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Approximation of Pettis Integrable Multifunctions with Values in Arbitrary Banach Spaces

Kazimierz Musiał*
University of Wrocław, Institute of Mathematics,
Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
musial@math.uni.wroc.pl

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Approximation of functions and multifunctions by simple functions and multifunctions plays an important role in the theory separable valued functions and multifunctions. Also in case of Pettis integrable functions with values in non-separable Banach spaces there exists a satisfactory approximation theory, but in case of Pettis integrable multifunctions with values in non-separable Banach spaces no such a theory exists. It is the aim of this paper to fill in that gap.

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Introduction

One of the most exploited property of multifunctions taking as their values (weakly) compact subsets of a separable Banach space is approximation of multifunctions by simple multifunctions and by selections.

In case of multifunctions having their values in non-separable Banach spaces the classical approximation is no more valid. First of all, multifunctions may be not Effros measurable and the selections are in general not strongly measurable. This excludes general approximation approach via simple functions. The second reason is that the weak topology restricted to weakly compact sets may be now not metrizable and this immediately eliminates some methods of proofs that used to be applied in case of separable Banach spaces. Consequently, the technique applied in this paper is closer to the methods used in the theory of Pettis integration of functions with values in non-separable Banach spaces.

Following the existing approximation and convergence theory for Pettis integrable

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functions (see [24] and [26]) I prove the corresponding approximation and convergence results for Pettis integrable multifunctions. The results are proved without invoking the existence of any selections. The main tool applied in the convergence part of the paper is a scalar equi-convergence in measure of a sequence of multifunctions. In case of strongly measurable functions on a finite measure space it is obvious that a.e. convergent sequence is convergent in measure. When functions fail to be strongly measurable, it is clear only that each composition with functionals is convergent in measure. Our approach shows that even in case of non-measurable functions and multifunctions, the a.e. convergence in the norm topology or in the Hausdorff metric yields a certain type of convergence in measure that is essentially stronger than the scalar convergence in measure. From a formal point of view this is simply the convergence in measure that is uniform on the unit ball of X^*, but it seems that that type of convergence has been totally overlooked, so far.

Here are the most essential results of the paper.

A characterization of $cb(X)$-valued Pettis integrable multifunctions that can be approximated in the Pettis metric by simple multifunctions (Theorems 2.3 and 4.6).

A characterization of $cb(X)$-valued Pettis integrable multifunctions that can be approximated in the Hausdorff metric by simple multifunctions (Theorems 3.4, 3.7, 4.3 and 4.4).

A characterization of $cb(X)$-valued multifunctions that satisfy the strong law of large numbers (Theorems 5.6 and 5.8).

1. Basic facts

This section contains definitions, notation and a few facts that are already mostly known.

Throughout (Ω, Σ, μ) is a complete probability space, Σ_μ^+ is the collection of all sets of positive measure, $\mathcal{N}(\mu) := \{E \in \Sigma : \mu(E) = 0\}$ and $A \cap \Sigma := \{A \cap E \colon E \in \Sigma\}$. $\sigma(\mathcal{E})$ is the σ-algebra generated by a family $\mathcal{E}$ of sets. (Ω, Σ, μ) is said to be separable, if it is separable in the Fréchet-Nikodým pseudometric. $\mathcal{L}$ denotes the collection of all Lebesgue measurable subsets of the set of real numbers $\mathbb{R}$ or $[0, 1]$ and, $\mathbb{N}$ is the set of all positive integers. X is a Banach space with its dual X^* and the closed unit ball of X is denoted by $B(X)$.

$c(X)$ denotes the collection of all nonempty closed convex subsets of X, $cb(X)$ is the collection of all bounded members of $c(X)$, $cwk(X)$ denotes the family of all weakly compact elements of $cb(X)$ and $ck(X)$ is the collection of all compact members of $cb(X)$. For every $C \in c(X)$ the *support function of* C is denoted by $s(\cdot, C)$ and defined on X^* by $s(x^*, C) = \sup\{\langle x^*, x\rangle \colon\ x \in C\}$, for each $x^* \in X^*$.

I assume that on $cb(X)$ the Hausdorff metric d_H is defined. It is known that $cb(X)$ endowed with d_H is a complete metric space. If $C, D \in cb(X)$, then we have Hörmander's formula $d_H(C, D) = \sup_{\|x^*\| \le 1} |s(x^*, C) - s(x^*, D)|$ (cf. [7, Théorème

II.18]). $cb(X)$ is considered with the Minkowski addition: $A \oplus B := \overline{A+B}$. If $A, B, C, D \subset \Omega$ are nonempty, then (cf. [19, Proposition I.1.17]) $d_H(A \oplus B, C \oplus D) \le d_H(A, C) + d_H(B, D)$.

Besides the metric convergence I will consider also the so called scalar convergence (cf. [29]). A sequence of sets $A_n \in cb(X)$ is *scalarly convergent* to $A \in cb(X)$ if $\lim_n s(x^*, A_n) = s(x^*, A)$, for every $x^* \in X^*$. A set $\emptyset \neq S \subset cb(X)$ is said to be *scalarly separable* in $cb(X)$, if there exists at most countable family $\{A_n : n \in \mathbb{N}\} \subset S$ such that for each $A \in S$ there is a subsequence $\{A_{n_k} : k \in \mathbb{N}\}$ that is scalarly convergent to A.

The weak* topology of X^* will be denoted by $\sigma(X^*, X)$.

Any map $\Gamma\colon \Omega \to c(X)$ is called a *multifunction.* I associate with each Γ the set

$$\mathscr{Z}_\Gamma := \{s(x^*, \Gamma) : \|x^*\| \le 1\},$$

where I consider functions, not equivalence classes of a.e. equal functions. The set of equivalence classes of functions scalarly equivalent will be denoted by $\mathbb{Z}_\Gamma$.

In the proofs I will often identify functions with their equivalence classes.

A function $f\colon \Omega \to X$ is called a *selection of* Γ if $f(\omega) \in \Gamma(\omega)$, for every $\omega \in \Omega$. If $A \subset X$ is nonempty, then I write $|A| := \sup\{\|x\|\colon x \in A\}$.

A map $M\colon \Sigma \to cb(X)$ is additive, if $M(A \cup B) = M(A) \oplus M(B)$ for every pair of disjoint elements of Σ. An additive map $M\colon \Sigma \to cb(X)$ called a *multimeasure* if $s(x^*, M(\cdot))$ is a finite measure, for every $x^* \in X^*$. If M is a point map, then I talk about measure. If $M\colon \Sigma \to cb(X)$ is countably additive in the Hausdorff metric, then it is called an *h-multimeasure.* It is known that if $M\colon \Sigma \to cwk(X)$, then M is a multimeasure if and only if it is an h-multimeasure (cf. [19, Theorem 8.4.10]). A multimeasure $M\colon \Sigma \to c(X)$ is said to be *μ-continuous* if $\mu(E) = 0$ yields $M(E) = \{0\}$, for every $E \in \Sigma$. $M(\Sigma)$ will denote the set $\{M(E) : E \in \Sigma\}$ and $\bigcup M(\Sigma)$ will denote the set $\bigcup\{M(E) : E \in \Sigma\}$. Let me point out here that in my paper [27] the set $\bigcup\{M(E) : E \in \Sigma\}$ was denoted by $M(\Sigma)$. I think that the just proposed notation is better. If $m : \Sigma \to X$ is a vector measure, then its range $\{m(E) : E \in \Sigma\}$ is denoted by $m(\Sigma)$ and is identified with $\bigcup m(\Sigma)$.

A vector measure $m\colon \Sigma \to X$ such that $m(A) \in M(A)$, for every $A \in \Sigma$, is called a *selection* of M. $\mathcal{S}(M)$ will denote the set of all countably additive selections of M.

$\mathcal{S}_\Gamma$ denotes the family of all scalarly measurable selections of a multifunction Γ.

A family $W \subset L_1(\mu)$ (or just a family W of integrable functions, not equivalence classes) is *uniformly integrable* if W is bounded in $L_1(\mu)$ and for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(A) < \delta$, then $\sup_{f \in W} \int_A |f|\, d\mu < \varepsilon$. Equivalently, W is uniformly integrable if and only if $\lim_{C \to 0} \sup_{f \in W} \int_{\{|f| > C\}} |f|\, d\mu = 0$.

I shall apply also the following fact (cf. [11, Theorem I.2.4]): Let $\{f_t\colon t \in T\}$ be a bounded subset of $L_1(\mu)$ and let $\mu_t : \Sigma \to \mathbb{R}$ be defined by $\mu_t(E) := \int_E f_t\, d\mu$, for each $E \in \Sigma$ and $t \in T$. Then $\{f_t : t \in T\}$ is uniformly integrable if and only if $\{\mu_t : t \in T\}$ is uniformly σ-additive.

Definition 1.1. A multifunction $\Gamma : \Omega \to c(X)$ is said to be *scalarly measurable* if for every $x^* \in X^*$, the map $s(x^*, \Gamma(\cdot))$ is measurable with respect to Σ. Sometimes I shall consider also multifunctions that are scalarly measurable with respect to a smaller σ-algebra. A multifunction is *simple* if it scalarly measurable and takes finitely many values. Scalarly measurable multifunctions $\Gamma, \Delta : \Omega \to c(X)$ are *scalarly equivalent* if $s(x^*, \Gamma) = s(x^*, \Delta)$ a.e., for each $x^* \in X^*$ separately. A multifunction $\Gamma\colon \Omega \to c(X)$ is *scalarly integrable* if $s(x^*, \Gamma)$ is integrable for every $x^* \in X^*$. $\Gamma\colon \Omega \to c(X)$ is *scalarly bounded* if there is a constant $M \geq 0$ such that for every $x^* \in X^*$

$$|s(x^*, \Gamma)| \leq M\|x^*\| \quad \text{a.e.}$$

I say that a space $Y \subset X$ *determines* a multifunction $\Gamma\colon \Omega \to c(X)$ (see [27, Definition 2.1]) if $s(x^*, \Gamma) = 0$ μ-a.e. for each $x^* \in Y^\perp$ (the exceptional sets depend on x^*). □

It is known (cf. [27]) that each scalarly measurable multifunction Γ can be represented as $\Gamma = \sum_{n=1}^{\infty} \Gamma_n \chi_{E_n}$, where Γ_n's are scalarly bounded and $\Sigma \ni E_n$'s are pairwise disjoint.

Definition 1.2. Denote by $\mathcal{C}$ an arbitrary nonvoid subfamily of $cb(X)$. A scalarly integrable multifunction $\Gamma\colon \Omega \to c(X)$ is *Pettis μ-integrable* in $\mathcal{C}$, if for each $A \in \Sigma$ there exists a set $M_\Gamma(A) \in \mathcal{C}$ such that

$$s(x^*, M_\Gamma(A)) = \int_A s(x^*, \Gamma)\, d\mu \quad \text{for every } x^* \in X^*. \tag{1}$$

I call $M_\Gamma(A)$ the *Pettis integral of* Γ over A and set $(P)\int_A \Gamma\, d\mu := M_\Gamma(A)$. If no confusion is possible, I shall write simply $\int_A \Gamma\, d\mu$. □

It follows from (1) that M_Γ is a μ-continuous multimeasure. Moreover,

$$\sup_{\|x^*\| \leq 1} \int_\Omega |s(x^*, \Gamma)|\, d\mu < \infty\,. \tag{2}$$

Indeed,

$$\int_\Omega |s(x^*, \Gamma)|\, d\mu \leq 2 \sup_{E \in \Sigma} \left| \int_E s(x^*, \Gamma)\, d\mu \right| = 2 \sup_{E \in \Sigma} |s(x^*, M_\Gamma(E))| < \infty,$$

where the last inequality follows from the fact that $s(x^*, M_\Gamma(\cdot))$ is for each x^* a scalar measure.

Hence, by the Banach–Steinhaus Theorem, $\bigcup M_\Gamma(\Sigma)$ is bounded. This yields

$$\sup_{\|x^*\| \leq 1} \int_\Omega |s(x^*, \Gamma)|\, d\mu \leq 2 \sup \left\{ \|x\| \colon x \in \bigcup M_\Gamma(\Sigma) \right\} < \infty\,.$$

Definition 1.3. Denote by $\mathcal{P}(\mu, \mathcal{C})$ the collection of all multifunctions $\Gamma\colon \Omega \to c(X)$ that are Pettis μ-integrable in $\mathcal{C}$ and by $\mathcal{P}(\mu, X)$ the space of X-valued Pettis

μ-integrable functions. I define in the usual way the multiplication by a number and the addition by $\Gamma(\omega) \oplus \Delta(\omega) := \overline{\Gamma(\omega) + \Delta(\omega)}$. Identifying scalarly equivalent elements of $\mathcal{P}(\mu, \mathcal{C})$, one gets the space $\mathbb{P}(\mu, \mathcal{C})$. $\mathbb{P}(\mu, cb(X))$ can be endowed with a (Pettis) metric (see [6, p. 392]) defined by:

$$d_P(\Gamma, \Delta) := \sup_{x^* \in B(X^*)} \int_\Omega |s(x^*, \Gamma) - s(x^*, \Delta)|\, d\mu \tag{3}$$

and $\mathbb{P}(\mu, X)$ can be furnished with the norm defined by $\|f\|_P := \sup_{\|x^*\| \le 1} \int_\Omega |x^* f|\, d\mu$. □

d_P is properly defined due to (2). Let us notice also that (see [6, Lemme 7])

$$\sup_{E \in \Sigma} d_H(M_\Gamma(E), M_\Delta(E)) \le d_P(\Gamma, \Delta) \le 2 \sup_{E \in \Sigma} d_H(M_\Gamma(E), M_\Delta(E))\,. \tag{4}$$

If Γ and Δ are only scalarly integrable, then (3) defines a metric on the collection of all scalarly integrable multifunctions (scalarly equivalent ones are identified). The proof of this fact will be presented in Section 6 since it needs some additional knowledge that is not in the main stream of the paper.

It is known that $\mathbb{P}(\mu, X)$ is in general not complete. The same holds true also in case of $\mathbb{P}(\mu, cb(X))$ and d_P (sequences divergent in $\mathbb{P}(\mu, X)$ remain divergent also in $\mathbb{P}(\mu, cb(X))$).

One can also easily check (just applying Hörmander's formula) that if $\Gamma : \Omega \to cb(X)$ is Pettis μ-integrable in $cb(X)$, then M_Γ is an h-multimeasure if and only if the family $\{s(x^*, M_\Gamma) : \|x^*\| \le 1\}$ of scalar measures is uniformly σ-additivite. In view of [11, Theorem I.2.4] this property is equivalent to the uniform integrability of $\mathbb{Z}_\Gamma$ with respect to μ. But one should remember that even in case of a function the uniform integrability - in general - does not guarantee its Pettis integrability (see [15]).

Let $\Xi \subset \Sigma$ be a σ-algebra and let Γ and Δ be two $c(X)$-valued multifunctions such that Γ is scalarly measurable with respect to Σ and Δ is scalarly measurable with respect to Ξ. Assume also that $\Gamma \in \mathbb{P}(\mu, cb(X))$ and $\Delta \in \mathbb{P}(\mu|\Xi, cb(X))$. If $\int_E \Gamma\, d\mu = \int_E \Delta\, d\mu$ for every $E \in \Xi$, then Δ is called the *conditional expectation* of Γ with respect to Ξ and is denoted by $\mathbb{E}(\Gamma|\Xi)$. If $\{\Sigma_n : n \in \mathbb{N}\}$ is an increasing sequence of σ-algebras $\Sigma_n \subset \Sigma$ and $\{\Gamma_n : \Omega \to cb(X) : n \in \mathbb{N}\}$ are multifunctions such that each Γ_n is scalarly measurable with respect to Σ_n and Pettis integrable on Σ_n, then $\{(\Gamma_n, \Sigma_n) : n \in \mathbb{N}\}$ is called a martingale, if $\mathbb{E}(\Gamma_{n+1}|\Sigma_n) = \Gamma_n$ for every $E \in \Sigma_n$ and $n \in \mathbb{N}$.

Let $R : cb(X) \to l_\infty(B(X^*))$ be the canonical Rådström isometry given by $R(C)(x^*) := s(x^*, C)$. If $M : \Sigma \to cb(X)$ is an h-multimeasure, then $R \circ M : \Sigma \to l_\infty(B(X^*))$ is a vector measure. If $M \ll \mu$, then also $R \circ M \ll \mu$. Following [3], I define an embedding of X^* into $l_\infty^*(B(X^*))$ by $x^* \to e_{x^*}$, where $\langle e_{x^*}, g \rangle := g(x^*)$, for arbitrary $g \in l_\infty(B(X^*))$. Notice that $s(x^*, W) = \langle e_{x^*}, R(W) \rangle$ for every $W \in cb(X)$ and $\{e_{x^*} : \|x^*\| \le 1\}$ is norming for $l_\infty(B(X^*))$.

2. Approximation by simple multifunctions. Compactness

It is my aim to characterize those multifunctions that can be – in some sense – approximated by sequences of simple multifunctions. First a lemma being a particular case of [6, Corollaire 4].

