the space of all real-valued measurable functions on (Ω, Σ, μ), functions which coincide μ-a.e. are not identified. τ_p is the topology of pointwise convergence in the space of all real-valued functions. $\mathcal{L}_1(\mu)$ is the space of all real-valued μ-integrable functions. A set $\mathcal{Z} \subset \mathcal{L}_1(\mu)$ is uniformly integrable if $\lim_{\mu(A)\to 0} \sup_{f\in\mathcal{Z}} \int_A |f|\,d\mu = 0$. $\mathcal{L}$ is the σ-algebra of Lebesgue measurable sets on the unit interval $[0, 1]$ or on the real line and λ is the Lebesgue measure on $\mathcal{L}$. $\mathcal{P}(S)$ denotes the collection of all subsets of S. If Q is a non-empty set with a topology $\mathcal{T}$, then the σ-algebra of Borel sets on Q is denoted by $\mathcal{B}o(Q, \mathcal{T})$. $\mathcal{B}a(Q, \mathcal{T})$ is the σ-algebra of Baire sets. Compact spaces are always assumed to be non-empty and Hausdorff. X is always an *infinite dimensional Banach space* (unless otherwise stated), $B(X)$ is its closed unit ball and X^* is the conjugate of X. If $H \subset X$ then $H^\perp$ denotes the annihilator of H in X^*. If $L \subseteq X$ then $\overline{L}^{**}$ is the $\sigma(X^{**}, X^*)$-closure of L in X^{**} and $\operatorname{lin} L$ is the linear space generated by L in X. If $f : \Omega \to X$, then the composition of f with a functional x^* is denoted as x^*f or $\langle x^*, f\rangle$.

We say that X has Mazur's property if each sequentially weak*-continuous functional on X^* is in X. X has the property (C) or X is a Corson space if each family of closed convex subsets of X with the countable intersection property has a nonempty intersection (see Corson (1961), Pol (1980) and Drewnowski (1986)). X has the property (K) if each sequence of points $x_n \in X$ converging to zero contains a subsequence $\langle x_{n_k}\rangle$ with convergent series $\sum_{k=1}^\infty x_{n_k}$.

By $B_c(X^{**})$ we denote the set of all $x^{**} \in X^{**}$ which are weak*-cluster points of countable subsets of $B(X)$.

If $\nu : \Sigma \to X$ is a measure, then we say that ν is μ-continuous if $\lim_{\mu(A)\to 0} \|\nu(A)\| = 0$. ν is *dominated* by μ if there exists $M > 0$ such that $\|\nu(E)\| \leqslant M\mu(E)$ for all $E \in \Sigma$. As a rule we will assume in the proofs that $M = 1$. $|\nu|$ denotes the variation of ν and $cabv(\mu, X)$ is the Banach space of all μ-continuous X-valued measures of finite variation endowed with the variation norm. $ca(\mu, X)$ is the Banach space of all μ-continuous X-valued measures equipped with the semivariation norm.

If $\nu : \Sigma \to X$ is a μ-continuous measure, then for every $E \in \Sigma_\mu^+$ the set

$$\mathcal{A}_\nu(E) := \left\{ \frac{\nu(F)}{\mu(F)} : F \in \Sigma_\mu^+ \text{ and } F \subseteq E \right\}$$

is said to be the *average range* of ν on E.

AXIOM L (*cf. Fremlin and Talagrand* (1979)). [0, 1] cannot be covered by less then the continuum closed sets of the Lebesgue measure zero.

AXIOM K (*cf. Fremlin and Talagrand* (1979)). There is a cardinal κ possessing the following properties:

(a) there is a set $U \subset [0, 1]$ of cardinality κ such that $\lambda^*(U) = 1$;
(b) $[0, 1]$ is not a union of κ Lebesgue negligible sets.

It is known (cf. Fremlin and Talagrand (1979)) that Axiom L is a consequence of Martin's Axiom and Axiom K follows from the existence of real measurable cardinals.

2. Measurable functions

DEFINITION 2.1. Let Γ be a linear total subset of X^*. A function $f:\Omega \to X$ is said to be Γ-scalarly *μ-measurable*, if x^*f is μ-measurable for each $x^* \in \Gamma$. If $\Gamma = X^*$, then f is called *scalarly μ-measurable*. If $X = Y^*$ and $\Gamma = Y$ then f is called *weak* scalarly μ-measurable*. f is strongly measurable if there is a sequence of X-valued simple functions $f_n = \sum_{i=1}^{k(n)} x_{ni}\chi_{E_{ni}}$ with $E_{ni} \in \Sigma$, which is μ-a.e. converging to f. If μ is fixed the reference to it will be suppressed. This will concern also all further definitions.

The following theorem, due to Pettis (1938), explains the relationship between the strong and weak measurability.

THEOREM 2.2 (Pettis's measurability theorem). *A function $f:\Omega \to X$ is strongly μ-measurable if and only if*

(i) *f is scalarly μ-measurable, and*
(ii) *f is μ-essentially separably valued, i.e., there exists $E \in \mathcal{N}(\mu)$ such that $f(\Omega \setminus E)$ is a separable subset of X.*

DEFINITION 2.3. We say that two Γ-scalarly μ-measurable functions $f, g:\Omega \to X$ are Γ-scalarly *μ-equivalent* if $x^*f = x^*g$ μ-a.e. for each $x^* \in \Gamma$. If $\Gamma = X^*$, we say about *scalarly μ-equivalent* functions and, if $X = Y^*$ and $\Gamma = Y$ then f and g are said to be *weak** scalarly *μ-equivalent*. Two strongly measurable functions f and g are *μ-equivalent* if $f = g$ μ-a.e.

Below are presented some classical examples of Γ-measurable functions.

EXAMPLE 2.4. A *scalarly measurable function that is not strongly measurable but is scalarly equivalent to a strongly measurable function.* Let $\alpha:[0, 1] \to [0, \infty[$ be a function such that the set $H := \{t \in [0, 1]: \alpha(t) > 0\}$ is of positive Lebesgue outer measure and let $\{e_t: t \in [0, 1]\}$ be the canonical basis for the nonseparable Hilbert space $l_2([0, 1])$. Define $f_\alpha:[0, 1] \to l_2([0, 1])$ by $f_\alpha(t) = \alpha(t)e_t$. It is a consequence of the Riesz Representation Theorem that $x^*f = 0$ λ-a.e. for each $x^* \in l_2([0, 1])^*$ (i.e., f is scalarly λ-equivalent to the zero function). On the other hand, if $E \subset H$ is of positive outer measure then $f_\alpha(E)$ is nonseparable. In virtue of Theorem 2.2, f_α is not strongly λ-measurable. If α is non-measurable then also the function $\|f_\alpha\| : t \to \|f_\alpha(t)\|$ is not measurable.

As a specific example one can take $\alpha = \chi_V$, where V is the Vitali set.

EXAMPLE 2.5 (Ryll-Nardzewski). *A weak* scalarly measurable function that is not scalarly measurable and not weak* scalarly equivalent to any scalarly measurable function.* Define $f : [0, 1] \to C^*[0.1]$ by $f(s) = \delta_s$. f is obviously weak*-λ-measurable, since for $y \in C[0, 1]$, we have $\langle y, f(s)\rangle = y(s)$. To see that f is not scalarly measurable denote by μ_a the atomic part of $\mu \in C^*[0, 1]$, and let V be a non-λ-measurable subset of $[0, 1]$. Define $x^{**} \in C^{**}[0, 1]$ by $x^{**}(\mu) = \mu_a(V)$. Since $x^{**}(f) = \chi_V$ the function f is not scalarly measurable with respect to λ.

Since $C[0, 1]$ is separable, f is not weak* scalarly equivalent to any scalarly (hence also strongly) measurable function.

It is worth to notice that the norm of f is a measurable function.

EXAMPLE 2.6 (Hagler). *A scalarly measurable function that is not scalarly equivalent to a strongly measurable one.* Let (A_n) be a sequence of nonempty subintervals of $[0.1]$, such that:

(i) $A_1 = [0, 1]$,

(ii) $A_n = A_{2n} \cup A_{2n+1}$ for each $n \in \mathbb{N}$,

(iii) $A_i \cap A_j = 0$ if $i \neq j$ and $2^n \leqslant i, j < 2^{n+1}$,

(iv) $\lim_n \lambda(A_n) = 0$.

Define $f : [0, 1] \to l_\infty$ by $f(t) = (\chi_{A_n}(t))$ for $t \in [0, 1]$. Then f is scalarly measurable (cf. Diestel and Uhl (1977)).

To prove that f is not scalarly equivalent to a strongly measurable function it is enough to show that f itself is not strongly measurable (because l_1 is separable). This immediately follows from Pettis's Measurability Theorem. Indeed, if $\lambda(E) > 0$ and t, s are two distinct points of E then there is n such that $t \in A_n$ but $s \notin A_n$. Hence $\|f(t) - f(s)\| = 1$.

One can ask when a scalarly measurable function is scalarly equivalent to a strongly measurable one. The first global non-trivial result of this type is due to D.R. Lewis (see Stegall (1975/76a, 1975/76b)), who proved that scalarly measurable functions taking their values in a WCG space are scalarly equivalent to strongly measurable functions. Edgar (1977) has undertaken an effort to characterize the Banach spaces with the property that each X-valued scalarly measurable function is scalarly equivalent to a strongly measurable function. In case of a single function we have

THEOREM 2.7 (Edgar (1977)). *Let $f : \Omega \to X$ be a scalarly measurable function. Then, f is scalarly equivalent to a strongly measurable function if and only if the image measure $f(\mu) : \mathcal{B}a(X, weak) \to \mathbb{R}$ is tight in the weak topology (i.e., for each $\varepsilon > 0$ there is a weakly compact $W \subset X$ with $\mu^*(W) > \mu(X) - \varepsilon$).*

As a direct consequence of the above theorem, one obtains:

THEOREM 2.8 (Edgar (1977)). *Let X be a Banach space. Given any (Ω, Σ, μ), each scalarly measurable $f : \Omega \to X$ is scalarly equivalent to a strongly measurable function if and only if $(X, weak)$ is measure compact (i.e., each measure on $(X, weak)$ is τ-additive, what in the case of the weak topology means the tightness of each measure, that is given $\varepsilon > 0$ there is a norm compact $K \subset X$ with $\mu^*(K) > \mu(X) - \varepsilon$).*

DEFINITION 2.9. A function $f : \Omega \to X$ is *Γ-scalarly μ-bounded* provided there is $M \geqslant 0$ such that for each $x^* \in \Gamma$ the inequality $|x^* f| \leqslant M\|x^*\|$ holds μ-a.e. If $\Gamma = X^*$ then we say about *scalarly μ-bounded* function, and in the case of $X = Y^*$ and $\Gamma = Y$- about *weak* scalarly μ-bounded* function.

An easy calculation proves that if $f : \Omega \to X$ is strongly measurable and scalarly bounded, then it is *bounded* (i.e., there is $M > 0$ such that $\sup\{\|f(\omega)\|: \omega \in \Omega\} \leqslant M$ μ-a.e.).

The following fact (usually presented in the context of a family of measurable scalar functions) permits often to reduce the general situation to the case of scalarly bounded functions.

PROPOSITION 2.10 (cf. Musiał (1979)). *If $f : \Omega \to X$ is Γ-measurable then there exists a non-negative measurable function φ_f^Γ with the following properties:*

(i) *For each $x^* \in \Gamma$ we have $|\langle x^*, f(\omega)\rangle| \leqslant \varphi_f^\Gamma(\omega)\|x^*\|$ μ-a.e.,*

(ii) *$\varphi_f^\Gamma(\omega) \leqslant \|f(\omega)\|_\Gamma$ $(= \sup\{|\langle x^*, f(\omega)\rangle|: x^* \in \Gamma \cap B(X^*)\})$ μ-a.e.,*

(iii) *If $\phi : \Omega \to [0, \infty)$ is a measurable function satisfying* (i) *and* (ii) *(with φ_f^Γ replaced by ϕ), then $\varphi_f^\Gamma \leqslant \phi$ μ-a.e.*

PROOF. Consider the set $\Omega \times \mathbb{R}$ endowed with the σ-algebra $\sigma(\Sigma \times \mathcal{L})$, and the product measure $\mu \times \kappa$, where κ is any probability measure on $\mathcal{L}$ such that $\mathcal{N}(\kappa) = \mathcal{N}(\lambda)$. Let $S(x^*) = \{(\omega, s): |\langle x^*, f(\omega)\rangle| \geqslant s\|x^*\|\}$ for $x^* \in \Gamma$, and let $a = \sup\{(\mu \times \kappa)[\bigcup_{n=1}^\infty S(x_n^*)]: x_n^* \in \Gamma \cap B(X^*), n \in \mathbb{N}\}$. Since $a < \infty$ there are $x_1^*, x_2^*, \ldots \in \Gamma \cap B(X^*)$ such that $a = (\mu \times \kappa)[\bigcup_{n=1}^\infty S(x_n^*)]$. Now, it is enough to put $\varphi_f^\Gamma = \sup_n |x_n^* f|$, where the supremum is taken pointwise. □

COROLLARY 2.11. *If $f : \Omega \to X$ is Γ-scalarly measurable, then there exists a sequence of pairwise disjoint sets $E_n \in \Sigma$ covering Ω and such that for each n the function f is Γ-scalarly bounded on E_n.*

It is easy to give an example of a scalarly bounded function that is not bounded (e.g., setting $\alpha(t) = t$ in Example 2.4). It turns out however that the things can be more complicated. Edgar (see Talagrand (1984)) proved that there exists a scalarly bounded function which is even not scalarly equivalent to any bounded function. Weak*-scalarly bounded functions behave much better. To formulate the result we will introduce a notion that will be in constant use further.

Let f be an X^*-valued function which is weak* scalarly measurable and weak*-scalarly bounded. If ρ is a lifting on $L_\infty(\mu)$ then we denote by $\rho_0(f)$ the unique X^*-valued function defined by

$$\langle x, \rho_0(f)(\omega)\rangle := \rho(xf)(\omega),$$

where x and ω run across X and Ω, respectively.

It is known (see A. and C. Ionescu Tulcea (1969)) that $\rho_0(f)$ is weak*-Borel measurable (i.e., $\rho(f)^{-1}(B) \in \Sigma$ for all weak*-Borel $B \subset X^*$) and the measure $\xi_0 := \mu\rho_0(f)^{-1}$ is a Radon measure on the completion Ξ_f^0 of the σ-algebra of weak*-Borel subsets of X^* (see von Weizsäcker (1978) and Edgar (1978)). It is however totally not obvious whether $\rho_0(f)$ is in general scalarly measurable.

PROPOSITION 2.12. *If $f:\Omega \to X^*$ is weak*-scalarly bounded and weak*-scalarly measurable, then for an arbitrary lifting ρ the function f is weak*-scalarly equivalent to the bounded and weak* Borel measurable function $\rho_0(f)$.*

3. Scalar integrals, basic properties

DEFINITION 3.1. Let Γ be a linear total subspace of X^*. A function $f:\Omega \to X$ is *Γ-scalarly μ-integrable* if $x^*f \in L_1(\mu)$ for each $x^* \in \Gamma$. If $\Gamma = X^*$, then f is called *scalarly μ-integrable*, and in the case of $f:\Omega \to X^*$ and $\Gamma = X \subseteq X^{**}$, the function f is said to be *weak*-scalarly μ-integrable.*

DEFINITION 3.2. A Γ-scalarly μ-integrable $f:\Omega \to X$ is *Γ–μ-integrable* if for each $E \in \Sigma$ there exists $\nu_f(E) \in X$ such that

$$x^*\nu_f(E) = \int_E x^* f\,d\mu$$

for each $x^* \in \Gamma$. The set function $\nu_f:\Sigma \to X$ is called the *indefinite Γ-integral* of f with respect to μ, and $\nu_f(E)$ is called the *Γ-integral* of f over $E \in \Sigma$ with respect to μ. An X^*-integrable function is called *Pettis μ-integrable* and an X-integrable function (if $f:\Omega \to X^*$ and $\Gamma = X$) is called *weak* μ-integrable* (or Gelfand μ-integrable). The Gelfand integral of f will be denoted by $^*\nu_f$. If $f:\Omega \to X$ is considered as an X^{**}-valued function then its weak* integral in X^{**} is called the *Dunford integral* and it is denoted by ν_f^{**}. It is clear that each Γ-integral is uniquely determined and it is an additive set function (provided it exists). The Γ-integral is also a Γ-measure, i.e., $x^*\nu_f$ is σ-additive for each $x^* \in \Gamma$. Sometimes, we shall use the following notations: $P - \int_E f\,d\mu$, weak$^* - \int_E f\,d\mu$ and $D - \int_E f\,d\mu$.

If $f:\Omega \to X$ is scalarly μ-integrable, then an operator $T_f:X^* \to L_1(\mu)$ associated with f is defined by $T_f(x^*) = x^*f$.

From the integral point of view, the functions with the same indefinite Pettis integrals are non-distinguishable, they are scalarly equivalent. We shall denote by $\mathbb{P}(\mu, X)$ (or by $\mathbb{P}((\Omega, \Sigma, \mu), X)$ if necessary) the space of classes of scalarly μ-equivalent Pettis μ-integrable X-valued functions. $\mathbb{P}(\mu, X)$ is a linear space with ordinary algebraic operations. One defines a norm on $\mathbb{P}(\mu, X)$ by

$$|f| = \sup\left\{\int_\Omega |\langle x^*, f\rangle| \, d\mu \colon x^* \in B(X^*)\right\} = \sup\{|x^*\nu_f|(\Omega) \colon x^* \in B(X^*)\}.$$

It is known that

$$\sup\left\{\left\|\int_E f \, d\mu\right\| \colon E \in \Sigma\right\}$$

defines an equivalent norm on $\mathbb{P}(\mu, X)$.

We will investigate also the space of all Pettis integrable functions with norm relatively compact range of their integrals: $\mathbb{P}_c(\mu, X) := \{f \in \mathbb{P}(\mu, X) \colon \nu_f(\Sigma)$ is norm relatively compact$\}$. Besides $\mathbb{P}(\mu, X)$ we will be often considering its subsets $\mathbb{P}(\mu, K)$ consisting of functions taking their values in a set $K \subset X$.

It is one of the main problems in the theory of vector integration to find conditions guaranteeing the existence of the Pettis integral. We shall start with two classical results.

PROPOSITION 3.3 (Gelfand (1936)). *Each weak* scalarly μ-integrable $f : \Omega \to X^*$ is weak* μ-integrable.*

As an immediate corollary we get the following fact

PROPOSITION 3.4 (Dunford (1937)). *Each scalarly μ-integrable function $f : \Omega \to X$ is Dunford μ-integrable.*

If X is reflexive then the Dunford and Pettis integrals coincide. When X is not reflexive, this may not be the case.

