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A VARIATIONAL HENSTOCK INTEGRAL CHARACTERIZATION OF THE RADON–NIKODÝM PROPERTY

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ABSTRACT. A characterization of Banach spaces possessing the Radon–Nikodým property is given in terms of finitely additive interval functions. We prove that a Banach space X has the RNP if and only if each X-valued finitely additive interval function possessing absolutely continuous variational measure is a variational Henstock integral of an X-valued function. Due to that characterization several X-valued set functions that are only finitely additive can be represented as integrals.

Introduction

There are known several characterizations of Banach spaces possessing the Radon–Nikodým property (cf. [4]). Those based on measure theory always use countably additive measures. On the other hand, there are no characterizations in terms of purely additive set functions. Of course, there are many publications concerning the Radon–Nikodým theorem for finitely additive set functions but none of them is then used to get a characterization of Banach spaces. It is the objective of this paper to fill up that gap, at least partially. During the last twenty years, the theory of gauge integrals have been intensively developed, and we took into our consideration the Henstock type integrals. More precisely, we investigate the variationally Henstock integrable functions that seem to be the most convenient for this topic. Their primitives are additive interval functions and in case the target Banach space is the real line, they have been characterized by Bongiorno, Di

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Piazza and Skvortsov as the functions which variational measure (see Definition 1.1) is absolutely continuous with respect to the Lebesgue measure (see [1]).

In [14, Theorem 2], Skvortsov and Solodov attempted to extend this theorem to Banach space valued functions, under the hypothesis that the Banach space possesses the Radon–Nikodym property. The integral involved in their extension is the Denjoy–Bochner integral, but Solodov [15, Theorem 2.2.] had shown that the Denjoy–Bochner and the variational Henstock integral are equivalent. Unfortunately, the proof of [14, Theorem 2] has a gap. In fact, it is not proved (and no reference is given to such a proof) that RNP implies that each X-valued BVG_*-function is almost everywhere differentiable. Only then it would have been possible to apply previous Theorem 1, where the required differentiability had been assumed.

In Theorem 3.6 of this paper, we supplement the above mentioned gap (this is the implication (i) $\Rightarrow$ (ii)), and our proof is not quite simple. In any case, such a proof cannot be obtained just by the replacement of the absolute value of numbers in the classical proof by the norm. In this paper, we also show that the theorem formulated by Skvortsov and Solodov really holds true (in our terminology it means that conditions (iv) and (vi) are equivalent, provided X has RNP) but our proof is different from that in [14]. Simply we prove the result directly for the variational Henstock integral without any appealing to the Denjoy–Bochner integral. Our result is also much more general, among others we prove that in fact RNP is equivalent to each of these two above mentioned conditions. In particular, (iv) and (vi) are direct consequences of RNP. Because we use only the variational Henstock integral, we do not need to refer to the paper of Solodov [15].

We obtain also other interesting characterizations of Banach spaces possessing the RNP, in terms of the differentiation of functions whose variational measure is absolutely continuous with respect to the Lebesgue measure (Theorem 3.6). This form is similar to the classical characterization of the RNP via differentiation of absolutely continuous functions (cf. [4, p. 217]) but now it is based on a different collection of interval functions. It is worth to notice that integrals with absolutely continuous variational measures, even when they define a vector measure on the Borel subsets of $[0,1]$, they do not coincide with the Pettis integrals, in general. In each infinite dimensional Banach space, there are Pettis integrals possessing nonabsolutely continuous variational measure (see Remark 4.3). And conversely, for each Banach space, there are variationally H-integrable functions which are not Pettis integrable.

We think that it is quite surprising that in spite of the fact that there are no satisfactory Radon–Nikodým type theorems for finitely additive set functions (only some approximative ones), the RNP can be totally described in terms of finitely additive interval functions.

1. Preliminaries

Let $[0,1]$ be the unit interval of the real line equipped with the usual topology and the Lebesgue measure λ. We denote by $\mathcal{I}$ the family of all nontrivial closed subintervals of $[0,1]$ and by $\mathcal{L}$ the family of all Lebesgue measurable subsets of $[0,1]$.

If $E \subset [0,1]$, we denote by $|E|_e$ and by $|E|$ its outer Lebesgue measure and its Lebesgue measure, in case $E \in \mathcal{L}$, respectively. Throughout this paper, X is a Banach space with dual X^*. The closed unit balls of X and X^* are denoted, respectively, by $B(X)$ and $B(X^*)$. If μ is a measure on $\mathcal{L}$, then by $\mu \ll \lambda$ we mean that $|E| = 0$ implies $\mu(E) = 0$. A mapping $\nu : \mathcal{L} \to X$ is said to be an X-*valued measure* if ν is countable additive in the norm topology of X. An X-valued measure ν is said to be λ-*continuous* if $|E| = 0$ implies $\nu(E) = 0$. The *variation* of an X-valued measure ν is denoted by $|\nu|$. A function $f : [0,1] \to X$ is said to be *weakly measurable* if for each $x^* \in X^*$ the real function x^*f is measurable; f is said to be *strongly measurable*, or simply *measurable*, if there is a sequence of simple functions f_n with $\lim_n \|f_n(t) - f(t)\| = 0$, for almost all $t \in [0,1]$.