Lemma 2.1. *Let $M : \Sigma \to cb(X)$ be a μ-continuous multimeasure. If $M_\Gamma(\Sigma)$ is relatively compact in the Hausdorff metric, then for each $\varepsilon > 0$ there exists a finite partition π of Ω into pieces of positive measure such that the simple multifunction $\Gamma = \sum_{E\in\pi} \frac{M(E)}{\mu(E)}\chi_E$ satisfies the inequality*

$$\sup_{E\in\Sigma} d_H\left(M(E), \int_E \Gamma\, d\mu\right) < \varepsilon. \tag{5}$$

Equivalently, $\sup_{E\in\Sigma}\sup_{\|x^*\|\le 1}\left|s(x^*, M(E)) - s(x^*, \int_E \Gamma\, d\mu)\right| < \varepsilon$.

More precisely, there exists a martingale of simple multifunctions $\Gamma_n : \Omega \to cb(X)$ such that

$$\sup_{E\in\Sigma} d_H\left(M(E), \int_E \Gamma_n\, d\mu\right) < 1/n \quad \text{for each } n \in \mathbb{N}.$$

If M is $cwk(X)$ or $ck(X)$-valued, then the same holds true for Γ and Γ_n's. □

It is well known that each $cwk(X)$-valued multimeasure is an h-multimeasure (cf. [19, Theorem 8.4.10]). I present another general example.

Corollary 2.2. *If $M : \Sigma \to cb(X)$ is a multimeasure and $M(\Sigma)$ is relatively compact in the Hausdorff metric, then M is an h-multimeasure.*

Proof. Let $\varepsilon > 0$ be arbitrary and Γ be a simple multifunction satisfying the inequality (5). If $\bigcup_k F_k = F \in \Sigma$ is a decomposition of F into pairwise disjoint elements of Σ, then

$$\begin{aligned}
&d_H\left(M(F), M\left(\bigcup_{k=1}^m F_k\right)\right)\\
&= d_H\left(M\left(\bigcup_{k=1}^m F_k\right) \oplus M\left(\bigcup_{k>m} F_k\right), M\left(\bigcup_{k=1}^m F_k\right)\right)\\
&= d_H\left(M\left(\bigcup_{k=1}^m F_k\right) + M\left(\bigcup_{k>m} F_k\right), M\left(\bigcup_{k=1}^m F_k\right)\right) \le d_H\left(M\left(\bigcup_{k>m} F_k\right), \{0\}\right)\\
&\le d_H\left(M\left(\bigcup_{k>m} F_k\right), \int_{\bigcup_{k>m} F_k} \Gamma\, d\mu\right) + d_H\left(\int_{\bigcup_{k>m} F_k} \Gamma\, d\mu, \{0\}\right)\\
&< \varepsilon + d_H\left(\int_{\bigcup_{k>m} F_k} \Gamma\, d\mu, \{0\}\right) < 2\varepsilon
\end{aligned}$$

for sufficiently large m, because Γ is a simple multifunction and so the map $E \longrightarrow \int_E \Gamma\, d\mu$ is an h-multimeasure. □

As a direct consequence one obtains the following approximation of multifunctions by a sequence of simple multifunctions:

Theorem 2.3. *Let $\varGamma : \varOmega \to c(X)$ be a scalarly integrable multifunction. Then the following conditions are equivalent:*

(*i*) *$\varGamma$ is Pettis integrable in $cb(X)$ and $M_\varGamma(\varSigma)$ is relatively compact in the Hausdorff metric;*

(*ii*) *There exists a (martingale) sequence of simple multifunctions $\varGamma_n : \varOmega \to cb(X)$ such that*

$$\lim_n d_P(\varGamma, \varGamma_n) = 0; \tag{6}$$

(*iii*) *$\varGamma$ is Pettis integrable in $cb(X)$ and $\mathbb{Z}_\varGamma$ is norm relatively compact in $L_1(\mu)$.*

In (i) and (ii), if $\varGamma$ is Pettis integrable in $cwk(X)$ (or in $ck(X)$), then $\varGamma_n$'s can be chosen to be also $cwk(X)$ (or $ck(X)$) valued.

Proof. (*i*) $\Rightarrow$ (*ii*). It is a direct consequence of Lemma 2.1 that there exists a martingale of simple multifunctions $\varGamma_n : \varOmega \to cb(X)$ such that

$$\sup_{E\in\varSigma} d_H\left(M_\varGamma(E), \int_E \varGamma_n\,d\mu\right) < 1/n, \quad \text{for every } n \in \mathbb{N}.$$

Hence

$$d_P(\varGamma, \varGamma_n) = \sup_{\|x^*\|\le 1} \int_\varOmega |s(x^*, \varGamma) - s(x^*, \varGamma_n)|\,d\mu < 2/n.$$

This yields the required result.

(*ii*) $\Rightarrow$ (*iii*). It follows from (*ii*) that given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\mathbb{Z}_\varGamma \subset \varepsilon B(L_1(\mu)) + \overline{\mathrm{conv}}\mathbb{Z}_{\varGamma_n}$. Since $\mathbb{Z}_{\varGamma_n}$ is norm relatively compact, it follows that also $\mathbb{Z}_\varGamma$ is norm relatively compact in $L_1(\mu)$. Moreover, it follows from (6) and (4) that for each $E \in \varSigma$ the sequence $\langle M_{\varGamma_n}(E)\rangle_n$ is Cauchy in the Hausdorff metric d_H. Consequently, it is d_H-convergent to a set that is the Pettis integral of $\varGamma$ on E.

(*iii*) $\Rightarrow$ (*i*). Assume that (*iii*) is fulfilled. Since $\mathbb{Z}_\varGamma$ is separable in $L_1(\mu)$, there exists a sequence (x_n^*) in $B(X^*)$, such that $\{s(x_n^*, \varGamma) : n \in \mathbb{N}\}$ is dense in $\mathbb{Z}_\varGamma$. If $\widetilde{\varSigma}$ is the σ-algebra generated by all $s(x_n^*, \varGamma)$ and by $\mathcal{N}(\mu)$ then, clearly $\mu|\widetilde{\varSigma}$ is separable and each $s(x^*, \varGamma)$ is $\widetilde{\varSigma}$–measurable. Assume that $\sigma(\{E_n : n \in \mathbb{N}\})$ is μ-dense in $\widetilde{\varSigma}$. Moreover, let π_n be the partition of $\varOmega$ generated by the sets $E_1, \ldots, E_n$.

Put for each n

$$\varGamma_n = \sum_{E\in\pi_n} \frac{M_\varGamma(E)}{\mu(E)}\chi_E \quad \text{with the convention } \{0\}/0 = \{0\}.$$

One can easily check that $\{\varGamma_n, \sigma(\pi_n), n \in \mathbb{N}\}$ is a $cb(X)$–valued martingale; in particular, for each $x^* \in X^*$, the sequence $\{s(x^*, \varGamma_n), \sigma(\pi_n), n \in \mathbb{N}\}$ is a real valued uniformly integrable martingale. Moreover, $\mathbb{E}(s(x^*,\Gamma)|\sigma(\pi_n)) = s(x^*,\Gamma_n)$ μ–a.e. for every $n\in\mathbb{N}$. Hence $\lim_n s(x^*,\Gamma_n) = \mathbb{E}(s(x^*,\Gamma)|\widetilde{\Sigma}) = s(x^*,\Gamma)$ in $L_1(\mu|\widetilde{\Sigma})$ and $\mu|\widetilde{\Sigma}$ –a.e. (cf. [11]).

Fix now $\varepsilon>0$, $x^*\in B(X^*)$ and $E\in\Sigma$. By the assumption there exists a set $\{z_1^*,\ldots,z_p^*\}\subset B(X^*)$ such that $\{s(z_i^*,\Gamma): i\le p\}$ forms an ε-mesh in $\mathbb{Z}_\Gamma$. If $F\in\Sigma$, $i\le p$ and $m\in\mathbb{N}$, then

$$
\begin{aligned}
&|s(x^*,M_\Gamma(E)) - s(x^*,M_\Gamma(F))|\\
&\le |s(x^*,M_\Gamma(E)) - s(z_i^*,M_\Gamma(E))|\\
&\qquad + |s(z_i^*,M_\Gamma(E)) - s(z_i^*,M_\Gamma(F))| + |s(z_i^*,M_\Gamma(F)) - s(x^*,M_\Gamma(F))|\\
&\le 2\int_\Omega |s(x^*,\Gamma)-s(z_i^*,\Gamma)|\,d\mu + |s(z_i^*,M_\Gamma(E)) - s(z_i^*,M_{\Gamma_m}(E))|\\
&\qquad + |s(z_i^*,M_{\Gamma_m}(E)) - s(z_i^*,M_{\Gamma_m}(F))| + |s(z_i^*,M_{\Gamma_m}(F)) - s(z_i^*,M_\Gamma(F))|\\
&\le 2\int_\Omega |s(x^*,\Gamma)-s(z_i^*,\Gamma)|\,d\mu + 2\int_\Omega |s(z_i^*,\Gamma)-s(z_i^*,\Gamma_m)|\,d\mu\\
&\qquad + |s(z_i^*,M_{\Gamma_m}(E)) - s(z_i^*,M_{\Gamma_m}(F))|
\end{aligned}
$$

The first part of the proof implies the existence of $m\in\mathbb{N}$ such that $\int_\Omega |s(z_i^*,\Gamma) - s(z_i^*,\Gamma_m)|\,d\mu<\varepsilon$, for every $i\le p$. Moreover, there exists $i\le p$ with $\int_\Omega |s(x^*,\Gamma) - s(z_i^*,\Gamma)|\,d\mu<\varepsilon$. For those i and m we have now the inequality

$$
|s(x^*,M_\Gamma(E)) - s(x^*,M_\Gamma(F))| \le 4\varepsilon + |s(z_i^*,M_{\Gamma_m}(E)) - s(z_i^*,M_{\Gamma_m}(F))|. \tag{7}
$$

But Γ_m is simple and so the set $\{M_{\Gamma_m}(F): F\in\Sigma\}$ is d_H relatively compact. If $\{F_1,\ldots,F_q\}$ is an ε-mesh in $\{M_{\Gamma_m}(F): F\in\Sigma\}$, then there exists $j\le q$ with

$$
|s(z_i^*,M_{\Gamma_m}(E)) - s(z_i^*,M_{\Gamma_m}(F_j))| < d_H(M_{\Gamma m}(E),M_{\Gamma m}(F_j)) < \varepsilon\,.
$$

Thus, setting in (7) $F=F_j$, we have

$$
|s(x^*,M_\Gamma(E)) - s(x^*,M_\Gamma(F_j))| \le 5\varepsilon
$$

and as the sets $\{F_1,\ldots,F_q\}$ are chosen independently of E and x^*, we have the required inequality

$$
d_H(M_\Gamma(E),M_\Gamma(F_j)) \le 5\varepsilon\,. \qquad \square
$$

Corollary 2.4. *If $\Gamma:\Omega\to c(X)$ is Pettis integrable in $cb(X)$, then $M_\Gamma(\Sigma)$ is relatively compact in the Hausdorff metric if and only if Γ can be approximated in d_P by a sequence of simple multifunctions $\Gamma_n:\Omega\to cb(X)$.*

3. Approximation by simple multifunctions. Separability

Lemma 3.1. *Let $\Gamma_n:\Omega\to c(X)$, $n\in\mathbb{N}$, be a sequence of scalarly integrable multifunctions such that $\sup_n\sup_{x^*\in B(X^*)}\int_\Omega |s(x^*,\Gamma_n)|\,d\mu<\infty$. Then*

$$
\lim_{C\to+\infty}\sup_n\sup_{x^*\in B(X^*)}\mu\{|s(x^*,\Gamma_n)|>C\}=0.
$$

In particular, if $\Gamma : \Omega \to c(X)$ is an arbitrary scalarly integrable multifunction fulfilling the inequality (2)*, then*

$$\lim_{C\to+\infty} \sup_{x^*\in B(X^*)} \mu\{s(x^*,\Gamma)| > C\} = 0. \tag{8}$$

Proof. If $L := \sup_n \sup_{x^*\in B(X^*)} \int_\Omega |s(x^*,\Gamma_n)|\,d\mu$, then for each $n \in \mathbb{N}$ and $x^* \in B(X^*)$, we have

$$L \geq \int_\Omega |s(x^*,\Gamma_n)|\,d\mu \geq C\mu\{|s(x^*,\Gamma_n)| > C\}.$$

It follows that

$$\sup_n \sup_{x^*\in B(X^*)} \mu\{|s(x^*,\Gamma_n)| > C\} \leq L/C$$

and so $\lim_{C\to+\infty} \sup_n \sup_{x^*\in B(X^*)} \mu\{|s(x^*,\Gamma_n)| > C\} = 0$.
Setting $\Gamma_n = \Gamma$, for all n, we obtain (8). □

Proposition 3.2. *Let $(\Gamma_n, \Sigma_n)_{n\in\mathbb{N}}$, be a martingale of multifunctions $\Gamma_n : \Omega \to c(X)$ that are Pettis integrable in $cb(X)$, $[cwk(X),\ ck(X)]$. If the family $\bigcup_n \mathbb{Z}_{\Gamma_n}$ is uniformly integrable, then there exists an h-multimeasure $M : \sigma(\mathcal{N}(\mu) \cup \bigcup_n \Sigma_n) \to cb(X)$, $[cwk(X),\ ck(X)]$ such that*

$$\lim_n d_H(M_{\Gamma_n}(E), M(E)) = 0, \quad \textit{for every } E \in \sigma\left(\mathcal{N}(\mu) \cup \bigcup_n \Sigma_n\right). \tag{9}$$

Proof. To prove (9) notice first that for every $E \in \bigcup_n \Sigma_n$ and $x^* \in X^*$ we have $\int_E s(x^*,\Gamma_n)\,d\mu = \int_E s(x^*,\Gamma)\,d\mu$, for sufficiently large n, depending only on E. This follows from the equality $\int_E \Gamma_n\,d\mu = \int_E \Gamma_m\,d\mu$, valid for every $E \in \Sigma_n$ and $m \geq n$. Let us fix $\varepsilon > 0$ and $\delta > 0$ such that $\mu(F) < \delta$ yields $\int_F |s(x^*,\Gamma_n)|\,d\mu < \varepsilon$, for every $n \in \mathbb{N}, F \in \Sigma$ and $x^* \in B(X^*)$. If $E \in \sigma(\mathcal{N}(\mu) \cup \bigcup_n \Sigma_n)$ then there exist $n_0 \in \mathbb{N}$ and $F \in \Sigma_{n_0}$ such that $\mu(E \triangle F) < \delta$. We have then for all $m \geq n \geq n_0$

$$\begin{aligned}
&|s(x^*, M_{\Gamma_n}(E)) - s(x^*, M_{\Gamma_m}(E))| \\
&\leq |s(x^*, M_{\Gamma_n}(E)) - s(x^*, M_{\Gamma_n}(F))| + |s(x^*, M_{\Gamma_n}(F)) - s(x^*, M_{\Gamma_m}(F))| \\
&\qquad + |s(x^*, M_{\Gamma_m}(F)) - s(x^*, M_{\Gamma_m}(E))| \\
&\leq \int_{E\triangle F} |s(x^*,\Gamma_n)|\,d\mu + |s(x^*, M_{\Gamma_n}(F)) - s(x^*, M_{\Gamma_m}(F))| \\
&\qquad + \int_{E\triangle F} |s(x^*,\Gamma_m)|\,d\mu < 2\varepsilon.
\end{aligned}$$

It follows that for each $E \in \sigma(\mathcal{N}(\mu) \cup \bigcup_n \Sigma_n)$ the sequence $\langle M_{\Gamma_n}(E)\rangle_n$ is Cauchy in the metric d_H. Consequently, it is convergent to a set $M(E)$. It is obvious that M is an h-multimeasure on $\sigma(\mathcal{N}(\mu) \cup \bigcup_n \Sigma_n)$. □

Proposition 3.3. *Let $(\Gamma_n, \Sigma_n)_{n\in\mathbb{N}}$, be a martingale of multifunctions $\Gamma_n : \Omega \to c(X)$ that are Pettis integrable in $cb(X)$, $[cwk(X), ck(X)]$ and, let $\Gamma : \Omega \to c(X)$ be a multifunction that is also Pettis integrable in $cb(X)$, $[cwk(X), ck(X)]$. Assume moreover that for each $n \in \mathbb{N}$ and each $x^* \in B(X^*)$*

$$\mathbb{E}(s(x^*, \Gamma)|\Sigma_n) = s(x^*, \Gamma_n) \quad a.e.$$

If $\mathbb{Z}_\Gamma$ is uniformly integrable, then the family $\bigcup_n \mathbb{Z}_{\Gamma_n}$ is uniformly integrable and

$$\lim_n d_H(M_{\Gamma_n}(E), M_\Gamma(E)) = 0, \quad \textit{for every } E \in \sigma\left(\mathcal{N}(\mu) \cup \bigcup_n \Sigma_n\right). \tag{10}$$

If Γ is scalarly measurable with respect to $\sigma\left(\mathcal{N}(\mu) \cup \bigcup_n \Sigma_n\right)$, then (10) is valid for every $E \in \Sigma$.