EXAMPLE 3.5 (cf. Diestel and Uhl (1977)). *A Dunford integrable function that is not Pettis integrable.* Define $f : (0, 1] \to c_0$ by

$$f(t) = \left(2\chi_{(2^{-1},1]}(t), 2^2\chi_{(2^{-2},2^{-1}]}(t), \ldots, 2^n\chi_{(2^{-n},2^{-n+1}]}(t), \ldots\right).$$

If $x^* = (\alpha_1, \alpha_2, \ldots) \in l_1 = c^*$, then

$$x^*f = \sum_{n=1}^{\infty} \alpha_n 2^n \chi_{(2^{-n},2^{-n+1}]} \quad \text{and} \quad \int_0^1 |x^*f| \, d\lambda \leqslant \sum_{n=1}^{\infty} |\alpha_n| < \infty.$$

It follows from Proposition 3.4 that f is Dunford λ-integrable. On the other hand, it is easily seen that for each $E \in \mathcal{L}$

$$D - \int_E f \, d\lambda = \{2\lambda(E \cap (2^{-1}, 1]), \ldots, 2^n\lambda(E \cap (2^{-n}, 2^{-n+1}]), \ldots\}.$$

In particular

$$D-\int_{(0,1]} f\,d\lambda=(1,1,1,\ldots,1,\ldots)\neq c_0$$

and so f is not λ-Pettis integrable.

At this place one should recall that such a phenomenon cannot happen if X is a separable Banach space not containing any isomorphic copy of c_0. Unfortunatelly, the result does not extend in this form to non-separable Banach spaces (cf. Edgar (1979)).

THEOREM 3.6 (Dimitrov (1971), Diestel (1973)). *If X is a separable Banach space without an isomorphic copy of c_0, then each X-valued Dunford integrable function is Pettis integrable.*

The function f_α considered in Example 2.4 is Pettis integrable with $\nu_{f_\alpha}=0$.
But there are also non-trivial examples of Pettis integrable functions.

EXAMPLE 3.7. Let f be the function considered in Example 2.6. Since $\|f(t)\|\leqslant 1$ everywhere, f is Dunford integrable and for each $E\in\mathcal{L}$ and $\eta=\eta_1+\eta_2\in l_\infty^*=l_1\oplus c_0^\perp$, we have

$$\begin{aligned}\int_E\eta(f)\,d\lambda&=\int_E\eta_1(f)\,d\lambda+\int_E\eta_2(f)\,d\lambda=\int_E\sum_{n=1}^\infty\chi_{A_n}\eta_1(\{n\})\,d\lambda\\&=\sum_{n=1}^\infty\eta_1(\{n\})\lambda(E\cap A_n)=\langle\eta_1,\langle\lambda(E\cap A_n)\rangle\rangle=\langle\eta,\langle\lambda(E\cap A_n)\rangle\rangle.\end{aligned}$$

The last equality follows from the fact that $\lim_n\lambda(A_n)=0$ and so $\nu(E)=\langle\lambda(E\cap A_n)\rangle\in c_0$. But η_2 considered as a functional on l_∞ belongs to $c_0^\perp$ and so $\eta_2\nu(E)=0$ for each $E\in\Sigma$. It follows that f is Pettis λ-integrable.

In case of a quite arbitrary Γ nobody has been too much interested in describing the properties of the Γ-integral. The most interesting case is when $\Gamma=X^*$.

THEOREM 3.8. *If f is Pettis μ-integrable, then ν_f is a μ-continuous measure of σ-finite variation. Moreover, $|\nu_f|(E)=\int_E\varphi_f\,d\mu$ for each $E\in\Sigma$ (we put here φ_f instead of $\varphi_f^{X^*}$ for simplicity). In particular, the collection $\{x^*f\colon\ \|x^*\|\leqslant 1\}$ is uniformly μ-integrable.*

The σ-additivity and continuity of ν_f is due to Pettis (1938). The σ-finiteness of the variation was proved by Rybakov only in 1968.

REMARK 3.9. A result similar to that in Theorem 3.8 for an arbitrary total $\Gamma\subseteq X^*$ is false. If Γ is norming (i.e., for each $x\in X$ the equality $\|x\|=\sup\{|\langle x^*,x\rangle|\colon\ \|x^*\|\leqslant 1,$

$x^* \in \Gamma\}$ holds) and f is Γ-integrable, then $|\nu_f|$ is a σ-finite measure and $|\nu_f|(E) = \int_E \varphi_f^\Gamma \, d\mu$ for all $E \in \Sigma$ (see Musiał (1979)). In particular, if $f : \Omega \to X^*$ is weak*-bounded and satisfies for a lifting ρ the equations $xf = \rho(xf)$ for all $x \in X$, then $|\nu_f|(E) = \int_E \|f\| \, d\mu$ for all $E \in \Sigma$. This clearly generalizes the well known equality for the Bochner integral. It may however happen that ν_f is not countably additive in the norm topology of X, and it is not λ-continuous.

It is so for the function presented in Example 3.5, when considered as an l_∞-valued function. $\nu_f : \mathcal{L} \to l_\infty$ is weak*-countably additive but not countably additive in the norm topology and not λ-continuous.

4. Pettis integral

We shall start describing conditions equivalent to the Pettis integrability with a classical result that is always a starting point, when one wants to find conditions guaranteeing the Pettis integrability of a single function.

THEOREM 4.1. *Let $f : \Omega \to X$ be scalarly integrable. Then f is Pettis μ-integrable if and only if $T_f : X^* \to L_1(\mu)$ is weak*-weakly continuous if and only if $T_f : B(X^*) \to L_1(\mu)$ is weak*-weakly continuous.*

The following corollary is a simple consequence of Theorem 4.1.

COROLLARY 4.2. *A scalarly integrable $f : \Omega \to X$ is Pettis integrable if and only if the set $\{x^* \in X^*\colon\ x^*f = 0 \ \mu\text{-a.e.}\}$ is weak* closed.*

COROLLARY 4.3. *If $f : \Omega \to X$ is Pettis μ-integrable, then $T_f : X^* \to L_1(\mu)$ is weakly compact and the set $\mathcal{Z}_f := \{x^*f\colon\ \|x^*\| \leqslant 1\}$ is weakly closed in $L_1(\mu)$.*

To formulate the next characterization of Pettis integrable functions we are going to use an idea which is a generalization of Huff's (1986) conception of separable-like functions.

DEFINITION 4.4. A scalarly measurable function $f : \Omega \to X$ is determined by a space $Y \subseteq X$ if for every $x^* \in Y^\perp$ the equality $x^*f = 0$ holds μ-a.e.

The characterization presented below has been discovered by Drewnowski (1986) (without (iii)). Stefannson (1992) proved then the equivalence of (i) and (iii).

THEOREM 4.5. *If $f : \Omega \to X$ is scalarly integrable and the induced operator $T_f : B(X^*) \to L_1(\mu)$ is weakly compact, then the following conditions are equivalent:*

(i) *$f \in \mathbb{P}(\mu, X)$;*
(ii) *f is determined by a weakly compactly generated space $Y \subseteq X$;*
(iii) *f is determined by a space $Y \subseteq X$ possessing Mazur's property;*
(iv) *f is determined by a Corson space $Y \subseteq X$.*

PROOF (*Sketch*). If $f \in \mathbb{P}(\mu, X)$, then one can take as Y the space generated by $\nu_f(\Sigma)$. Since each WCG space has Mazur's property (cf. Diestel (1975, p. 148)) and is a Corson space, we have (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv). To prove (iii) $\Rightarrow$ (i) we are going to use a simple idea of Stefannson. Let $f : \Omega \to X$ be scalarly measurable and determined by a weakly compactly generated $Y \subseteq X$. First denote by T_Y the operator $T_Y : Y^* \to L_1(\mu)$ defined by $T_Y(y^*) := T_f(y^*_{\text{ext}})$, where y^*_{ext} is an arbitrary extension of y^* to the whole X. In a standard way one proves that T_Y is weak*-weakly continuous if and only if T_f is weak*-weakly continuous if and only if T_Y is $\sigma(X^*, Y)$-weakly continuous. Since T_f is weakly compact and f is determined by Y, one can prove, applying Mazur's theorem on the equality of weak and norm closures of convex sets that T_f is sequentially $\sigma(X^*, Y)$-weakly continuous. Hence T_Y is sequentially $\sigma(X^*, Y)$-weakly continuous. Now, since Y has Mazur's property and T_Y is sequentially $\sigma(X^*, Y)$-weakly continuous, we get the weak*-weak continuity of T_Y and hence the Pettis integrability of f. For the proof of (iv) $\Rightarrow$ (i) we refer to Drewnowski (1986). □

Notice that each of the conditions (ii)–(iv) of Theorem 4.5 can be equivalently written as $\nu_f^{**}(\Sigma) \subset \overline{Y}^{**}$, where Y is respectively WCG, Mazur or Corson. Example 3.5 shows that the weak compactness of T_f in the above theorem is necessary.

To formulate more general result, we need a notion introduced by Drewnowski (1986).

DEFINITION 4.6. If $\mathcal{C}_X$ is the collection of all Corson subspaces of X then the Corson envelope of X is defined by

$$\widetilde{X} := \bigcup_{Y \in \mathcal{C}_X} \overline{Y}^{**}.$$

Drewnowski (1986) strengthened the implication (iv) $\Rightarrow$ (i) of the above theorem to the following form:

THEOREM 4.7. *If $f : \Omega \to X$ is scalarly integrable, $T_f : X^* \to L_1(\mu)$ is weakly compact and $\nu_f^{**}(\Sigma) \subset \widetilde{X}$, then $f \in \mathbb{P}(\mu, X)$.*

Geitz (1981) introduced a notion related to Rieffel's essential range of strongly measurable function (Rieffel (1968)).

DEFINITION 4.8. Let $f : \Omega \to X$ be a function and let $E \in \Sigma$ be an arbitrary set. The *core* of f over E, denoted by $\operatorname{cor}_f(E)$, is the set defined by

$$\operatorname{cor}_f(E) := \bigcap \{ \overline{\operatorname{conv}} f(E \setminus N) : N \in \mathcal{N}(\mu) \}.$$

Geitz (1981) also noticed that if $f \in \mathbb{P}(\mu, X)$, then for each $E \in \Sigma_\mu^+$

$$\operatorname{cor}_f(E) = \overline{\operatorname{conv}} \mathcal{A}_{\nu_f}(E).$$

The following lemma is quite useful:

LEMMA 4.9. *If f is weakly measurable and* $\mathrm{cor}_f(E) \neq \emptyset$ *for all* $E \in \Sigma_\mu^+$, *then* $x^* f = 0$ *μ-a.e. if and only if* $x^* = 0$ *on* $\mathrm{cor}_f(\Omega)$.

Then Andrews (1985) introduced the *weak*-core* of a weak* scalarly measurable function $f : \Omega \to X^*$:

$$\mathrm{cor}_f^*(E) := \bigcap \{\overline{\mathrm{conv}}^* f(E \setminus N) \colon N \in \mathcal{N}(\mu)\}.$$

Drewnowski (1986) introduced the *weak**-core* (he called it the weak*-core) setting

$$\mathrm{cor}_f^{**}(E) = \bigcap \{\overline{\mathrm{conv}}^{**} f(E \setminus N) \colon N \in \mathcal{N}(\mu)\}$$

and proved that if $f \in \mathbb{P}(\mu, X)$, then

$$\mathrm{cor}_f^{**}(E) = \overline{\mathrm{cor}_f(E)}^{**} = \overline{\mathcal{A}_{\nu_f}(E)}^{**}.$$

THEOREM 4.10 (Talagrand (1984)). *Let $f : \Omega \to X$ be scalarly integrable. Then f is Pettis integrable if and only if $T_f : X^* \to L_1(\mu)$ is weakly compact and* $\mathrm{cor}_f(E) \neq \emptyset$ *for each* $E \in \Sigma_\mu^+$.

PROOF. Assume that $T_f : X^* \to L_1(\mu)$ is weakly compact, $\mathrm{cor}_f(E) \neq \emptyset$ for each $E \in \Sigma_\mu^+$ and $f \notin \mathbb{P}(\mu, X)$. Then, according to Lemma 5-1-2 of Talagrand (1984) (which remains true for functions with weakly compact T_f), there exist functionals $y^*, x^* \in B(X^*)$ such that $\mu\{\omega \in \Omega \colon y^* f(\omega) \neq z^* f(\omega)\} > 0$ and $y^* f$ is in the pointwise closure of the set

$$\mathcal{W}_f := \{x^* f \in \mathcal{Z}_f \colon x^* f = z^* f \ \mu\text{-a.e.}\}.$$

Let $\langle x_\alpha^* \rangle \subset \mathcal{W}_f$ be pointwise convergent to $y^* f$. Without loss of generality, we may assume that $x_\alpha^* \to y^*$ in $\sigma(X^*, X)$. Since for every α we have $x_\alpha^* f = z^* f$ μ-a.e., we have $x_\alpha^*|_{\mathrm{cor}_f(\Omega)} = z^*|_{\mathrm{cor}_f(\Omega)}$. Hence $y^*|_{\mathrm{cor}_f(\Omega)} = z^*|_{\mathrm{cor}_f(\Omega)}$ and consequently $y^* f = z^* f$ μ-a.e.

This however contradicts our assumption, and so $f \in \mathbb{P}(\mu, X)$. □

Another proofs of the above result were presented by Edgar (1982) and Huff (1986). Certainly the above theorem has been inspired by earlier characterizations of Pettis integrability obtained by Geitz (1982) in case of a perfect measure (that was a little bit more technical) and by Sentilles's (1981) characterization (also for perfect measures only) of Pettis integrability in terms of the Stonian transform.

Talagrand (1984) formulated also an equivalent condition guaranteeing Pettis integrability of f, which has been then reformulated and reproved by Drewnowski (1986) in the following form:

THEOREM 4.11 (Drewnowski (1986)). *Let $f : \Omega \to X$ be scalarly integrable. If* $\mathrm{cor}_f^{**}(E) \cap \widetilde{X} \neq \emptyset$ *for each* $E \in \Sigma_\mu^+$ *and T_f is weakly compact, then f is Pettis integrable.*

DEFINITION 4.12 (*Riddle, Saab and Uhl* (1983)). A family $\mathcal{H}$ of real-valued functions on (Ω, Σ, μ) has the *Bourgain property* if for each $E \in \Sigma_\mu^+$ and each $\varepsilon > 0$ there is a finite collection $\mathcal{F} \subseteq \mathcal{P}(E) \cap \Sigma_\mu^+$ such that for each function $h \in \mathcal{H}$ one can find $F \in \mathcal{F}$ with $\sup h(F) - \inf h(F) < \varepsilon$.

In case of a uniformly bounded $\mathcal{H}$ an equivalent definition is as follows: for each $E \in \Sigma_\mu^+$ and each pair $a < b$ of reals, there is a finite collection $\mathcal{F} \subseteq \mathcal{P}(E) \cap \Sigma_\mu^+$ such that for each function $h \in \mathcal{H}$ one can find $F \in \mathcal{F}$ with $\inf\{h(\omega)\colon \omega \in F\} \geqslant a$ or $\sup\{h(\omega)\colon \omega \in F\} \leqslant b$.

It is easy to see that the Bourgain property of $\mathcal{H}$ yields the same property of the pointwise closure of $\mathcal{H}$. Moreover the Bourgain property of a function constrains its measurability.

PROPOSITION 4.13 (Riddle, Saab and Uhl (1983)). *If $\mathcal{H}$ satisfies the Bourgain property, then each function in $\mathcal{H}$ is measurable and each function in the pointwise closure of $\mathcal{H}$ is the almost everywhere pointwise limit of a sequence from $\mathcal{H}$.*

PROOF. In order to prove the proposition take $f \in \overline{\mathcal{H}}^{\tau_p}$ and an ultrafilter $\mathcal{U}$ on $\mathcal{H}$ which has f as a cluster point. Then, put for $E \in \Sigma_\mu^+$ and $\varepsilon > 0$

$$\mathcal{H}(E, \varepsilon) = \{h \in \mathcal{H} : \sup h(E) - \inf h(E) < \varepsilon\}.$$

Each $E \in \Sigma_\mu^+$ contains $F \in \Sigma_\mu^+$ with $\mathcal{H}(F, \varepsilon) \in \mathcal{U}$. Using the Zorn–Kuratowski lemma, one can find for each positive ε a maximal family $\mathcal{P}_\varepsilon$ of pairwise disjoint sets in Σ_μ^+ such that $\mathcal{H}(F, \varepsilon) \in \mathcal{U}$ for each $F \in \mathcal{P}_\varepsilon$. It is obvious that $\mu(\Omega \setminus \bigcup_{F \in \mathcal{P}_\varepsilon} F) = 0$. Moreover, if $\mathcal{R}_\varepsilon$ is the family of all finite subcollections of $\mathcal{P}_\varepsilon$, then

$$f \in \overline{\bigcap_{\mathcal{R} \in \mathcal{R}_\varepsilon} \bigcap_{E \in \mathcal{R}} \mathcal{H}(E, \varepsilon)}^{\tau_p} \in \mathcal{U}.$$

Now let for each $m \in \mathbb{N}$ the sequence $\{A_{m,n}\colon n \in \mathbb{N}\}$ be an enumeration of $\mathcal{P}_{1/m}$ and let $\omega_{m,n} \in A_{m,n}$ be arbitrary. Taking for each $m \in \mathbb{N}$ a function

$$f_m \in \bigcap_{k=1}^{m} \bigcap_{n=1}^{m} \mathcal{H}(A_{k,n}; 1/k)$$

such that

$$|f_m(\omega_{k,n}) - f(\omega_{k,n})| < 1/k$$

for each $1 \leqslant k, n \leqslant m$, we get a sequence $(f_m) \subseteq \mathcal{H}$ that is μ-a.e. convergent to f. $\square$

DEFINITION 4.14. Let S be a topological space and let μ be a positive finite measure defined on a σ-algebra $\mathcal{B}$ containing all Borel subsets of S. μ is said to be selfsupporting, if for each $B \in \mathcal{B}_\mu^+$ there exists $A \in \mathcal{P}(B) \cap \mathcal{B}_\mu^+$ such that $\mu|A$ is strictly positive (i.e., for each open U, we have $U \cap A = \emptyset$ or $U \cap A \in \mathcal{B}_\mu^+$).

It is easily seen that if μ is selfsupporting, then for each $B \in \mathcal{B}_\mu^+$ there exists $A \in \mathcal{P}(B) \cap \mathcal{B}_\mu^+$ such that $\mu|A$ is strictly positive and $\mu(A) = \mu(B)$. We shall write $A = \operatorname{supp}(\mu|B)$.

It is well known that Radon measures are selfsupporting. Also if ρ is a lifting on Σ, $\mathcal{T}_\rho$ is the topology defined by taking as its basis, the family $\{A \in \Sigma\colon\ A \subseteq \rho(A)\}$ (cf. A. and C. Ionescu Tulcea (1969, p. 59)), then μ is selfsupporting with respect to $\mathcal{T}_\rho$.