In the sequel, the symbol $\int f \, d\lambda$ denotes, respectively, the Lebesgue integral of f, if f is a scalar function, and the Bochner integral of f, if f is a vector valued function.

A *tagged partition in* $[0,1]$, or simply a *partition in* $[0,1]$ is a finite collection of pairs $\mathcal{P} = \{(I_1,t_1),\ldots,(I_p,t_p)\}$, where $I_1,\ldots,I_p$ are nonoverlapping subintervals of $[0,1]$ and $t_i \in I_i$, $i = 1,\ldots,p$. Given a subset E of $[0,1]$, we say that the partition $\mathcal{P}$ is *anchored on* E if $t_i \in E$ for each $i = 1,\ldots,p$. If $\bigcup_{i=1}^p I_i = [0,1]$, we say that $\mathcal{P}$ is a *partition of* $[0,1]$. A *gauge* on $E \subset [0,1]$ is a positive function on E. For a given gauge δ, we say that a partition $\{(I_1,t_1),\ldots,(I_p,t_p)\}$ is δ-*fine* if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1,\ldots,p$.

Definition 1.1. A function $f : [0,1] \to X$ is said to be *variationally Henstock integrable*, or simply *variationally H-integrable*, if there exists an additive function $\Phi : \mathcal{I} \to X$, satisfying the following condition: given $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that

$$\sum_{i=1}^{p} \big\| f(t_i)|I_i| - \Phi(I_i) \big\| < \varepsilon, \tag{1.1}$$

for each δ-fine partition $\{(I_i,t_i) : i = 1,\ldots,p\}$ of $[0,1]$.

We set $\Phi(I) = (vH)\int_I f \, d\lambda$ and call the function Φ the *variational H-primitive of* f.

Replacing (1.1) by

$$\left\| \sum_{i=1}^{p} f(t_i)|I_i| - \Phi[0,1] \right\| < \varepsilon, \tag{1.2}$$

we obtain the definition of a *Henstock integrable* function.

We set $\Phi[0,1] := (vH)\int_0^1 f\,d\lambda$ (resp. $\Phi[0,1] := (H)\int_0^1 f\,d\lambda$) and call Φ the *variational Henstock* (resp. *Henstock*) *primitive* of f.

By $vH([0,1],X)$, we denote the set of all vH-integrable functions $f : [0,1] \to X$. In case of $X = \mathbb{R}$, the variational Henstock and the Henstock integrals coincide (cf. [8]). Moreover, rather the name of Henstock–Kurzweil instead of Henstock exists in the literature, we will denote by $HK([0,1])$ the space of all HK-integrable functions $f : [0,1] \to \mathbb{R}$, and by $H([0,1],X)$ the space of all H-integrable functions $f : [0,1] \to X$.

It is well known that if $f : [0,1] \to \mathbb{R}$ is HK-integrable on $[0,1]$, then f is HK-integrable on each $I \in \mathcal{I}$. We call the additive interval function $F(I) := (HK)\int_I f\,d\lambda$, $I \in \mathcal{I}$, the *HK-primitive* of f. It has been proven in [3] that a variationally H-integrable function is strongly measurable. Moreover, its variational H-primitive is continuous and differentiable a.e.

Let us recall that an interval function $\Phi : \mathcal{I} \to X$ is *differentiable* at $t \in [0,1]$ and $\Phi'(t)$ is its *derivative* at t if

$$\lim_{h\to 0}\left\|\frac{\Phi\langle t,t+h\rangle}{|h|} - \Phi'(t)\right\| = 0,$$

where $\langle a,b\rangle = [\min\{a,b\},\max\{a,b\}]$.

Definition 1.2. A function $f : [0,1] \to X$ is said to be *scalarly* measurable (*scalarly* integrable) if, for every $x^* \in X^*$, the function x^*f is Lebesgue measurable (integrable). A scalarly integrable function $f : [0,1] \to X$ is said to be *Pettis integrable* if, for each set $A \in \mathcal{L}$ there exists a vector $w_A \in X$ such that for every $x^* \in X^*$

$$\langle x^*, w_A\rangle = \int_A x^* f(t)\,dt.$$

We call w_A the *Pettis integral* of f over A and we write $w_A := (P)\int_A f(t)\,dt$.

For further information concerning the Pettis integral, we refer to [10] and [11].