Proof. Since the conditional expectation operator is a contraction on $L_1(\mu)$, we have for every $x^* \in X^*$ and $n \in \mathbb{N}$

$$\int_E |s(x^*, \Gamma_n)| d\mu \le \int_E |s(x^*, \Gamma)| d\mu \quad \text{for every } E \in \Sigma_n\,.$$

In particular

$$\sup_n \sup_{x^* \in B(X^*)} \int_\Omega |s(x^*, \Gamma_n)|\, d\mu \le \sup_{x^* \in B(X^*)} \int_\Omega |s(x^*, \Gamma)|\, d\mu < \infty.$$

We have to prove that

$$\lim_{C\to\infty} \sup_n \sup_{x^* \in B(X^*)} \int_{\{|s(x^*,\Gamma_n)|>C\}} |s(x^*, \Gamma_n)|\, d\mu = 0.$$

But $\{|s(x^*, \Gamma_n)| > C\} \in \Sigma_n$, for every $n \in \mathbb{N}$, $x^* \in X^*$ and $C \in \mathbb{R}$ and consequently,

$$\int_{\{|s(x^*,\Gamma_n)|>C\}} |s(x^*, \Gamma_n)|\, d\mu \le \int_{\{|s(x^*,\Gamma_n)|>C\}} |s(x^*, \Gamma)|\, d\mu.$$

Let $\varepsilon >$ be arbitrary and $\delta > 0$ be adapted to ε in such a way that $\mu(E) < \delta$ yields $\int_E |s(x^*, \Gamma)|\, d\mu < \varepsilon$, for every $x^* \in B(X^*)$.

We have then

$$\begin{aligned} &\sup_n \sup_{x^* \in B(X^*)} \int_{\{|s(x^*,\Gamma_n)|>C\}} |s(x^*, \Gamma_n)|\, d\mu \\ &\le \sup_n \sup_{x^* \in B(X^*)} \int_{\{|s(x^*,\Gamma_n)|>C\}} |s(x^*, \Gamma)|\, d\mu \le \varepsilon \end{aligned}$$

for sufficiently large C (in virtue of Lemma 3.1) and so the expected uniform integrability takes place.

(10) is a consequence of Proposition 3.2.

Assume now that Γ is scalarly measurable with respect to $\widetilde{\Sigma} := \sigma\left(\mathcal{N}(\mu) \cup \bigcup_n \Sigma_n\right)$.

Claim.

$$\lim_n \sup_{x^*\in B(X^*)} \left|\int_\Omega hs(x^*,\Gamma_n)\,d\mu - \int_\Omega hs(x^*,\Gamma)\,d\mu\right| = 0 \quad \text{for each } h\in L_\infty(\mu|\widetilde{\Sigma}). \tag{11}$$

Proof. It is a consequence of Proposition 3.2 that

$$\lim_n \sup_{x^*\in B(X^*)} \left|\int_E s(x^*,\Gamma)\,d\mu - \int_E s(x^*,\Gamma_n)\,d\mu\right| = 0 \quad \text{for every } E\in\widetilde{\Sigma}\,. \tag{12}$$

Fix $\varepsilon>0$ and $h\in L_\infty(\mu|\widetilde{\Sigma})$ and, let $h_\varepsilon\in L_\infty(\mu|\widetilde{\Sigma})$ be a simple function such that $\|h-h_\varepsilon\|_\infty<\varepsilon$. We have then

$$\begin{aligned}
&\left|\int_\Omega hs(x^*,\Gamma_n)\,d\mu - \int_\Omega hs(x^*,\Gamma)\,d\mu\right| \\
&\le \int_\Omega |h-h_\varepsilon|\,|s(x^*,\Gamma_n)|\,d\mu + \left|\int_\Omega h_\varepsilon s(x^*,\Gamma_n)\,d\mu - \int_\Omega h_\varepsilon s(x^*,\Gamma)\,d\mu\right| \\
&\qquad + \int_\Omega |h-h_\varepsilon|\,|s(x^*,\Gamma)|\,d\mu \\
&\le \varepsilon \sup_n \int_\Omega |s(x^*,\Gamma_n)|\,d\mu + \left|\int_\Omega h_\varepsilon s(x^*,\Gamma_n)\,d\mu - \int_\Omega h_\varepsilon s(x^*,\Gamma)\,d\mu\right| \\
&\qquad + \varepsilon\int_\Omega |s(x^*,\Gamma)|\,d\mu \\
&\le 2\varepsilon\int_\Omega |s(x^*,\Gamma)|\,d\mu + \left|\int_\Omega h_\varepsilon s(x^*,\Gamma_n)\,d\mu - \int_\Omega h_\varepsilon s(x^*,\Gamma)\,d\mu\right|
\end{aligned}$$

Since, in virtue of (12), the second term is as small as we need, for sufficiently large n and independently of $x^*\in B(X^*)$, we have the required equality (11). □

In order to obtain the convergence (10) for every $E\in\Sigma$, it suffices to notice that if $H\in\Sigma$ and $n\in\mathbb{N}$, then

$$\begin{aligned}
\int_H s(x^*,\Gamma_n)\,d\mu &= \int_\Omega \chi_H s(x^*,\Gamma_n)\,d\mu = \int_\Omega \mathbb{E}\left(\chi_H s(x^*,\Gamma_n)|\widetilde{\Sigma}\right)\,d\mu \\
&= \int_\Omega \mathbb{E}(\chi_H|\widetilde{\Sigma})s(x^*,\Gamma_n)\,d\mu.
\end{aligned}$$

Similarly,

$$\int_H s(x^*,\Gamma)\,d\mu = \int_\Omega \mathbb{E}(\chi_H|\widetilde{\Sigma})s(x^*,\Gamma)\,d\mu,$$

because Γ is scalarly measurable with respect to $\widetilde{\Sigma}$. The Claim completes the proof. □

It has been proven in [23, Theorem 10.1] that if $f : \Omega \to X$ is a Pettis integrable function, then $\mathbb{Z}_f$ is separable in $L_1(\mu)$ if and only if the range $\nu_f(\Sigma)$ of the integral is separable in X. While investigating integrals of multifunctions one meets a different situation. As long as the set $\mathbb{Z}_\Gamma$ is weakly relatively compact, there is a positive correlation between the separability of $M_\Gamma(\Sigma)$ and of $\mathbb{Z}_\Gamma$. In general however, the separability of $\mathbb{Z}_\Gamma$ is a weaker property.

Let us notice also that if $\Gamma : \Omega \to c(X)$ is scalarly integrable and $R \circ \Gamma$ is Pettis integrable, then $\mathbb{Z}_{R\circ\Gamma}$ is uniformly integrable (this is an elementary fact about vector measures). But $s(x^*, \Gamma(\omega)) = \langle e_{x^*}, R\circ\Gamma(\omega)\rangle$ (for every ω and $x^* \in X^*$) and so $\mathbb{Z}_\Gamma \subset \mathbb{Z}_{R\circ\Gamma}$ is also uniformly integrable.

Theorem 3.4. *Let X be an arbitrary Banach space and let $\Gamma \colon \Omega \to c(X)$ be a multifunction that is Pettis integrable in $cb(X)$, $[cwk(X), ck(X)]$ and M_Γ is an h-multimeasure. Then the following conditions are equivalent:*

(j) *$M_\Gamma(\Sigma)$ is separable in the Hausdorff metric;*

(jj) *$\mathbb{Z}_\Gamma$ is a separable subset of $L_1(\mu)$;*

(jjj) *There exists a martingale $(\Gamma_n, \Sigma_n)_{n\in\mathbb{N}}$ (or a sequence $(\Gamma_n)_{n\in\mathbb{N}}$) of $cb(X)$ $[cwk(X), ck(X)]$–valued simple multifunctions, such that one of the following conditions is fulfilled, for each $x^* \in X^*$:*
- (α) *$\{s(x^*, \Gamma_n) : n \in \mathbb{N}\}$ is μ–a.e. convergent to $s(x^*, \Gamma)$;*
- (β) *$\{s(x^*, \Gamma_n) : n \in \mathbb{N}\}$ is convergent in μ–measure to $s(x^*, \Gamma)$;*
- (γ) *$\{s(x^*, \Gamma_n) : n \in \mathbb{N}\}$ is convergent to $s(x^*, \Gamma)$ in $L_1(\mu)$;*

(jv) *There exists a martingale $(\Gamma_n, \Sigma_n)_{n\in\mathbb{N}}$ (or a sequence $(\Gamma_n)_{n\in\mathbb{N}}$) of $cb(X)$ $[cwk(X), ck(X)]$–valued simple multifunctions, such that*

$$\lim_n s(x^*, M_{\Gamma_n}(E)) = s(x^*, M_\Gamma(E)) \quad \textit{for every } E \in \Sigma \textit{ and } x^* \in X^*;$$

(v) *There exists a σ–algebra $\widetilde{\Sigma} \subseteq \Sigma$ such that $(\Omega, \widetilde{\Sigma}, \mu|\widetilde{\Sigma})$ is separable and Γ is scalarly measurable with respect to $\widetilde{\Sigma}$;*

(vj) *There exists a martingale $(\Gamma_n, \Sigma_n)_{n\in\mathbb{N}}$ (or a sequence $(\Gamma_n)_{n\in\mathbb{N}}$) of $cb(X)$ $[cwk(X), ck(X)]$–valued simple multifunctions, such that*

$$\lim_n d_H(M_\Gamma(E), M_{\Gamma_n}(E)) = 0, \quad \textit{for every } E \in \Sigma.$$

If Γ is Pettis integrable in $ck(X)$, then the above conditions are equivalent to the following one:

(vjj) *$M_\Gamma(\Sigma)$ is scalarly separable in $cb(X)$.*

Proof. (j) $\Rightarrow$ (jj). If $R : cb(X) \to l_\infty(B(X^*))$ is the Rådström embedding, then $\nu : \Sigma \to l_\infty(B(X^*))$ defined by $\nu(E) = R\circ M_\Gamma(E)$ is countably additive in the norm topology of $l_\infty(B(X^*))$ and $\nu(\Sigma)$ is a separable set.

It follows from [23, Theorem 3] (see also [30, Theorem 5-3-2], [24, Theorem 10.1] and [26, Theorem 6.8]) that the set $\{\frac{dz^*\nu}{d\mu} : z^* \in l^*_\infty(B(X^*))\}$ is separable in $L_1(\mu)$ (the original proof needs only obvious modifications).

Notice that $s(x^*, \Gamma(\omega)) = \langle e_{x^*}, R\circ\Gamma(\omega)\rangle$ for every ω and so the separability of $\mathbb{Z}_\Gamma$ follows from the separability of the set $\{\frac{dz^*\nu}{d\mu} : z^* \in l_\infty^*(B(X^*))\} \supset \mathcal{Z}_\Gamma$ in $L_1(\mu)$.

$(jj) \Rightarrow (jjj)\alpha$. Assume, that the set $\mathbb{Z}_\Gamma$ is separable. Then there exists a sequence (x_n^*) in $B(X^*)$, such that $\{s(x_n^*, \Gamma) : n \in \mathbb{N}\}$ is dense in $\mathbb{Z}_\Gamma$. If $\widetilde{\Sigma}$ is the σ–algebra generated by all $s(x_n^*, \Gamma)$ and by $\mathcal{N}(\mu)$ then, clearly $\mu|\widetilde{\Sigma}$ is separable and each $s(x^*, \Gamma)$ is $\widetilde{\Sigma}$–measurable. Assume that $\sigma(\{E_n : n \in \mathbb{N}\})$ is μ-dense in $\widetilde{\Sigma}$. Moreover, let π_n be the partition of Ω generated by the sets $E_1, \ldots, E_n$.

Put for each n

$$\Gamma_n = \sum_{E\in\pi_n} \frac{M_\Gamma(E)}{\mu(E)}\chi_E \ \text{ with the convention } \{0\}/0 = \{0\}.$$

One can easily check that $\{(\Gamma_n, \sigma(\pi_n)) : n \in \mathbb{N}\}$ is a $cb(X)$–valued martingale; in particular, for each $x^* \in X^*$, the sequence $\{(s(x^*, \Gamma_n), \sigma(\pi_n)) : n \in \mathbb{N}\}$ is a real valued uniformly integrable martingale. Moreover, $\mathbb{E}(s(x^*, \Gamma)|\sigma(\pi_n)) = s(x^*, \Gamma_n)$ μ–a.e. for every $n \in \mathbb{N}$. Hence $\lim_n s(x^*, \Gamma_n) = \mathbb{E}(s(x^*, \Gamma)|\widetilde{\Sigma}) = s(x^*, \Gamma)$ in $L_1(\mu|\widetilde{\Sigma})$ and $\mu|\widetilde{\Sigma}$ –a.e. (cf. [11]).

The implications $(jjj)\alpha \Rightarrow (jjj)\beta \Rightarrow (jjj)\gamma \Rightarrow (jv)$ are obvious.

$(jv) \Rightarrow (v)$. Let $(\Gamma_n)_{n=1}^\infty$ be a sequence of simple multifunctions fulfilling the condition (jv). If $\mathcal{N}(\mu) \subset \Sigma_n \subset \Sigma$ is a separable σ-algebra such that Γ_n is scalarly measurable with respect to Σ_n, then Γ is scalarly measurable with respect to $\sigma(\bigcup_n \Sigma_n)$ that is also a separable σ-algebra.

$(v) \Rightarrow (vj)$. Assume that $\widetilde{\Sigma}$ is μ-complete and $\sigma(\{E_n : n \in \mathbb{N}\})$ is μ-dense in $\widetilde{\Sigma}$. Moreover, let π_n be the partition of Ω generated by the sets $E_1, \ldots, E_n$.

Put for each n

$$\Gamma_n = \sum_{E\in\pi_n} \frac{M_\Gamma(E)}{\mu(E)}\chi_E \ \text{ with the convention } \{0\}/0 = \{0\}.$$

One can easily check that $\{\Gamma_n, \sigma(\pi_n), n \in \mathbb{N}\}$ is a $cb(X)$–valued martingale and $\mathbb{E}(s(x^*, \Gamma)|\sigma(\pi_n)) = s(x^*, \Gamma_n)$ a.e. We apply now Proposition 3.3.

$(vj) \Rightarrow (j)$. Each family $M_{\Gamma_n}(\Sigma)$ is separable and so also $M_\Gamma(\Sigma)$ is separable.

$(vjj) \Rightarrow (j)$. We use the Rådström embedding $R : cb(X) \longrightarrow C[B(X^*, \sigma(X^*, X))]$, into the space of weak*-continuous functions defined $B(X^*)$. The required conclusion follows then from [19, Theorem 7.2.11].

$(j) \Rightarrow (vjj)$. is obvious. □

Remark 3.5. According to a result of Bartle-Dunford-Schwartz [2] the range of each Banach space valued measure is weakly relatively compact. Thus, the set $(R\circ M_\Gamma)(\Sigma) = M_\Gamma(\Sigma)$ in Theorem 3.4 is a weakly relatively compact subset of $l_\infty(B(X^*))$. □

The next example (taken from [16]) shows that there are $ck(c_0)$-valued Pettis integrable multifunctions such that $\mathbb{Z}_\Gamma$ is separable without $M_\Gamma(\Sigma)$ being separable. Since c_0 is separable, also $ck(c_0)$ is separable in the Hausdorff metric. It is well known (cf. [3]) that if X is separable, then each $ck(X)$-valued multifunction Γ which is Pettis integrable in $cwk(X)$ generates Pettis integrability of $R\circ\Gamma$. In our example Γ is Pettis integrable in $cb(c_0)$ and $R\circ\Gamma$ fails to be Pettis integrable.

It is worth to notice that Γ is generated by a Henstock-Kurzweil-Pettis integrable function (see [13]).

Example 3.6. Consider a sequence of intervals $A_n = [a_n, b_n] \subseteq [0,1]$ such that $a_1 = 0$, $b_n < a_{n+1}$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} b_n = 1$ and define $f : [0,1] \to c_0$ by

$$f(t) = \left\langle \frac{1}{|A_{2n-1}|}\chi_{A_{2n-1}}(t) - \frac{1}{|A_{2n}|}\chi_{A_{2n}}(t) \right\rangle_{n=1}^{\infty}.$$

The function f is scalarly integrable but it is not Pettis integrable (see [16]). In particular, $\mathbb{Z}_f$ is not uniformly integrable but being a subset of $L_1[0,1]$ it is separable. The Dunford integral of f over $E \in \mathcal{L}$ is given by

$$\int_E f(t)\,dt = \left\langle \frac{|E\cap A_{2n-1}|}{|A_{2n-1}|} - \frac{|E\cap A_{2n}|}{|A_{2n}|} \right\rangle_{n=1}^{\infty}.$$

Define $\Gamma : [0,1] \to cb(c_0)$ by $\Gamma(t) := \mathrm{conv}\{0, f(t)\}$. Γ is scalarly integrable and as the zero function is a Pettis integrable selection of Γ, it follows from [14, Theorem 3.7] that Γ is Pettis integrable in $cb(c_0)$. Being a subset of $L_1[0,1]$ the set $\mathbb{Z}_\Gamma$ is a separable set. But $\mathbb{Z}_\Gamma$ is not uniformly integrable and so $M_\Gamma : \mathcal{L} \to ck(c_0)$ fails to be an h-multimeasure. Since f is a non-Pettis integrable selection of Γ, the multifunction Γ is not Pettis integrable in $cwk(c_0)$ (see [14, Theorem 5.4]). Hence, it follows from [3, Proposition 3.5] that $R\circ\Gamma$ is a non-Pettis integrable function. Thus, even in case of a separable Banach space, Pettis integrability of Γ in $cb(X)$ does not guarantee – in general – Pettis integrability of $R\circ\Gamma$. On the other hand, it is known that if X is separable, then also $ck(X)$ is separable and so (cf. [3]) each $ck(X)$-valued multifuncion that is Pettis integrable in $cwk(X)$ generates Pettis integrability of $R\circ\Gamma$.