PROPOSITION 4.15. *Let S be a topological space and let $\mathcal{H}$ be a uniformly bounded family of real-valued continuous functions on S. Moreover, let μ be a selfsupporting measure on a σ-algebra $\mathcal{B} \supseteq \mathcal{B}o(S)$. Then, if $\mathcal{H}$ does not contain any sequence equivalent to the standard unit vector basis of l_1 (in the sup norm), then $\mathcal{H}$ has the Bourgain property.*

PROOF. Suppose that $\mathcal{H}$ does not have the Bourgain property. Then, there exists $T \in \mathcal{B}_\mu^+$ such that $T = \operatorname{supp}(\mu|T)$ and there exist $a < b$ such that for each finite collection $\mathcal{R} \subseteq \mathcal{P}(T) \cap \mathcal{B}_\mu^+$ there is $f \in \mathcal{H}$ with $\inf f(R) < a$ and $\sup f(R) > b$, for every $R \in \mathcal{R}$.

We shall now construct inductively a collection $\{A_{n,m}\colon\ m = 1, \ldots, 2^n;\ n \in \mathbb{N}\}$ of sets from $\mathcal{B}_\mu^+$ and a sequence $(f_n) \subseteq \mathcal{H}$ satisfying the following properties:

$$A_{n+1,2m-1} \cup A_{n+1,2m} \subseteq A_{n,m},$$

$$f_{n+1}(s) < a \quad \text{if } s \in A_{n+1,2m-1},$$

$$f_{n+1}(s) > b \quad \text{if } s \in A_{n+1,2m}.$$

Assume, we have already constructed $\{f_m\colon\ m = 1, \ldots, k\}$ and $\{A_{n,m}\colon\ m = 1, \ldots, 2^n;\ n = 1, \ldots, k\}$. By the assumption, we can find for each $m \in \{1, \ldots, 2^k\}$ a set

$$T_{k,m} \in \mathcal{B}_\mu^+ \cap \mathcal{P}(A_{k,m})$$

such that $T_{k,m} = \operatorname{supp}(\mu|T_{k,m})$ (for the first inductive step set $T_{01} = T_{02} = T$). Moreover, there is $f_{k+1} \in \mathcal{H}$ with

$$\inf f_{k+1}(T_{k,m}) < a \quad \text{and} \quad \sup f_{k+1}(T_{k,m}) > b$$

for every $m \in \{1, \ldots, 2^k\}$.

Put now

$$A_{k+1,2m-1} = \{s \in T_{k,m}\colon\ f_{k+1}(s) < a\},$$

$$A_{k+1,2m} = \{s \in T_{k,m}\colon\ f_{k+1}(s) > b\}.$$

It follows that $A_{k+1,m} \in \mathcal{B}_\mu^+$ for every $m \in \{1, \ldots, 2^{k+1}\}$. Rosenthal's argument (1974) shows that the sequence (f_n) is equivalent to the standard basis of l_1 in the sup norm. □

DEFINITION 4.16. Let $K \subset X^*$ be a nonvoid set. A set $W \subseteq X$ is weakly precompact with respect to K (or weakly K-precompact) if each bounded sequence in $\langle x_n \rangle \subset W$ has a subsequence $\langle x_{n_k} \rangle$ such that for each $x^* \in K$ the sequence $\langle x^*, x_{n_k} \rangle$ is convergent. If $K = B(X^*)$, then W is called weakly precompact (equivalently, no bounded sequence in W is equivalent, in the norm topology, to the unit vector basis of l_1).

As particular instances of Proposition 4.15 we get the following results:

COROLLARY 4.17. *Let ρ be a lifting on $L_\infty(\Omega, \Sigma, \mu)$ and let $\mathcal{H}$ be a uniformly bounded family of real-valued functions defined on Ω and such that $\mathcal{H}$ is weakly precompact in $L_\infty(\mu)$. Then $\rho(\mathcal{H})$ has the Bourgain property.*

COROLLARY 4.18 (Riddle and Saab (1985)). *If $\{xf\colon \|x\| \leqslant 1\}$ has the Bourgain property, then $f \in \mathbb{P}(\mu, X^*)$ and the range of ν_f is norm relatively compact.*

PROOF. Fix $\varepsilon > 0$, $E \in \Sigma$ and $x^{**} \in B(X^{**})$. Then put

$$\mathcal{H} = \left\{\langle x, f\rangle\colon x \in B(X),\ \left|\langle x^{**} - x, {}^*\nu_f(E)\rangle\right| < \varepsilon\right\}.$$

Clearly $\mathcal{H}$ has the Bourgain property.

It follows from the Goldstine theorem that $\langle x^{**}, f\rangle$ is in the pointwise closure of $\mathcal{H}$ and so (Proposition 4.13) there are $x_n \in X$, $n \in \mathbb{N}$, such that $\langle x_n, f\rangle \in \mathcal{H}$ for each n and $\lim_n \langle x_n, f\rangle = \langle x^{**}, f\rangle$ μ-a.e.

Hence, we get

$$\left|\langle x^{**}, {}^*\nu_f(E)\rangle - \int_E \langle x^{**}, f\rangle\, d\mu\right| \leqslant 2\varepsilon$$

by the Lebesgue Convergence Theorem, and this proves that $f \in \mathbb{P}(\mu, X^*)$. The norm relative compactness of $\nu_f(\Sigma) = {}^*\nu_f(\Sigma)$ follows from the properties of T_f: if $\langle x_n\rangle$ is weakly convergent, then $\langle T_f(x_n)\rangle$ is a.e. convergent (cf. Riddle, Saab and Uhl (1983)). □

COROLLARY 4.19 (Haydon (1976)). *Let X be a Banach space containing no isomorphic copy of l_1. If μ is a complete finite Radon measure on $B(X^*)$ equipped with the weak* topology, then the identity function on $B(X^*)$ is μ-Pettis integrable.*

PROOF. In view of Proposition 4.15 the family $\mathcal{H} = B(X)$ has the Bourgain property. Hence Corollary 4.18 yields the Pettis integrability of the identity function. □

Among several questions concerning Pettis integrability the following seems to be quite interesting: assume that $f \in \mathbb{P}(\mu, X^*)$ is weak*-scalarly bounded and ρ is a lifting on $L_\infty(\mu)$. When is the function $\rho_0(f)\colon \Omega \to X^*$ Pettis integrable?

The next result is due to Talagrand (1984) but his proof is different from that presented here.

PROPOSITION 4.20. *Let $f\colon \Omega \to X^*$ be a weak*-scalarly bounded and weak*-scalarly measurable function. Assume that for each $\delta > 0$ there is $E \in \Sigma$ with $\mu(\Omega \setminus E) < \delta$ and such that the set $\{\langle f, x\rangle \chi_E\colon \|x\| \leqslant 1\}$ is weakly precompact in $L_\infty(\mu)$. Then $\rho_0(f) \in \mathbb{P}(\mu, X^*)$. In particular, if X does not contain any isomorphic copy of l_1, then $\rho_0(f) \in \mathbb{P}(\mu, X^*)$.*

PROOF. The assertion is an immediate consequence of Corollaries 4.17 and 4.18. □

As a consequence we get a result due to Riddle, Saab and Uhl (1983) and deduced by them from Theorem 9.10 via Theorem 9.7.

COROLLARY 4.21. *Let X be a separable Banach space and let $f:\Omega\to X^*$ be a weak*-scalarly bounded and weak*-scalarly measurable function. Assume that for each $\delta>0$ there is $E\in\Sigma$ with $\mu(\Omega\setminus E)<\delta$ and such that the set $\{\langle f,x\rangle\chi_E\colon \|x\|\leqslant 1\}$ is weakly precompact in $L_\infty(\mu)$. Then $f\in\mathbb{P}(\mu,X^*)$.*

THEOREM 4.22 (Macheras and Musiał (2000)). *Let $f:\Omega\to X^*$ be a scalarly bounded Pettis integrable function. If each $x^{**}\in X^{**}$ is ξ_0-measurable, then $\rho_0(f)\in\mathbb{P}(\mu,X^*)$.*

PROOF. According to Theorem 4.5 and the weak*-Borel measurability of $\rho_0(f)$ we have only to show that for an arbitrary $z\in\nu_f(\Sigma)^\perp$ the equality $\langle z,\rho_0(f)\rangle=0$ holds true μ-a.e. Suppose that there exists $z\in\nu_f(\Sigma)^\perp$ such that $\mu\{\omega\in\Omega\colon \langle z,\rho_0(f)(\omega)\rangle>0\}>0$ and $\|z\|=1$ and let K_f^0 be the the weak*-closure of $\rho_0(f)(\Omega)$ in X^*. Then, since ξ_0 is a Radon measure and $\xi_0(K_f^0)=\mu(\Omega)$, there exist a weak*-compact set $L\subseteq\{x^*\in K_f^0\colon \langle z,x^*\rangle>0\}$ and a positive real number a such that

$$\xi_0(L)>0,\quad z\text{ is continuous on }L\quad\text{and}\quad \langle z,x^*\rangle>a\text{ for each }x^*\in L.$$

Take now an arbitrary net $(x_\alpha)_{\alpha\in A}$ in $B(X)$ that is $\tau(X^{**},X^*)$-convergent to z. Then, the convergence to z is uniform on $\nu_f(\Sigma)$. To avoid unnecessary complications, we assume at once that the initial net $(x_\alpha)_{\alpha\in A}$ is Mackey convergent to z. Then, for each $n\in\mathbb{N}$ there exists $\alpha_n\in A$ such that

$$\left|\langle x_\alpha,\nu_f(E)\rangle\right|\leqslant 1/n\quad\text{for all }E\in\Sigma\text{ and all }\alpha\geqslant\alpha_n.$$

If $A_n:=\{\alpha\in A\colon \alpha\geqslant\alpha_n\}$, then for each collection of points $x_1^*,\dots,x_n^*\in L$ there is an index $\alpha_{x_1^*,\dots,x_n^*}\in A_n$ such that

$$\left|\langle z,x_i^*\rangle-\langle x_{\alpha_{x_1^*,\dots,x_n^*}},x_i^*\rangle\right|<1/n\quad\text{for each }i\leqslant n.$$

Equivalently,

$$L^n\subseteq\bigcup_{\alpha\in A_n}\left\{x^*\colon \left|\langle z,x^*\rangle-\langle x_\alpha,x^*\rangle\right|<1/n\right\}^n.$$

Now, as a consequence of the compactness of L and the continuity of $z|L$, there exists a finite set $B_n\subset A_n$ such that the inclusion

$$L^n\subseteq\bigcup_{\alpha\in B_n}\left\{x^*\colon \left|\langle z,x^*\rangle-\langle x_\alpha,x^*\rangle\right|<1/n\right\}^n$$

holds true. It follows that $z|L$ is a pointwise cluster point of the countable set $\{x_\alpha|L\colon x_\alpha\in\bigcup_{n=1}^\infty B_n\}$. Consequently, there exists $x_0^{**}\in X^{**}$ that is a weak*-cluster point of the set $\{x_\alpha\colon \alpha \in \bigcup_{n=1}^{\infty} B_n\}$ and $x_0^{**}|L = z|L$. It follows from the construction of x_0^{**} that $x_0^{**} \in \nu_f(\Sigma)^{\perp}$ and so

$$\int_{\rho_0(f)^{-1}(L)} \langle x_0^{**}, \rho_0(f)\rangle \, d\mu \geqslant a\mu\big(\rho_0(f)^{-1}(L)\big) > 0 = \big\langle x_0^{**}, \nu_f\big(\rho_0(f)^{-1}(L)\big)\big\rangle.$$

On the other hand, one can easily show that if $f : \Omega \to X^*$ is a scalarly bounded Pettis integrable function, then

$$\langle x^{**}, \nu_f(E)\rangle = \int_E \langle x^{**}, \rho_0(f)\rangle \, d\mu$$

for each $x^{**} \in B_c(X^{**})$ and each $E \in \Sigma$.

This gives the required contradiction. □

The next corollary is part of Proposition 4.20 but can be also derived from Theorem 4.22 if one takes into account Haydon (1976) who proved that if $l_1 \not\subset X$ isomorphically, then each functional $x^* \in X^*$ is universally measurable with respect to the weak* Borel structure of X^* (this is in fact contained in the assertion of Corollary 4.19).

COROLLARY 4.23. *If $l_1 \not\subset X$ isomorphically and $f : \Omega \to X^*$ is weak* scalarly measurable and weak* scalarly bounded, then for each lifting ρ the function $\rho_0(f)$ is Pettis integrable.*

Trying to overcome difficulties met while characterizing Pettis integrable functions Fremlin imposed an additional requirement besides the scalar integrability. The property was called by him "proper relative compactness" (Fremlin (1982)) and then "stability" by Talagrand (1984).

DEFINITION 4.24. Let $\mathcal{H}$ be a collection of real valued functions defined on Ω. $\mathcal{H}$ is said to be stable if for each $A \in \Sigma_\mu^+$ and arbitrary reals $\alpha < \beta$ there exist $k, l \in \mathbb{N}$ satisfying the inequality

$$\mu_{k+l}^*\bigg(\bigcup_{f\in\mathcal{H}} \{f < \alpha\}^k \times \{f > \beta\}^l \cap A^{k+l}\bigg) < \mu(A)^{k+l},$$

where μ_{k+l} is the direct product of $k + l$ copies of μ.

One may assume in the above definition that $k = l$. Moreover (Talagrand (1984)), if $\mathcal{H}$ is stable and pointwise bounded then it is pointwise relatively compact in $\mathcal{L}_0(\mu)$ and its pointwise closure is also stable.

DEFINITION 4.25 (*Talagrand* (1984)). A function $f : \Omega \to X$ is properly measurable if the set

$$\mathcal{Z}_f := \{x^*f\colon \|x^*\| \leqslant 1\}$$

is stable.

THEOREM 4.26 (Talagrand (1984)). *If $f:\Omega \to X$ is properly measurable and $\mathcal{Z}_f$ is uniformly integrable, then $f \in \mathbb{P}(\mu, X)$ and the range of ν_f is norm relatively compact.*

If μ is perfect, Axiom L holds and $B(X^*)$ is weak-separable, then each scalarly measurable $f:\Omega \to X$ is properly measurable (and hence it is Pettis integrable if $\mathcal{Z}_f$ is uniformly integrable).*

PROOF (*Sketch*). We have to prove only that the map $x^* \to x^*f$ from the unit ball of X^* into $L_1(\mu)$ is weak*-norm continuous. Let $(x_s^*f)_{s\in S}$ be a net that is pointwise convergent to x_0^*f and $\|x_s^*\| \leqslant 1$ for all $s \in S$. According to Talagrand (1984, Theorem 9-5-2), the net $(x_s^*f)_{s\in S}$ is convergent to x_0^*f in μ-measure. Because of the boundedness of the net in $L_\infty(\mu)$ the Lebesgue dominated convergence theorem may be applied, yielding the convergence in the norm of $L_1(\mu)$.

The second assertion is a direct consequence of Axiom L (Talagrand (1984, Theorem 9-3-3)). □

In case of a weak*-scalarly bounded and weak*-scalarly measurable function $f:\Omega \to X^*$ we have the following two results:

THEOREM 4.27. *Let $\Delta \subseteq X$ be a bounded norming subset of X and let $f:\Omega \to X^*$ be a function. If each countable subset of the set $\{xf\colon x \in \Delta\}$ is stable then for each lifting ρ on $L_\infty(\mu)$ the function $\rho_0(f)$ is an element of $\mathbb{P}(\mu, X^*)$ and the range of its Pettis integral is norm relatively compact. If ρ is consistent, then $\rho_0(f)$ is properly measurable.*

PROOF. If ρ is a lifting, then each countable subfamily of $\{x\rho_0(f)\colon x \in \Delta\}$ is stable. If moreover ρ is consistent, then the whole collection $\{x\rho_0(f)\colon x \in \Delta\}$ is stable (see Musiał (2000)). The Pettis integration of $\rho_0(f)$ and the norm relative compactness of its Pettis integral are now consequences of Theorem 4.26.

Let now σ be an arbitrary lifting on $L_\infty(\mu)$. Then again each countable subfamily of $\{x\sigma_0(f)\colon x \in \Delta\}$ is stable. Due to the countability of the set we may apply Lemma 9-3-2 of Talagrand (1984) to get the stability of each countable subset of Δ with respect to $\mu\sigma_0(f)^{-1}$ defined on the completion of weak*-Borel subsets of X^* with respect to $\mu\sigma_0(f)^{-1}$. But as all elements of Δ are weak*-continuous, we may apply Theorem 9-4-2 of Talagrand (1984) obtaining the $\mu\sigma_0(f)^{-1}$-measurability of all elements of X^{**} (notice that in the proof of the implication (c) $\Rightarrow$ (a) of 9-4-2 Axiom F is not applied). Since $\sigma_0(f)$ is weak*-scalarly measurable, we get the weak scalar measurability of $\sigma_0(f)$. The Pettis integrability of $\sigma_0(f)$ follows now from Theorem 4.22. Since the Pettis integral of $\sigma_0(f)$ coincides with that of $\rho_0(f)$, this completes the whole proof. □

THEOREM 4.28 (Talagrand (1984)). (Axiom L) *Let (Ω, Σ, μ) be perfect, $f:\Omega \to X^*$ be a function and let $\Delta \subseteq X$ be a bounded norming set. Moreover, let A_Δ be the union of the weak*-closures in X^{**} of countable subsets of Δ. Then the following conditions are equivalent:*

(a) *For each $x^{**} \in A_\Delta$ the function $x^{**}f$ is measurable;*
(b) *If ρ is a consistent lifting on $L_\infty(\mu)$, then $\rho_0(f)$ is properly measurable;*

(c) *If ρ is a lifting on $L_\infty(\mu)$, then $\rho_0(f) \in \mathbb{P}(\mu, X^*)$ and the range of its Pettis integral is norm relatively compact;*
(d) *If ρ is a lifting on $L_\infty(\mu)$, then $\rho_0(f)$ is scalarly measurable.*

PROOF. If μ is perfect, Axiom L holds true and (a) is satisfied, then each countable subset of $\{xf: x \in \Delta\}$ is stable by Talagrand (1984, Theorem 9-3-3). □

5. Limit theorems

It is the purpose of this section to prove the convergence theorems of Vitali and Lebesgue type for the Pettis integral. The theorems had been proved first by Geitz (1981) under the assumption of perfectness of the basic measure space and then they were proved in full generality by Musiał (1985). Geitz's proofs were based on a James's characterization of weakly compact subsets of a Banach space (see James (1964)) and on Fremlin's subsequence theorem (see Fremlin (1975). Applying the classical Mazur theorem on the equality of weak and norm closures of convex sets instead of Fremlin's theorem I was able to get rid of perfectness of the measure space (Musiał (1985)). Below I am presenting still another proof based on a Grothendieck characterization of weakly compact sets that seems to be more elementary than James's characterization (cf. Holmes (1975, p. 157 for a simple proof)). In fact one can obtain the result directly from Theorem 4.5 but the proof presented here seems to be more exciting.

PROPOSITION 5.1 (Grothendieck (1952)). *A bounded nonvoid set $W \subset X$ is relatively weakly compact if and only if for every two sequences $\langle x_n \rangle \subset W$ and $\langle x_n^* \rangle \subset B(X^*)$ the equality*

$$\lim_n \lim_m \langle x_m^*, x_n \rangle = \lim_m \lim_n \langle x_m^*, x_n \rangle$$

holds true, provided all the limits exist.