2. Variational measures

Throughout this paper, the letter Φ will denote an arbitrary additive interval function $\Phi : \mathcal{I} \to X$. We will identify an interval function Φ with the point function $\Phi(t) = \Phi([0,t])$, $t \in [0,1]$; and conversely, we will identify a point function $\Phi : [0,1] \to X$ with the interval function $\Phi([a,b]) = \Phi(b) - \Phi(a)$, $[a,b] \in \mathcal{I}$.

Definition 2.1. Given $\Phi : \mathcal{I} \to X$, a gauge δ and a set $E \subset [0,1]$, we define

$$\mathrm{Var}(\Phi,\delta,E) = \sup\left\{\begin{array}{r}\sum_{i=1}^p \|\Phi(I_i)\| : \{(I_i,t_i) : i = 1,\ldots,p\}\ \delta\text{-fine}\\ \text{partition anchored on } E\end{array}\right\}.$$

Then we set

$$\mathrm{V}_{\Phi}(E) = \inf\{\mathrm{Var}(\Phi, \delta, E) : \delta \text{ gauge on } E\}.$$

If Φ is continuous, then $V_{\Phi}(I) \leq |\Phi|(I)$ for every $I \in \mathcal{I}$, where

$$|\Phi|(I) = \sup\left\{\sum_i \|\Phi(I_i)\| : I_i \text{ are nonoverlapping subintervals of } I\right\}.$$

We call V_{Φ} the *variational measure generated by* Φ. It is known that V_{Φ} is a metric outer measure on $[0,1]$ (see [16]). In particular, V_{Φ} is a measure over all Borel sets of $[0,1]$.

Definition 2.2. We say that the variational measure V_{Φ} is *σ-finite* if there is a sequence of (pairwise disjoint) sets F_n covering $[0,1]$ and such that $V_{\Phi}(F_n) < \infty$, for every $n \in \mathbb{N}$.

As V_{Φ} is also a regular (in Thomson's sense) measure (see [16, Theorem 3.15]), the previous definition is equivalent to that in which the sets F_n are in $\mathcal{L}$.

Corollary 2.3. *If $V_{\Phi} \ll \lambda$, then Φ is continuous on $[0,1]$ and V_{Φ} is σ-finite.*

Proof. Let t_0 be a point in $[0,1]$. As $V_{\Phi}(\{t_0\}) = 0$, for every $\varepsilon > 0$ there is a gauge δ such that for each δ-fine interval I containing t_0 we have $\|\Phi(I)\| < \varepsilon$ and so Φ is continuous in t_0.

By repeating, with obvious changes, the proof of [6, Theorem 1] (see also [1, Theorems 1 and 5]) we infer that V_{Φ} is σ-finite. ☐

Definition 2.4. A function $\Phi\colon [0,1] \to X$ is said to be BV_* on a set $E \subseteq [0,1]$ if $\sup \sum_{i=1}^{n} \omega(\Phi(J_i)) < +\infty$, where the supremum is taken over all finite collections $\{J_1, \ldots, J_n\}$ of nonoverlapping intervals in $\mathcal{I}$ with end-points in E, and the symbol $\omega(\Phi(J))$ stands for $\sup\{\|\Phi(u) - \Phi(z)\| : u, z \in J\}$. The function Φ is said to be BVG_* on $[0,1]$ if $[0,1] = \bigcup_n E_n$ and Φ is BV_* on each E_n.

In case of a continuous real valued function Φ, the next result has been proven in [16, Theorem 7.8], using arguments related to the structure of the real line. Here, we give a direct proof for vector valued functions.

Theorem 2.5. *Φ is BVG_* on $[0,1]$ if and only if V_{Φ} is σ-finite.*

Proof. "Only if" part. It is enough to prove that if Φ is BV_* on a set E, then V_{Φ} is σ-finite on E. So, let $M > 0$ be such that $\sum_{i=1}^{n} \omega(\Phi(J_i)) < M$, for each collection $\{J_1, \ldots, J_n\}$ of non overlapping intervals in $\mathcal{I}$ with end-points in E. The assertion is obvious in case the set E is countable. If E is uncountable, then we may always find $F \subset E$ such that $E \setminus F$ is at most countable and the points $\inf F$ and $\sup F$ are not isolated in F. Fix such an F. For each $k \in \mathbb{N}$ with $\inf(F) + 1/k < \sup(F) - 1/k$, we define

$F_k = F \cap (\inf(F)+1/k, \sup(F)-1/k)$. We will prove that $V_\Phi(F_k) \le 2M$, for each k such that $\inf(F)+1/k < \sup(F)-1/k$. This completes the proof, since $F = \bigcup_k F_k$.