For each $t \in [0,1]$ we have $\Gamma(t) = \mathrm{conv}\{0, f(t)\} = \{\alpha(t)f(t) \colon 0 \le \alpha(t) \le 1\}$. If h is a strongly measurable selection of Γ, then $\alpha(t) = \|h(t)\|/\|f(t)\|$ for each $t \in \bigcup_n A_n$ and zero otherwise. It follows that $\alpha : [0,1] \to [0,1]$ is a measurable function. Thus, if a strongly measurable function $h : [0,1] \to c_0$ is a selection selection of Γ, then there exists a measurable function $\alpha : [0,1] \to [0,1]$ such that $h = \alpha f$.

One can easily check that αf is a Pettis integrable selection of Γ if and only if

$$\lim_n \frac{1}{|A_n|}\int_{A_n} \alpha(t)\,dt = 0.$$

Denote by $\mathbb{A}$ the collection of all $\alpha : [0,1] \to [0,1]$ such that $\lim_n \frac{1}{|A_n|}\int_{A_n} \alpha(t)\,dt =$

0. Then

$$M_\Gamma(E) = \int_E \Gamma(t)\,dt$$
$$= \overline{\left\{\left\langle \frac{1}{|A_{2n-1}|}\int_{E\cap A_{2n-1}} \alpha(t)\,dt - \frac{1}{|A_{2n}|}\int_{E\cap A_{2n}} \alpha(t)\,dt\right\rangle_{n=1}^{\infty} : \alpha \in \mathbb{A}\right\}}$$

e are going to prove that the set $M_\Gamma(\mathcal{L})$ is non-separable in the Hausdorff metric. To show it, let us recall first that according to a result of Sierpiński [28] there exists continuum infinite subsequences of $\mathbb{N}$ with the property that every two different subsequences have only finitely many common terms. Denote the whole collection by Ξ. If $\Delta = \{n_1, n_2, \ldots, n_k, \ldots\} \in \Xi$, then $E_\Delta := \bigcup_{k=1}^{\infty} A_{2n_k-1}$. We have then

$$M_\Gamma(E_\Delta) = \overline{\left\{\left\langle \frac{1}{|A_{2n-1}|}\int_{E_\Delta\cap A_{2n-1}} \alpha(t)\,dt\right\rangle_{n=1}^{\infty} : \alpha \in \mathbb{A}\right\}}$$

If $\Delta_1 \neq \Delta_2$ and $\alpha \in \mathbb{A}$, then

$$\sup_{\alpha\in\mathbb{A}}\inf_{\beta\in\mathbb{A}}\left\|\left\langle \frac{1}{|A_{2n-1}|}\int_{E_{\Delta_1}\cap A_{2n-1}} \alpha(t)\,dt\right\rangle_{n=1}^{\infty} - \left\langle \frac{1}{|A_{2n-1}|}\int_{E_{\Delta_2}\cap A_{2n-1}} \beta(t)\,dt\right\rangle_{n=1}^{\infty}\right\|_{c_0}$$
$$= \sup_{\alpha\in\mathbb{A}}\inf_{\beta\in\mathbb{A}}\sup_n\left|\frac{1}{|A_{2n-1}|}\int_{E_{\Delta_1}\cap A_{2n-1}} \alpha(t)\,dt - \frac{1}{|A_{2n-1}|}\int_{E_{\Delta_2}\cap A_{2n-1}} \beta(t)\,dt\right|$$
$$\geq \sup_{\alpha\in\mathbb{A}}\sup_{n_k\in\Delta_1\setminus\Delta_2}\frac{1}{|A_{2n_k-1}|}\int_{A_{2n_k-1}} \alpha(t)\,dt = 1$$

It follows that $d_H(M_\Gamma(E_{\Delta_1}), M_\Gamma(E_{\Delta_2})) \geq 1$ and so $M_\Gamma(\Sigma)$ is non-separable. $\square$

In the general case separability of $\mathbb{Z}_\Gamma$ is equivalent to local separability of $M_\Gamma(\Sigma)$.

One should remember that in the next theorem, multifunctions that are Pettis integrable in $cwk(X)$ or $ck(X)$ come under Theorem 3.4.

Theorem 3.7. *Let X be an arbitrary Banach space and let $\Gamma\colon \Omega \to c(X)$ be a multifunction that is Pettis integrable in $cb(X)$. Then the following conditions are equivalent:*

(i) *$\mathbb{Z}_\Gamma$ is a separable subset of $L_1(\mu)$;*

(ii) *There exists a martingale $(\Gamma_n, \Sigma_n)_{n\in\mathbb{N}}$ (or a sequence $(\Gamma_n)_{n\in\mathbb{N}}$) of $cb(X)$–valued simple multifunctions, such that one of the following conditions is fulfilled for each $x^* \in X^*$:*
   - (a) *$\{s(x^*, \Gamma_n) : n \in \mathbb{N}\}$ is uniformly integrable and μ–a.e. convergent to $s(x^*, \Gamma)$;*
   - (b) *$\{s(x^*, \Gamma_n) : n \in \mathbb{N}\}$ is uniformly integrable and convergent in μ–measure to $s(x^*, \Gamma)$;*
   - (c) *$\{s(x^*, \Gamma_n) : n \in \mathbb{N}\}$ is convergent to $s(x^*, \Gamma)$ in $L_1(\mu)$;*

(*iii*) *There exists a martingale* $(\Gamma_n, \Sigma_n)_{n\in\mathbb{N}}$ *(or a sequence* $(\Gamma_n)_{n\in\mathbb{N}}$*) of* $cb(X)$–*valued simple multifunctions, such that*

$$\lim_n s(x^*, M_{\Gamma_n}(E)) = s(x^*, M_\Gamma(E)) \quad \text{for every } E \in \Sigma \text{ and } x^* \in X^*\,;$$

(*iv*) *There exists a* σ–*algebra* $\widetilde{\Sigma} \subseteq \Sigma$ *such that* $(\Omega, \widetilde{\Sigma}, \mu|\widetilde{\Sigma})$ *is separable and* Γ *is scalarly measurable with respect to* $\widetilde{\Sigma}$*;*

(*v*) *There exists a decomposition* $\Omega = \bigcup_k H_k$ *of* Ω *into pairwise disjoint sets of positive measure such that each family* $M_\Gamma(H_k \cap \Sigma)$ *is separable in the Hausdorff metric;*

(*vi*) $M_\Gamma(\Sigma)$ *is locally* d_H*-separable, that is each set* $E \in \Sigma^+_\mu$ *contains a subset* $F \in \Sigma^+_\mu$ *such that* $M_\Gamma(F \cap \Sigma)$ *is* d_H*-separable.*

Each of the above conditions yields scalar separability of $M_\Gamma(\Sigma)$ *in* $cb(X)$*.*

Proof. The proofs of the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ coincide with the corresponding proofs in Theorem 3.4.

$(iv) \Rightarrow (v)$. Let $\Omega = \bigcup_k H_k$ be a decomposition of Ω into pairwise disjoint sets of positive measure such that Γ is scalarly bounded on every H_k. Since M_Γ is an h-multimeasure on each set H_k, it follows from Theorem 3.4 that each family $M_\Gamma(H_k \cap \Sigma)$ is d_H-separable.

$(v) \Rightarrow (i)$. It follows from Theorem 3.4 that each family $\mathbb{Z}_{\Gamma|H_k}$ is separable in $L_1(\mu|H_k)$. Hence $\mathbb{Z}_\Gamma$ is also separable in $L_1(\mu)$. Indeed, if $x^* \in X^*$, then $\lim_n \int_\Omega |s(x^*, \Gamma) - s(x^*, \Gamma\chi_{\bigcup_{i=1}^n H_i})|\, d\mu = 0$ and $s(x^*, \Gamma\chi_{\bigcup_{i=1}^n H_i})$ can be approximated by a fixed countable subset of $\mathbb{Z}_{\Gamma|\bigcup_{i=1}^n H_i}$.

$(v) \Rightarrow$ *scalar separability of* $M_\Gamma(\Sigma)$. Let the sets H_k, $k \in \mathbb{N}$, be defined as in (iv). By the assumption, for each $k \in \mathbb{N}$ there exists a countable family $\mathcal{F}_k := \{E_{kn} : n \in \mathbb{N}\} \subset \Sigma \cap H_k$ such that $\{M_\Gamma(E_{kn}) : n \in \mathbb{N}\}$ is d_H-dense in $M_\Gamma(H_k \cap \Sigma)$.

I claim that $M_\Gamma(\Sigma)$ is scalarly separable. Let $E \in \Sigma$ and $\varepsilon > 0$ be arbitrary. For each $k \in \mathbb{N}$ there exists $E_{k,n_k} \in \mathcal{F}_k$ such that $d_H(M_\Gamma(E \cap H_k), M_\Gamma(E_{k,n_k})) < \varepsilon/2^k$. Hence, if $\|x^*\| \le 1$, then

$$\left| s(x^*, M_\Gamma(E)) - s\left(x^*, M_\Gamma\left(\bigcup_{k=1}^m E_{k,n_k}\right)\right)\right|$$
$$\le \varepsilon \sum_{k=1}^m 2^{-k} + \sum_{k=m+1}^\infty |s(x^*, M_\Gamma(E \cap H_k))| < \varepsilon$$

for sufficiently large m. Thus, $\lim_m s(x^*, M_\Gamma(\bigcup_{k=1}^m E_{k,n_k})) = s(x^*, M_\Gamma(E))$.

The equivalence of (v) and (vi) is obvious. □

The example below presents still another multifunction $\Gamma : [0,1] \to ck(c_0)$ that is Pettis integrable in $cb(c_0)$, $M_\Gamma(\mathcal{L})$ is not d_H-separable but $\mathbb{Z}_\Gamma$ is separable. Its idea is taken from [11], p. 53. Contrary to Example 3.6 the function generating Γ is not Henstock-Kurzweil-Pettis integrable.

Example 3.8. Consider a sequence of intervals $B_n = (a_n, b_n] \subseteq [0,1]$ such that $b_1 = 1$, $a_{n+1} < b_{n+1} \leq a_n$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} a_n = 0$. Let $f\colon [0,1] \to c_0$ be given by

$$f(t) = \left(\frac{1}{|B_1|}\chi_{B_1}(t), \frac{1}{|B_2|}\chi_{B_2}(t), \dots, \frac{1}{|B_n|}\chi_{B_n}(t), \dots\right)$$

for each $t \in [0,1]$. f is a scalarly integrable function (with respect to the Lebesgue measure) and its Dunford integral over $E \in \mathcal{L}$ is given by

$$\int_E f(t)\,dt = \left(\frac{|E\cap B_1|}{|B_1|}, \frac{|E\cap B_2|}{|B_2|}, \dots, \frac{|E\cap B_n|}{|B_n|}, \dots\right)_{n=1}^{\infty}.$$

The Pettis integral and the Henstock-Kurzweil-Pettis integral of f do not exist, because the Dunford integral $\int_0^1 f(t)\,dt = (1,1,\dots,1,\dots) \notin c_0$ (compare with [11], p. 53). In particular the set $\mathbb{Z}_f$ is not uniformly integrable. It is however separable as a subset of $L_1[0,1]$. Define a $ck(c_0)$-valued multifunction by $\varGamma(t) := \operatorname{conv}\{0, f(t)\}$, $t \in [0,1]$. One can easily see that $\varGamma$ is scalarly integrable and as the zero function is a Pettis integrable selection of $\varGamma$, it follows from [14, Theorem 3.7] that $\varGamma$ is Pettis integrable in $cb(c_0)$ and if $\mathcal{S}_P(\varGamma)$ is the set of all Pettis integrable selections of $\varGamma$, then

$$M_\varGamma(E) = \overline{\left\{\int_E f(t)\,dt\colon f \in \mathcal{S}_P(\varGamma)\right\}}.$$

Since $\mathbb{Z}_\varGamma$ is not uniformly integrable, $M_\varGamma : \mathcal{L} \to ck(c_0)$ fails to be an h-multimeasure. Since f is a non-Pettis integrable selection of $\varGamma$, the multifunction $\varGamma$ is not Pettis integrable in $cwk(c_0)$ (see [14, Theorem 5.4]). Hence, it follows from [3, Proposition 3.5] that $R\circ\varGamma$ is a non-Pettis integrable function.

For each $t \in [0,1]$ we have $\varGamma(t) = \operatorname{conv}\{0, f(t)\} = \{\alpha(t)f(t)\colon 0 \leq \alpha(t) \leq 1\}$. If h is a strongly measurable selection of $\varGamma$, then $\alpha(t) = \|h(t)\|/\|f(t)\|$ for each $t \in \bigcup_n B_n$. Without loss of generality, we may assume that $\alpha(t) = 0$ if $t \notin \bigcup_n B_n$ and so $\alpha : [0,1] \to [0,1]$ is a measurable function. Thus, if a strongly measurable function $h : [0,1] \to c_0$ is a selection of $\varGamma$, then there exists a measurable function $\alpha : [0,1] \to [0,1]$ such that $h = \alpha f$.

It is clear that a selection αf is Pettis integrable if and only if $\lim_n \frac{1}{|B_n|}\int_{B_n} \alpha(t)\,dt = 0$.

Let $\mathbb{A} := \{\alpha\colon [0,1] \to [0,1]\colon \alpha f$ is Pettis integrable $\}$. As a direct consequence of the above equality and the separable theory (cf. [14]) we have for each $E \in \mathcal{L}$ the equality

$$\int_E \varGamma(t)\,dt = \overline{\operatorname{conv}\left\{\int_E \alpha(t)f(t)\,dt\colon \alpha \in \mathbb{A}\right\}} = \overline{\left\{\int_E \alpha(t)f(t)\,dt\colon \alpha \in \mathbb{A}\right\}}$$

where the last equality is a consequence of the convexity of $\mathbb{A}$.

Thus,

$$M_\varGamma(E) = \overline{\left\{\left\langle\frac{1}{|B_n|}\int_{E\cap B_n} \alpha(t)\,dt\right\rangle_{n=1}^{\infty} : \alpha \in \mathbb{A}\right\}},$$

for every $E \in \mathcal{L}$. Let us prove now that $M_\Gamma(\mathcal{L})$ is d_H non-separable.

Denote by Ξ the collection of all increasing infinite subsequences of $\mathbb{N}$ such that if $\Delta_1 \in \Xi$ and $\Delta_2 \in \Xi$, then Δ_1 and Δ_2 have only finitely many common terms. It is well known that the cardinality of Ξ equals continuum. If $\Delta = \{n_1, n_2, \ldots, n_k, \ldots\} \in \Xi$, then let $F_\Delta := \bigcup_{k=1}^\infty B_{n_k}$.

If $\Delta_1 \neq \Delta_2$ and $\alpha \in \mathbb{A}$, then

$$\begin{aligned}
&\sup_{\alpha\in\mathbb{A}}\inf_{\beta\in\mathbb{A}}\left\|\left\langle \frac{1}{|B_n|}\int_{F_{\Delta_1}\cap B_n}\alpha(t)\,dt\right\rangle_{n=1}^{\infty} - \left\langle \frac{1}{|B_n|}\int_{F_{\Delta_2}\cap B_n}\beta(t)\,dt\right\rangle_{n=1}^{\infty}\right\|_{c_0}\\
&= \sup_{\alpha\in\mathbb{A}}\inf_{\beta\in\mathbb{A}}\sup_n\left|\frac{1}{|B_n|}\int_{F_{\Delta_1}\cap B_n}\alpha(t)\,dt - \frac{1}{|B_n|}\int_{F_{\Delta_2}\cap B_n}\beta(t)\,dt\right|\\
&\geq \sup_{\alpha\in\mathbb{A}}\sup_{n_k\in\Delta_1\setminus\Delta_2}\frac{1}{|B_{n_k}|}\int_{B_{n_k}}\alpha(t)\,dt = 1
\end{aligned}$$

Hence $d_H(M_\Gamma(F_{\Delta_1}), M_\Gamma(F_{\Delta_2})) \geq 1$ and the range of $R\circ M_\Gamma$ is non-separable. $\square$

Remark 3.9. Separability of $\mathbb{Z}_\Gamma$ and the range of $M_\Gamma(\Sigma)$ are not hereditary properties. That is, there are multifunctions Δ and Γ such that $\Delta(\omega) \subset \Gamma(\omega)$ for every $\omega \in \Omega$, $\mathbb{Z}_\Gamma$ is separable and $\mathbb{Z}_\Delta$ is not. Similarly, the range of M_Γ is d_H-separable, whereas the range of M_Δ is not.