The theorem we are going to present is an analogue of Vitali's convergence theorem. Conditions (a) and (b) of this theorem guarantee that for each $x^* \in X^*$ and $E \in \Sigma$ the sequence $\{\int_E x^* f_n \, d\mu : n \in \mathbb{N}\}$ is convergent to $\int_E x^* f \, d\mu$, and that the set $\{x^* f: x^* \in B(X^*)\}$ is weakly relatively compact in $L_1(\mu)$. They may be replaced by any others guaranteeing the above weak compactness and the convergence of the appropriate sequence of scalar integrals.

THEOREM 5.2 (Geitz, Musiał). *Let $f : \Omega \to X$ be a function. If there exists a sequence of Pettis integrable functions $f_n : \Omega \to X$ such that:*
(a) *The set $\{x^* f_n : \|x^*\| \leqslant 1, n \in \mathbb{N}\}$ is uniformly μ-integrable,*
(b) *$\lim_n x^* f_n = x^* f$ in μ-measure, for each $x^* \in X^*$,*
then f is Pettis μ-integrable and,

$$\lim_n \int_E f_n \, d\mu = \int_E f \, d\mu$$

weakly in X, for each $E \in \Sigma$.

PROOF. Fix $E \in \Sigma$. Since the classical Vitali convergence theorem yields the convergence $\lim_n \int_E x^* f_n \, d\mu = \int_E x^* f \, d\mu$, for each $x \in X^*$, we see that the sequence $(\int_E f_n \, d\mu)$ is weakly Cauchy.

In order to prove our assertion, it is sufficient to show that given $\langle f_{n_m} \rangle_m$ and $\langle x_k^* \rangle_k \subset B(X^*)$, we have

$$\alpha := \lim_k \lim_m \left\langle x_k^*, \int_E f_{n_m} \, d\mu \right\rangle = \beta := \lim_m \lim_k \left\langle x_k^*, \int_E f_{n_m} \, d\mu \right\rangle,$$

provided all the limits exist.

It follows from (a) and (b) that the sequence $\langle x_k^*, f \rangle_k$ is uniformly integrable and bounded in $L_1(\mu)$. Hence it is weakly relatively compact. This yields the existence of a subsequence $\langle z_k^* \rangle_k$ of $\langle x_k^* \rangle_k$ and a function $h \in L_1(\mu)$ such that $z_k^* f \to h$ weakly in $L_1(\mu)$. Mazur's theorem yields now the existence of functionals $w_k^* \in \operatorname{conv}\{z_i^* \colon i \geqslant k\}$ such that

$$\lim_k \int_\Omega |w_k^* f - h| \, d\mu = 0 \quad \text{and} \quad \lim w_k^* f = h \ \mu\text{-a.e.}$$

If w_0^* is a weak*-cluster point of $\langle w_k^* \rangle_k$, then $h = w_0^* f$ μ-a.e. Consequently,

$$\alpha = \int_E w_0^* f \, d\mu.$$

On the other hand, we have

$$\begin{aligned} \lim_k \left\langle x_k^*, \int_E f_{n_m} \, d\mu \right\rangle &= \lim_k \left\langle z_k^*, \int_E f_{n_m} \, d\mu \right\rangle = \lim_k \left\langle w_k^*, \int_E f_{n_m} \, d\mu \right\rangle \\ &= \left\langle w_0^*, \int_E f_{n_m} \, d\mu \right\rangle = \int_E \langle w_0^*, f_{n_m} \rangle \, d\mu. \end{aligned}$$

It follows from the classical Vitali theorem that

$$\beta = \lim_m \int_E \langle w_0^*, f_{n_m} \rangle \, d\mu = \int_E \langle w_0^*, f \rangle \, d\mu.$$

This proves the required equality $\alpha = \beta$. □

As an immediate consequence of Theorem 5.2, we get the following generalization of the classical Lebesgue Dominated Convergence Theorem.

THEOREM 5.3 (Geitz, Musiał). *Let $f : \Omega \to X$ be a function satisfying the following two conditions*:

(α) *There exists a sequence of Pettis μ-integrable functions $f_n : \Omega \to X$ such that $\lim_n x^* f_n = x^* f$ in μ-measure, for each $x^* \in X^*$*,

(β) *There exists $h \in L_1(\mu)$ such that for each $x^* \in B(X^*)$ and each $n \in \mathbb{N}$, the inequality $|x^* f_n| \leqslant h$ holds μ-a.e. (the exceptional sets depend on x^*).*

Then $f \in \mathbb{P}(\mu, X)$ *and*

$$\lim_n \int_E f_n \, d\mu = \int_E f \, d\mu$$

weakly in X, *for all* $E \in \Sigma$.

PROPOSITION 5.4. *Assume that the assumptions of Theorem* 5.2 *or* 5.3 *are satisfied. Assume moreover that* μ *is perfect and the functions* f_n *are scalarly bounded. Then* $\lim_{n\to\infty} f_n = f$ *weakly in* $\mathbb{P}(\mu, X)$.

PROOF. According to Collins and Ruess (1983) if $\int_E x^* f \, d\mu \to \int_E x^* f \, d\mu$ for all $E \in \Sigma$ and all x^* being extreme points of $B(X^*)$, then the sequence $\langle f_n \rangle$ is weakly convergent to f in $\mathbb{P}(\mu, X)$. $\square$

It would be interesting to know whether the unit ball of X^* in the assumptions of Theorems 5.2 or 5.3 may be replaced by its extreme points.

It is natural to ask when the condition (α) of Theorem 5.3 is sufficient for the Pettis integrability of a function. If X contains an isomorphic copy of c_0, then (α) is too weak to guarantee the integrability. Indeed, let f be the function considered in Example 3.5. With the same notation, we have

$$x^* f = \sum_{n=1}^{\infty} \alpha_n 2^{-n} \chi_{(2^{-n}, 2^{-n+1}]}.$$

If

$$f_n(t) = \left(2\chi_{(2^{-1},1]}(t), \ldots, 2^n \chi_{(2^{-n},2^{-n+1}]}(t), 0, 0, 0, \ldots\right)$$

then clearly

$$\lim_n \int_E x^* f_n \, d\mu = \int_E x^* f \, d\mu$$

for all $x^* \in X^*$ and $E \in \Sigma$, but $f \notin \mathbb{P}(\mu, X)$

It turns out however that c_0 is the only exceptional Banach space.

THEOREM 5.5. *Let* X *be without any isomorphic copy of* c_0. *If* $f : \Omega \to X$ *is scalarly* μ*-integrable and there are functions* $f_n \in \mathbb{P}(\mu, X)$ *such that*

$$\lim_n \int_E x^* f_n \, d\mu = \int_E x^* f \, d\mu$$

for all $E \in \Sigma$ *and* $x^* \in X^*$, *then* $f \in \mathbb{P}(\mu, X)$ *and*

$$\lim_n \int_E f_n \, d\mu \in \int_E f \, d\mu$$

weakly in X *for all* $E \in \Sigma$.

PROOF. In virtue of Corollary 2.10 the set Ω can be decomposed into pairwise disjoint set $\Omega_n \in \Sigma$, such that for each $n \in \mathbb{N}$ and $x^* \in X^*$ the inequality $|x^* f \chi_{\Omega_n}| \leqslant n\|x^*\|$ holds μ-a.e. It follows from the comments preceding Theorem 5.2 that f is Pettis integrable on each Ω_n. Now it is sufficient to apply the series characterization of a Banach space not containing c_0 (Bessaga and Pełczyński (1958)). $\square$

Notice that if X is separable, then the assumption of scalar integrability itself yields the required assertion (Theorem 3.6).

6. The range of the Pettis integral

For several years it has been an open question of Pettis whether for each Pettis integrable function the range of ν_f is always norm relatively compact.

Fremlin and Talagrand (1979) gave a negative answer to that question. Before providing their example let us first present a positive result of Stegall (see Fremlin and Talagrand (1979)).

THEOREM 6.1 (Stegall). *If μ is perfect then for each scalarly integrable $f : \Omega \to X$ such that T_f is weakly compact, the range $\nu_f^{**}(\Sigma)$ of the Dunford integral of f in X^{**} is norm relatively compact.*

PROOF. We may assume that f is scalarly bounded. It is obvious that for each $E \in \Sigma$ the equality $T_f^* \chi_E = \nu_f^{**}(E)$ holds, so in order to prove the norm relative compactness of $\nu_f(\Sigma)$ it is sufficient to show the compactness of T_f. To do it, choose any sequence $(x_n) \subseteq B(X^*)$. Since f is scalarly measurable, $(x_n^* f)$ has a subsequence $(x_{n_k}^* f)$ that converges a.e. Otherwise, we could apply Fremlin's subsequence theorem (Fremlin (1975)), to get a subsequence without measurable cluster points, in the space of all real-valued functions endowed with the topology of pointwise convergence.

If x^* is a weak* cluster point of $\langle (x_{n_k}^*) \rangle$, then $x_{n_k}^* f \to x^* f$ pointwise, and hence in $L_1(\mu)$, because of the Lebesgue theorem.

Thus, T_f is compact, and the assertion is proved. $\square$

Since perfectness of the basic measure space in not necessary for a Pettis integral to have norm relatively compact range there is an obvious question when it can happen. It turns out that the answer depends on approximation of Pettis integrable functions by simple functions.

THEOREM 6.2 (Musiał (1985)). *If $f \in \mathbb{P}(\mu, X)$, then $\nu_f(\Sigma)$ is norm relatively compact if and only if f is a limit of a sequence of X-valued simple functions, in the norm topology of $\mathbb{P}(\mu, X)$.*

We say that X has the *μ-PCP* (*Pettis compactness property*) if for each $f \in \mathbb{P}(\mu, X)$ the set $\nu_f(\Sigma)$ is norm relatively compact. If this property is satisfied for an arbitrary (Ω, Σ, μ), then X has the PCP. No general description of PCP is known but there are interesting partial solutions. In particular we have the following result of Talagrand:

THEOREM 6.3 (Talagrand (1980)). (MA) *If l_∞ is not a quotient of X, then X has the* PCP.

Talagrand (1980) presents also an example of X possessing PCP and having l_∞ among its quotients.

More can be said about the following stronger property.

DEFINITION 6.4. A set $\emptyset \neq K \subseteq X$ has the *μ-Compact Range Property* (*CRP*) if every X-valued μ-continuous measure of finite variation with its average range contained in K has norm relatively compact range.

The global property in this context (i.e., CRP of X or, equivalently, of $B(X)$) has been studied by Musiał (1979). Then it was localized by Riddle, Saab and Uhl (1983). They called K to be a *set of complete continuity*.

DEFINITION 6.5 (*Riddle, Saab and Uhl* (1983)). A subset $K \neq \emptyset$ of a Banach space X is called a *μ-weak Radon–Nikodým set* (respectively, a *μ-weak**-RN set*) if for every μ-continuous measure of finite variation $\nu : \Sigma \to X$ satisfying for all sets $E \in \Sigma$ the inclusion $\mathcal{A}_\nu(E) \subseteq K$ there exists a K-valued (respectively, a $\overline{K}^{**}$-valued) Pettis integrable density of ν with respect to μ. In the above definitions the μ-continuity may be replaced by μ-domination. If K is a μ-weak Radon–Nikodým set for all finite complete μ, then it is called a weak Radon–Nikodým set. Similarly for μ-weak**-RN set. Sometimes we will say also that K has the weak (or weak**) Radon–Nikodým property. If $K = B(X)$ then we say about the weak RNP (or weak** RNP) of X.

WRNP was introduced in Musiał (1979) and W**RNP by Janicka (see Musiał (1980)). It is an immediate consequence of the complementability of X^* in X^{***} that X^* has WRNP if and only if it has W**RNP.

REMARK 6.6. $L_1[0,1]$ is an example of a Banach space without the W**RNP. Indeed, $L_1[0,1]$ is complementable in $L^{**}[0,1]$, so the W**RNP of $L_1[0,1]$ would imply the RNP of the space, and it is well known that $L_1[0,1]$ does not enjoy the last property.

Since each measure space can be embedded as a thick subset of a perfect measure space (cf. Musiał (1979)), we get the following conclusion from Theorem 6.1:

THEOREM 6.7 (Musiał (1979) in case of $K = X$). *If K has the W**RNP (in particular WRNP), then K has also the CRP.*

In case of the separable range of ν_f the complete characterization is given by the following theorem that has been obtained independently by Talagrand (1984) and Musiał (1985):

THEOREM 6.8. *If $f \in \mathbb{P}(\mu, X)$, then the following are equivalent:*

(i) *$\{x^*f \colon x^* \in B(X^*)\}$ is a separable subset of $L_1(\mu)$;*

(ii) *There exists a σ-algebra $\widetilde{\Sigma} \subseteq \Sigma$ such that $(\Omega, \widetilde{\Sigma}, \mu|\widetilde{\Sigma})$ is separable, and f is scalarly measurable with respect to $\widetilde{\Sigma}$;*

(iii) *There exists a sequence $\langle f_n \rangle$ of X-valued simple functions, such that $\{x^* f_n\colon n \in \mathbb{N}, \|x^*\| \leqslant 1\}$ is uniformly integrable and for each $x^* \in X^*$ the sequence $\langle x^* f_n \rangle_{n \in \mathbb{N}}$ is μ-a.e. convergent to $x^* f$;*

(iv) *There exists a sequence $\langle f_n \rangle$ of X-valued simple functions, such that for each $x^* \in X^*$ the sequence $\langle x^* f_n \rangle_{n \in \mathbb{N}}$ is convergent to $x^* f$ weakly in $L_1(\mu)$;*

(v) *$\nu_f(\Sigma)$ is a separable subset of X.*

PROOF. (i) $\Rightarrow$ (ii) Assume that the set $\{x^* f\colon x^* \in B(X^*)\}$ is separable. Then there exists a sequence $\langle x_n^* \rangle$ in $B(X^*)$, such that $\{x_n^* f\colon n \in \mathbb{N}\}$ is dense in $\{x^* f\colon x^* \in B(X^*)\}$. If $\widetilde{\Sigma}$ is the σ-algebra generated by all $x_n^* f$ and by $\mathcal{N}(\mu)$ then, clearly $\mu|\widetilde{\Sigma}$ is separable and each $x^* f$ is $\widetilde{\Sigma}$-measurable.

(ii) $\Rightarrow$ (iii) Assume that f is scalarly measurable with respect to a separable measure space $(\Omega, \Sigma_0, \mu|\Sigma_0)$ and let $\widetilde{\Sigma} = \sigma\{(E_n\colon n \in \mathbb{N}\}) \subseteq \Sigma_0$ be a countably generated σ-algebra that is $\mu|\Sigma_0$-dense in Σ_0. Moreover, let π_n be the partition of Ω generated by the sets $E_1, \ldots, E_n$.

Put for each n

$$f_n = \sum_{E \in \pi_n} \frac{\nu_f(E)}{\mu(E)} \chi_E$$

with the convention $0/0 = 0$.

Since $\{x^* f\colon \|x^*\| \leqslant 1\}$ is uniformly integrable, this yields the uniform integrability of $\{x^* f_n\colon n \in \mathbb{N}, \|x^*\| \leqslant 1\}$. As by the assumption $\widetilde{\Sigma}$ is dense in Σ_0, we have $E(x^* f|\widetilde{\Sigma}) = x^* f$ μ-a.e., and so $\lim_n x^* f_n = x^* f$ μ-a.e..

(iv) $\Rightarrow$ (v) The condition (iv) means that for each $E \in \Sigma$ the sequence $(\nu_{f_n}(E))$ is weakly convergent to $\nu_f(E)$. Hence, $\nu_f(\Sigma)$ is contained in the weak closure of the set $\bigcup_{n=1}^{\infty} \nu_{f_n}(\Sigma)$ and the last set is separable, since the ranges of all ν_{f_n}-s are finite dimensional.

(v) $\Rightarrow$ (i) Suppose that $\{x^* f\colon x^* \in B(X^*)$ is non-separable and take an arbitrary $x_1^* \in S(X^*)$ and $h_1 \in L_\infty(\mu)$ such that $\langle h_1, x_1^* f \rangle = 1$. Assume then that we have already constructed for an ordinal $\beta < \omega_1$ a family $\{(x_\alpha^*, h_\alpha)\colon \alpha < \beta\}$ with the following properties:

(α) $x_\alpha^* \in S(x^*)$,

(β) $h_\alpha \in L_\infty(\mu)$,

(γ) $x_\gamma^* f \in \overline{\text{lin}}\{x_\alpha^* f\colon \alpha < \gamma\}$ for each $\gamma < \beta$,

(δ) $\langle h_\gamma, x_\alpha^* f \rangle = \begin{cases} 1 & \text{if } \alpha = \gamma < \beta, \\ 0 & \text{if } \alpha < \gamma < \beta. \end{cases}$

Since $\{x^* f\colon x^* \in B(X^*)\}$ is non-separable, one can find $x_\beta^* \in S(X^*)$ such that $x_\beta^* f \notin \overline{\text{lin}}\{x_\alpha^* f\colon \alpha < \beta\}$. Then, applying the Hahn–Banach theorem we get $h_\beta \in L_\infty(\mu)$ such that $\langle h_\beta, x_\beta^* f \rangle = 1$ and $\langle h_\beta, x_\alpha^* f \rangle = 1$ for all $\alpha < \beta$.

Consequently, we get a net $\{(x_\alpha^*, h_\alpha)\colon \alpha < \omega_1\}$ satisfying (α)–(δ) for all α, β, γ less than ω_1.

It follows that

$$\|T_f^*(h_\beta) - T_f^*(h_\alpha)\| \geqslant 1$$

whenever $\alpha < \beta$, and so the set $T_f^*(L_\infty(\mu))$ is non-separable in X^{**}. But $\mathrm{lin}\{\chi_E\colon E \in \Sigma\}$ is norm dense in $L_\infty(\mu)$ and so $\mathrm{lin}\,\nu(\Sigma)$ is norm dense in $T_f^*(L_\infty(\mu))$. It follows that $\nu(\Sigma)$ is non-separable. □

Stefansson (1992) noticed that one can add an additional equivalent condition to Theorem 6.8:

f is determined by a separable space.

Combining Theorem 5.2 with Theorem 6.8 we get the following characterization of Pettis integrability:

THEOREM 6.9 (Geitz, Musiał). *Let $f:\Omega \to X$ be a function. Then, $f \in \mathbb{P}(\mu, X)$ and $\nu_f(\Sigma)$ is a separable set if and only if there exists a sequence $\langle f_n\rangle$ of X-valued simple functions, such that:*

(j) *The family $\{x^* f_n\colon n \in \mathbb{N},\ x^* \in B(X^*)\}$ is uniformly integrable,*

(jj) *For each $x^* \in X^*$ we have $\lim_n x^* f_n = x^* f$ μ-a.e.*

If f is scalarly bounded, then $\langle f_n\rangle$ can be taken to be bounded (i.e., $\sup_n \|f_n(\omega)\| \leqslant M$ μ-a.e.).

The first result of the above type was obtained by Geitz (1981) who proved it in case of a perfect measure μ. It was then generalized by Musiał (1985).