Fix a gauge δ on F_k and take a δ-fine partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ anchored in F_k. We can assume that $t_1 < t_2 < \cdots < t_p$. Now take $t_0, t_{p+1} \in F$ with $t_0 < t_1$, $t_{p+1} > t_p$, and define $J_1 = (t_0,t_1), J_2 = (t_1,t_2),\dots,J_{p+1} = (t_p,t_{p+1})$. Then

$$\sum_{i=1}^{p} \|\Phi(I_i)\| \le 2\sum_{i=1}^{p+1} \omega(\Phi(J_i)) < 2M.$$

Hence, $V_\Phi(F_k) \le V(\Phi,\delta,F_k) \le 2M$.

"If" part. Let $A \subset [0,1]$ be such that $V_\Phi(A) < \infty$. That is, there is $M > 0$ and a gauge δ such that for each δ-fine partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ anchored in A, we have

$$\sum_{i=1}^{p} \|\Phi(I_i)\| \le M. \tag{2.1}$$

Then for each $k \in \mathbb{N}$ define $A_k = \{t \in A : \delta(t) > 1/k\}$. Since $A = \bigcup_k A_k$ and $A_k = \bigcup_{s=0}^{k-1}(A_k \cap [\frac{s}{k}, \frac{s+1}{k}])$, it is enough to prove that the function Φ is BV_* on $A_k \cap [\frac{s}{k}, \frac{s+1}{k}]$, for each $k = 1,2,\dots,$ and for each $s = 0,\dots,k-1$. Fix s and k and set $B_{ks} = A_k \cap [\frac{s}{k}, \frac{s+1}{k}]$. Now take any finite family of nonoverlapping intervals $\{(\alpha_1,\beta_1),\dots,(\alpha_p,\beta_p)\}$ with endpoints in A_k, and let $\alpha_j < u_j < v_j < \beta_j$, for each j. Then the families $\{(\alpha_j,\beta_j),\alpha_j\}$, $\{(\alpha_j,u_j),\alpha_j\}$, and $\{(v_j,\beta_j),\beta_j\}$ are δ-fine partitions anchored in B_{ks}. Hence, according to (2.1), we have

$$\sum_{j=1}^{p} \|\Phi(\beta_j)-\Phi(\alpha_j)\| \le M, \qquad \sum_{j=1}^{p} \|\Phi(u_j)-\Phi(\alpha_j)\| \le M,$$
$$\sum_{j=1}^{p} \|\Phi(\beta_j)-\Phi(v_j)\| \le M.$$

Thus,

$$\sum_{j=1}^{p} \|\Phi(v_j)-\Phi(u_j)\| \le 3M.$$

So $\sum_{j=1}^{p} \omega(\Phi(\alpha_j-\beta_j)) \le 3M$, and Φ is BV_* on B_{ks}. □

3. The main result

In the sequel, we need the following proposition.

COROLLARY 3.1 (13, Theorem 7.5.1). *Let $f : [0,1] \to X$ be a variationally H-integrable function and let $\Phi : \mathcal{I} \to X$ be its variational H-primitive. Then $V_\Phi \ll \lambda$.*

REMARK 3.2. There exists an interval function $\Phi : \mathcal{I} \to X$ such that $V_\Phi \ll \lambda$, but Φ is neither a variational H-primitive nor an H-primitive.

Proof of Corollary 3.1. Let $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(t) = \begin{cases} 1, & \text{if } \frac{2k}{2^n} \le t \le \frac{2k+1}{2^n}, k = 0, \dots, 2^{n-1}-1; \\ -1, & \text{if } \frac{2k+1}{2^n} \le t \le \frac{2k+2}{2^n}, k = 0, \dots, 2^{n-1}-1. \end{cases}$$

The primitives of f_n's are given by

$$\Phi_n(t) = \begin{cases} t - \frac{2k}{2^n}, & \text{if } \frac{2k}{2^n} \le t \le \frac{2k+1}{2^n}, k = 0, \dots, 2^{n-1}-1; \\ \frac{2k+2}{2^n} - t, & \text{if } \frac{2k+1}{2^n} \le t \le \frac{2k+2}{2^n}, k = 0, \dots, 2^{n-1}-1. \end{cases}$$

We have always $|\Phi_n(t)| \le 2^{-n}$ and so, if we define a set function Φ by setting $\Phi(I) := \langle \Phi_n(I) \rangle_{n \in \mathbb{N}}$, then $\Phi : \mathcal{I} \to c_0$. Take now $E \in \mathcal{L}$ with $|E| = 0$ and a positive ε. Then fix an open set $U \supset E$ with $|U| < \varepsilon$. We have then for each partition $\{(I_1,t_1), \dots, (I_p,t_p)\}$ anchored in E and such that $\bigcup_{i=1}^p I_i \subset U$

$$\begin{aligned} \sum_{i=1}^m \|\Phi(I_i)\| &= \sum_{i=1}^m \left\| \sum_n \Phi_n(I_i) e_n \right\| \\ &\le \sum_{i=1}^m \sup_n |\Phi_n(I_i)| \le \sum_{i=1}^m \lambda(I_i) < \lambda(U) < \varepsilon, \end{aligned}$$

where $\{e_n : n \in \mathbb{N}\}$ is the standard basis of c_0. Hence, $V_\Phi \ll \lambda$.

