Gelfand Integral of Multifunctions

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It has been proven by Cascales, Kadets and Rodríguez in J. Convex Analysis 18 (2011) 873–895 that each weak∗ scalarly integrable multifunction (with respect to a probability measure µ), whose values are compact convex subsets of a conjugate Banach space X∗ and the family of support functions determined by X is order bounded in L1(µ), is Gelfand integrable in the family of weakly compact convex subsets of X∗. A question has been posed whether a similar result holds true for multifunctions with weakly compact convex values. We prove that the answer is affirmative if X does not contain any isomorphic copy of l1. If moreover the multifunction is compact valued, then it is Gelfand integrable in the family of compact convex subsets of X∗.

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Introduction.

Throughout this paper Ω is a non-empty set, Σ is a σ-algebra of its subsets and (Ω, Σ, µ) is a complete probability space. X is a Banach space with its dual X∗. If A ⊂ X∗, then $\overline{A}$ is the weak∗ closure of A and $\overline{A}^w$ is its weak closure. $P(X)$ denotes the collection of all subsets of X and the closed unit ball of X is denoted by $B(X)$. c(X) denotes the collection of all nonempty closed convex subsets of X, $cb(X)$ is the collection of all bounded members of c(X), $cwk(X)$ denotes the family of all weakly compact elements of $cb(X)$ and $ck(X)$ is the collection of all compact members of $cb(X)$. Moreover, $cw^k(X^*)$ denotes the family of all nonempty convex weak∗ compact subsets of $X^*$. For every $C \in c(X)$ the support function of C is denoted by $s(\cdot, C)$ and defined on $X^*$ by $s(x^*, C) = \sup\{\langle x^*, x \rangle : x \in C\}$, for each $x^* \in X^*$. If $C = \emptyset$, $s(\cdot, C)$ is identically $-\infty$. Otherwise $s(\cdot, C)$ does not take value $-\infty$.

A map $\Gamma : \Omega \to c(X)$ is called a multifunction. A multifunction $\Gamma$ is said to be

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scalarly measurable if for every $x^* \in X^*$, the map $s(x^*, \Gamma(\cdot))$ is measurable. $\Gamma$ is said to be scalarly integrable if, for every $x^* \in X^*$, the function $s(x^*, \Gamma(\cdot))$ is integrable. We set $Z_\Gamma := \{ s(x^*, \Gamma) : \|x^*\| \leq 1 \}$. 

A multifunction $\Gamma : \Omega \to c(X^*)$ is said to be weak* scalarly measurable if for every $x \in X$, the map $s(x, \Gamma(\cdot))$ is measurable. We set $\widehat{Z}_\Gamma := \{ s(x, \Gamma) : \|x\| \leq 1 \}$. $\Gamma : \Omega \to c(X^*)$ is said to be weak* scalarly integrable if, for every $x \in X$, the function $s(x, \Gamma(\cdot))$ is integrable.

A function $f : \Omega \to X$ is called a selection of $\Gamma$ if, for almost every $\omega \in \Omega$, one has $f(\omega) \in \Gamma(\omega)$.

A function $\gamma : \Omega \to X^*$ is said to be a weak* quasi-selection of a multifunction $\Gamma : \Omega \to c(X^*)$ if $\gamma$ is weak* scalarly measurable and we have $(x, \gamma(\omega)) \in (x, \Gamma(\omega))$ for $\mu$-almost every $\omega \in \Omega$ (the exceptional sets depend on $x$). In particular, $s(x^*, \Gamma - \gamma) \geq 0$ a.e., for each $x^* \in X^*$. The collection of all weak* quasi-selections of $\Gamma$ will be denoted by $W^*QS_\Gamma$.

If moreover $f(\omega) \in \Gamma(\omega)$, for almost every $\omega \in \Omega$, then $\gamma$ called a weak* selection of $\Gamma$. We denote the set of all weak* selections of $\Gamma$ by $W^*S_\Gamma$.

One can easily check that a weak* scalarly measurable $\gamma : \Omega \to X^*$ is a weak* quasi-selection of $\Gamma$ if and only if for every $x \in X$ $(x, \gamma) \leq s(x, \Gamma)$ a.e.

A family $W \subset L_1(\mu)$ is uniformly integrable if $W$ is bounded in $L_1(\mu)$ and for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(A) < \delta$, then $\sup_{f \in W} \int_A |f| \, d\mu < \varepsilon$.