Plebanek (1993) introduced the following notion: a weak Baire measure μ on X (i.e., a measure on the σ-algebra of Baire sets in the weak topology of a Banach space X) is scalarly concentrated on a subspace Y of X if $x^*|Y = 0$ implies $x^* = 0$ μ-a.e., for all functionals $x^* \in X^*$. Then he proved the following characterization of Pettis integrable functions with separable range of their integrals:

PROPOSITION 6.10. *If $f:\Omega \to X$ is scalarly bounded and scalarly measurable then $f \in \mathbb{P}(\mu, X)$ and $\nu_f(\Sigma)$ is separable if and only if the measure $f(\mu)$ is scalarly concentrated on a separable subspace of X.*

In the particular case of spaces of continuous functions on compact spaces a few further results are known. Rosenthal (1970, Theorem 4.5) proved that if (and only if) a compact space K satisfies the countable chain condition (CCC), then all weakly compact subsets of $C(K)$ are separable. Hence all $C(K)$-valued Pettis integrals have then norm separable ranges. In particular it is so in case of K carrying a strictly positive Radon measure. Another type of a sufficient condition was formulated by Plebanek (1993).

PROPOSITION 6.11. *If every sequentially continuous function $f: K \to \mathbb{R}$ is continuous, then every Pettis integral of a $C(K)$-valued function has a separable range.*

EXAMPLE 6.12 (*Fremlin and Talagrand* (1979)). Let I be an arbitrary nonvoid set and let μ be the product measure on $K_I := \{0, 1\}^I$ of the measure $1/2\delta_{\{0\}} + 1/2\delta_{\{1\}}$. If Σ is the completion of the product σ-algebra, then let $\overline{\Sigma}$ be the σ-algebra on K_I defined in the following way: $E \in \overline{\Sigma}$ if and only if there exists a free non-measurable filter W on I and a set $F \in \Sigma$ such that $E \cap W = F \cap W$. It can be proved then that there is a unique extension of μ to a complete measure $\overline{\mu}$ on $\overline{\Sigma}$ (cf. Talagrand (1984, Theorem 13-2-1)). Let φ be the canonical injection of K_I into $l_\infty(I)$. Then, φ is scalarly $\overline{\mu}$-integrable but it is not $\overline{\mu}$-Pettis integrable.

The detailed proof of these facts can be found in Fremlin and Talagrand (1979) or in Talagrand (1984, Chapter 13).

EXAMPLE 6.13 (*Fremlin and Talagrand* (1979)). Let $\overline{\mu}$ be the measure described in Example 6.12 and let $\kappa = \overline{\mu} \mathbin{\widehat{\otimes}} \overline{\mu}$ be its complete product on $K_I \times K_I$. If $f : K_I \times K_I \to l_\infty(I)$ is defined by

$$f(x, y) = \varphi(x) - \varphi(y)$$

then $f \in \mathbb{P}(\kappa, l_\infty(I))$, but $\nu_f(\overline{\Sigma} \mathbin{\widehat{\otimes}} \overline{\Sigma})$ is not relatively compact if I is infinite, and non-separable if I is uncountable.

7. Universal integrability

DEFINITION 7.1. Let $K \neq \emptyset$ be a compact space. A function $h : K \to \mathbb{R}$ is said to be universally measurable if it is measurable with respect to the completion of each Radon measure defined on K. $f : K \to X$ is Γ-universally scalarly measurable if $x^* f$ is universally measurable for every $x^* \in \Gamma \subset X^*$.

THEOREM 7.2 (Riddle, Saab and Uhl (1983)). *Let μ be a Radon measure on a compact space K and let $f : K \to X^*$ be a scalarly bounded and scalarly universally measurable function. If X is weakly compactly generated and f takes its values in a weak*-separable subspace of X^*, then $f \in \mathbb{P}(\mu, X^*)$.*

PROOF. Assume first the separability of X and let $\delta > 0$ be arbitrary. Since X is separable, there exists a compact set $L \subset K$ such that $\mu(K \setminus L) < \delta$ and $\langle f, x\rangle$ is continuous on L for each $x \in X$. Let

$$A = \{\langle f, x\rangle|_L : \ \|x\| \leqslant 1\}$$

and $M_r(L)$ be the set of all real-valued universally measurable functions on L equipped with the pointwise convergence topology. As f is universally measurable, the set A is relatively compact in $M_r(L)$. According to Theorem 2F of Bourgain, Fremlin and Talagrand (1978), every sequence in A has a pointwise convergent subsequence and so, it is weakly precompact in $C(L)$. A direct application of Rosenthal's theorem (Rosenthal (1974)) says that A contains no copy of the standard unit vector basis of l_1. Since, the canonical embedding of $C(L)$ into $L_\infty(K,\mu)$ is a contraction, the set $\{\langle f,x\rangle\chi_L\colon \|x\|\leqslant 1\}$ contains no copy of the l_1-basis in the $L_\infty(K,\mu)$-norm either. Thus, it is weakly precompact and Lemma 4.20 completes the proof. The non-separable case is a consequence of separable complementability of X. □

Assuming the continuum hypothesis, Plebanek (1998) proved that the weak* separability of the range cannot be omitted. He proved namely that if μ is the Haar measure on 2^{ω_1}, then $L_1(\mu)$ is a non-separable WCG space and there is a bounded function $f:[0,1]\to L_\infty(\mu)$ such that $\langle x^*, f\rangle$ is Borel measurable for every $x^*\in L_\infty(\mu)^*$ and $f\notin \mathbb{P}(\mu, L_\infty(\mu))$.

Applying some consequences of Proposition 4.20 Bator (1988a) and Stefansson (1995) have got the following decomposition properties of universally measurable functions:

PROPOSITION 7.3. *Let $K\neq\emptyset$ be a compact space and let $f:K\to X^*$ be bounded and universally scalarly measurable. Then:*

(a) (Bator) *for each Radon measure μ on K there exist functions g and h such that g has the Bourgain property for μ, h is weak* scalarly μ-null and $f=g+h$;*
(b) (Stefansson) *if X is a WCG space, then for each Radon measure μ on K there exist functions f_1 and f_2 such that f_1 is universally Pettis integrable, f_2 is weak* scalarly equivalent to zero and $f=f_1+f_2$.*

As an immediate consequence of Proposition 7.3 and Corollary 4.17 we get the following result:

COROLLARY 7.4. *Let μ be a Radon measure on a compact space K and let $f:K\to X^*$ be bounded and universally scalarly measurable. Then for each lifting ρ on $L_\infty(\mu)$ the function $\rho_0(f)$ is Pettis μ-integrable.*

The following result of E. Saag generalizes Corollary 4.19 of Haydon (1976).

PROPOSITION 7.5. *Let $\emptyset\neq K\subset X^*$ be a convex weak*-compact set equipped with the weak* topology. If $B(X)$ is weakly precompact with respect to K, then the identity function on K is universally Pettis integrable and the integrals take their values in K.*

PROOF. In view of Proposition 4.15 if μ is a Radon measure on K, then the family $\mathcal{H}=B(X)$ has the Bourgain property. Hence Corollary 4.18 yields the Pettis integrability of the identity function. The Hahn–Banach theorem applied to the weak* topology of X^* proves that the integral takes its values in K. □

8. Pettis integral property

DEFINITION 8.1. X has the μ-Pettis integral property if each X-valued, scalarly bounded and scalarly μ-measurable function is μ-Pettis integrable. If such a property holds true for all complete measures, then X is said to have the Pettis Integral Property (PIP). If μ

is the Lebesgue measure on the unit interval, then we say about the Lebesgue PIP. If X has the μ-PIP with respect to all Radon measures, then it is said to have the universal PIP. A measure space (Ω, Σ, μ) has PIP if for every Banach space X each bounded scalarly μ-measurable X-valued function is Pettis integrable.

X has the universal PIP if for every compact K each bounded universally scalarly measurable function $f : K \to X$ is universally Pettis integrable.

The first example of a Banach space without the Lebesgue PIP was given by Pettis (1938). He constructed an example of a bounded scalarly Borel measurable $f : [0, 1] \to l_\infty[0, 1]$ that is not λ-Pettis integrable. Then, Edgar (1979), assuming (CH), published an example of $C[0, \omega_1]$-valued function, which is also not λ-Pettis integrable.

No complete characterization of Banach spaces possessing PIP is known, but there are several partial answers.

THEOREM 8.2 (Fremlin and Talagrand (1979)). *If Axiom K is true, then every Banach space has the Lebesgue PIP. If either Axiom K or Axiom L is true, then l_∞ has the μ-PIP for every perfect μ.*

As observed by Plebanek (1993) the original proof of the first part of the above result gives in fact more. To formulate it let us assume that μ is non-atomic and denote by $\mathrm{non}(\mathcal{N}_\mu)$ the minimal cardinality of a set $\Theta \subset \Omega$ which is not in $\mathcal{N}_\mu$. Then, let $\mathrm{cov}(\mathcal{N}_\mu)$ be the minimal cardinality of a subfamily of $\mathcal{N}_\mu$ covering Ω. Moreover, let λ_κ be the standard product measure on 2^κ.

THEOREM 8.3. *If* $\mathrm{non}(\mathcal{N}_{\lambda_\kappa}) < \mathrm{cov}(\mathcal{N}_{\lambda_\kappa})$, *then the measure λ_κ has PIP.*

The main positive result is an immediate consequence of Theorem 4.10.

THEOREM 8.4 (Talagrand (1984)). *If X is Corson, then X has PIP.*

As a direct consequence of Theorem 2.8 we get PIP of measure compact spaces (Edgar (1979)). Since each weakly countably determined space X (see Vašák (1981)) (in particular every K-analytic or WCG) is Lindelöf in the weak topology of X (Vašák (1981)), and every Lindelöf space is measure compact (see Varadarajan (1961)), all such spaces have PIP (Edgar (1979)). If card $\Gamma = \aleph_1$ then, $l_1(\Gamma)$ is measure compact, hence it has PIP. On the other hand, $l_1(\aleph_1)$ is not Corson (see Edgar (1979), where several other important observations concerning PIP can be found).

It turns out also that there is a strong connection between PIP and real-valued measurable cardinals.

THEOREM 8.5 (Edgar (1979)). *$l_1(\Gamma)$ has PIP if and only if* card Γ *is not a real-valued measurable cardinal.*

THEOREM 8.6 (Andrews (1985)). *If the least real-measurable cardinal is not less than the continuum, then $l_1(\Gamma)$ has UPIP for all Γ.*

Andrews (1985) presented also a few other conditions guaranteeing UPIP of conjugate Banach spaces with metrizable weak* compact and separable subsets.

Edgar (1979) observed also that Mazur's property implies PIP, and since each X with angelic $(X^*, weak^*)$ is Mazur, also such spaces possesses PIP. As noticed by Plebanek (1993), if K is a first countable compact space, then $C(K)$ is Mazur. Consequently, we get the following result.

THEOREM 8.7 (Plebanek (1993)). *If K is a first countable compact space, then $C(K)$ has PIP and every $C(K)$-valued Pettis integral has a separable range.*

In case of UPIP certainly the most important result has been obtained by Riddle, Saab and Uhl and it is simply a reformulation of the main part of Theorem 7.2.

THEOREM 8.8 (Riddle, Saab and Uhl (1983)). *If X is separable, then X^* has UPIP.*

No general characterization of UPIP even for conjugate spaces is known.

9. Weak Radon–Nikodým property and related properties

The following two results due, independently, to Dinculeanu (1967) in case of Banach space valued measures of finite variation and Rybakov (1968) in case of σ-finite variation are the starting points of all non-separable Radon–Nikodým type theorems.

THEOREM 9.1. *Let $\nu : \Sigma \to X^*$ be a weak* measure. If $|\nu|$ is a σ-finite measure, such that $\mathcal{N}(\mu) \subseteq \mathcal{N}(|\nu|)$, then there exists a weak* scalarly integrable function $f : \Omega \to X^*$ such that*

$$f(\Omega) \subseteq \overline{\text{conv}}^* \mathcal{A}_\nu(\Omega) \quad \text{and} \quad \langle x, \nu(E) \rangle = \int_E \langle x, f \rangle \, d\mu$$

for each $x \in X$ and each $E \in \Sigma$.

If μ dominates ν and ρ is a lifting on $L_\infty(\mu)$ then f can be chosen to satisfy also the equality $\langle x, f \rangle = \rho(\langle x, f \rangle)$ for all $x \in X$.

PROOF. Assume first that ν satisfies the inequality $|\nu|(E) \leqslant M\mu(E)$ for all $E \in \Sigma$. Denote for $x \in X$ by f_x the Radon–Nikodým derivative of the measure $\langle x, \nu \rangle$ with respect to μ. Clearly, $|f_x| \leqslant M$ μ-a.e., and so $|\rho(f_x)| \leqslant M$ everywhere.

Defining $f : \Omega \to X^*$ by

$$\langle x, f(\omega) \rangle = \rho(f_x)(\omega)$$

for each $\omega \in \Omega$ and $x \in X$ we get the equality

$$\int_E \langle x, f \rangle \, d\mu = \int_E \rho(f_x) \, d\mu = \int_E f_x \, d\mu = \langle x, \nu(E) \rangle$$

for each $E \in \Sigma$ and each $x \in X$.

Consider now the family $\widetilde{\Pi}$ of all finite partitions π of Ω such that if $\pi = \{E_1, \dots, E_n\}$ then $\rho(E_i) = E_i$ for all $i \leqslant n$. We say that $\pi_1 \preceq \pi_2$ if each element of π_1 is a union of elements of π_2. Then, let

$$f_\pi := \sum_{E \in \pi} \frac{\nu(E)}{\mu(E)} \chi_E.$$

Applying Lemma VI.8.3 of Dunford and Schwartz (1958), we see that $xf(\omega) = \lim_{\pi \in \widetilde{\Pi}} xf_\pi(\omega)$ for all $\omega \in \Omega$ and all $x \in X$ (cf. Kupka (1972)).

The general case follows by decomposing of Ω into pairwise disjoint sequence of sets $\Omega_n \in \Sigma$, such that $|\nu|(E \cap \Omega_n) \leqslant n\,\mu(E \cap \Omega_n)$, for all $E \in \Sigma$ and $n \in \mathbb{N}$. □

The property saying that for each X^*-valued measure ν of σ-finite variation that is μ-continuous there is a weak* measurable $f : \Omega \to X^*$ such that

$$\langle x, \nu(E) \rangle = \int_E \langle x, f \rangle \, d\mu \tag{1}$$

for each $E \in \Sigma$ and $x \in X$, may be called the *μ-weak* Radon–Nikodým Property* (μ-W*RNP). Without loss of generality one may assume that ν is μ dominated. So it is a consequence of Theorem 9.1 that each conjugate Banach space has the W*RNP (please notice that we use this name in a different meaning than it is used in Talagrand (1984)). f will be called a *weak* density* (*or a weak* Radon–Nikodým derivative*) of ν with respect to μ.

More generally, a set $\emptyset \neq K \subseteq X^*$ has weak* RNP if for each (Ω, Σ, μ) and each $\nu : \Sigma \to X^*$ satisfying $\nu(E) \in \mu(E) \cdot K$ for every $E \in \Sigma$ there is $f : \Omega \to K$ such that (1) is satisfied. Then weak* compact convex subsets of X^* have W*RNP.

As a consequence of Theorem 9.1 we obtain the following result:

THEOREM 9.2. *Let $\nu : \Sigma \to X$ be a μ-continuous measure of σ-finite variation. Then, there exists a weak* measurable $f : \Omega \to X^{**}$ such that*

$$f(\Omega) \subseteq \overline{\text{conv}}^{**} \mathcal{A}_\nu(\Omega) \quad \textit{and} \quad \langle x^*, \nu(E) \rangle = \int_E \langle x^*, f \rangle \, d\mu$$

for each $x^* \in X^*$ and $E \in \Sigma$. If μ dominates ν and ρ is a lifting on $L_\infty(\mu)$ then f can be chosen to satisfy also the equality $\langle x^*, f \rangle = \rho(\langle x^*, f \rangle)$ for all $x^* \in X^*$.

It is interesting and useful to know that the W**RNP and the WRNP are determined by a single measure space.

THEOREM 9.3 (Musiał (1982) if $K = B(X)$). *Let $K = \overline{\text{conv}}\, K \neq \emptyset$ be a bounded subset of X. If K has the λ-W**RNP (respectively λ-WRNP), then it has also the W**RNP (respectively WRNP).*

PROOF. Let (Ω, Σ, μ) be an arbitrary complete probability measure space and let $\nu : \Sigma \to X$ be a nonatomic measure satisfying for every $E \in \Sigma$ the relation $\nu(E) \in \mu(E) \cdot K$.

(A) Assume first that Σ is the completion of a countably generated σ-algebra $\widetilde{\Sigma} \subseteq \Sigma$ with respect to $\mu | \widetilde{\Sigma}$.

Let $(E_n) \subseteq \Sigma$ be a sequence generating $\widetilde{\Sigma}$ and let $\chi : \Omega \to [0, 1]$ be its Marczewski function:

$$\chi(\omega) = 2 \sum_{n=1}^{\infty} 3^{-n} \chi_{E_n}(\omega).$$

It can be easily checked that $\chi^{-1} : \mathcal{L} \cap \chi(\Omega) \to \mathcal{P}(\Omega)$ is a Boolean σ-isomorphism of $\mathcal{B}_{[0,1]} \cap \chi(\Omega)$ onto $\widetilde{\Sigma}$. Let $\overline{\mu} : \mathcal{L} \to [0, 1]$ be the image of μ under χ and, let $\theta : [0, 1] \to [0, 1]$ be the function defined by $\lambda([0, \theta(t)]) = \overline{\mu}([0, t])$. If $\xi = \theta \circ \chi$, then for each $E \in \mathcal{L}$, we have $\mu[\xi^{-1}(E)] = \lambda(E)$ and the measure algebras of μ and λ are isomorphic.

Let now $\overline{\nu} : \mathcal{L} \to X$ be given by

$$\overline{\nu}(B) = \nu\big[\xi^{-1}(B)\big].$$

We have $\|\overline{\nu}(B)\| \leqslant \lambda(B)$ for each $B \in \mathcal{L}$. Hence, by the assumption, there is $f \in \mathbb{P}(\lambda, \overline{K}^{**})$ (respectively $\mathbb{P}(\lambda, K)$), such that $\overline{\nu}(B) = \int_E f \, d\lambda$.

It follows that for each $E \in \widetilde{\Sigma}$

$$\nu(E) = P - \int_E f \circ \xi \, d\mu.$$

Moreover, in view of Theorem 6.7, the set $\nu(\widetilde{\Sigma})$ is norm relatively compact.

(B) Assume now that Σ is arbitrary and notice that $\nu(\Sigma)$ is a norm relatively compact subset of X^{**} (respectively X). Denote by Ξ the collection of all complete measure spaces $(\Omega, \Delta, \mu|\Delta)$ with $\Delta \subseteq \Sigma$ being the completion of a countably generated σ-algebra Σ_Δ with respect to $\mu | \Sigma_\Delta$ and order Ξ upwards by inclusion.

In view of (A), for each Δ there is $f_\Delta \in \mathbb{P}(\mu|\Delta, \overline{K}^{**})$ (respectively $\mathbb{P}(\mu|\Delta, K)$), such that

$$\nu(E) = P - \int_E f_\Delta \, d\mu$$

for each $E \in \Delta$.