Φ cannot be neither an H-primitive nor a vH-primitive of any c_0-valued function, because $f := \langle f_n \rangle$ is not c_0-valued. □

This remark implies that condition $V_\Phi \ll \lambda$ in Proposition 3.1 is only necessary, in general. To characterize the Banach spaces for which it is also sufficient we need the following two lemmata.

LEMMA 3.3. *Let X be a Banach space and let $\nu : \mathcal{L} \to X$ be a λ-continuous measure of finite variation. If $\Phi : \mathcal{I} \to X$ is defined by $\Phi(I) := \nu(I)$, for all $I \in \mathcal{I}$, then V_Φ is finite, $V_\Phi \ll \lambda$ and $V_\Phi(E) \le |\nu|(E)$, whenever $E \in \mathcal{L}$.*

Proof. Since ν is an X-valued measure and λ is finite, the λ-continuity of ν implies $\lim_{\lambda(A)\to 0} \|\nu(A)\| = 0$. It follows that the function Φ is continuous, and $V_\Phi(I) \le |\nu|(I)$, for $I \in \mathcal{I}$. Consequently $V_\Phi(U) \le |\nu|(U)$, if U is open. Now let $E \in \mathcal{L}$ and let $U \supseteq E$ be open; then $V_\Phi(E) \le V_\Phi(U) \le |\nu|(U)$. So by the outer regularity of $|\nu|$, we have $V_\Phi(E) \le |\nu|(E)$. This easily implies $V_\Phi[0,1] < \infty$, and $V_\Phi \ll \lambda$. □

DEFINITION 3.4. A function $f : [0,1] \to X$ is said to be *Lipschitz at the point* $t \in [0,1]$ if there exist two positive constants C and δ such that

$$\|f(t+h) - f(t)\| \le C|h|,$$

for all $h \in \mathbb{R}$, with $|h| < \delta$.

The next lemma has been proven in [2, Proposition 1]. Here, we enclose it with the proof in order to make the paper self-contained.

Lemma 3.5. *Let X have the RNP and let $f:[0,1]\to X$. Denote by G the set of all points $t\in[0,1]$ at which f is Lipschitz. Then f is differentiable a.e., in G.*

Proof. For each natural n, let G_n denote the set of all $t\in G$ such that

$$\|f(t+h)-f(t)\|\le n|h|, \quad \text{whenever } |h|<\frac{1}{n}.$$

Clearly, $G=\bigcup G_n$ and it is easy to see that each G_n is a closed set.

Let f_n be the extension of $f\,|_{G_n}$ to $[0,1]$, such that f_n is linear on each contiguous interval of G_n. It is easy to prove that f_n is a Lipschitz function on $[0,1]$. Since X has the RNP, every function $h:[0,1]\to X$ of bounded variation is differentiable a.e., in $[0,1]$ (see [4, p. 217]). Then if $\Gamma_n\subset[0,1]$ is the set of all differentiability point of f_n, $|[0,1]\setminus\Gamma_n|=0$. Now denote by $\tilde{G}_n$ the set of all points $t\in G_n$ at which the distance function $\operatorname{dist}(t,G_n)$ is differentiable. Since $\operatorname{dist}(t,G_n)$ is Lipschitz, then $|G_n\setminus\tilde{G}_n|=0$. Hence, $|G_n\setminus(\Gamma_n\cap\tilde{G}_n)|=0$.

Define $N=\bigcup_n(G_n\setminus(\Gamma_n\cap\tilde{G}_n))$ and let $t\in G\setminus N$. Then there exists n such that $t\in\Gamma_n\cap\tilde{G}_n$. We are proving that f is differentiable at the point t.

Let $0<\varepsilon<2n$. By the differentiability of f_n and $\operatorname{dist}(t,G_n)$ at the point t, there exists $\delta_\varepsilon\in(0,\frac{1}{n})$ such that

$$\left\|\frac{f_n(t+h)-f_n(t)}{h}-f_n'(t)\right\|<\frac{\varepsilon}{2}, \tag{3.1}$$

and

$$\operatorname{dist}(t+h,G_n)<\frac{\varepsilon}{2(\|f_n'(t)\|+n)}|h|,$$

for each $0<|h|<\delta_\varepsilon$.