As an example let $V = B(l_\infty(W))$, where W is an uncountable set and let (Ω, Σ, μ) be such that there exists a Pettis integrable function $f : \Omega \to B(l_\infty(W))$ with non-separable range of its integral (see [15, 2D]). If $\Gamma \equiv V$, then $\mathbb{Z}_\Gamma$ is compact in $L_1(\mu)$ and $M_\Gamma(\Sigma)$ is d_H-compact, whereas $\mathbb{Z}_f$ amd $\nu_f(\Sigma)$ are not separable in $L_1(\mu)$ and in X, respectively (see [24] or [26]).

Question 3.10. If V is a weakly compact set, then each V-valued scalarly measurable function is scalarly equivalent to a strongly measurable function (see [20]) and so the above example does not work there. This leads to the following two questions.

Let Γ and $M_\Gamma : \Sigma \to cwk(X)$ be as in Theorem 3.7. Assume that $f \in \mathcal{S}_\Gamma$ is arbitrary and the condition (i) is fulfilled. Is the set $\mathbb{Z}_f$ separable in $L_1(\mu)$?

Assume that $m \in \mathcal{S}(M_\Gamma)$ is arbitrary and the set $M_\Gamma(\Sigma)$ is separable in the Hausdorff metric or it is scalarly separable. Is the set $m(\Sigma)$ separable?

Question 3.11. Assume that Γ is $cb(X)$ valued. Suppose that for each $f \in \mathcal{S}_\Gamma$ the set $\mathbb{Z}_f$ is separable (or norm relatively compact) in $L_1(\mu)$ (it is so if X has the WRNP, for instance). Is $\mathbb{Z}_\Gamma$ also separable (or norm relatively compact)?

Assume that M is a multimeasure such that $M(\Sigma)$ is scalarly separable. When is $M(\Sigma)$ separable in the Hausdorff metric?

4. Convergence theorems

The following theorem has been proven in [27, Theorem 5.1].

Theorem 4.1. *Let $\Gamma\colon \Omega \to cwk(X)$ be scalarly integrable and let $\{\Gamma_n\colon \Omega \to cwk(X)\colon n \in \mathbb{N}\}$ be a sequence of multifunctions Pettis integrable in $cwk(X)$. If the family $\bigcup_n \mathbb{Z}_{\Gamma_n}$ is uniformly integrable and $\lim_n s(x^*, \Gamma_n) = s(x^*, \Gamma)$ in μ–measure, for each $x^* \in X^*$ (or weakly in $L_1(\mu)$, for each $x^* \in X^*$), then Γ is Pettis integrable in $cwk(X)$ and,*

$$(SC) \qquad \lim_n s(x^*, M_{\Gamma_n} E) = s(x^*, M_\Gamma(E)) \quad \text{for every } x^* \in X^* \text{ and } E \in \Sigma.$$

If Γ_n's are assumed to be $c(X)$-valued, determined by a WCG space and Pettis integrable in $cb(X)$, then Γ is Pettis integrable in $cb(X)$ and (SC) holds true.

One may ask if the scalar convergence (SC) can be replaced by a stronger one. In general the answer is negative even for Pettis integrable functions (see [27, Remark 5.4]). It is my aim to show that assuming stronger properties of the multifunctions one may obtain also a stronger convergence of the corresponding integrals. Our first proposition is to assume that $(\Gamma_n)_n$ is a martingale. Then we will replace the scalar convergence $s(x^*, \Gamma_n) \to s(x^*, \Gamma)$ in measure by the uniform convergence in measure.

Theorem 4.2. *Let $\Gamma\colon \Omega \to cwk(X)$ be scalarly integrable and let $(\Gamma_n, \Sigma_n)_{n\in\mathbb{N}}$ be a martingale of multifunctions $\Gamma_n\colon \Omega \to cwk(X)$, $n \in \mathbb{N}$, that are Pettis integrable in $cwk(X)$. If the family $\bigcup_n \mathbb{Z}_{\Gamma_n}$ is uniformly integrable and $\lim_n s(x^*, \Gamma_n) = s(x^*, \Gamma)$ in μ–measure, for each $x^* \in X^*$ (or weakly in $L_1(\mu)$, for each $x^* \in X^*$), then Γ is Pettis integrable in $cwk(X)$, $\mathbb{E}(\Gamma|\Sigma_n) = \Gamma_n$, $n \in \mathbb{N}$, in the sense of scalar equivalence, and*

$$(HC) \qquad \lim_n d_H(M_\Gamma(E), M_{\Gamma_n}(E)) = 0 \quad \text{for every } E \in \Sigma\,.$$

If Γ_n's are assumed to be $c(X)$-valued, determined by a WCG space and Pettis integrable in $cb(X)$, then Γ is Pettis integrable in $cb(X)$ and (HC) holds true.

Proof. In virtue of Theorem 4.1 Γ is Pettis integrable in $cwk(X)$ (or in $cb(X)$). For each x^* the sequence $\{(s(x^*, \Gamma_n), \sigma(\Gamma_n)) : n \in \mathbb{N}\}$ is a scalar martingale converging in measure to $s(x^*, \Gamma)$. Being uniformly integrable, it is convergent in $L_1(\mu)$. Hence $\mathbb{Z}_\Gamma$ is contained in the weak closure of the relatively weakly compact set $\bigcup_n \mathbb{Z}_{\Gamma_n} \subset L_1(\mu)$. Consequently, $\mathbb{Z}_\Gamma$ is uniformly integrable and M_Γ is an h-multimeasure. At this stage we apply (11) that holds true for arbitrary martingale. □

As a direct consequence of Theorems 3.4, 4.1 and 3.7 we obtain the following characterizations of multifunctions approximated by simple multifunctions:

Theorem 4.3. *Let $\Gamma : \Omega \to c(X)$ be a scalarly integrable multifunction. Then Γ is Pettis integrable in $cwk(X)$ and $M_\Gamma(\Sigma)$ is separable in the Hausdorff metric if and only if there exists a (martingale) sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of $cwk(X)$–valued simple multifunctions, such that $\bigcup_n \mathbb{Z}_{\Gamma_n}$ is uniformly integrable and, one of the following conditions is satisfied for each $x^* \in X^*$:*

(α) $\{s(x^*,\Gamma_n) : n\in\mathbb{N}\}$ *is* μ*–a.e. convergent to* $s(x^*,\Gamma)$*;*
(β) $\{s(x^*,\Gamma_n) : n\in\mathbb{N}\}$ *is convergent in* μ*–measure to* $s(x^*,\Gamma)$*;*
(γ) $\{s(x^*,\Gamma_n) : n\in\mathbb{N}\}$ *is convergent to* $s(x^*,\Gamma)$ *in* $L_1(\mu)$*;*
(δ) $\{s(x^*,\Gamma_n) : n\in\mathbb{N}\}$ *is convergent to* $s(x^*,\Gamma)$ *weakly in* $L_1(\mu)$.

Proof. $\Leftarrow$. It follows from Theorems 4.1 that Γ is Pettis integrable in $cwk(X)$ and so M_Γ is an h-multimeasure, what is equivalent to the uniform integrability of the set $\mathbb{Z}_\Gamma$. Theorem 3.4 yields the d_H-separability of $M_\Gamma(\Sigma)$.

$\Rightarrow$. If $\Gamma : \Omega\to c(X)$ is Pettis integrable in $cwk(X)$, then $\mathbb{Z}_\Gamma$ is uniformly integrable (see [27, Proposition 2.2]) and so M_Γ is an h-multimeasure. According to Theorem 3.4 there exists a martingale $(\Gamma_n,\Sigma_n)_{n=1}^\infty$ satisfying the conditions (α)–(δ). It follows from Proposition 3.3 that the family $\bigcup_n \mathbb{Z}_{\Gamma_n}$ is uniformly integrable. □

Theorem 4.4. *Let* $\Gamma : \Omega\to c(X)$ *be a scalarly integrable multifunction. Assume that there exists a (martingale) sequence* $(\Gamma_n)_{n\in\mathbb{N}}$ *of* $cb(X)$*–valued simple multifunctions, such that* $\bigcup_n \mathbb{Z}_{\Gamma_n}$ *is uniformly integrable and, one of the following conditions is satisfied for each* $x^*\in X^*$*:*

(α) $\{s(x^*,\Gamma_n) : n\in\mathbb{N}\}$ *is* μ*–a.e. convergent to* $s(x^*,\Gamma)$*;*
(β) $\{s(x^*,\Gamma_n) : n\in\mathbb{N}\}$ *is convergent in* μ*–measure to* $s(x^*,\Gamma)$*;*
(γ) $\{s(x^*,\Gamma_n) : n\in\mathbb{N}\}$ *is convergent to* $s(x^*,\Gamma)$ *in* $L_1(\mu)$*;*
(δ) $\{s(x^*,\Gamma_n) : n\in\mathbb{N}\}$ *is convergent to* $s(x^*,\Gamma)$ *weakly in* $L_1(\mu)$.

Then Γ *is Pettis integrable in* $cb(X)$ *and* $M_\Gamma(\Sigma)$ *is locally separable in the Hausdorff metric.*

Proof. The proof is similar to the proof of Theorem 4.3 but instead of Theorem 3.4 we apply now Theorem 3.7. □

I say that a sequence of scalarly measurable multifunctions $\Gamma_n\colon\Omega\to cb(X)$, $n\in\mathbb{N}$, is *scalarly equi-convergent in measure* to a scalarly measurable multifunction $\Gamma\colon\Omega\to cb(X)$ if for every $\delta>0$

$$\lim_n \sup_{\|x^*\|\le 1} \mu\{\omega\in\Omega\colon |s(x^*,\Gamma_n(\omega))-s(x^*,\Gamma(\omega))|>\delta\}=0.$$

If $\Gamma\colon\Omega\to cb(X)$ and $\Delta\colon\Omega\to cb(X)$ are scalarly measurable, then let

$$\rho(\Gamma,\Delta):=\inf\left\{\lambda : \sup_{\|x^*\|\le 1}\mu\{|s(x^*,\Gamma)-s(x^*,\Delta)|\ge\lambda\}\le\lambda\right\}.$$

One can check that ρ is a pseudometric on $\mathcal{P}(\mu,cb(X))$ such that $\rho(\Gamma,\Delta)=0$ if and only if Γ and Δ are scalarly equivalent.Thus, ρ is a metric on $\mathbb{P}(\mu,cb(X))$. Moreover, $\rho(\Gamma_n,\Gamma)\to 0$ if and only if the sequence $\Gamma_n\colon\Omega\to cb(X)$, $n\in\mathbb{N}$, is scalarly equi-convergent in measure to $\Gamma\colon\Omega\to cb(X)$.

If $\Gamma\colon\Omega\to cb(X)$ and $\Delta\colon\Omega\to cb(X)$ are scalarly integrable then, $\rho(\Gamma,\Delta)\le d_P(\Delta,\Gamma)$.

It is also easy to see that if f_n's are strongly measurable functions, then it may happen that $f_n \to f$ in the sense of scalar equi-convergence in measure, but $f_n \nrightarrow f$ in measure. But in the non-separable case, where in general the functions $\omega \longrightarrow \|f(\omega)\|$ are non-measurable, the scalar equi-convergence in measure is a very convenient substitute of the convergence in measure.

Lemma 4.5. *Let $\Gamma_n \colon \Omega \to cb(X)$, $n \in \mathbb{N}$, and $\Gamma : \Omega \to cb(X)$ be scalarly measurable multifunctions. Consider the following conditions:*

(A) $\lim_n d_H(\Gamma_n, \Gamma) = 0$, *$\mu$-a.e.;*

(B) $\forall\, \delta > 0 \ \lim_n \mu_*\{d_H(\Gamma_n, \Gamma) > \delta\} = 0$ *(μ_* is the inner measure induced by μ);*

(C) $\lim_n \rho(\Gamma_n, \Gamma) = 0$.

Then (A) $\Rightarrow$ (B) $\Rightarrow$ (C).

Proof. $(A) \Rightarrow (B)$. Given $\delta > 0$ and $n \in \mathbb{N}$ let

$$A_n^\delta := \{\omega \in \Omega : d_H(\Gamma_n(\omega), \Gamma(\omega)) > \delta\}.$$

Suppose that there exists $\delta > 0$ with $\limsup_n \mu_*(A_n^\delta) = a > 0$. If $B_n^\delta \subseteq A_n^\delta$ is a measurable kernel of A_n^δ, then there is an increasing sequence (n_k) of integers such that $\mu(B_{n_k}^\delta) > a/2$, for all $k \in \mathbb{N}$. Let

$$B^\delta := \limsup_k B_{n_k}^\delta = \bigcap_{m=1}^\infty \bigcup_{k=m}^\infty B_{n_k}^\delta.$$

Then $\mu(B^\delta) = \lim_m \mu\left(\bigcup_{k=m}^\infty B_{n_k}^\delta\right) \geq a/2$. It follows that if $\omega \in B^\delta$, then the inequality $d_H(\Gamma_{n_k}(\omega), \Gamma(\omega)) > \delta$ holds true for infinitely many k's. That contradicts the μ-a.e. convergence of the sequence $(\Gamma_n)_{n\in\mathbb{N}}$ to Γ in the Hausdorff metric.

$(B) \Rightarrow (C)$. Let A_n^δ be defined as before. Then

$$A_n^\delta = \bigcup_{\|x^*\|\leq 1} \{\omega : |s(x^*, \Gamma_n(\omega)) - s(x^*, \Gamma(\omega))| > \delta\}$$

and so

$$\mu_*(A_n^\delta) \geq \mu\{|s(x^*, \Gamma_n) - s(x^*, \Gamma)| > \delta\} \quad \text{for each } x^* \in B(X^*).$$

Consequently,

$$\mu_*(A_n^\delta) \geq \sup_{\|x^*\|\leq 1} \mu\{|s(x^*, \Gamma_n) - s(x^*, \Gamma)| > \delta\}$$

what yields the scalar equi-convergence of (Γ_n) in measure to Γ. □

The next result is a generalization of [27, Theorem 5.2].

Theorem 4.6. *Let $\Gamma_n \colon \Omega \to cwk(X)$ $[ck(X)]$, $n \in \mathbb{N}$, be Pettis integrable in $cwk(X)$ $[ck(X)]$ and satisfying the following two conditions:*

(a) *The set $\bigcup_n \mathbb{Z}_{\Gamma_n}$ is uniformly integrable,*

(b) *The sequence* $\{\Gamma_n\colon n\in\mathbb{N}\}$ *is scalarly equi-convergent in measure to a scalarly integrable* $\Gamma\colon \Omega\to c(X)$.

Then Γ *is Pettis integrable in* $cwk(X)$ $[ck(X)]$ *and* $\lim_n d_P(\Gamma_n,\Gamma)=0$. *In particular*

$$(PC)\qquad \lim_n d_H(M_{\Gamma_n}(E), M_\Gamma(E))=0 \quad \text{uniformly on } \Sigma\,.$$

If Γ_n*'s are assumed to be* $c(X)$*-valued, determined by a WCG space and Pettis integrable in* $cb(X)$*, then* Γ *is Pettis integrable in* $cb(X)$ *and* $\lim_n d_P(\Gamma_n,\Gamma)=0$.

Proof. Assume that Γ_n's are $c(X)$-valued. It follows from Theorem 4.1 that Γ is Pettis integrable in $cb(X)$. Due to the classical Vitali convergence theorem, the set $\mathbb{Z}_\Gamma$ is a subset of the norm closure of the set $\bigcup_n \mathbb{Z}_{\Gamma_n}$. Consequently, $\mathbb{Z}_\Gamma$ is uniformly integrable. Together with the condition (a) this yields the uniform integrability of the family $\{s(x^*,\Gamma_n)-s(x^*,\Gamma)\colon \|x^*\|\le 1,\ n\in\mathbb{N}\}$. We have then for arbitrary $E\in\Sigma$ and $\delta>0$

$$\begin{aligned}
d_H(M_{\Gamma_n}(E),M_\Gamma(E)) &= \sup_{\|x^*\|\le 1}|s(x^*,M_{\Gamma_n}(E))-s(x^*,M_\Gamma(E))| \\
&\le d_P(\Gamma_n,\Gamma)=\sup_{\|x^*\|\le 1}\int_\Omega |s(x^*,\Gamma_n)-s(x^*,\Gamma)|\,d\mu \\
&\le \sup_{\|x^*\|\le 1}\int_{\{|s(x^*,\Gamma_n)-s(x^*,\Gamma)|\le\delta\}}|s(x^*,\Gamma_n)-s(x^*,\Gamma)|\,d\mu \\
&\qquad + \sup_{\|x^*\|\le 1}\int_{\{|s(x^*,\Gamma_n)-s(x^*,\Gamma)|>\delta\}}|s(x^*,\Gamma_n)-s(x^*,\Gamma)|\,d\mu \\
&\le \delta + \sup_{\|x^*\|\le 1}\int_{\{|s(x^*,\Gamma_n)-s(x^*,\Gamma)|>\delta\}}|s(x^*,\Gamma_n)-s(x^*,\Gamma)|\,d\mu \quad (13)
\end{aligned}$$

Due to the scalar equi-convergence of $\{\Gamma_n\colon n\in\mathbb{N}\}$ to Γ and the uniform integrability of $\{s(x^*,\Gamma_n)-s(x^*,\Gamma)\colon \|x^*\|\le 1,\ n\in\mathbb{N}\}$, the second term of the inequality (13) is arbitrarily small for sufficiently large n's. The required convergence relations are consequences of the above inequalities.