We shall prove that the net (f_Δ) is Cauchy in the norm of $\mathbb{P}(\mu, X^{**})$ (respectively $\mathbb{P}(\mu, X)$).

To prove it, fix $\varepsilon > 0$ and take a simple function $h_\varepsilon : \Omega \to X$, such that

$$\sup_{E \in \Sigma} \left\| \nu(E) - \int_E h_\varepsilon \, d\mu \right\| \leqslant \varepsilon$$

(cf. Musiał (1980)).

Now fix $\overline{\Delta} \in \Xi$ such that h_ε is $\overline{\Delta}$-measurable. Then, for each $\Delta \geqslant \overline{\Delta}$

$$\left| f_\Delta - h_\varepsilon \right| \leqslant 4 \sup_{E \in \Sigma} \left\| \nu(E) - \int_E h_\varepsilon \, d\mu \right\| < \varepsilon.$$

It follows that for $\Delta_1, \Delta_2 \geqslant \overline{\Delta}$ the inequality $\left| f_{\Delta_1} - f_{\Delta_2} \right| < 2\varepsilon$ holds, and so the net is Cauchy, as required.

But Ξ is countably directed, so there exists Δ_0 such that for each $\Delta \geqslant \Delta_0$ we have $\left| f_\Delta - f_{\Delta_0} \right| = 0$.

It follows that each such f_Δ is scalarly μ-equivalent to f_{Δ_0} and so for each $E \in \Sigma$, we get the equality

$$\nu(E) = P - \int_E f_{\Delta_0} \, d\mu.$$

This completes the proof. □

DEFINITION 9.4. Given a directed set $(\Pi, \leqslant)$, a family of σ-algebras $\Sigma_\pi \subseteq \Sigma$, and functions $f_\pi \in \mathbb{P}((\Omega, \Sigma_\pi, \mu|\Sigma_\pi); X)$ with $\pi \in \Pi$, the system $\{f_\pi, \Sigma_\pi; \pi \in \Pi\}$ is a *martingale* if $\pi \leqslant \rho$ yields $\Sigma_\pi \subseteq \Sigma_\rho$ and $E(f_\rho|\Sigma_\pi) = f_\pi$. The martingale is bounded if there is $M > 0$ such, that for each $x^* \in X^*$ and each $\pi \in \Pi$ the inequality $|\langle x^*, f_\pi \rangle| \leqslant M\|x^*\|$ holds μ-a.e. The martingale is convergent in $\mathbb{P}(\mu, X)$ if there is $f \in \mathbb{P}(\mu, X)$ such that $\lim_\pi \left| f_\pi - f \right| = 0$. The collection of all finite Σ-partitions of Ω into sets of positive measure is denoted by Π_Σ. We order it in the following way: $\pi_1 \leqslant \pi_2$ if each element of π_1 is, except for a null set, a union of element of π_2.

The following theorem is a martingale characterization of the WRNP and the W**RNP.

THEOREM 9.5 (Musiał (1980) for $K = B(X)$). *For a bounded set $K = \overline{\text{conv}}\, K \neq \emptyset \subset X$ the following conditions are equivalent*:

(i) *K has the WRNP* (*respectively W**RNP*),

(ii) *Given any (Ω, Σ, μ) and any bounded martingale $\{f_n, \Sigma_n;\ n \in \mathbb{N}\}$ of K-valued Pettis μ-integrable* (*simple*) *functions, then $\{f_n, \Sigma_n;\ n \in \mathbb{N}\}$ is convergent in $\mathbb{P}(\mu, K)$* (*respectively $\mathbb{P}(\mu, \overline{K^{**}})$*).

PROOF. Assume (i) is satisfied and take a bounded martingale $\{f_n, \Sigma_n;\ n \in \mathbb{N}\}$ in $\mathbb{P}(\mu, K)$. Assume, that $M > 0$ is such that $|\langle x^*, f_n \rangle| \leqslant M\|x^*\|$ μ-a.e. (the exceptional sets depend on x^*).

Let $\widetilde{\Sigma}_0 = \bigcup_{n=1}^\infty \Sigma_n$ and let $\tilde{\nu} : \widetilde{\Sigma}_0 \to X$ be given by

$$\tilde{\nu}(E) = \lim_n \int_E f_n \, d\mu$$

for each $E \in \widetilde{\Sigma}_0$.

We have $\|\tilde{\nu}(E)\| \leqslant M\,\mu(E)$ and $\widetilde{\nu}(E) \in \mu(E) \cdot K$ for all $E \in \widetilde{\Sigma}$. The set function $\tilde{\nu}$ extends uniquely to a measure $\nu_1 : \widetilde{\Sigma} = \sigma(\widetilde{\Sigma}_0) \to X$ satisfying the similar conditions for all $E \in \widetilde{\Sigma}$. Setting for each $E \in \Sigma$

$$\nu(E) = \int_E E\big(\chi_E | \widetilde{\Sigma}\big)\, d\nu_1$$

(where the integral is in the sense of Bartle, Dunford and Schwartz (1955)) we get an extension of ν_1 to the whole Σ satisfying for all $E \in \Sigma$ the relations

$$\big\|\nu(E)\big\| \leqslant M\,\mu(E) \quad \text{and} \quad \nu(E) \in \mu(E) \cdot K.$$

Since K has the WRNP (or W**RNP), we get $f \in \mathbb{P}(\mu, K)$ (respectively $\mathbb{P}(\mu, \overline{K}^{**})$) being the density of ν with respect to μ.

Since K has the W**RNP, it has the CRP (by Theorem 6.7). Thus, in a similar way, as it has been done in the proof of Theorem 9.3, one can show that $\{(f_n, \Sigma_n);\, n \in \mathbb{N}\}$ is a Cauchy martingale in $\mathbb{P}(\mu, \overline{K}^{**})$.

Since $(\nu|\Sigma_n)(E) = \int_E f_n\, d\mu$ for each $n \in \mathbb{N}$, we have for each $x^* \in X^*$ $\langle x^*, f_n\rangle = E(\langle x^*, f\rangle | \Sigma_n)$ and this gives

$$\lim_n \int_\Omega \big|E\big(x^* f | \widetilde{\Sigma}\big) - x^* f_n\big|\, d\mu = 0.$$

Together with the Cauchy condition, this yields

$$\lim_n \big|f_n - f\big| = 0.$$

Assume now that (ii) is satisfied and take a measure $\nu : \Sigma \to X$ satisfying for each $E \in \Sigma$ the relations

$$\big\|\nu(E)\big\| \leqslant \mu(E) \quad \text{and} \quad \nu(E) \in \mu(E) \cdot K.$$

Define for each $\pi \in \Pi_\Sigma$ the function f_π by

$$f_\pi = \sum_{E \in \pi} \frac{\nu(E)}{\mu(E)} \chi_E$$

and let $\Sigma_\pi = \sigma(\pi)$. $\{(f_\pi, \Sigma_\pi);\, \pi \in \Pi_\Sigma\}$ is a bounded martingale in $\mathbb{P}(\mu, K)$. If $\pi_1 \leqslant \pi_2 \leqslant \cdots$ then, by the assumption, $\{(f_{\pi_n}, \Sigma_{\pi_n});\, n \in \mathbb{N}\}$ is convergent in $\mathbb{P}(\mu, K)$ (respectively $\mathbb{P}(\mu, \overline{K}^{**})$).

Let $f : \Omega \to X^{**}$ be a weak* density of ν with respect to μ:

$$\big\langle x^*, \nu(E)\big\rangle = \int_E \big\langle x^*, f\big\rangle\, d\mu$$

for each $x^* \in X^*$, $E \in \Sigma$.

One can easily see that there exists in Π_Σ a sequence $\pi_1 \leqslant \pi_2 \leqslant \cdots$ such that

$$\lim_n \sup\left\{\int_\Omega \left|\langle x^*, f - f_{\pi_n}\rangle\right| d\mu \colon x^* \in B(X^*)\right\} = 0$$

and so, in particular

$$\lim_n \int_E \langle x^*, f_{\pi_n}\rangle \, d\mu = \langle x^*, \nu(E)\rangle$$

for each $x^* \in X^*$ and $E \in \Sigma$.

Hence, if $g = \lim_n f_{\pi_n} \in \mathbb{P}(\mu, K)$ (respectively $\mathbb{P}(\mu, \overline{K}^{**})$), then for each $E \in \Sigma$, we get the required equality

$$\nu(E) = \int_E g \, d\mu. \qquad \square$$

PROPOSITION 9.6 (C. Ryll-Nardzewski, see Musiał (1979)). *l_∞ does not have the WRNP.*

PROOF. Let π_n be the dyadic partition of $[0, 1]$ into 2^n intervals and, let $\tilde{\pi}_n$ be the collection of all possible unions of elements taken from π_n. If (A_n) is an enumeration of $\bigcup_{n=1}^\infty \tilde{\pi}_n$, then clearly $\lim_n \lambda(A_n) = 0$. Define a measure $\nu : \mathcal{L} \to c_0 \subset l_\infty$ by setting

$$\nu(E) = \langle \lambda(E \cap A_n)\rangle_n.$$

Then, $\nu(\mathcal{L})$ is a norm relatively compact subset of c_0, $\|\nu(E)\| \leqslant \lambda(E)$ for each $E \in \mathcal{L}$ and ν is without Pettis λ-integrable derivative in l_∞. Indeed, let $f : [0, 1] \to l_\infty = l_1^*$ be a weak* density of ν with respect to λ. It means in particular that if (e_n) is the standard basis in l_1, then

$$\lambda(E \cap A_n) = \langle e_n, \nu(E)\rangle = \int_E \langle e_n, f\rangle \, d\lambda$$

for each $n \in \mathbb{N}$.

But the sequence (χ_{A_n}) is pointwise dense in $\{0, 1\}^{[0,1]}$. Thus, if χ_A is a non-λ-measurable cluster point of (χ_{A_n}), then χ_A is λ-a.e. equal to a pointwise cluster point of $(\langle e_n, f\rangle)$. Such a point is of the form $\langle x^*, f\rangle$ for a functional $x^* \in l_\infty^*$. This means that f is not scalarly measurable and hence it cannot be a Pettis integrable density of ν with respect to λ. $\square$

The next theorem has been first proved by Musiał (1976) (see also (1979)) for separably complementable X. Then, in full generality the necessity has been proved by Musiał and Ryll-Nardzewski (1978) and the sufficiency by Janicka (1979), Musiał (1979) and J. Bourgain.

THEOREM 9.7. *X^* has the weak Radon–Nikodým property if and only if X contains no isomorphic copy of l_1.*

Assume that X contains no copy of l_1. If $\nu : \Sigma \to X$ is a μ-dominated measure and ρ is a lifting on $L_\infty(\mu)$, then by the weak* Radon–Nikodým property of X^*, there exists a weak* density $f : \Omega \to X^*$ of ν with respect to μ such that

$$\rho(\langle x, f\rangle) = \langle x, f\rangle \quad \text{for each } x \in X.$$

Since X contains no isomorphic copy of l_1, it follows from Corollaries 4.17 and 4.18 that

$$\langle x^{**}, \nu(E)\rangle = \int_E \langle x^{**}, f\rangle \, d\mu$$

by the Lebesgue Convergence Theorem, and this proves the WRNP of X^*.

Assume now that X contains a subspace that is isomorphic to l_1 and let $T : l_1 \to X$ be an isomorphic embedding. Then $T^* : X^* \to l_\infty$ is a surjection. If $\nu : \mathcal{L} \to l_\infty$ is the measure constructed in the proof of Proposition 9.6, then according to Musiał and Ryll-Nardzewski (1978), there is a λ-dominated measure $\kappa : \mathcal{L} \to X^*$ such that $T^*\kappa = \nu$. Since ν is not Pettis differentiable, also κ cannot have a Pettis integrable density.

At this place it is also worth to recall a characterization of WRNP of X^* in terms of functions. The result is a direct consequence of the W*RNP of conjugate spaces.

COROLLARY 9.8 (Musiał (1979)). *X^* has WRNP if and only if for every complete (Ω, Σ, μ) and each weak*-bounded and weak*-scalarly measurable $f : \Omega \to X^*$ there exists $g \in \mathbb{P}(\mu, X^*)$ and a weak*-scalarly null $h : \Omega \to X^*$ with $f = g + h$.*

More or less at the same time Rybakov (1977) presented another characterization of Banach spaces not containing l_1 (see also Musiał (1979)).

THEOREM 9.9. *X does not contain any isomorphic copy of l_1 if and only if X^* has CRP.*

The next result is a generalization of Theorems 9.7 and 9.9. It was also an essential step towards localizing of the weak RNP.

THEOREM 9.10 (Riddle, Saab and Uhl (1983)). *For a given operator $T : X \to Y$ the following conditions are equivalent:*

(i) *The set $T(B(X))$ is weakly precompact;*
(ii) *T factors through a Banach space containing no isomorphic copy of l_1;*
(iii) *The set $T^*(B(Y^*))$ has WRNP;*
(iv) *The set $T^*(B(Y^*))$ has CRP;*
(v) *T^* factors through a Banach space with the WRNP.*

The most general description of sets possessing WRNP will be given in Theorem 9.15. In order to present an idea of its proof we need however yet a few additional facts.

LEMMA 9.11 (Rosenthal (1974)). *Let $\langle x_n\rangle$ be a pointwise bounded sequence of real-valued functions defined on a set S and having no pointwise convergent subsequence. Then there* *exists a subsequence $\langle x_{n_k}\rangle$ of $\langle x_n\rangle$, a real number r and $\delta > 0$ such that for every infinite subset M of $\{n_k\colon k \geqslant 1\}$, there is a point $s \in S$ with*

$$x_m(s) > r + \delta \quad \textit{for infinitely many } m \in M$$

and

$$x_m(s) < r \quad \textit{for infinitely many } m \in M.$$

THEOREM 9.12 (Rosenthal (1978)). *Let Q be an uncountable Polish space and let $(A_n, B_n)_{n\geqslant 1}$ be a sequence of pairs of closed and disjoint subsets of Z. Assume that the sequence $(A_n, B_n)_{n\geqslant 1}$ has no convergent subsequence. Then, there exists a compact subset L of Q, a homeomorphism h from L onto the Cantor set $\Delta = \{0, 1\}^{\omega}$, and an increasing sequence $\langle n_k\rangle$ such that*

$$A_{n_k} \cap L = h^{-1}(V_k) \quad \textit{and} \quad B_{n_k} \cap L = h^{-1}\big(V_k^c\big)$$

for all k (here $V_k = \{t = \langle t_n\rangle \in \Delta\colon t_k = 0\}$).

The next lemma is taken from Matsuda (1985).

LEMMA 9.13. *Let X be separable and let $Q \neq \emptyset$ be a weak*-compact convex subset of X^*. If $B(X)$ is not Q-weakly precompact, then there is a Radon measure on $(Q, weak^*)$ and a measure $\nu : \Sigma \to X^*$ such that*

(a) *$\nu(E) \in \mu(E) \cdot Q$ for each $E \in \Sigma$;*

(b) *$\nu(\Sigma)$ is not relatively compact in the norm topology.*

PROOF (*Sketch*). Let $\langle x_n\rangle$ be a sequence in $B(X)$ without subsequence pointwise convergent on Q. Without loss of generality we may assume that $\langle x_n\rangle$, r and $\delta > 0$ satisfy the conclusion of Lemma 9.11. Setting $A_n := \{x^* \in Q\colon \langle x^*, x_n\rangle \geqslant r + \delta\}$ and $B_n := \{x^* \in Q\colon \langle x^*, x_n\rangle \leqslant r\}$ we get a sequence $(A_n, B_n)_{n\geqslant 1}$ that has no convergent subsequences. In virtue of Theorem 9.12 there exists a compact subset L of Q, a homeomorphism h from L onto the Cantor set $\Delta = \{0, 1\}^{\omega}$, and an increasing sequence $\langle n_k\rangle$ such that

$$A_{n_k} \cap L = h^{-1}(V_k) \quad \text{and} \quad B_{n_k} \cap L = h^{-1}\big(V_k^c\big)$$

for all k (here $V_k = \{t = \langle t_n\rangle \in \Delta\colon t_k = 0\}$).

Let η be the normalized Haar measure on Δ and let $\overline{\eta}$ be the Radon measure on L such that $h(\overline{\eta}) = \eta$. If ξ is the extension of $\overline{\eta}$ to the whole Q. Then

$$\xi(A_{n_k}) = \overline{\eta}\big(h^{-1}(V_k)\big) = \eta(V_k) = 1/2 = \xi(B_{n_k})$$

and

$$\xi(A_{n_i} \cap B_{n_j}) = \eta\big(V_i \cap V_j^c\big) = 1/4$$

whenever $i \neq j$. Let ν be the weak*-integral of the identity function on Q with respect to ξ. ν is a vector measure on the σ-algebra of weak*-Borel subsets of Q.

The conclusion (a) of the lemma is an easy consequence of the separation theorem. To see that (b) is also fulfilled notice that if $i < j$ then

$$\begin{aligned}\left\|\nu(A_{n_i}) - \nu(A_{n_j})\right\| &\geqslant \left\langle \nu(A_{n_i}) - \nu(A_{n_j}), x_{n_i}\right\rangle \\ &\geqslant (r+\delta)\xi(A_{n_i} \cap B_{n_j}) - r\xi(A_{n_j} \cap B_{n_i}) = \delta/4. \qquad \square\end{aligned}$$

DEFINITION 9.14 (*Talagrand* (1984)). A weak* compact subset K of X^* is a Pettis set if the identity function is universally scalarly measurable (with respect to Radon measures on $(K, \textit{weak}^*)$).

The next theorem summarizes most of the results presented earlier in this paper and concerning WRNP.

THEOREM 9.15. *Let $K \neq \emptyset$ be a weak*-compact convex subset of X^*. Then the following conditions are equivalent:*

(i) *$B(X)$ is weakly precompact with respect to K;*
(ii) *the identity function is universally Pettis integrable on K;*
(iii) *K is a Pettis set;*
(iv) *K has WRNP;*
(v) *K has λ-WRNP;*
(vi) *K has CRP;*
(vii) *K has λ-CRP.*

PROOF (*Sketch*). (i) $\Rightarrow$ (ii) follows from Proposition 7.5. (iii) $\Rightarrow$ (i) is a consequence of Bourgain, Fremlin and Talagrand (1978). (i) $\Rightarrow$ (iv) follows from Corollaries 4.17 and 4.18. (iv) $\Rightarrow$ (vi) follows from Theorem 6.7 and (vii) $\Rightarrow$ (i) from Lemma 9.13. $\square$

Applying the result of Talagrand (1984) which says that if $K \subset X^*$ is a Pettis set then its weak* closure has WRNP, one gets further generalizations of Theorem 9.15.

10. Conditional expectation

Let $\Xi \subset \Sigma$ be a σ-algebra. If $f \in \mathbb{P}(\mu, X)$, then a function $E(f|\Xi): \Omega \to X$ is a conditional expectation of f with respect to Ξ if $E(f|\Xi) \in \mathbb{P}(\mu, \Xi, X)$ and

$$\int_E E(f|\Xi)\, d\mu = \int_E f\, d\mu \tag{2}$$

for all $E \in \Xi$.