Then for any fixed $0<|h|<\delta_\varepsilon$, we can find $\bar{t}\in G_n$ such that

$$|t+h-\bar{t}|<\frac{\varepsilon}{2(\|f_n'(t)\|+n)}|h|. \tag{3.2}$$

Now $f_n=f$ on G_n. Therefore, by (3.1) and (3.2), we have

$$\begin{aligned}
&\|f(t+h)-f(t)-f_n'(t)t\|\\
&\quad\le\|f(\bar{t})-f(t)-f_n'(t)(\bar{t}-t)\|+\|f(\bar{t})-f(t+h)\|+\|f_n'(t)\||t+h-\bar{t}|\\
&\quad\le\frac{\varepsilon}{2}|\bar{t}-t|+n|t+h-\bar{t}|+\|f_n'(t)\||t+h-\bar{t}|\\
&\quad<\frac{\varepsilon}{2}|h|+\frac{\varepsilon}{2}|h|=\varepsilon|h|.
\end{aligned}$$

Since this is true for any $0<|h|<\delta_\varepsilon$, we get the differentiability of f at the point t. □

The following characterization of the RNP is the main result of this paper.

THEOREM 3.6. *Let X be a Banach space. Then the following conditions are equivalent*:

(i) *X has the Radon–Nikodým property*;

(ii) *If $\Phi:\mathcal{I}\to X$ is BVG_* on $[0,1]$, then Φ is differentiable a.e. in $[0,1]$*;

(iii) *If V_Φ is σ-finite, then Φ is differentiable a.e. in $[0,1]$*;

(iv) *If $V_\Phi \ll \lambda$, then Φ is differentiable a.e. in $[0,1]$*;

(v) *If $V_\Phi \ll \lambda$, then Φ is differentiable a.e. in $[0,1]$, $\Phi' \in vH([0,1],X)$ and*

$$\Phi(I)=(vH)\int_I \Phi'(t)\,dt \quad \textit{for every } I\in\mathcal{I};$$

(vi) *If $V_\Phi \ll \lambda$, then there exists $f\in vH([0,1],X)$ such that*

$$\Phi(I)=(vH)\int_I f(t)\,dt \quad \textit{for every } I\in\mathcal{I}.$$

Proof. (i) ⇒ (ii) Let $[0,1]=\bigcup_n E_n$ be a decomposition of $[0,1]$ such that Φ is BV_* on each E_n. Then define $f(t)=\Phi([0,t])$ for $t\in[0,1]$, and set

$$G_n=\{t\in[0,1]:\|f(u)-f(t)\|<n|u-t|,\\ \text{for each } u\in[0,1] \text{ with } |u-t|<1/n\},$$

and $G=[0,1]\setminus\bigcup_n G_n$. By Lemma 3.5, it is enough to prove that $|G|=0$.

Assume, by contradiction, that $|G|_e>0$. Then there exists $k\in\mathbb{N}$ such that $|E_k\cap G|_e>0$. By definition of G, to each $t\in G$ and each $n\in\mathbb{N}$ there exists $u\in[0,1]$ such that $|u-t|<1/n$ and $\|f(u)-f(t)\|>n|u-t|$. Given $M>0$, take $n\in\mathbb{N}$ such that $n|E_k\cap G|_e>2M$ and let

$$\mathcal{F}=\{[t,u]:t\in E_k\cap G, |u-t|<1/n, \|f(t)-f(u)\|>n|u-t|\}.$$

It is easy to check that $\mathcal{F}$ is a Vitali covering of $E_k\cap G$. Then there exists a finite number of disjoint intervals $\{[t_i,u_i]\}$ such that $[t_i,u_i]\in\mathcal{F}$ and $|E_k\cap G|_e<2\sum_i|u_i-t_i|$. Consequently,

$$\sum_i\|F[t_i,u_i]\|=\sum_i\|f(u_i)-f(t_i)\|>n\sum_i|u_i-t_i|>n\frac{|E_k\cap G|_e}{2}>M.$$

By the arbitrariness of M, this implies that Φ is not BV_* on E_k, which is in contradiction with the hypothesis.

(ii) ⇒ (iii) If V_Φ is σ-finite, then Φ is BVG_* (by Theorem 2.5) and so the result is a consequence of (ii).

(iii) ⇒ (iv) Assume that $V_\Phi\ll\lambda$. According to Proposition 2.3, V_Φ is σ-finite. Then condition (iii) implies the required differentiability of Φ.

(iv) ⇒ (v) Assume that $V_\Phi\ll\lambda$ and denote by N the set of all $t\in[0,1]$ such that $\Phi'(t)$ does not exist. By hypothesis $|N|=0$ and $V_\Phi(N)=0$. Now define $f:[0,1]\to X$ as follows:

$$f(t)=\begin{cases}\Phi'(t), & \text{if exists;}\\ 0, & \text{otherwise.}\end{cases}$$

We are going to prove that f is vH-integrable with variational H-primitive Φ. Fix $\varepsilon > 0$. If $t \in [0,1] \setminus N$ define $\delta(t)$ such that

$$\big\|\Phi(I) - \Phi'(t)|I|\big\| < \varepsilon|I|, \tag{3.3}$$

for each interval $I \in \mathcal{I}$ having t as one of its end-points and with $|I| < \delta(t)$. If $t \in N$, taking into account that $V_\Phi(N) = 0$, define $\delta(t)$ such that

$$\sum_{j=1}^{s} \|\Phi(J_j)\| < \varepsilon, \tag{3.4}$$

for each δ-fine partition $\{(J_j, t_j) : j = 1, \ldots, s\}$ anchored in N.