Since metric spaces $\mathbb{P}(\mu,cwk(X))$ and $\mathbb{P}(\mu,ck(X))$ are complete, in case of $cwk(X)$ and $ck(X)$-valued multifunctions Γ_n, $n\in\mathbb{N}$, it follows from (PC) that also M_Γ takes its values in $cwk(X)$ or $ck(X)$, respectively. □

Corollary 4.7. *Let* $\Gamma\colon\Omega\to c(X)$ *be scalarly integrable and let* $\{\Gamma_n\colon\Omega\to cb(X)\colon n\in\mathbb{N}\}$ *be a sequence of simple multifunctions satisfying the following two conditions:*

(a) *The set* $\bigcup_n \mathbb{Z}_{\Gamma_n}$ *is uniformly integrable,*

(b) *The sequence* $\{\Gamma_n\colon n\in\mathbb{N}\}$ *is scalarly equi-convergent in measure to* Γ.

Then Γ *is Pettis integrable in* $cb(X)$ *and* $M_\Gamma(\Sigma)$ *is relatively compact in the Hausdorff metric.*

If Γ_n's are functions, then we get the following result, that is a generalization of [27, Corollary 5.3]:

Theorem 4.8 ([1]). *Let $f_n\colon \Omega \to X$, $n \in \mathbb{N}$, be a sequence of Pettis integrable functions scalarly equi-convergent in measure to a function $f\colon \Omega \to X$ (e.g. $\|f_n - f\| \to 0$ a.e.). If the family $\bigcup_n \mathbb{Z}_{f_n}$ is uniformly integrable, then f is Pettis integrable and*

$$\lim_n \|f_n - f\|_P = 0.$$

In particular

$$\lim_n \sup_{E\in\Sigma} \left\| \int_E f_n\, d\mu - \int_E f\, d\mu \right\| = 0\,.$$

Remark 4.9. In [22] Banach spaces possessing the weak Radon-Nikodým property are characterized in terms of martingales convergent in the Pettis norm. As a consequence, if X is without WRNP and μ is non-atomic, then there exists a scalarly bounded X-valued martingale which is divergent in $\mathbb{P}(\mu, X)$. In virtue of Theorem 4.8 that martingale cannot be scalarly equi-convergent in measure. But if X has WRNP, then it suffices to assume in Theorem 4.8 the uniform integrability of $\bigcup_n \mathbb{Z}_{f_n}$ only.

5. The strong law of large numbers

I shall start with a simple but useful fact concerning Pettis integrability on larger σ-algebra than the initial one. To prove it, we need to know how to integrate bounded non-negative real functions with respect to multifunctions, but such a theory is well known.

Lemma 5.1. *Assume that Ξ is a sub-σ-algebra of Σ and that $\Gamma : \Omega \to cb(X)$ is scalarly measurable with respect to Ξ. If $\Gamma \in \mathcal{P}(\mu|\Xi, cb(X))$, then we have also $\Gamma \in \mathcal{P}(\mu, cb(X))$.*

Proof. If $F \in \Sigma$, then we set $\widetilde{M_\Gamma}(F) := \int_\Omega \mathbb{E}(\chi_F|\Xi)\, dM_\Gamma$. Hence, if $x^* \in X^*$, then

$$\begin{aligned} s(x^*, \widetilde{M_\Gamma}(F)) &= \int_\Omega \mathbb{E}(\chi_F|\Xi)\, ds(x^*, M_\Gamma) = \int_\Omega \mathbb{E}(\chi_F|\Xi) s(x^*, \Gamma)\, d\mu \\ &= \int_\Omega \mathbb{E}(s(x^*, \Gamma)\chi_F|\Xi)\, d\mu = \int_\Omega s(x^*, \Gamma)\chi_F\, d\mu = \int_F s(x^*, \Gamma)\, d\mu \qquad \square \end{aligned}$$

For each $j \in \mathbb{N}$ let $\Omega_j := \Omega$ and π_j be the canonical projection of $\Omega^\infty = \prod_{n=1}^\infty \Omega_n$ onto Ω_j. Moreover, let μ_∞ be the countable direct product of μ on Ω^∞. Following Talagrand [31] (see also [25]), I say that a function $f : \Omega \to X$ satisfies the *strong law of large numbers* (briefly SLLN) if there exists $a_f \in X$ such that

$$\lim_{n\to\infty} \left\| a_f - \frac{1}{n} \sum_{j=1}^n f(\pi_j) \right\| = 0 \quad \mu_\infty - \text{a.e.}$$

Hoffmann-Jørgensen proved (see [17]) that if f satisfies the SLLN, then f is scalarly integrable and $a_f = \int_\Omega f\,d\mu$, where the integral is the Gelfand integral taking its value in X on the set Ω. Then Talagrand strengthened that result, proving (see [31]) that if f satisfies the SLLN, then f is Pettis integrable. Consequently $\mathbb{Z}_f$ is uniformly integrable.

By analogy, I formulate the corresponding property for multifunctions.

Definition 5.2. A multifunction $\Gamma : \Omega \to cb(X)$ satisfies the *strong law of large numbers* if there exists $A_\Gamma \in cb(X)$ such that

$$\lim_{n\to\infty} d_H\left(A_\Gamma, \frac{1}{n}\bigoplus_{j=1}^{n}\Gamma(\pi_j)\right) = 0 \quad \mu_\infty - \text{a.e.}$$

It should be observed that if Γ takes its values in $cwk(X)$ or in $ck(X)$, then also $A_\Gamma \in cwk(X)$ or $A_\Gamma \in ck(X)$, respectively. □

Proposition 5.3. *If $\Gamma : \Omega \to cb(X)$ satisfies the strong law of large numbers, then Γ is scalarly integrable and $\int_\Omega s(x^*,\Gamma)\,d\mu = s(x^*,A_\Gamma)$, for every $x^* \in X^*$.*

Proof. If $x^* \in X^*$, then $\lim_{n\to\infty}|s(x^*,A_\Gamma) - \frac{1}{n}\sum_{j=1}^{n} s(x^*,\Gamma(\pi_j))| = 0$ $\mu_\infty - \text{a.e.}$ Consequently, it follows from the classical theory that Γ is scalarly integrable and $\int_\Omega s(x^*,\Gamma)\,d\mu = s(x^*,A_\Gamma)$. □

Before we prove the main characterization of multifunctions that satisfy the SLLN, we have to recall the definition of a stable set of functions.

Definition 5.4 (Fremlin, Talagrand [30]). Let $\mathcal{H}$ be a collection of real valued functions defined on Ω. $\mathcal{H}$ is said to be *stable* if for each $A \in \Sigma_\mu^+$ and arbitrary reals $\alpha < \beta$ there exist $k, l \in \mathbb{N}$ satisfying the inequality

$$\mu_{k+l}^*\left(A^{k+l} \cap \bigcup_{f\in\mathcal{H}} \{f<\alpha\}^k \times \{f>\beta\}^l\right) < \mu(A)^{k+l},$$

where μ_{k+l} is the direct product of $k+l$ copies of μ.

If $\Gamma : \Omega \to cb(X)$ is such that $\mathcal{Z}_\Gamma$ is stable, then we say that Γ is *properly measurable* (see [30] for functions). □

If $\mathcal{H}$ is stable then all elements of $\mathcal{H}$ are measurable functions. If $\mathcal{H}$ is stable and pointwise bounded, then its pointwise closure is also stable (see [30, 9-1]).

Now I can formulate a result that partially generalizes Theorem 1.10 from [27], proved for multifunctions with weakly compact values, to arbitrary $cb(X)$-valued multifunctions. In that case it was assumed that $\mathbb{Z}_\Gamma$ is uniformly integrable. In the following theorem I assume that Γ is locally integrably bounded, what simply means that if $A \in \Sigma_\mu^+$, then there is $A \supset B \in \Sigma_\mu^+$ such that $|\Gamma|$ is pointwise dominated on B by an integrable function.

Theorem 5.5. *Let $\Gamma : \Omega \to cb(X)[cwk(X),\ ck(X)]$ be properly measurable and locally integrably bounded. If $\mathbb{Z}_\Gamma$ is uniformly integrable, then Γ is Pettis integrable in $cb(X)[cwk(X),\ ck(X)]$, $R\circ\Gamma$ is Pettis integrable and $M_\Gamma(\Sigma)$ is relatively compact in the Hausdorff metric.*

Proof. Assume first that Γ is integrably bounded and let $h \in \mathcal{L}_1(\mu)$ be such that $|s(x^*,\Gamma(\omega))| \leq h(\omega)$ for all $x^* \in B(X^*)$ and $\omega \in \Omega$. We have $s(x^*,\Gamma(\omega)) = \langle e_{x^*}, R\circ\Gamma(\omega)\rangle$ for every ω and $x^* \in X^*$. Thus, the set $\mathcal{Z}_{R\circ\Gamma}$ is also order bounded by h. Since $\{e_{x^*} : \|x^*\| \leq 1\}$ is norming for $l_\infty(B(X^*))$, its absolutely convex hull is weak*-dense in $B(l_\infty^*(B(X^*))$. This means that $\mathcal{Z}_{R\circ\Gamma}$ is contained in the pointwise closure of the absolutely convex hull of $\mathcal{Z}_\Gamma$, which is a stable set (see [30, Theorem 11-2-1]). Hence $\mathcal{Z}_{R\circ\Gamma}$ is stable. Consequently $R\circ\Gamma$ is Pettis integrable and $M_{R\circ\Gamma}(\Sigma)$ is norm relatively compact([30, Theorem 6-1-2]).

Then, we can argue exactly as in the proof of Proposition 4.4 in [4]. Since

$$\int_E R\circ\Gamma\, d\mu \in \mu(E)\,\overline{\mathrm{conv}} R\circ\Gamma(E) \quad \text{for every } E \in \Sigma$$

and $R(cb(X))$ is a closed convex cone, there exists a set $M(E) \in cb(X)$ with $R(M(E)) = \int_E R\circ\Gamma\, d\mu$. If $x^* \in X^*$ is arbitrary, then

$$\begin{aligned}\int_E s(x^*,\Gamma)\, d\mu &= \int_E \langle e_{x^*}, R\circ\Gamma\rangle\, d\mu = \langle e_{x^*}, \int_E R\circ\Gamma\, d\mu\rangle \\ &= \langle e_{x^*}, R(M(E))\rangle = s(x^*, M(E)).\end{aligned}$$

This proves the Pettis integrability of Γ in $cb(X)$. Since R is an isometry the set $M_\Gamma(\Sigma)$ is d_H-relatively compact.

Taking into account that $cwk(X)$ and $ck(X)$ are also closed cones, we obtain the other two conclusions.

If Γ is $cwk(X)$-valued, then its Pettis integrability in $cwk(X)$ follows also from [27, Theorem 1.10]. If Γ is $ck(X)$-valued, then its Pettis integrability in $ck(X)$ follows also from [27, Theorem 3.16]

Consider now the general situation. Let $\Omega = \bigcup_n \Omega_n$ be a partition of Ω into pairwise disjoint measurable sets in such a way that Γ is integrably bounded on each Ω_n. For each $n \in \mathbb{N}$ let $M_n : \Sigma \to cb(X)$ be defined by $M_n(E) = \int_{E\cap\Omega_n} \Gamma\, d\mu$. Since Γ is scalarly integrable we have for all $x^* \in X^*$ and $E \in \Sigma$

$$\sum_n |s(x^*, M_n(E\cap\Omega_n))| \leq \sum_n \int_{E\cap\Omega_n} |s(x^*,\Gamma)|\, d\mu = \int_\Omega |s(x^*,\Gamma)|\, d\mu < \infty\,.$$

Due to uniform integrability of $\mathcal{Z}_\Gamma$, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{\|x^*\|\leq 1} \int_{\bigcup_{n>n_0}\Omega_n} |s(x^*,\Gamma)|\, d\mu < \varepsilon.$$

It follows that the series $\sum_{n=1}^\infty \int_{E\cap\Omega_n} \Gamma\, d\mu$ is Cauchy in the metric d_H, hence it is convergent. □

Theorem 5.6. *For a multifunction $\Gamma : \Omega \to cb(X)$ the following conditions are equivalent:*

(*i*) Γ *satisfies the strong law of large numbers;*
(*ii*) Γ *is properly measurable and integrably bounded.*

Proof. $(i) \Rightarrow (ii)$. Let us notice first that if $\Gamma : \Omega \to cb(X)$ satisfies the strong law of large numbers, then also the function $R\circ\Gamma : \Omega \to Y = l_\infty(B(X^*))$ does. According to [31] $R\circ\Gamma$ satisfies the SLLN if and only if $R\circ\Gamma$ is properly measurable and $\int_\Omega^* \|R\circ\Gamma\|_Y \, d\mu < \infty$.

Notice that $s(x^*, \Gamma(\omega)) = \langle e_{x^*}, R\circ\Gamma(\omega)\rangle$ for every ω and $x^* \in X^*$. Thus, if the set $\mathcal{Z}_{R\circ\Gamma}$ is stable the same holds true also for $\mathcal{Z}_\Gamma$. We have also the equality $\|R\circ\Gamma\|_Y = |\Gamma|$ pointwise and so (ii) is fulfilled.

$(ii) \Rightarrow (i)$. Exactly as in the proof of Theorem 5.5 we conclude that $\mathcal{Z}_{R\circ\Gamma}$ is stable and $\int_\Omega^* \|R\circ\Gamma\|_Y \, d\mu < \infty$. Consequently $R\circ\Gamma$ satisfies the SLLN and is Pettis integrable. But that means the existence of a point $a_{R\circ\Gamma} \in Y$ such that

$$\lim_n \left\| a_{R\circ\Gamma} - \frac{1}{n}\sum_{j=1}^n R\circ\Gamma(\pi_j) \right\| = 0 \quad \mu_\infty - \text{a.e.} \tag{14}$$

Invoking once again to the proof of Theorem 5.5 we obtain a set $A_\Gamma \in cb(X)$ with $R(A_\Gamma) = a_{R\circ\Gamma} = \int_\Omega R\circ\Gamma \, d\mu$ and $\int_\Omega s(x^*, \Gamma)\, d\mu = s(x^*, A_\Gamma)$. The equality (14) turns now into

$$\lim_n d_H \left(A_\Gamma, \frac{1}{n} \bigoplus_{j=1}^n \Gamma(\pi_j) \right) = 0 \quad \mu_\infty - \text{a.e.} \qquad \square$$

As a consequence of Theorems 2.3, 5.5 and 5.6, we have the following

Corollary 5.7. *If $\Gamma : \Omega \to cb(X)\,[cwk(X),\ ck(X)]$ satisfies the SLLN, then Γ is Pettis integrable in $cb(X)\,[cwk(X),\ ck(X)]$, $M_\Gamma(\Sigma)$ is relatively compact in d_H and there exist simple multifunctions $\Gamma_n : \Omega \to cb(X)\ [cwk(X),\ ck(X)]$, $n \in \mathbb{N}$, such that* $\lim_n d_P(\Gamma, \Gamma_n) = 0$.

It is our aim now to prove that Cesaro averages of the multifunctions $\Gamma(\pi_j)$, $j \in \mathbb{N}$, are convergent not only a.e. but also in the Pettis norm.

Theorem 5.8. *If $\Gamma : \Omega \to cb(X)$ satisfies the strong law of large numbers, then*

$$\begin{aligned} &\lim_n d_P \left(A_\Gamma, \frac{1}{n} \bigoplus_{j=1}^n \Gamma(\pi_j) \right) \\ = &\sup_{x^* \in B(X^*)} \int_\Omega \left| s(x^*, A_\Gamma) - \frac{1}{n}\sum_{i=1}^n s(x^*, \Gamma(\pi_i)) \right| d\mu_\infty = 0 \,. \end{aligned}$$

Proof. Notice first that each multifunction $\Gamma(\pi_j) : \Omega^\infty \to cb(X)$ is scalarly measurable with respect to the σ-algebra $\pi_j^{-1}(\Sigma)$ and is Pettis integrable on $\pi_j^{-1}(\Sigma)$.

The measurability is obvious and the integrability is a simple consequence of the equality $\int_{\pi_j^{-1}(B)} s(x^*, \Gamma(\pi_j))\, d\mu_\infty = \int_B s(x^*, \Gamma)\, d\mu = s(x^*, M_\Gamma(B))$, valid for arbitrary $B \in \Sigma$ and $x^* \in X^*$.

It follows from Lemma 5.1 that each $\Gamma(\pi_j)$ is Pettis integrable on the product σ-algebra. Moreover, Theorem 5.6 yields the uniform integrability of $\mathcal{Z}_\Gamma$ and so each $\mathcal{Z}_{\Gamma(\pi_j)}$ is uniformly integrable with respect to μ_∞.