A map $M : \Sigma \to c(X)$ is additive, if $M(A \cup B) = \overline{M(A) + M(B)}$ for each pair of disjoint elements of $\Sigma$. An additive map $M : \Sigma \to c(X^*)$ is called a weak* multimeasure if $s(x, \Gamma(\cdot))$ is a measure, for every $x \in X$. If $M$ is a point map, then we talk about measure and weak* measure, respectively. If $m : \Sigma \to X^*$ is a weak* measure such that $m(A) \in M(A)$, for every $A \in \Sigma$, then $m$ is called a weak*-selection of $M$. $W^*S(M)$ will denote the set of all weak* countably additive selections of $M$. Costé [4] proved that $W^*S(M) \neq \emptyset$.

Denote by $C$ an arbitrary family of non-empty closed convex subsets of $X^*$. A weak* scalarly integrable multifunction $\Gamma : \Omega \to c(X^*)$ is said to be Gelfand integrable in $C$, if for each set $A \in \Sigma$ there exists a set $M^G(\Gamma)(A) \in C$ such that

$$s(x, M^G(\Gamma)(A)) = \int_A s(x, \Gamma) \, d\mu, \quad \text{for every } x \in X. \quad (1)$$

It is clear that $M^G(\Gamma)$ is a weak* multimeasure. $(G) \int_E \Gamma \, d\mu := M^G(\Gamma)(A)$ is called the Gelfand integral of $\Gamma$ on $A$. It follows from [4] that $W^*S(M^G(\Gamma)) \neq \emptyset$.

Denote by $D$ an arbitrary family of closed convex bounded subsets of $X$. A scalarly integrable multifunction $\Gamma : \Omega \to D$ is said to be Pettis integrable in $D$, if for each $A \in \Sigma$ there exists a set $M(\Gamma)(A) \in D$ such that

$$s(x^*, M(\Gamma)(A)) = \int_A s(x^*, \Gamma) \, d\mu \quad \text{for every } x^* \in X^*. \quad (2)$$
$M_\Gamma (A)$ is called the Pettis integral of $\Gamma$ over $A$.

It has been proven in [2] and in [17, Theorem 6.7] that each weak$^*$ scalarly integrable $\Gamma : \Omega \rightarrow cw^*k(X^*)$ is Gelfand integrable in $cw^*k(X^*)$. In view of [17, Proposition 6.5] the Gelfand integral of $\Gamma$ is a weak$^*$ multimeasure of $\sigma$-finite variation. It follows that each weak$^*$ measure being a selection of the Gelfand integral is of $\sigma$-finite variation. But then, it is a classical result (cf. [14, Theorem 11.1] or [15, Theorem 9.1]) that each such weak$^*$ measure has a weak$^*$ integrable density. Since, as we have already mentioned, $\mathcal{W}^*S(M_\Gamma^G) \neq \emptyset$, we have the following fact (proved also in [2, Theorem 4.5])

If $\Gamma : \Omega \rightarrow cw^*k(X^*)$ is Gelfand integrable in $cw^*k(X^*)$, then $\mathcal{W}^*QS_\Gamma \neq \emptyset$.

**1. The range of Gelfand’s integral.**

It has been proven in [2, Remark 6.9] that each $ck(X^*)$ valued scalarly integrable multifunction with order bounded $\underline{Z}_\Gamma \subset L_1(\mu)$ is Gelfand integrable in $cwk(X^*)$ (The result has been generalized in [17, Theorem 6.10] to $ck(X^*)$ valued multifunction with uniformly integrable $\underline{Z}_\Gamma$). A question has been posed in [2] whether the result can be extended to $cwk(X^*)$ valued multifunctions. If $X = l_1$, then the answer is negative (see [2]). But if $X$ does not contain any isomorphic copy of $l_1$, then the answer turns out to be affirmative (Theorem 1.4) in general. To achieve that result we apply [13, Theorem 3]:

Each $X^*$-valued weak$^*$-scalarly measurable (integrable) function is scalarly measurable (Pettis integrable) provided $l_1 \nsubseteq X$ and $X$ is separable.

It is perhaps worth to mention that [13, Theorem 3] is a consequence of the celebrated result of Odell and Rosenthal [18] saying that if $X$ is separable and $l_1 \nsubseteq X$, then $X$ is weak$^*$ sequentially dense in $X^{**}$.