Now let $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ be a δ-fine partition of $[0,1]$. We may assume that the tags t_i of the partition are end-points of the corresponding interval I_i. Therefore, by (3.3) and (3.4), we have

$$\begin{aligned} &\sum_{i=1}^{p} \big\|f(t_i)|I_i| - \Phi(I_i)\big\| \\ &\quad = \sum_{t_i \in N} \|\Phi(I_i)\| + \sum_{t_i \in [0,1]\setminus N} \big\|f(t_i)|I_i| - \Phi(I_i)\big\| \\ &\quad < 2\varepsilon. \end{aligned}$$

Thus, f and so also Φ', is variationally H-integrable and this gives (v).

(vi) $\Rightarrow$ (i) Assume that each additive function $\Phi : \mathcal{I} \to X$ such that $V_\Phi \ll \lambda$ is a vH-primitive and let $\nu : \mathcal{L} \to X$ be a λ-continuous measure of finite variation. Define $\Phi : \mathcal{I} \to X$ by $\Phi(I) := \nu(I)$. It follows from Lemma 3.3 that $V_\Phi \ll \lambda$. Hence, there is a variationally Henstock integrable $f : [0,1] \to X$, such that

$$\Phi(I) = (vH) \int_I f \, d\lambda \quad \text{for every } I \in \mathcal{I}.$$

Consequently, for $x^* \in X^*$ we have

$$x^*\nu(I) = x^*\Phi(I) = (HK) \int_I x^* f \, d\lambda \quad \text{for every } I \in \mathcal{I}, \tag{3.5}$$

and $V_{x^*\Phi} \ll \lambda$ (see [1]).

Moreover, since ν is countably additive and of finite variation, for every $x^* \in X^*$ the measure $x^*\nu$ is bounded and of finite variation. Therefore, $V_{x^*\Phi}([0,1]) \le |x^*\nu|([0,1]) < \infty$. Then by [12, Proposition 5], $x^* f \in L_1[0,1]$ for each $x^* \in X^*$.

Let us fix $x^* \in B(X^*)$ and let $\mathfrak{A}$ be the algebra generated by all the intervals $(a,b] \subset [0,1]$. Then it follows from (3.5) that

$$\int_A x^* f \, d\lambda = x^*\nu(A) \quad \text{for every } A \in \mathfrak{A}.$$

But both sides of the above equality are real set functions countably additive on the algebra $\mathfrak{A}$ and so they can be uniquely extended to a measure on $\sigma(\mathfrak{A}) = \mathcal{B}[0,1]$, where $\mathcal{B}[0,1]$ denotes the family of all Borel subsets of $[0,1]$. This means that

$$\int_E x^* f \, d\lambda = x^* \nu(E) \quad \text{for every } E \in \mathcal{B}[0,1]. \tag{3.6}$$

Since both sides of (3.6) have unique extensions to $\mathcal{L}$, the above equality holds for all $E \in \mathcal{L}$. This proves that ν has a Pettis integrable Radon–Nikodým density f.

Moreover, [3, Theorem 9] constrains the strong measurability of f. Then f is Bochner integrable, the variation of ν being finite by assumption. Thus, X has the RNP. □

4. Pettis integrals

If $\nu : \mathcal{L} \to X$ is a λ-continuous measure of finite variation and if Φ is the interval function defined by $\Phi(I) := \nu(I)$, then $V_\Phi \ll \lambda$ (see Lemma 3.3). So if X has RNP, then Φ is a.e., differentiable. In case when a λ-continuous measure ν is only of σ-finite variation (as in the case of Pettis primitives which are not Bochner primitives), the variational measure V_Φ may be not λ-continuous. We have the following result describing this situation.