The first example of a Pettis integrable function without conditional expectation was published by Rybakov (1971). It was an l_4-valued function. Heinich (1973) published then an example of l_1-valued Pettis integrable function on $[0, 1]^2$, which does not admit the conditional expectation with respect to a sub-σ-algebra of the σ-algebra of Lebesgue measurable sets. These examples can be classed in the following pattern: If $f \in \mathbb{P}(\mu, X)$ has infinite variation $|\nu_f|$ and σ-algebra $\Xi \subset \Sigma$ is such that $|\nu|\Xi|$ is not σ-finite then the existence of $E(f|\Xi)$ would contradict Theorem 3.8. Since for each non-atomic μ and infinite dimensional X there is $f \in \mathbb{P}(\mu, X)$ with infinite $|\nu_f|$, there are a lot of such examples.

The following global result is an obvious consequence of the above considerations:

PROPOSITION 10.1 (Musiał, (1976) for $K = B(X)$). *Let $K \subset X$ be a closed and convex set with WRNP. If $f \in \mathbb{P}(\mu, K)$ and Ξ is a $\mu|\Xi$-complete sub-σ-algebra of Σ, then f has the conditional expectation with respect to Ξ if and only if the measure $\nu_f|\Xi$ is of σ-finite variation.*

First examples of scalarly bounded Pettis integrable functions without conditional expectations were presented by Talagrand (1984, 6–4). One of them is $f \in \mathbb{P}(\kappa, l_\infty(I))$ defined in Example 6.13 (see also Musiał (1985)). Indeed, if Ξ is the completion of the Borel algebra of $K_I \times K_I$ with respect to $\mu \otimes \mu$, then Ξ is κ-dense in $\overline{\Sigma\widehat{\otimes}\Sigma}$. If there existed a Pettis integrable $E(f|\Xi)$, then the equality (2) would be true for all κ-measurable sets. This is however impossible in case of infinite I, since according to Stegall's result (Theorem 6.1) the range of $\nu_{E(f|\Xi)}$ is norm relatively compact.

Talagrand (1984) obtained the following interesting result covering the particular case of the conditional expectations:

THEOREM 10.2. (Axiom L) *Let $(\Theta, \mathcal{T}, \nu)$ be a complete probability space and let $T : L_1(\mu) \to L_1(\nu)$ be a bounded operator. Then, let $f : \Omega \to X^*$ be a weak*-scalarly bounded function such that each countable subset of $\{xf\colon \|x\| \leqslant 1\}$ is stable. Then there exists a properly measurable function $g : \Theta \to X^*$ such that $T(xf) = xg$ ν-a.e. for all $x \in X$. If $f \in \mathbb{P}(\mu, X^*)$, then $g = Tf \in \mathbb{P}(\nu, X^*)$.*

Riddle and Saab (1985) proved another sufficient condition guaranteeing the existence of conditional expectations.

THEOREM 10.3. *If $f \in \mathbb{P}(\mu, X^*)$ is scalarly bounded and the set $\{xf\colon \|x\| \leqslant 1\}$ is weakly precompact in $L_\infty(\mu)$, then f has conditional expectation with respect to all sub-σ-algebras of Σ.*

PROOF. Assume that $f \in \mathbb{P}(\mu, X^*)$ satisfies the above assumptions, let Ξ be a sub-σ-algebra of Σ and let ρ be a lifting on $L_\infty(\mu)$. Define $g : \Omega \to X^*$ by setting $xg = \rho(E_\Xi(xf))$, for each $x \in X$. Since $\{xf\colon \|x\| \leqslant 1\}$ is weakly precompact in $L_\infty(\mu)$ and the conditional expectation operator $E_\Xi : L_\infty(\mu) \to L_\infty(\mu|\Xi)$ is a contraction, the set $\{xg\colon \|x\| \leqslant 1\}$ is weakly precompact in $L_\infty(\mu|\Xi)$. In view of Corollaries 4.17 and 4.18 we have $g \in \mathbb{P}(\mu|\Xi, X^*)$. Now, since the equality $\int_E xg\, d\mu = \int_E xf\, d\mu$ for all $x \in X$ and the functions f and g are Pettis integrable, we get the same equality for all $x \in X^{**}$. $\square$

11. Differentiation

Already Pettis (1938) noticed that if $f \in \mathbb{P}(\lambda, X)$ then for each $x^* \in X^*$ there exists a set $A(x^*) \in \mathcal{L}$ of measure one, such that the equality

$$\lim_{h\to 0} h^{-1} \int_t^{t+h} \langle x^*, f(\omega)\rangle \, d\mu = x^* f(t)$$

holds true for all $t \in A(x^*)$. Pettis asked then whether in case of a strongly measurable $f \in \mathbb{P}(\lambda, X)$ the sets $A(x^*)$ could be replaced by a single set of full measure, i.e., whether the Pettis integral of f is a.e. weakly differentiable.

The answer is negative in case $X = l_2$ (Phillips (1940)) and $X = C[0, 1]$ (Munroe (1946a)). Thomas (1976) conjectured that such a counterexample could be constructed in every infinite dimensional X.

In (1994) Kadets proved that for every infinite dimensional Banach space X there is a strongly measurable and Pettis integrable function $f : [0, 1] \to X$ for which $g(t) := \int_0^t f(s)\, ds$ is non-differentiable on a set of positive measure.

Then Dilworth and Girardi (1995) generalized the above result proving that always there exist functions that have nowhere weakly differentiable Pettis integrals. To formulate the main result more precisely let $\boldsymbol{\Psi}$ be the collection of all increasing functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the growth condition

$$\sum_{n=1}^{\infty} \psi\left(2^{-p_{n-1}}\right)\sqrt{2^{p_n}} < \infty, \tag{3}$$

for some increasing sequence $(p_n)_{n=0}^{\infty}$ of integers and such that $\psi(0) = 0$.

THEOREM 11.1 (Dilworth and Girardi (1995)). *For each X and $\psi \in \boldsymbol{\Psi}$ there exists a strongly measurable $f \in \mathbb{P}(\lambda, X)$ such that*

$$\left\| \int_I f \, d\lambda \right\| \geqslant \psi\left(\lambda(I)\right)$$

for every non-degenerated interval $I \subseteq [0, 1]$.

If moreover

$$\sum_{n=1}^{\infty} \psi\left(2^{-p_{n-1}}\right) 2^{p_n} = \infty,$$

then $f \notin L_1(\lambda, X)$.

PROOF (*Sketch*). Let $\{I_k^n : n = 0, 1, 2, \ldots;\ k = 1, 2, \ldots, 2^n\}$ be the dyadic intervals on $[0, 1]$, i.e.,

$$I_k^n = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right[.$$

Define inductively a collection $\{A_k^n: n = 0, 1, 2, \ldots;\ k = 1, 2, \ldots, 2^n\}$ of pairwise disjoint sets of positive measure such that $A_k^n \subset I_k^n$.

Fix $K > 1$. By a theorem of Mazur there is a basic sequence $\{x_n\}$ in X with basis constant at most K. Take a blocking $\{F_n\}$ of $\{x_n\}$ with each subspace F_n of large enough dimension to find (using the finite-dimensional version of Dvoretzky's Theorem (Dvoretzky, 1961)) a 2^n-dimensional subspace E_n of F_n such that the Banach–Mazur distance between E_n and $l_2^{2^n}$ is less than 2. Let $T_n : l_2^{2^n} \to E_n$ be for each n an operator such that $\|T_n\| \leqslant 2$ and $\|T_n^{-1}\| = 1$. If $\{u_k^n: k = 1, 2, \ldots, 2^n\}$ are the standard unit vectors in $l_2^{2^n}$, then let $e_k^n := T_n u_k^n$.

Let $\langle p_n \rangle$ be an increasing sequence of integers, with $p_0 = 0$ satisfying

$$\sum_{n=1}^{\infty} \psi\big(4 \cdot 2^{-p_{n-1}}\big)\sqrt{2^{p_n}} < \infty.$$

Define $f : [0, 1] \to X$ by

$$f(\omega) = 2K \sum_{n=1}^{\infty} \psi\big(4 \cdot 2^{-p_{n-1}}\big) \sum_{k=1}^{2^{p_n}} \chi_{A_k^{p_n}} \lambda\big(A_k^{p_n}\big)^{-1} e_k^{p_n}.$$

One can easily check that for each $E \in \mathcal{L}$

$$\int_E f\, d\lambda = 2K \sum_{n=1}^{\infty} \psi\big(4 \cdot 2^{-p_{n-1}}\big) \sum_{k=1}^{2^{p_n}} \lambda\big(A_k^{p_n}\big)^{-1} \int_E \chi_{A_k^{p_n}}\, d\lambda e_k^{p_n}.$$

Since $\langle E_n \rangle$ form a finite-dimensional decomposition there is a projection of $\bigoplus_n E_n$ onto E_{p_n} of norm not exceeding $2K$.

Now, if I is an arbitrary interval in $[0, 1]$ one can find first $I_j^m \subset I$ such that $4\lambda(I_j^m) \geqslant \lambda(I)$ and then n satisfying $p_{n-1} \leqslant m < p_n$.

It follows that

$$\left\| \int_I f\, d\lambda \right\| \geqslant \psi\big(4 \cdot 2^{-p_{n-1}}\big) \left[\sum_{k=1}^{2^{p_n}} \lambda\big(A_k^{p_n}\big)^{-1} \left| \int_I \chi_{A_k^{p_n}}\, d\lambda \right|^2 \right]^{1/2}$$

and so since $A_k^{p_n} \subset I_k^{p_n} \subset I_j^m \subset I$ we have

$$\left\| \int_I f\, d\lambda \right\| \geqslant \psi\big(4 \cdot 2^{-p_{n-1}}\big).$$

But ψ is increasing and $4 \cdot 2^{-p_{n-1}} \geqslant 4 \cdot 2^{-m} \geqslant \lambda(I)$ and so

$$\left\| \int_I f\, d\lambda \right\| \geqslant \psi\big(\lambda(I)\big).$$

If $\sum_{n=1}^{\infty}\psi(2^{-p_{n-1}})2^{p_n}=\infty$, then $\sum_{n=1}^{\infty}\psi(4\cdot 2^{-p_{n-1}})\sum_{k=1}^{2^{p_n}}\|e_k^{p_n}\|=\infty$ and so $f\notin L_1(\lambda,X)$ (according to Diestel and Uhl (1977, p. 55)). □

As a corollary we get an answer to the original Pettis's question.

COROLLARY 11.2 (Dilworth and Girardi (1995)). *There exists a strongly measurable $f\in\mathbb{P}(\mu,X)$ that has nowhere weakly differentiable integral.*

PROOF. Taking in Theorem 11.1 $\psi(t)=t^{3/4}$ we get a function f satisfying for each $t\in[0,1]$ the equality

$$\lim_{h\to 0}\left\|\frac{1}{h}\int_t^{t+h}f(\omega)\,d\omega\right\|=\infty.$$

If the Pettis integral of f were weakly differentiable at some t, then the above limit would be finite. □

As noticed in Dilworth and Girardi (1995)) it follows from Theorem 11.1 the existence of a Pettis integrable function such that its integral is of infinite variation on every subinterval of $[0,1]$. This generalizes earlier result of Janicka and Kalton (1977) where the Banach space valued measure possessing that property could not be represented as a Pettis integral (A good reference to the circle of problems concerning vector measures of infinite variation on all sets of positive measure is the work of Drewnowski and Lipecki (1995)).

With some further effort one can get other results connecting the differentiability problem with absolutely summing operators and cotype of X (see Dilworth and Girardi (1995)).

12. Fubini theorem

The classical Fubini theorem holds true not only for real functions but also for Bochner integrable functions on an arbitrary measure space. Thomas (1976) observed that for every infinite dimensional Banach space X does exist a strongly measurable and λ^2-Pettis integrable function $f:[0,1]^2\to X$ with the property that for every $x\in[0,1]$ the function $y\to f(x,y)$ is not λ-Pettis integrable. Recently Michalak (2000) attempted to resolve the problem in case of arbitrary bounded Pettis integrable functions.

THEOREM 12.1 (Michalak (2000)). *Let (Ω,Σ,μ) and (Θ,T,ν) be complete probability spaces and let X be a WCG space not containing any isomorphic copy of l_1. If $f\in\mathbb{P}(\mu\widehat{\otimes}\nu,X^*)$ is such that the set $\{|xf|:\ x\in B(X)\}$ is order bounded in $L_1(\mu\widehat{\otimes}\nu)$, then there exists a function $g:\Omega\times\Theta\to X^*$ which is scalarly equivalent to f and*

(i) *the function $g(\cdot,\theta)\in\mathbb{P}(\mu,X^*)$ for ν-a.e. $\theta\in\Theta$;*
(ii) *the function $g(\omega,\cdot)\in\mathbb{P}(\nu,X^*)$ for μ-a.e. $\omega\in\Omega$;*

(iii) $\int_{A\times B} g\, d\mu \otimes \nu = \int_A (\int_B g(\omega,\theta)\, d\nu(\theta))\, d\mu(\omega) = \int_B (\int_A g(\omega,\theta)\, d\mu(\omega))\, d\nu(\theta)$ *for all* $A \in \Sigma$ *and* $B \in T$.

The above theorem cannot however be extended to all Banach spaces. If $l_2(\Omega, X)$ is the space of all functions $f : \Omega \to X$ such that $\sum_{\omega\in\Omega} \|f(\omega)\|^2 < \infty$, then one gets the following result:

THEOREM 12.2 (Michalak (2000)). *Let* (Ω, Σ, μ) *and* (Θ, T, ν) *be complete probability spaces and let* X *be a Banach space. Assume that* μ *vanishes on points and there is a nonmeasurable subset of* Θ. *Then for every bounded Pettis integrable function* $f : \Omega \times \Theta \to l_2(\Omega, X)$ *there exists a bounded function* $g : \Omega \times \Theta \to l_2(\Omega, X)$ *which is scalarly equivalent to* f *and*

$$\mu\big(\big\{\omega \in \Omega\colon\ g(\omega,\cdot) \text{ is not scalarly } \nu\text{-measurable}\big\}\big) \geqslant 1/2.$$

13. Spaces of Pettis integrable functions

13.1. *Space of all Pettis integrable functions*

Pettis (1938) noticed that $\mathbb{P}(\lambda, L_2(\lambda))$ is non-complete. Rybakov (1970) proved that $\mathbb{P}(\lambda, c_0)$ is non-complete. Then Thomas (1976) proved that $\mathbb{P}(\mu, X)$ is non-complete in case of an arbitrary not purely atomic μ and infinite dimensional X. Janicka and Kalton (1977) proved the same in case of the Lebesgue measure on $[0, 1]$.

Modifying the example of Pettis, Drewnowski, Florencio and Paúl (1992) showed that $\mathbb{P}(\lambda, L_2(\lambda))$ does not have property (K). Then, Drewnowski and Lipecki (1995) proved that $\mathbb{P}(\mu, X)$ is never ultrabarrelled if μ is non-atomic, and so it is neither Baire nor has property (K). However, the following holds true:

THEOREM 13.1 (Drewnowski, Florencio and Paúl (1992)). $\mathbb{P}(\mu, X)$ *is always barrelled.*

PROOF (*Sketch*). Let us say that a locally convex space Z admits an (Ω, Σ, μ)-Boolean algebra of projections (see Drewnowski et al. (1992) for details) if there exists a set $\{P_A\colon A \in \Sigma\}$ of linear projections in Z such that:

(1) P_Ω is the identity on Z, $P_{A\cap B} = P_A \cdot P_B$ for all $A, B \in \Sigma$, and $P_{A\cup B} = P_A + P_B$ for all disjoint $A, B \in \Sigma$;

(2) P_A is continuous for every $A \in \Sigma$;

(3) for every $x \in Z$, the vector measure $F_x : \Sigma \to Z$ defined by $F_x(A) := P_A(x)$ is μ-continuous (that is $P_A(x) = 0$ if $\mu(A) = 0$).

Now we need the following

PROPOSITION. *Let* Z *be a metrizable locally convex space admitting an* (Ω, Σ, μ)-*Boolean algebra of projections* $\{P_A\colon A \in \Sigma\}$. *Assume also that the projections satisfy the following condition*:

(4) *If* $\langle A_n \rangle$ *is a sequence of pairwise disjoint elements of* Σ *and* $\langle x_n \rangle$ *is a null sequence in* Z *such that* $P_{A_n}(x_n) = x_n$, $n = 1, 2, \ldots$, *then there exists a sequence* $\langle n_k \rangle$ *in* $\mathbb{N}$ *such that the series* $\sum_k x_{n_k}$ *is convergent.*

If W *is a closed subspace of* Z *such that* $P_A(W) \subseteq W$ *for all* $A \in \Sigma$ *and* (Ω, Σ, μ) *is atomless, then* W *is barrelled.*

Now we are ready to present the proof. Assume first that (Ω, Σ, μ) is atomless. For $A \in \Sigma$ and $f \in \mathbb{P}(\mu, X)$, define $P_A(f) := \chi_A \cdot f$. One can easily check that the conditions (1)–(3) are fulfilled. We have to check yet the condition (4) of the just formulated proposition. So let $\langle A_n \rangle$ be a pairwise disjoint sequence in Σ and let $\langle f_n \rangle$ be a null sequence in $\mathbb{P}(\mu, X)$. Without loss of generality, we may assume that $M := \sum_n \| f_n \| < \infty$. Let $f(\omega) := \sum_n f_n(\omega)$ be defined for all $\omega \in \Omega$. Then, for every $x^* \in X^*$ we have

$$\int_\Omega \left|\langle x^*, f(\omega) \rangle\right| d\mu = \int_\Omega \sum_n \left|\langle x^*, f_n(\omega) \rangle\right| d\mu \leqslant M \|x^*\|$$

and so f is scalarly integrable. Since for every $A \in \Sigma$, we have $\sum_n \| \int_A f_n \, d\mu \| < \infty$, the series $\sum_n \int_A f_n \, d\mu$ is convergent to an element $\nu(A) \in X$. One easily see now that

$$\langle x^*, \nu(A) \rangle = \sum_n \int_{A \cap A_n} \langle x^*, f \rangle \, d\mu = \int_A \langle x^*, f \rangle \, d\mu.$$

This completes the proof in case of an atomless μ.

If μ is atomic, then $\mathbb{P}(\mu, X)$ is a Banach space. In the general case one decomposes the measure space into atomless and atomic parts obtaining $\mathbb{P}(\mu, X)$ as a direct sum of a barrelled space and a Banach space, which is again barrelled. □

The above result has been then generalized by Díaz et al. (1995) to the following form:

THEOREM 13.2. $\mathbb{P}(\mu, X)$ *is always ultrabornological.*

PROOF (*Sketch*). The proof is based on the following fact (see Díaz et al. (1995) for details).