**Lemma 1.1.** Let $\Gamma : \Omega \rightarrow cwk(X^*)$ be weak$^*$ scalarly measurable multifunction. If $l_1 \nsubseteq X$ and $X$ is separable, then $\Gamma$ is scalarly measurable.

**Proof.** Let us observe that due to Castaing representation (see [3, Theorem III.7] or [20, Proposition 7]) there exists a sequence of weak$^*$ scalarly measurable selections $\{f_n : \Omega \rightarrow X^* : n \in \mathbb{N}\}$ of $\Gamma$ such that $\Gamma(\omega) = \{f_n(\omega) : n \in \mathbb{N}\}^\omega$ for every $\omega \in \Omega$. Since $l_1 \nsubseteq X$ and $X$ is separable all the functions $f_n$ are scalarly measurable. Moreover, $\Gamma(\omega) = \{f_n(\omega) : n \in \mathbb{N}\}^\omega$ for every $\omega \in \Omega$, because $\{f_n(\omega) : n \in \mathbb{N}\}$ is weakly relatively compact. Thus, we have for each $x^{**} \in X^{**}$ $s(x^{**}, \Gamma) = \sup_n \langle x^{**}, f_n \rangle$ everywhere and so $\Gamma$ is scalarly measurable.

**Theorem 1.2.** Let $\Gamma : \Omega \rightarrow cwk(X^*)$ be a weak$^*$ scalarly integrable multifunction. If $l_1 \nsubseteq X$ and $X$ is separable, then $\Gamma$ is Pettis integrable in $cwk(X^*)$.

**Proof.** Since $X$ is separable, it follows from the Kuratowski and Ryll-Nardzewski selection theorem ([12]) that $\mathcal{W}^*S_\Gamma \neq \emptyset$. Hence according to [1, Theorem 4.2]
In order to prove Pettis integrability of $\Gamma$, it suffices to show that each scalarly measurable selection of $\Gamma$ is Pettis integrable. But if $f \in W^*S_{\Gamma}$, then $f$ is Gelfand integrable and since $l_1 \not\subseteq X$ and $X$ is separable, it is Pettis integrable ([13]). □

In order to obtain a generalization of Theorem 1.2 to non-separable Banach spaces, we need a simple fact that is certainly known.

**Lemma 1.3.** A set $W \subset X^*$ is weakly relatively compact if and only if for each separable subspace $Y$ of $X$ the range of $W$ under the canonical projection of $X^*$ onto $Y^*$ is weakly relatively compact.

**Proof.** It is enough to show that if all ranges of $W$ in spaces $Y^*$ are weakly relatively compact, then $W$ is weakly relatively compact. In order to prove the weak relative compactness of $W$ in $X^*$ it suffices to show that for an arbitrary sequence $\{w_i : i \in \mathbb{N}\} \subset W$ there exist vectors $v_n \in \text{conv}\{w_i : i \geq n\}$ such that the sequence $(v_n)_n$ is norm convergent (cf. [11, §24 3(8)]). Let $\{w_i : i \in \mathbb{N}\} \subset W$ be arbitrary. According to [5, Lemma VI.8.8] there exists a separable subspace $Y \subset X$ such that the closed linear space $Z$ generated by $\{w_i : i \in \mathbb{N}\} \subset W$ is isometrically embedded in $Y^*$. Let $j : Y \to X$ be the canonical embedding. By the assumption $j^*(W) \supset j^*(\{w_i : i \in \mathbb{N}\})$ is weakly relatively compact and so there exist vectors $v_n \in \text{conv}\{w_i : i \geq n\}$ such that the sequence $(v_n)_n$ is norm convergent in $Y^*$. But the norm convergence in $Y^*$ coincides on $Z = j^*(Z)$ with the norm convergence in $X^*$.

**Theorem 1.4.** Let $\Gamma : \Omega \to \text{cw}k(X^*)$ be a weak* scalarly integrable multifunction. If $l_1 \not\subseteq X$, then $\Gamma$ is Gelfand integrable in $\text{cw}k(X^*)$.

**Proof.** According to Lemma 1.3, a set $W \subset X^*$ is weakly relatively compact provided for each separable subspace $Y$ of $X$ the range of $W$ under canonical projection of $X^*$ onto $Y^*$ is weakly relatively compact.