It follows from Lemma 4.5 that the sequence $\{\frac{1}{n}\bigoplus_{j=1}^n \Gamma(\pi_j) : n \in \mathbb{N}\}$ is scalarly equi-convergent in measure μ_∞ to A_Γ. Taking into account the uniform integrability of $\mathcal{Z}_{\Gamma(\pi_j)}$ we are able to prove that the set $\bigcup_{n=1}^\infty \mathcal{Z}_{\frac{1}{n}\oplus_{j=1}^n \Gamma(\pi_j)}$ is also uniformly integrable with respect to μ_∞. To show it, given $\varepsilon > 0$ let $\delta > 0$ be fixed in such a way that $\mu_\infty(A) < \delta$ yields

$$\sup\left\{\int_A |s(x^*, \Gamma(\pi_1))|\, d\mu_\infty : x^* \in B(X^*)\right\} < \varepsilon\,.$$

Moreover, let $\eta > 0$ be arbitrary. Then

$$\begin{aligned}
&\lim_n \sup_{x^* \in B(X^*)} \mu_\infty\left\{\left|s\left(x^*, \frac{1}{n}\bigoplus_{j=1}^n \Gamma(\pi_j)\right)\right| > \eta + |A_\Gamma|\right\}\\
\le\ &\lim_n \sup_{x^* \in B(X^*)} \mu_\infty\left\{\left|s\left(x^*, \frac{1}{n}\bigoplus_{j=1}^n \Gamma(\pi_j)\right)\right| > \eta + |s(x^*, A_\Gamma)|\right\}\\
=\ &\lim_n \sup_{x^* \in B(X^*)} \mu_\infty\left\{\left|s\left(x^*, \frac{1}{n}\bigoplus_{j=1}^n \Gamma(\pi_j)\right)\right| - |s(x^*, A_\Gamma)| > \eta\right\}\\
\le\ &\lim_n \sup_{x^* \in B(X^*)} \mu_\infty\left\{\left|s\left(x^*, \frac{1}{n}\bigoplus_{j=1}^n \Gamma(\pi_j)\right) - s(x^*, A_\Gamma)\right| > \eta\right\} = 0.
\end{aligned}$$

If $C_1 > |A_\Gamma|$, then we have for $\eta = C_1 - |A_\Gamma|$

$$\lim_n \sup_{x^* \in B(X^*)} \mu_\infty\left\{\left|s\left(x^*, \frac{1}{n}\bigoplus_{j=1}^n \Gamma(\pi_j)\right)\right| > C_1\right\} = 0.$$

Let $n_0 \in \mathbb{N}$ be such that

$$\forall\, n > n_0 \ \sup_{x^* \in B(X^*)} \mu_\infty\left\{\left|s\left(x^*, \frac{1}{n}\bigoplus_{j=1}^n \Gamma(\pi_j)\right)\right| > C_1\right\} < \delta.$$

Since the family $\left\{\frac{1}{n}s\big(x^*, \bigoplus_{j=1}^n \Gamma(\pi_j)\big) : x^* \in B(X^*)\ \&\ n \le n_0\right\}$ is uniformly integrable, there exists $C_2 > 0$ such that

$$\forall\, n \le n_0 \ \sup_{x^* \in B(X^*)} \mu_\infty\left\{\left|s\left(x^*, \frac{1}{n}\bigoplus_{j=1}^n \Gamma(\pi_j)\right)\right| > C_2\right\} < \delta.$$

Thus,

$$\forall\, C > \max\{C_1, C_2\} \quad \sup_n \sup_{x^* \in B(X^*)} \mu_\infty \left\{ \left| s\left(x^*, \frac{1}{n} \bigoplus_{j=1}^n \Gamma(\pi_j) \right) \right| > C \right\} < \delta. \quad (15)$$

To prove the required uniform integrability we have to show that

$$\lim_{C\to\infty} \sup_{x^*\in B(X^*)} \sup_n \int_{\left\{\left|s\left(x^*, \frac{1}{n}\oplus_{j=1}^n \Gamma(\pi_j)\right)\right| > C\right\}} \left| s\left(x^*, \frac{1}{n} \bigoplus_{j=1}^n \Gamma(\pi_j) \right) \right| d\mu_\infty = 0.$$

But notice that for each $n \in \mathbb{N}$ and $x^* \in B(X^*)$ the set $\left\{\left|s\left(x^*, \frac{1}{n}\bigoplus_{j=1}^n \Gamma(\pi_j)\right)\right| > C\right\}$ and the expression $\sum_{j=1}^n |s(x^*, \Gamma(\pi_j)|$ are permutation invariant with respect to $\{\pi_1, \ldots, \pi_n\}$. Consequently, if $C > \max\{C_1, C_2\}$, then

$$\begin{aligned} &\int_{\left\{\left|s\left(x^*, \frac{1}{n}\oplus_{j=1}^n \Gamma(\pi_j)\right)\right| > C\right\}} \left| s\left(x^*, \frac{1}{n} \bigoplus_{j=1}^n \Gamma(\pi_j) \right) \right| d\mu_\infty \\ &\leq \frac{1}{n} \int_{\left\{\left|s\left(x^*, \frac{1}{n}\oplus_{j=1}^n \Gamma(\pi_j)\right)\right| > C\right\}} \sum_{j=1}^n |s(x^*, \Gamma(\pi_j))|\, d\mu_\infty \\ &= \int_{\left\{\left|s\left(x^*, \frac{1}{n}\oplus_{j=1}^n \Gamma(\pi_j)\right)\right| > C\right\}} |s(x^*, \Gamma(\pi_1))|\, d\mu_\infty\,. \end{aligned}$$

Applying (15) and the uniform integrability of $\mathcal{Z}_{\Gamma(\pi_1)}$, we obtain the inequality

$$\int_{\left\{\left|s\left(x^*, \frac{1}{n}\oplus_{j=1}^n \Gamma(\pi_j)\right)\right| > C\right\}} \left| s\left(x^*, \frac{1}{n} \bigoplus_{j=1}^n \Gamma(\pi_j) \right) \right| d\mu_\infty < \varepsilon.$$

which proves the uniform integrability of $\bigcup_{n=1}^\infty \mathbb{Z}_{\frac{1}{n}\oplus_{j=1}^n \Gamma(\pi_j)}$.

Now the convergence $\lim_n d_P\left(A_\Gamma, \frac{1}{n}\bigoplus_{j=1}^n \Gamma(\pi_j)\right) = 0$ is a direct consequence of Theorem 4.6. □

As a particular case of Theorem 5.8, we have

Theorem 5.9. *Let $f : \Omega \to X$ be a function. If f satisfies the strong law of large numbers, then*

$$\lim_n \left\| a_f - \frac{1}{n} \sum_{j=1}^n f(\pi_j) \right\|_P = 0\,.$$

6. Gelfand and Dunford integrable multifunctions

It is the purpose of this section to prove that formula (3) produces a metric also in case of scalarly integrable multifunctions. To this aim I need a few basic facts concerning the Gelfand and the Dunford integrals for multifunctions.

A multifunction $\Gamma : \Omega \to c(X^*)$ is said to be *weak* scalarly measurable*, if for every $x \in X$, the support function $s(x, \Gamma(\cdot))$ is measurable. $\Gamma : \Omega \to c(X^*)$ is said to be *weak* scalarly integrable*, if for every $x \in X$, the function $s(x, \Gamma(\cdot))$ is *integrable.* A multifunction $\Gamma : \Omega \to c(X^*)$ is *weak* scalarly bounded*, if there is a constant $M > 0$ such that for every $x \in X$ $|s(x, \Gamma)| \leq M\|x\|$ a.e.

A function $f : \Omega \to X^*$ is said to be a *weak* quasi-selection* of a multifunction $\Gamma : \Omega \to c(X^*)$, if f is weak* scalarly measurable and we have $xf(\omega) \in x\Gamma(\omega)$ for μ-almost every $\omega \in \Omega$ (the exceptional sets depend on x). The collection of all weak* quasi-selections of Γ will be denoted by $\mathcal{W}^*\mathcal{QS}(\Gamma)$.

If moreover $f(\omega) \in \Gamma(\omega)$, for every $\omega \in \Omega$, then f is called a *weak* selection* of Γ.

$cw^*k(X^*)$ denotes the family of all weak*-compact members of $cb(X^*)$. $\overline{H}^*$ denotes the closure of H in the weak* topology.

If $\Gamma : \Omega \to cw^*k(X^*)$ is weak* scalarly integrable, then Γ can be represented in the form

$$\Gamma(\omega) = \sum_n \Gamma_n(\omega)\chi_{E_n}(\omega) \tag{16}$$

where the sets $E_n \in \Sigma$ are pairwise disjoint, $\mu\left(\bigcup_n E_n\right) = \mu(\Omega)$ and each $\Gamma_n \colon \Omega \to cw^*k(X^*)$ is weak* scalarly bounded.

$M : \Sigma \to cw^*k(X^*)$ is additive, if $M(A \cup B) = M(A) + M(B)$. An additive map $M : \Sigma \to cw^*k(X^*)$ is called a *weak* multimeasure*, if $s(x, M(\cdot))$ is a finite measure, for every $x \in X$. If M is a point map, then we talk about weak* measure. If $m : \Sigma \to X^*$ is a weak* measure such that $m(A) \in M(A)$, for every $A \in \Sigma$, then m is called a *weak* selection* of M. $\mathcal{W}^*\mathcal{S}(M)$ will denote the set of all weak* selections of M.

If $M : \Sigma \to cw^*k(X^*)$ is a weak* multimeasure, then the weak* semivariation of M on a set $E \in \Sigma$ is defined by $\|M\|^*(E) := \sup\{|s(x, M)|(E) : x \in B(X)\}$, where $|s(x, M)|(E)$ is the ordinary variation of the measure $s(x, M)$ on the set E.

If $M : \Sigma \to cb(X)$ is a multimeasure, then the semivariation of M on a set $E \in \Sigma$ is defined by $\|M\|(E) := \sup\{|s(x^*, M)|(E) : x^* \in B(X^*)\}$.

Proposition 6.1. *If $M : \Sigma \to cw^*k(X^*)$ is a weak* multimeasure, then $\|M\|^*(\Omega) < \infty$. If $M : \Sigma \to cb(X)$ is a multimeasure, then $\|M\|(\Omega) < \infty$. In particular the set $\bigcup M(\Sigma)$ is bounded.*

Proof. Assume that $M : \Sigma \to cw^*k(X^*)$ is a weak* multimeasure. If $x \in B(X)$, then it follows from the classical measure theory that $|s(x, M)|(\Omega) \leq 2\sup\{|s(x, M(E))| : E \in \Sigma\} < \infty$. It follows that the set $\bigcup M(\Sigma) = \bigcup\{M(E) : E \in \Sigma\}$ is weak* bounded and since X is a Banach space, it is norm bounded. Hence, $\sup\{|s(x, M)|(\Omega) : x \in B(X)\} \leq 2|\bigcup M(\Sigma)| < \infty$. The second part can be proved in a similar way. $\square$

If M is a weak* multimeasure or a multimeasure, then the variation of M is defined by

$$|M|(E) := \sup_{\mathcal{P}} \left\{ \sum_{A \in \mathcal{P}} |M(A)| : \mathcal{P} \text{ is a finite measurable partition of } E \right\}.$$

It turns out that sometimes a weak* multimeasure is an h-multimeasure.

Proposition 6.2. *If $M : \Sigma \to cw^*k(X^*)$ is a weak* multimeasure (or $M : \Sigma \to cb(X)$ is a multimeasure) of finite variation, then it is an h-multimeasure.*

Proof. Let $\{A_i : i \in \mathbb{N}\}$ be a sequence of pairwise disjoint elements of Σ_μ^+. We have then

$$\begin{aligned}
& d_H \left(\overline{\sum_{i \le k} M(A_i)}^*, M\left(\bigcup_{j=1}^{\infty} A_j \right) \right) \\
&= \sup_{\|x\| \le 1} \left| s\left(x, \overline{\sum_{i \le k} M(A_i)}^* \right) - s\left(x, \overline{M\left(\bigcup_{i \le k} A_i \right) + M\left(\bigcup_{j > k} A_j \right)}^* \right) \right| \\
&= \sup_{\|x\| \le 1} \left| s\left(x, \sum_{i \le k} M(A_i) \right) - s\left(x, M\left(\bigcup_{i \le k} A_i \right) + M\left(\bigcup_{j > k} A_j \right) \right) \right| \\
&= \sup_{\|x\| \le 1} s\left(x, M\left(\bigcup_{j > k} A_j \right) \right) \le \sup_{\|x\| \le 1} \sum_{j > k} |s(x, M(A_j))| \\
&\le \sum_{j > k} |M|(A_j).
\end{aligned}$$

Since the variation of M is finite the last term tends to zero as $k \to \infty$ and that proves the required σ-additivity of M. $\square$

The next result proved by Costé [8] shows that weak* multimeasures are rich with weak* measures, being selections of the initial weak* multimeasure.

Proposition 6.3. *If $M : \Sigma \to cw^*k(X^*)$ is an arbitrary weak* multimeasure, then $\mathcal{W}^*\mathcal{S}(M) \neq \emptyset$ and for each $E \in \Sigma$*

$$M(E) = \overline{\{m(E) : m \in \mathcal{W}^*\mathcal{S}(M)\}}^*.$$

Proof. I shall recall the proof of the second part only. Suppose there is $E \in \Sigma$ such that

$$M(E) \setminus \overline{\{m(E) : m \in \mathcal{W}^*\mathcal{S}(M)\}}^* \neq \emptyset.$$

Then, due to the Hahn-Banach theorem, there are $x_0 \in X$ and $\alpha \in \mathbb{R}$ such that

$$s(x_0, M(E)) > \alpha > \langle x_0, m(E) \rangle \quad \text{for every } m \in \mathcal{W}^*\mathcal{S}(M). \tag{17}$$

Define $M'_{x_0} : \Sigma \to cw^*k(X^*)$ by

$$M'_{x_0}(A) := \{x^* \in M(A) : x_0(x^*) = s(x_0, M(A))\}.$$

Notice that (17) yields

$$M'_{x_0}(E) \cap \overline{\{m(E) : m \in \mathcal{W}^*\mathcal{S}(M)\}}^* = \emptyset. \tag{18}$$

According to [10, Theorem 2.3] M'_{x_0} is a weak* multimeasure and, by the first part of the current proposition, it has a selectiom $\widetilde{m} : \Sigma \to X^*$ that is a weak* measure. Clearly we have $\widetilde{m}(E) \in M'_{x_0}(E)$. But $\widetilde{m}$ is also a selection of M and so $\widetilde{m}(E) \in \overline{\{m(E) : m \in \mathcal{W}^*\mathcal{S}(M)\}}^*$. This however contradicts (18). □

Definition 6.4. Denote by $\mathcal{C}$ an arbitrary family of nonempty convex subsets of X^*. A weak* scalarly integrable multifunction $\Gamma : \Omega \to c(X^*)$ is *Gelfand integrable* in $\mathcal{C}$, if for each set $A \in \Sigma$ there exists a set $M^G_\Gamma(A) \in \mathcal{C}$ such that

$$s(x, M^G_\Gamma(A)) = \int_A s(x, \Gamma)\, d\mu, \quad \text{for every } x \in X.$$

$M^G_\Gamma(A)$ is called the Gelfand integral of Γ on A and we write $(G)\int_A \Gamma\, d\mu := M^G_\Gamma(A)$. □

One can immediately see that if $\mathcal{C} \in \{cw^*k(X^*),\ cwk(X^*),\ ck(X^*)\}$, then M^G_Γ is a weak* multimeasure. By Proposition 6.1 $\|M^G_\Gamma\|^*(\Omega) < \infty$.

One can easily check that $|M^G_\Gamma|$ is always a measure. Indeed, if $\{A_i : i \in \mathbb{N}\}$ are pairwise disjoint elements of Σ, then

$$|M^G_\Gamma|\left(\bigcup_{i=1}^{\infty} A_i\right) \geq |M^G_\Gamma|\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} |M^G_\Gamma|(A_i),$$

for every $n \in \mathbb{N}$. Consequently,

$$|M^G_\Gamma|\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} |M^G_\Gamma|(A_i)\,.$$

And conversely, if $B_1, \ldots, B_k$ are pairwise disjoint measurable subsets of $\bigcup_{n=1}^{\infty} A_n$ and $x_i \in B(X)$, $i = 1, \ldots, k$, are arbitrary, then

$$\begin{aligned}
&\sum_{j=1}^{k} \left|s(x_j, M^G_\Gamma(B_j))\right| \\
&= \sum_{j=1}^{k} \left| s\left(x_j, M^G_\Gamma\left(B_j \cap \bigcup_{i=1}^{\infty} A_i\right)\right)\right| = \sum_{j=1}^{k} \left| s\left(x_j, \sum_{i=1}^{\infty} M^G_\Gamma(B_j \cap A_i)\right)\right| \\
&\leq \sum_{i=1}^{\infty}\sum_{j=1}^{k} \left|s(x_j, M^G_\Gamma(B_j \cap A_i))\right| \leq \sum_{i=1}^{\infty}\sum_{j=1}^{k} \left|M^G_\Gamma(B_j \cap A_i)\right| \\
&\leq \sum_{i=1}^{\infty} |M^G_\Gamma|(A_i).
\end{aligned}$$

Given $\varepsilon > 0$ one can always find $x_j \in B(X)$ such that $|M_\Gamma^G(B_j)| \le s(x_j, M_\Gamma^G(B_j)) + \varepsilon/k$. Hence we have

$$\sum_{j=1}^{k} \left|M_\Gamma^G(B_j)\right| \le \sum_{j=1}^{k} \left|s(x_j, M_\Gamma^G(B_j))\right| + \varepsilon \le \sum_{i=1}^{\infty} |M_\Gamma^G|(A_i) + \varepsilon.$$

It follows that

$$|M_\Gamma^G| \left(\bigcup_{i=1}^{\infty} A_i \right) \le \sum_{i=1}^{\infty} |M_\Gamma^G|(A_i).$$

The following fact is essential for our further investigations:

Proposition 6.5. *If $\Gamma : \Omega \to cw^*k(X^*)$ is Gelfand integrable in $cw^*k(X^*)$, then $|M_\Gamma^G|$ is a σ-finite measure and*

$$|M_\Gamma^G|(E) = \int_E \psi_\Gamma \, d\mu, \quad \textit{for every } E \in \Sigma,$$

where ψ_Γ is a non-negative measurable function satisfying the following properties:

(α) $|s(x, \Gamma(\omega))| \le \psi_\Gamma(\omega)\|x\|$ *for every x and almost every ω;*

(β) $\psi_\Gamma(\omega) \le |\Gamma(\omega)|$ *for every ω;*

(γ) *If φ is another measurable function satisfying the conditions (α) and (β) (with ψ_Γ replaced by φ), then $\psi_\Gamma \le \varphi$ a.e.*

Proof. The existence of ψ_Γ is well known (cf. [27] or [24]).