PROPOSITION. *Let* $\{P_A \colon A \in \Sigma\}$ *be an* (Ω, Σ, μ)*-Boolean algebra of projections in a metrizable locally convex space* Z. *Assume that* $\{P_A \colon A \in \Sigma\}$ *is an equicontinuous family of operators and satisfies the following condition*:

(5) *If* $\langle \Omega_n \rangle$ *is a decreasing sequence in* Σ *with* $\mu(\bigcap \Omega_n) = 0$, $\langle x_n \rangle$ *is a bounded sequence in* Z *such that* $P_{\Omega_n}(x_n) = x_n$ *for all* n, *and* $\langle \alpha_n \rangle \in l_1$, *then the series* $\sum_n \alpha_n x_n$ *is convergent in* Z.

If $P_A(Z)$ *is ultrabornological for each* μ*-atom* A, *then* Z *is also ultrabornological.*

The rest of the proof is similar to the previous one. $P_A(f) := \chi_A \cdot f$ with $f \in \mathbb{P}(\mu, X)$ and $A \in \Sigma$. This is an equicontinuous family of projections forming an (Ω, Σ, μ)-Boolean algebra of projections in $\mathbb{P}(\mu, X)$. One can easily check the validity of the condition (5). If A is an atom then $P_A(\mathbb{P}(\mu, X))$ is isomorphic to X. Hence the assumptions of the proposition are satisfied and so $\mathbb{P}(\mu, X)$ is ultrabornological. $\square$

There was also an attempt to introduce a complete metric on $\mathbb{P}(\mu, X)$ (necessarily not equivalent to the original one). Heiliö (1988) defined such a topology composing the original semivariation norm with the convergence in measure. Setting for an arbitrary X-valued function f

$$|f|_\mu := \inf\{a\colon \mu^*\{\omega \in \Omega\colon \|f(\omega)\| \geqslant a\} \leqslant a\}.$$

Heiliö set then for each scalarly integrable f

$$|||f||| := \|f\| + |f|_\mu$$

and proved that $\mathbb{P}(\mu, X, |||\cdot|||)$ is complete. One of the consequences of this fact is another proof of the incompleteness of $\mathbb{P}(\mu, X)$.

There is a wide class of problems concerning the possibility of embedding of a space Y into $\mathbb{P}(\mu, X)$ provided Y is an isomorphic subspace of X. Diestel proved in 1988 (unpublished) that if the range of μ is infinite and X contains an isomorphic copy of c_0, then the completion of $\mathbb{P}_c(\mu, X)$ contains a complemented copy of c_0. Emmanuele (1992) generalized it to $\mathbb{P}(\mu, X)$. Díaz et al. (1993) proved that if X contains an isomorphic copy of c_0 and μ is non-atomic and perfect, then $\mathbb{P}(\mu, X)$ contains a complemented copy of c_0. This has been then generalized by Freniche.

THEOREM 13.3 (Freniche (1998)). *If the range of μ is infinite, then the following are equivalent*:

(i) *X contains a copy of c_0*;
(ii) *$\mathbb{P}(\mu, X)$ contains a copy of c_0*;
(iii) *$\mathbb{P}(\mu, X)$ contains a complemented copy of c_0*.

The next result concerns $\mathbb{P}(\lambda, X)$ as a subspace of the space $ca(\mathcal{L}, \lambda, X)$ of all X-valued λ-continuous measures equipped with the semivariation norm.

THEOREM 13.4 (Drewnowski and Lipecki (1995)). *If X is separable then $\mathbb{P}(\lambda, X)$ is an $\mathcal{F}_{\sigma\delta}$ but not $\mathcal{F}_\sigma$ subset of $ca(\mathcal{L}, \lambda, X)$.*

13.2. *Functions satisfying the strong law of large numbers*

Hoffmann-Jørgensen (1985) and Talagrand (1987) introduced the space $LLN(\mu, X)$ of X-valued functions satisfying the law of large numbers.

$$LLN(\mu, X) = \left\{ f : \Omega \to X\colon \exists a_f \in X \lim_{n\to\infty} \left\| a_f - \frac{1}{n}\sum_{i=1}^{n} f(\omega_i) \right\| = 0 \text{ for } \mu^\infty\text{-a.e. } (\omega_i) \in \Omega^\infty \right\}$$

where μ^∞ is the countable direct product of μ on Ω^∞ – the countable product of Ω.

Talagrand (1987) defined the Glivenko–Cantelli seminorm on $LLN(\mu, X)$ setting for an arbitrary function $f : \Omega \to X$

$$\|f\|_{GC} = \limsup_n \int^* h_n \, d\mu^\infty,$$

where

$$h_n(\omega) = \sup_{\|x^*\| \leqslant 1} \frac{1}{n} \sum_{i \leqslant n} \left| x^*(f(\omega_i)) \right|.$$

It has been proved by Beck (1963) that Bochner integrable functions satisfy the strong law of large numbers. If X is separable and a function $f : \Omega \to X$ satisfies the strong law of large numbers, then $f \in L_1(\mu, X)$, i.e., f is Bochner integrable (Hoffmann-Jørgensen (1985)).

Talagrand (1987) described completely the class of functions satisfying the law of large numbers. His proof is however too long to be presented here.

THEOREM 13.5 (Talagrand (1987)). *For a function $f : \Omega \to X$ and $\overline{\omega} = \langle \omega_n \rangle \in \Omega^\infty$ set $S_n(\overline{\omega}) = (1/n) \sum_{i=1}^n f(\omega_i)$. Then the following conditions are equivalent*:

(a) *f satisfies the strong law of large numbers*;
(b) *f is properly measurable and $\int_\Omega^* \|f\| \, d\mu < \infty$*;
(c) *For almost all $\overline{\omega} \in \Omega^\infty$, the sequence $(1/n)S_n(\overline{\omega})$ converges in norm*;
(d) *For each $\varepsilon > 0$ there is a simple function $g : \Omega \to X$ such that $\|f - g\|_{GC} \leqslant \varepsilon$.*

It easily follows from the above theorem that the Glivenko–Cantelli norm coincides on $LLN(\mu, X)$ with the ordinary $|\cdot|$ norm and each function satisfying the law of large numbers is Pettis integrable. Functions in $LLN(\mu, X)$ that are scalarly equivalent are not distinguishable by the GC-norm. This permits us to identify scalarly equivalent elements of $LLN(\mu, X)$ and investigate the quotient space (denoted also by $LLN(\mu, X)$).

In the context of $LLN(\mu, X)$ Talagrand (1987) successfully applied the concept of stable sets to description of the so called Glivenko–Cantelli classes of functions.

Dobric (1990) posed a question about completeness of $LLN(\mu, X)$. It turned out that the space is almost never complete and in general it is even not barrelled.

THEOREM 13.6 (Musiał (2001a)). *Assume that μ is not purely atomic. Then the space $LLN(\mu, X)$ is non-complete. If moreover $LLN(\mu, X, var)$ is complete, then $LLN(\mu, X)$ is even not barrelled.*

PROOF. $\mathbb{P}_c(\mu, X)$ endowed with the Pettis norm is non-complete (see Thomas (1976)). Let (f_n) be a Cauchy sequence in $\mathbb{P}_c(\mu, X)$ that is not convergent in $\mathbb{P}_c(\mu, X)$. It follows from Musiał (1979, Proposition 3) that for each $n \in \mathbb{N}$ there exists a simple function $h_n : \Omega \to X$ with $|f_n - h_n| < 1/n$. Since simple functions are properly measurable and $LLN(\mu, X) \subseteq \mathbb{P}_c(\mu, X)$, the sequence (h_n) is also Cauchy in $LLN(\mu, X)$.

Clearly, the sequence (h_n) is divergent in $LLN(\mu, X)$. This completes the proof of the noncompleteness.

If $LLN(\mu, X)$ were barrelled then applying the closed graph theorem to the identity map from $LLN(\mu, X)$ to $LLN(\mu, X, var)$ one would get its continuity. Hence the two norms would be equivalent. This is however impossible if X is infinite dimensional and μ is not purely atomic. $\square$

If each X-valued Pettis integrable function is weakly equivalent to a strongly measurable function (it is in case of X possessing RNP or in case of a measure compact X), then $LLN(\mu, X, var) = L_1(\mu, X)$ and so $LLN(\mu, X)$ is not barrelled.

13.3. *Functions with integrals of bounded variation*

We denote by $\mathbb{P}V(\mu, X)$ the set of those $f \in \mathbb{P}(\mu, X)$ for which $|\nu_f|$ is finite. $\mathbb{P}V(\mu, X)$ can be considered as a subspace of $cabv(\mu, X)$, endowed with the variation norm. We define a $\|\cdot\|_V$ norm on $\mathbb{P}V(\mu, X)$ by setting

$$\|f\|_V := |\nu_f|(\Omega).$$

Clearly $\left|f\right| \leqslant \|f\|_V$ for each $f \in \mathbb{P}(\mu, X)$.

Exactly as in the proof of Theorem 13.1 one can obtain

THEOREM 13.7 (Musiał (2001a)). *$\mathbb{P}V(\mu, X)$ is barrelled.*

PROPOSITION 13.8 (Musiał (2001a)). *If X has the weak Radon–Nikodým property, then $\mathbb{P}V(\mu, X)$ is a Banach space. In particular if Y is a Banach space not containing any isomorphic copy of l_1, then $\mathbb{P}V(\mu, Y^*)$ is complete.*

It is not known whether the space $\mathbb{P}V(\mu, X^*)$ is always complete. Assuming the validity of Axiom L, one can prove however the completeness of $\mathbb{P}V(\mu, X^*)$ for an arbitrary perfect measure μ.

THEOREM 13.9 (Musiał (2001a)). (Axiom L) *If μ is perfect then $\mathbb{P}V(\mu, X^*)$ is complete.*

PROOF (*Sketch*). Let (f_n) be a Cauchy sequence in $\mathbb{P}V(\mu, X^*))$ and let ν be the limit of (ν_{f_n}) in $cabv(\mu, X^*)$. The classical Radon–Nikodým theorem guarantees the existence for each $n \in \mathbb{N}$ of a function $h_n \in L_1(\mu)$ such that

$$|\nu - \nu_{f_n}|(E) = \int_E h_n \, d\mu, \tag{4}$$

for each $E \in \Sigma$. Moreover, since X^* has the W*RNP, there exists a weak* measurable function $f : \Omega \to X^*$ such that for each $x \in X$ and each $E \in \Sigma$

$$x\nu(E) = \int_E xf \, d\mu.$$

The theorem will be proved if we show that there exists a Pettis integrable function $\tilde{f}$ that is weak*-equivalent to f. To do it let us notice first that for each x from the closed unit ball of X and for each $n \in \mathbb{N}$ the equality (4) yields the relation

$$\left|\langle x, f - f_n\rangle\right| \leqslant h_n \quad \mu\text{-a.e.} \tag{5}$$

It follows that for each $n \in \mathbb{N}$ and each $x^{**} \in B_c(X^{**})$ we have

$$\left|\langle x^{**}, f - f_n\rangle\right| \leqslant h_n \quad \mu\text{-a.e.}$$

Since the sequence $\langle h_n\rangle$ is convergent to zero in the norm of $L_1(\mu)$, we obtain the measurability of all functions $x^{**}f$, with $x^{**} \in B_c(X^{**})$. If f is weak*-scalarly bounded, then the Pettis integrability of $\rho_0(f)$ follows from Theorem 4.28 and we may put $\tilde{f} = \rho_0(f)$. If f is arbitrary, we apply the decomposition Corollary 2.11. This completes the proof. □

The next result shows that sometimes the assumptions concerning Axiom L and the measure space are superfluous. We omit a direct and easy proof.

PROPOSITION 13.10 (Musiał (2001a)). *Let X be a separable Banach space and let Y be a closed linear subspace of X^*. Then $\mathbb{P}V(\mu, Y)$ is complete for an arbitrary μ.*

It is not known whether the space $\mathbb{P}V(\mu, X^*)$ is always complete. There is however an example due to D. Fremlin of a non-conjugate X, such that $\mathbb{P}V(\lambda, X)$ is non-complete. Since λ is perfect, it follows that in general the structure of the Banach space X is more important, than the properties of μ.

EXAMPLE 13.11 (*Fremlin, see Musiał* (2001a)). For each $t \in]0, 1]$ let $e_t \in l_\infty[0, 1]$ be the unit vector at t: $e_t(s) = 0$ if $s \neq t$ and $e_t(s) = 1$ if $s = t$. Consider $\chi_{[0,t]}$ as an element of $L_\infty[0, 1]$ and set for each $t \in]0, 1]$

$$x(t) = \chi_{[0,t]} + e_t/t^2.$$

Then, let X be the closed linear subspace of $L_\infty[0, 1] \times l_\infty[0, 1]$ generated by all $x(t)$, $t \in]0, 1]$ and all $(w, 0)$ with $w \in C[0, 1]$.

THEOREM 13.12 (Musiał (2001a)). *The space $\mathbb{P}V(\lambda, X)$ does not have the (K) property.*

13.4. *$LLN(\mu, X)$ equipped with the variation norm of integrals*

It has been proved in Talagrand (1987) that Pettis integrals of functions from $LLN(\mu, X)$ are measures of finite variation. Thus, it makes sense to equip the space $LLN(\mu, X)$ with the variation norm of the integrals. It will be denoted by $LLN(\mu, X, var)$. Contrary to $\mathbb{P}V(\mu, X^*)$, the space $LLN(\mu, X^*, var)$ behaves quite well.

THEOREM 13.13 (Musiał (2001a)). *$LLN(\mu, X^*, var)$ is a Banach space.*

PROOF (*Sketch*). Let ρ be a consistent lifting on (Ω, Σ, μ) and, let (f_n) be a Cauchy sequence in $LLN(\mu, X^*, var)$. Moreover, let ν be the limit of (ν_{f_n}) in $cabv(\mu, X^*)$ and, let $f : \Omega \to X^*$ be a weak*-density of ν with respect to μ. We shall split the proof into two parts. Assume first that f is weak*-scalarly bounded. Without loss of generality, we may assume then that $f = \rho_0(f)$. Moreover, let for each n a function $\psi_n \in L_1(\mu)$ be the RN-density of the measure $|\nu_{f_n}|$ and let $\Omega_{n,m} := \rho(\{\omega \in \Omega\colon \psi_n(\omega) \leqslant m\})$. Since $\psi_n \in L_1(\mu)$, there exists for each $n \in \mathbb{N}$ a number m_n such that $\mu(\Omega^c_{n,m_n}) < 1/n$.

Since $f_n \chi_{\Omega_{n,m}}$ is scalarly bounded and properly measurable, it follows from the consistency of ρ that $g_n := \rho_0(f_n \chi_{\Omega_{n,m_n}})$ is bounded and properly measurable, for each $n \in \mathbb{N}$. Hence, $g_n \in LLN(\mu, X^*, var)$ and since $f - g_n = \rho_0(f - g_n)$, we see that for each $n \in \mathbb{N}$ and some $M > 0$

$$\|f - g_n\|_{GC} \leqslant |\nu - \nu_{f_n}|(\Omega) + M\mu\big(\Omega^c_{n,m_n}\big).$$

That is f is approximated in the Glivenko–Cantelli norm by elements of $LLN(\mu, X^*)$. In virtue of Theorem 13.5, f satisfies the law of large numbers.

The general case follows by an appropriate decomposition of Ω. □

13.5. *Bounded Pettis integrable functions*

We denote by $P_\infty(\mu, X)$ the linear space

$$\Big\{f \in \mathbb{P}(\mu, X)\colon \|f\|_{P_\infty} := \sup_{\|x^*\| \leqslant 1} \|x^* f\|_\infty < \infty\Big\},$$

where $\|x^* f\|_\infty$ is the $L_\infty(\mu)$-norm of $x^* f$. One can easily check that $\|\cdot\|_{P_\infty}$ is a norm. Then, let $P^c_\infty(\mu, X) := \{f \in P_\infty(\mu, X)\colon \nu_f(\Sigma)$ is norm relatively compact$\}$.

Identifying weakly equivalent functions – we denote by $LLN_\infty(\mu, X)$ the linear space

$$\Big\{f \in LLN(\mu, X)\colon \|f\|_{P_\infty} := \sup_{\|x^*\| \leqslant 1} \|x^* f\|_\infty < \infty\Big\}.$$

PROPOSITION 13.14. *If X has the WRNP, then $P_\infty(\mu, X)$ is complete.*

THEOREM 13.15 (Musiał (2001b)). (Axiom L) *If μ is perfect, then $P_\infty(\mu, X^*)$ is complete.*

PROOF. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $P_\infty(\mu, X^*)$. Then for each $m, n \in \mathbb{N}$

$$\sup_{\|x^{**}\| \leqslant 1} \big\|x^{**} f_n - x^{**} f_m\big\|_\infty = \sup_{\|x\| \leqslant 1} \|x f_n - x f_m\|_\infty$$

and

$$\sup_{\|x\|\leqslant 1} \|xf_n - xf_m\|_\infty = \sup_\omega \left\| \rho_0(f_n)(\omega) - \rho_0(f_m)(\omega) \right\|, \tag{6}$$

where ρ is a lifting on $L_\infty(\mu)$. Consequently, the sequence $(\rho_0(f_n))$ is uniformly convergent to a function $h : \Omega \to X^*$ such that $h = \rho_0(h)$. Since for each $x^{**} \in B_c(X^{**})$ the functions $\rho_0(f_n)$ are measurable, according to Talagrand (1984, Theorem 6-2-1) (where the Axiom L is used), the functions $\rho_0(f_n)$ are in $P_\infty(\mu, X^*)$ and so h is scalarly measurable. Then, it is a consequence of Theorem 5.2 that $h \in P_\infty(\mu, X^*)$.

Thus, using (6), with h rather then f_m, we get

$$\begin{aligned} \lim_n \|f_n - h\|_{P_\infty} &= \lim_n \sup_{\|x\|\leqslant 1} \|xf_n - xh\|_\infty \\ &= \lim_n \sup_\omega \left\| \rho_0(f_n)(\omega) - \rho_0(h)(\omega) \right\| = 0. \end{aligned}$$

This proves the completeness of $P_\infty(\mu, X^*)$. □

The above proof makes it obvious that in fact the following more general result holds true:

THEOREM 13.16 (Musiał (2001b)). *Let μ and X be arbitrary. If for each countable family $\mathcal{F} \subset P_\infty(\mu, X^*)$ there exists a lifting ρ such that $\rho_0(f)$ is μ-Pettis-integrable for each $f \in \mathcal{F}$, then $P_\infty(\mu, X^*)$ and $P_\infty^c(\mu, X^*)$ are complete.*

COROLLARY 13.17. *If X is separable, then for each μ the spaces $P_\infty^c(\mu, X^*)$ and $P_\infty(\mu, X^*)$ are complete.*

Analysis of the proof of Theorem 13.15 when ρ is consistent shows the validity of the next result.

THEOREM 13.18 (Musiał (2001b)). *The space $LLN_\infty(\mu, X^*)$ is complete.*

Considering each X-valued function as an X^{**}-valued function we get the following result in case of an arbitrary Banach space X:

THEOREM 13.19 (Musiał (2001b)). *The completion of the space $LLN_\infty(\mu, X)$ is a subspace of $LLN_\infty(\mu, X^{**})$. If Axiom L is satisfied and μ is perfect then the completion of $P_\infty(\mu, X)$ is a subspace of $P_\infty(\mu, X^{**})$.*

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