Let $j : Y \to X$ be the canonical embedding of a separable Banach space $Y$ into $X$ and $M : \Sigma \to \text{cw}^*k(X^*)$ be the Gelfand integral of $\Gamma$. We have then

$$s(x, M(A)) = \int_A s(x, \Gamma) \, d\mu \text{ for every } x \in X \text{ and } A \in \Sigma.$$  

In particular, if $x = j(y)$, then

$$s(j(y), M(A)) = s(y, j^*M(A))$$

and

$$s(j(y), \Gamma(\omega)) = s(y, j^*\Gamma(\omega)) \text{ for every } \omega \in \Omega.$$  

Hence, $j^*\Gamma$ is weak* scalarly integrable and

$$s(y, j^*M(A)) = \int_A s(y, j^*\Gamma(\omega)) \, d\mu.$$
Thus, we may apply Theorem 1.2 obtaining the weak compactness of every set $j^* M(A)$. Thus, $M(A)$ is weakly relatively compact. We should prove yet that it is weakly compact. But $M(A) \in cw^* k(X^*)$ and so

$$M(A) \subset \overline{M(A) \cap M(A)^*} = M(A).$$

This completes the proof of the theorem.

As a corollary from the above theorem we obtain the following result

**Proposition 1.5.** Let $\Gamma : \Omega \to cw^k(X)$ be scalarly integrable. If $l_1 \not\subseteq X^*$, then $\Gamma$ is Gelfand integrable in $cw(X^{**})$.

**Proof.** $\Gamma$ can be treated as a weak* scalarly integrable $X^{**}$-valued multifunction. Then we apply Theorem 1.4.

**Remark 1.6.** Assume that $\text{card } T$ is real measurable. Then $l_1(T)$ is not measure compact and does not have PIP (see [7, page 575]). In particular, there exists a measure space $(\Omega, \Sigma, \mu)$ and a scalarly measurable function $f : \Omega \to l_1(T)$ that is not scalarly equivalent to any strongly measurable function (see [6, Proposition 5.4]). Hence, there is a set $E \in \Sigma$ of positive measure such that $f|_E$ is scalarly bounded but not Pettis integrable. This proves that in Theorem 1.2, if $X$ is non-separable (contrary to the separable case), then even uniform integrability of $\mathcal{Z}_\Gamma$ does not guarantee Pettis integrability of $\Gamma$.

To the best of my knowledge no ZFC example of a Banach space $X$ not containing any isomorphic copy of $l_1$ and such that $X^*$ does not possesses PIP is known. But it is consistent with ZFC to assume that each such an $X^*$ has Lebesgue PIP ([9, Proposition 3C]).

The rest of the paper is devoted to compact valued Gelfand integrable multifunctions.

**Definition 1.7.** Following [8] we say that a non-empty set $H \subset X^*$ is an $L$-set if each $\sigma(X, X^*)$-convergent sequence $\langle x_n \rangle$ is uniformly convergent on $H$.

Let us mention that L-sets coincide with $\tau(X^*, X)$ relatively compact sets (see Grothendieck [10, p. 134]). L-sets are also called sometimes $X$-limited sets.

**Lemma 1.8.** If $H$ is a bounded subset of $X^*$, then $s(\cdot, H)$ is weakly sequentially continuous on $X$ if and only if $H$ is an $L$-set in $X^*$.

**Proof.** If $H$ is an $L$-set, the weak sequential continuity of $s(\cdot, H)$ is immediate. Assume now that $s(\cdot, H)$ is weakly sequentially continuous and take $x_n \to 0$ weakly. If $x^* \in H$, then

$$-s(-x_n, H) \leq x^*(x_n) \leq s(x_n, H) \quad \text{for every } n \in \mathbb{N}.$$  

It follows that $x_n \to 0$ uniformly on $H$. 

\[\square\]
Proposition 1.9. Let \( \Gamma : \Omega \to c(A(X^*)) \) be weak* scalarly integrable and let \( T_{\Gamma} : X \to L_1(\mu) \) be defined by \( T_{\Gamma}(x) = s(x, \Gamma) \). Then \( T_{\Gamma} \) is sequentially weakly-weakly continuous if and only if \( \Gamma \) is Gelfand integrable in the collection of \( L \)-sets.