Then, if $E_n := \{\omega : n - 1 \le \psi_\Gamma(\omega) < n\}$ and $\Gamma_n(\omega) = \Gamma(\omega)$ whenever $\omega \in E_n$, then we get the representation (16). If Γ is Gelfand integrable and $E \in \Sigma$, then

$$|s(x, M_\Gamma^G(E \cap E_n))| \le \int_{E \cap E_n} |s(x, \Gamma)| \, d\mu \le \|x\| \int_{E \cap E_n} \psi_\Gamma \, d\mu \le n\|x\|\mu(E \cap E_n).$$

Moreover,

$$\begin{aligned} |M_\Gamma^G(E \cap E_n)| &= \sup\{\|x^*\| : x^* \in M_\Gamma^G(E \cap E_n)\} = \sup_{x^* \in M_\Gamma^G(E \cap E_n)} \sup_{\|x\| \le 1} |x(x^*)| \\ &= \sup_{\|x\| \le 1} |s(x, M_\Gamma^G(E \cap E_n))| \le \int_{E \cap E_n} \psi_\Gamma \, d\mu \end{aligned}$$

and so $|M_\Gamma^G|(E \cap E_n) \le \int_{E \cap E_n} \psi_\Gamma \, d\mu \le n\mu(E \cap E_n) < \infty$. Thus, M_Γ^G is of σ-finite variation and

$$|M_\Gamma^G|(E) \le \int_E \psi_\Gamma \, d\mu \, .$$

By the classical Radon–Nikodým theorem there exists a nonnegative measurable function h on Ω such that

$$|M_\Gamma^G|(E) = \int_E h \, d\mu \ \text{ for each } E \in \Sigma.$$

The inequality $|M_\Gamma^G|(E) \le \int_E \psi_\Gamma d\mu$ yields now the relation $h \le \psi_\Gamma$ μ–a.e.. If $\|x\| \le 1$, then

$$|s(x, M_\Gamma^G)|(E) = \int_E |s(x,\Gamma)| d\mu \le |M_\Gamma^G|(E) = \int_E h\, d\mu\,.$$

Hence $|s(x,\Gamma)| \le h$ μ–a.e.. It follows from the properties of ψ_Γ that $\psi_\Gamma \le h$ μ–a.e., and this completes the proof of the equality

$$|M_\Gamma^G|(E) = \int_E \psi_\Gamma\, d\mu, \quad \text{for every } E \in \Sigma\,. \qquad \square$$

Proposition 6.6. *Let $\Gamma : \Omega \to cw^*k(X^*)$ be Gelfand integrable in $cw^*k(X^*)$. Then $\mathcal{W}^*\mathcal{QS}(\Gamma) \neq \emptyset$ and a weak* scalarly integrable $\gamma : \Omega \to X^*$ is a Gelfand integrable weak* quasi-selection of Γ if and only if for every $x \in X$ $\langle x, \gamma\rangle \le s(x,\Gamma)$ a.e., if and only if $\Sigma \ni A \longrightarrow \int_A \gamma\, d\mu$ is a weak* selection of M_Γ^G.*

Proof. It is quite obvious that a weak* scalarly integrable $\gamma : \Omega \to X^*$ is a Gelfand integrable weak* quasi-selection of Γ if and only if for every $x \in X$ $\langle x, \gamma\rangle \le s(x,\Gamma)$ a.e. so I will prove only that $\mathcal{W}^*\mathcal{QS}(\Gamma) \neq \emptyset$.

Let $M_\Gamma^G : \Sigma \to cw^*k(X^*)$ be the Gelfand integral of Γ. Then, according to Propositions 6.3 and 6.5 there exists a weak* selection $m : \Sigma \to X^*$ of M_Γ^G that is of σ-finite variation. Moreover $s(x, M_\Gamma^G(\cdot)) \ll \mu$ for every x and so (see [24], Theorem 11.1) for every $m \in \mathcal{W}^*\mathcal{S}(M)$ there is a Gelfand integrable function $\gamma : \Omega \to X^*$ satisfying for every x and every $A \in \Sigma$ the equality

$$\langle x, m(A)\rangle = \int_A \langle x, \gamma\rangle\, d\mu.$$

Thus, we have for every x and every $A \in \Sigma$ the relation

$$\int_A \langle x, \gamma\rangle\, d\mu \le s(x, M_\Gamma^G(A)) = \int_A s(x,\Gamma)$$

and so $\langle x, \gamma\rangle \le s(x,\Gamma)$ a.e., for every x separately. This means that $\gamma \in \mathcal{W}^*\mathcal{QS}(\Gamma)$.

The rest of the proof is also easy. $\square$

The following theorem has been independently proven in [5]. I present a simpler proof.

Theorem 6.7. *If $\Gamma : \Omega \to cw^*k(X^*)$ is weak* scalarly integrable, then Γ is Gelfand integrable in $cw^*k(X^*)$ and for every $E \in \Sigma$*

$$M_\Gamma^G(E) = \overline{\left\{(G)\int_E \gamma\, d\mu : \gamma \in \mathcal{W}^*\mathcal{QS}(\Gamma)\right\}}^{*}\,. \tag{19}$$

Proof. Assume at the beginning that Γ is scalarly bounded. Let us fix $A \in \Sigma$ and define a sublinear functional on X setting

$$\varphi(x) := \int_A s(x, \Gamma)\, d\mu.$$

Denote by X' the space X^* endowed with the topology $\sigma(X^*, X)$. Then $X'^* = X$. In order to prove the existence of a weak*-closed convex set $M_A \subset X^*$ with

$$s(x, M_\Gamma^G(A)) = \int_A s(x, \Gamma)\, d\mu \quad \text{for every } x \in X,$$

we have to show first, according to the Hörmander theorem [18], that the functional is $\sigma(X'^*, X')$-lower semicontinuous (cf. [7], p. 48). That is we need the $\sigma(X, X^*)$-lower semicontinuity of φ. Equivalently, we have to prove that for each real α

$$Q(\alpha) := \{x \in X : \varphi(x) \le \alpha\}$$

is weakly closed. As it is convex it is enough to show that it is norm closed. So let $\{x_n : n \in \mathbb{N}\} \subset Q(\alpha)$ be such that $\|x_n - x\| \to 0$. Since for each ω the set $\Gamma(\omega)$ is norm bounded, the support function $s(\cdot, \Gamma(\omega))$ is norm-continuous for every ω. Consequently,

$$s(x_n, \Gamma(\omega)) \to s(x, \Gamma(\omega)) \quad \text{for every } \omega\,.$$

Consequently, due to the Lebesgue dominated convergence theorem,

$$\int_A s(x, \Gamma)\, d\mu = \int_A \lim_n s(x_n, \Gamma)\, d\mu = \lim_n \int_A s(x_n, \Gamma)\, d\mu \le \alpha.$$

This proves the Gelfand integrability of Γ in the family of weak*-closed convex subsets of X^*. Since Γ is weak* scalarly integrable, the set $M_\Gamma^G(A)$ is also weak* bounded. Thus, Γ is Gelfand integrable in $cw^*k(X^*)$.

If Γ is arbitrary, then according to (16) $\Gamma(\omega) = \sum_n \Gamma_n(\omega)\chi_{E_n}(\omega)$, where all Γ_n's are weak* scalarly bounded and E_n's are pairwise disjoint. As we have just proven each Γ_n is Gelfand integrable in $cw^*k(X^*)$. By Proposition 6.6, each $\Gamma_n\chi_{E_n}$ has a weak* quasi-selection δ_n. Then $\delta := \sum_n \delta_n$ is a Gelfand integrable weak* quasi-selection of Γ.

Let $\Delta(\omega) := \Gamma(\omega) - \delta(\omega)$. We have now $s(x, \Delta) \ge 0$ a.e., for each $x \in X$. In order to prove the Gelfand integrability of Δ in $cw^*k(X^*)$, we follow exactly as in the first part of the proof, proving the weak* lower semicontinuity of the functional

$$\psi(x) := \int_A s(x, \Delta)\, d\mu.$$

But now we apply the Fatou lemma instead of the Lebesgue dominated convergence theorem.

The Gelfand integrability of Γ in $cw^*k(X^*)$ is a simple consequence of the equalities (for each $x \in X$ and each $E \in \Sigma$)

$$\begin{aligned}\int_E s(x,\Gamma)\,d\mu &= \int_E s(x,\Delta)\,d\mu + \int_E x\delta\,d\mu \\ &= s\left(x,(G)\int_E \Delta\,d\mu\right) + x\left((G)\int_E \delta\,d\mu\right) \\ &= s\left(x,(G)\int_E \Delta\,d\mu + (G)\int_E \delta\,d\mu\right).\end{aligned}$$

We are going to prove now the equality (19). According to Propositions 6.3 and 6.5 for each $E \in \Sigma_\mu^+$

$$M_\Gamma^G(E) = \overline{\{m(E) : m \in \mathcal{W}^*\mathcal{S}(M_\Gamma^G)\}}^* \tag{20}$$

and each weak* countably additive selection $m : \Sigma \to X^*$ is of σ-finite variation. Moreover $s(x, M_\Gamma^G(\cdot)) \ll \mu$ for every x and so (see [24], Theorem 11.1) for every $m \in \mathcal{W}^*\mathcal{S}(M_\Gamma^G)$ there is a Gelfand integrable function $\gamma : \Omega \to X^*$ satisfying for every x and every $A \in \Sigma$ the relations

$$\int_A \langle x,\gamma\rangle\,d\mu = \langle x, m(A)\rangle \le s(x, M_\Gamma^G(A)) = \int_A s(x,\Gamma)\,d\mu\,, \tag{21}$$

what means that γ is a weak* quasi-selection of Γ. (19) is a direct consequence of (20) and (21). □

Definition 6.8. Denote by $\mathcal{C}$ an arbitrary family of nonempty convex subsets of X^{**}. A scalarly integrable multifunction $\Gamma : \Omega \to c(X)$ is *Dunford integrable* in $\mathcal{C}$, if for every set $A \in \Sigma$ there exists a set $M_\Gamma^D(A) \in \mathcal{C}$ such that

$$s(x^*, M_\Gamma^D(A)) = \int_A s(x^*,\Gamma)\,d\mu, \quad \text{for every } x^* \in X^*.$$

$M_\Gamma^D(A)$ is called the Dunford integral of Γ on A. □

As a consequence of Theorem 6.7 we obtain

Theorem 6.9. *Each scalarly integrable multifunction $\Gamma : \Omega \to cb(X)$ is Dunford integrable in $cw^*k(X^{**})$.*

Proof. We want to apply Theorem 6.7 but Γ is not necessarily $cw^*k(X^{**})$-valued. So we define a $cw^*k(X^{**})$-valued multifunction $\widetilde{\Gamma}$ by setting $\widetilde{\Gamma}(\omega) := \overline{\Gamma(\omega)}^*$, $\omega \in \Omega$, where the closure is taken with respect to the weak* topology of X^{**}. We have then $s(x^*, \widetilde{\Gamma}(\omega)) = s(x^*, \Gamma(\omega))$, for each $x^* \in X^*$ and each $\omega \in \Omega$. According to Theorem 6.7, $\widetilde{\Gamma}$ is Gelfand integrable in $cw^*k(X^{**})$ and so for each $A \in \Sigma$ there exists $M_{\widetilde{\Gamma}}^D(A) \in cw^*k(X^{**})$ satisfying the equality

$$s(x^*, M_{\widetilde{\Gamma}}^D(A)) = \int_A s(x^*,\widetilde{\Gamma})\,d\mu = \int_A s(x^*,\Gamma)\,d\mu \ \text{ for every } x^* \in X^*.$$

This proves the required Dunford integrability of Γ. $\square$

Proof of the correctness of the definition of d_P for scalarly integrable multifunctions. Setting in Proposition 6.1 $M = M^D_{\tilde{\Gamma}}$ one obtains $\sup\{\int_\Omega |s(x,\Gamma)|\, d\mu : \|x\| \le 1\} < \infty$ and that proves that the definition of d_P by (3) is proper. $\square$

I am going to take the opportunity and present yet a result that is not connected with the main topic of this paper but generalizes [5, Theorem 6.8 and Remark 6.9] that described Gelfand integrability of compact valued multifunctions.

Theorem 6.10. *Let $\Gamma : \Omega \to ck(X^*)$ be a weak* scalarly integrable multifunction and let the set $\{s(x,\Gamma)\colon x \in B(X)\}$ be uniformly integrable. Then Γ is Gelfand integrable in $cwk(X^*)$ and $\bigcup M^G_\Gamma(\Sigma)$ is a weakly relatively compact set. If the zero function is a weak* quasi-selection of Γ, then $\bigcup M^G_\Gamma(\Sigma)$ is a weakly compact set.*

Proof. Consider a partition of Ω of the form $\Omega = \bigcup_n \Omega_n$ such that $(n-1)\|x\| \le |s(x,\Gamma)| < n\|x\|$ a.e., for every $x \in X$ and $n \in \mathbb{N}$ separately. According to [5] each multifunction Γ_n defined by $\Gamma_n(E) := \Gamma(E \cap \Omega_n)$ is Gelfand integrable in $cwk(X^*)$.

Let $\varepsilon >$ be fixed and $\delta > 0$ be adapted to ε in such a way that $\mu(E) < \delta$ yields $\int_E s(x,\Gamma)\, d\mu < \varepsilon$, for every $x \in B(X)$ and $E \in \Sigma$. Let $m \in \mathbb{N}$ be such that $\mu\big(\bigcup_{n \ge m} \Omega_n\big) < \delta$ and let $E \in \Sigma$ be fixed.

According to [5] Γ is Gelfand integrable in $cwk(X^*)$ on $\bigcup_{i<m} \Omega_i$, so let $W \in cwk(X^*)$ be such that $M^G_\Gamma(E \cap \bigcup_{i<m} \Omega_i) \subset W$. We have then for each $x \in X$ and $m > n_0$

$$
\begin{aligned}
s(x, M^G_\Gamma(E)) &= s\Big(x, M^G_\Gamma\big(E \cap \bigcup_{i<m} \Omega_i\big)\Big) + s\Big(x, M^G_\Gamma\big(E \cap \bigcup_{i\ge m} \Omega_i\big)\Big) \\
&\le s\Big(x, M^G_\Gamma\big(E \cap \bigcup_{i<m} \Omega_i\big)\Big) + \int_{\bigcup_{i\ge m} \Omega_i \cap E} |s(x,\Gamma)|\, d\mu \\
&\le s\Big(x, M^G_\Gamma\big(E \cap \bigcup_{i<m} \Omega_i\big)\Big) + \varepsilon \le s(x,W) + \varepsilon\|x\| \\
&= s(x, W + \varepsilon B(X^*)).
\end{aligned}
$$

It follows that

$$M^G_\Gamma(E) \subseteq W + \varepsilon B(X^*).$$

Due to Grothendieck's characterization of weak compactness (cf. [12, Lemma XIII.2]) the set $M^G_\Gamma(E)$ is weakly relatively compact.

Now, let $f : \Omega \to X^*$ be a weak* quasi-selection of Γ and $G := \Gamma - f$.

According to the first part of the proof the multifunction G is Gelfand integrable in $cwk(X^*)$ and since $\bigcup M^G_G(\Sigma) = M^G_G(\Omega)$, the total range of M^G_G is weakly compact. As a weak* quasi-selection of Γ the function f satisfies for each $x \in X$ the inequality $xf \le s(x,\Gamma)$ and so the set $\{xf : \|x\| \le 1\}$ is weakly relatively compact.

It is known, and easily seen, that $M_f^G(\Sigma)$ is a weakly relatively compact set. Consequently the set $M_\Gamma^G(\Sigma) \subset M_G^G(\Sigma) + M_f^G(\Sigma)$ is weakly relatively compact. This completes the proof. $\square$

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