Corollary 4.1. *Let $f : [0,1] \to X$ be a strongly measurable Pettis integrable function, let ν be its Pettis integral and let $\Phi : \mathcal{I} \to X$ be the interval function defined by $\Phi(I) := \nu(I)$. Then $V_\Phi \ll \lambda$ if and only if $f \in vH([0,1], X)$.*

Proof. The "if" part follows by Proposition 3.1. To prove the "only if" part we observe that if f is Pettis integrable, then for each $x^* \in X^*$ we have $(x^*\Phi)' = x^* f$ a.e. Therefore, Φ is a continuous function with separable valued scalar derivative f. Since by hypothesis $V_\Phi \ll \lambda$, applying [9, Theorem 8] we get $f \in vH([0,1], X)$ with vH-primitive Φ. □

Corollary 4.2. *Let $f : [0,1] \to X$ be a strongly measurable Pettis integrable function, let ν be its Pettis integral and let $\Phi : \mathcal{I} \to X$ be the interval function defined by $\Phi(I) := \nu(I)$. If Φ is not differentiable on a set of positive outer Lebesgue measure, then $V_\Phi \not\ll \lambda$.*

Proof. It follows at once from the previous Corollary and from the observation that each variational H-primitive is differentiable a.e. □

Remark 4.3. Examples of such strongly measurable X-valued Pettis integrable functions, for an arbitrary infinite dimensional Banach space X, can be found in [5]. Their indefinite Pettis integrals are nowhere weakly differentiable. Hence, their indefinite Pettis integrals are nowhere strongly differentiable. In particular, none of them has absolutely continuous variational measure. For some of them, the upper derivative of the function $I \to \|\Phi(I)\|$ is infinite on a set of full Lebesgue measure. Then it follows that $|\Phi|(I) = \infty$, for every $I \in \mathcal{I}$.

Thus, in each infinite dimensional Banach space there are Pettis integrable functions which are not variationally Henstock integrable. And conversely, for each Banach space, there are variationally H-integrable functions which are not Pettis integrable.

It is worth to notice also that for every infinite dimensional Banach space there are functions that are Pettis and variationally Henstock integrable (hence their integrals are almost everywhere differentiable), and still they are not Bochner integrable (see [7]).

Added in proof. The following condition can be added to Theorem 3.6: (vii) If $V_\phi \ll \lambda$, then there exists a strongly measurable $f \in H([0,1], X)$ such that

$$\Phi(I) = (H) \int_I f(t)\, dt, \quad \text{for every } I \in \mathcal{I}.$$

The proof is contained in the proof of the implication (vi) $\Rightarrow$ (i).

We do not know if the strong measurability of f may be omitted.

References

[1] B. Bongiorno, L. Di Piazza and V. Skvortsov, *A new full descriptive characterization of Denjoy-Perron integral*, Real Analysis Exchange **21** (1995/96), 256–263. MR 1407278

[2] D. Bongiorno, *Stepanoff's theorem in separable Banach spaces*, Comment. Math. Univ. Carolinae **39** (1998), 323–335. MR 1651959

[3] S. S. Cao, *The Henstock integral for Banach-valued functions*, SEA Bull. Math. **16** (1992), 35–40. MR 1173605

[4] J. Diestel and J. J. Uhl, *Vector measures*, Math. Surveys, vol. 15, Amer. Math. Soc., Providence, RI, 1977. MR 0453964

[5] S. J. Dilworth and M. Girardi, *Nowhere weak differentiability of the Pettis integral*, Quest. Math. **18** (1995), 365–380. MR 1354118

[6] L. Di Piazza, *Varational measures in the theory of the integration in R^m*, Czechoslovak. Math. J. **51** (2001), 95–110. MR 1814635

[7] L. Di Piazza and V. Marraffa, *The McShane, PU and Henstock integrals of Banach valued functions*, Czechoslovak Math. J. **52** (2002), 609–633. MR 1923266

[8] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Graduate Studies in Math., vol. 4, Amer. Math. Soc., Providence, RI, 1994. MR 1288751

[9] V. Marraffa, *A descriptive characterization of the variational Henstock integral*, Proceedings of the International Mathematics Conference (Manila, 1998), Matimyás Mat. **22** (1999), 73–84. MR 1770168

[10] K. Musiał, *Topics in the theory of Pettis integration*, Rend. Istit. Mat. Univ. Trieste **23** (1991), 177–262. MR 1248654

[11] ———, *Pettis integral*, Handbook of Measure Theory I, Elsevier, Amsterdam, 2002, pp. 531–586. MR 1954622

[12] W. F. Pfeffer, *The Lebesgue and Denjoy-Perron integrals from a descriptive point of view*, Ricerche di Mat. **XLVIII** (1999), 211–223. MR 1760817

[13] S. Schwabik and Y. Guoju, *Topics in Banach space integration*, Series in Real Analysis, vol. 10, World Scientific, Hackensack, NJ, 2005. MR 2167754

[14] V. A. Skvortsov and A. P. Solodov, *A descriptive characterization of the Denjoy-Bochner integral and its generalizations*, Moscow University Mathematics Bulletin **57** (2002), 36–39. MR 1934075

[15] A. P. Solodov, *Riemann-type definition for the restricted Denjoy-Bochner integral*, Fundamentalnaya i Prikladnaya Matematika **7** (2001), 887–895. MR 1879304

[16] B. Thomson, *Derivatives of interval functions*, Mem. Amer. Math. Soc. **452** (1991). MR 1078198

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