Proof. For arbitrary \( x \in X \) and \( E \in \Sigma \) we have
\[
s(x, M_{\Gamma}^G(E)) = \int_E s(x, \Gamma) \, d\mu = \langle \chi_E, T_{\Gamma}(x) \rangle. \tag{3}
\]
According to Lemma 1.8 \( M_{\Gamma}^G(E) \) is an \( L \)-set if and only if \( s(\cdot, M_{\Gamma}^G(E)) \) is weakly sequentially continuous. It follows from (3) that \( s(\cdot, M_{\Gamma}^G(E)) \) is weakly sequentially continuous if and only if \( \langle \chi_E, T_{\Gamma}(\cdot) \rangle \) is weakly sequentially continuous. \( \square \)

Theorem 1.10. Let \( \Gamma : \Omega \to c(A(X^*)) \) be weak* scalarly integrable and \( \overline{G}_\Gamma \) be uniformly integrable. Then \( \Gamma \) is Gelfand integrable in the \( L \)-sets. If the zero function is a weak* quasi-selection of \( \Gamma \), then \( \bigcup M_{\Gamma}^G(\Sigma) := \bigcup_{E \in \Sigma} M_{\Gamma}^G(E) \) is an \( L \)-set.

Proof. Fix \( E \in \Sigma \) and let \( \langle x_n \rangle \) be weakly convergent to zero. Since each \( \Gamma(\omega) \) is compact the sequence \( \langle x_n \rangle \) is on \( \Gamma(\omega) \) uniformly convergent to zero. By the assumption the sequence \( \langle s(x_n, \Gamma) \rangle \) is uniformly integrable, and so, applying the Vitali convergence theorem we obtain
\[
s(x_n, M_{\Gamma}^G(E)) = \int_E s(x_n, \Gamma) \, d\mu \to 0.
\]
According to Lemma 1.8 the set \( M_{\Gamma}^G(E) \) is an \( L \)-set.

Now let zero function be a weak* quasi-selection of \( \Gamma \). If \( E \in \Sigma \), then \( M_{\Gamma}^G(E) \subseteq M_{\Gamma}^G(\Omega) \), because \( s(x, \Gamma) \geq 0 \) a.e., for every \( x \in X \). By the first part of the proof \( M_{\Gamma}^G(\Omega) \) is an \( L \)-set. It follows that \( \bigcup M_{\Gamma}^G(\Sigma) \subseteq M_{\Gamma}^G(\Omega) \) is an \( L \)-set. \( \square \)

I do not know if a similar result for Gelfand integrable functions has been announced somewhere.

The following result is a strengthening of [2, Theorem 6.8] for Banach spaces not containing \( l_1 \). In its proof we apply the result of Emmanuele [8]: if \( l_1 \not\subseteq X \), then \( L \)-sets are norm relatively compact. Let us mention that Emmanuele’s result follows from Odell’s observation that every Dunford-Pettis operator defined on a Banach space not containing \( l_1 \) is compact (see [19, p. 377]).

Theorem 1.11. Let \( \Gamma : \Omega \to c(A(X^*)) \) be weak* scalarly integrable and \( \overline{G}_\Gamma \) be uniformly integrable. If \( X \) does not contain any isomorphic copy of \( l_1 \), then \( \Gamma \) is Gelfand integrable in \( c(A(X^*)) \) and \( \bigcup M_{\Gamma}^G(\Sigma) \) is norm relatively compact.

Proof. If \( g \) is a weak* quasi-selection of \( \Gamma \) and \( G : \Gamma = \Gamma - g \), then \( (G) \int_E \Gamma \, d\mu \subseteq (G) \int_E G \, d\mu + (G) \int_E g \, d\mu \), for each \( E \in \Sigma \). In particular
\[
\bigcup M_{\Gamma}^G(\Sigma) \subseteq \bigcup M_{G}^G(\Sigma) + \nu_g(\Sigma) \tag{4}
\]
where $\nu_g(\Sigma) := \{(G) \int_{E} g \, d\mu : E \in \Sigma\}$. But $l_1 \nsubseteq X$ and so the range of each Gelfand integral is norm relatively compact (cf. [13, Lemma 2]). It follows that $\overline{Z}_g$ is uniformly integrable and consequently also $\overline{Z}_G$ is uniformly integrable. We may apply now Theorem 1.10 and the result of Emmanuele [8] to obtain the relative compactness of $\bigcup M^G_\Sigma(\Sigma)$. The inclusion (4) yields Gelfand integrability of $\Gamma$ in $ck(X^*)$ and the relative compactness of $\bigcup M^G_\Sigma(\Sigma)$.

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**References**


