J. Math. Anal. Appl. 408 (2013) 452–464

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Henstock–Kurzweil–Pettis integrability of compact valued multifunctions with values in an arbitrary Banach space ☆

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ARTICLE INFO

Article history:
Received 5 November 2012
Available online 13 June 2013
Submitted by Bernardo Cascales

Keywords:
Multifunction
Set-valued Pettis integral
Set-valued Henstock–Kurzweil–Pettis integral
Henstock integral
Support function, selector, convergence theorems

ABSTRACT

The aim of this paper is to describe Henstock–Kurzweil–Pettis (HKP) integrable compact valued multifunctions. Such characterizations are known in case of functions (see Di Piazza and Musiał (2006) [16]). It is also known (see Di Piazza and Musiał (2010) [19]) that each HKP-integrable compact valued multifunction can be represented as a sum of a Pettis integrable multifunction and of an HKP-integrable function. Invoking to that decomposition, we present a pure topological characterization of integrability. Having applied the above results, we obtain two convergence theorems, that generalize results known for HKP-integrable functions. We emphasize also the special role played in the theory by weakly sequentially complete Banach spaces and by spaces possessing the Schur property.

0. Introduction

We assume that the unit interval of the real line is endowed with the Lebesgue measure and X is an arbitrary Banach space (in particular X may be non-separable). The Henstock–Kurzweil–Pettis integral is the natural generalization of the Pettis integral for a function, obtained by replacing the Lebesgue integrability for scalar functions by the Henstock–Kurzweil integrability. It integrates essentially more functions than the Pettis (see [25, Example 44] and [24, Proposition 2]) and the Henstock ones (see [15, Example 1]). The authors have proven in [16, Theorem 3] that a scalarly Henstock–Kurzweil integrable function $f : [0, 1] \to X$, is Henstock–Kurzweil–Pettis integrable if and only if f is determined by a weakly compactly generated (WCG) subspace of X and on the set $\{x^*f : \|x^*\| \le 1\}$ the topology of convergence in measure coincides with the weak topology of Henstock–Kurzweil integrable real functions.

In the current paper the authors prove that a similar characterization holds true also in case of scalarly Henstock–Kurzweil integrable multifunctions that are compact valued (Theorem 2.5). The basic tools are the decomposition theorem [19, Theorem 2], the selector theorem [9, Theorem 3.8] and the results of [32].

We apply then the above characterization to obtain Vitali type convergence theorems for sequences of Henstock–Kurzweil–Pettis (Theorem 4.4) and Henstock (Theorem 5.3) integrable multifunctions.

Our results generalize the earlier convergence theorems proved for sequences of functions (see [16, Theorem 5]) and extend some of the convergence theorems from [32], proved there for compact valued Pettis integrable multifunctions, to non-absolute gauge integrals.

☆ The work of the authors has been partially supported by the Polish Ministry of Science and Higher Education, Grant No. N N201 416139, and by the grant prot. U 2012/000388 09/05/2012 of GNAMPA - INDAM (Italy).

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0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.05.073

If X contains no copy of c_0, it is well known that each X-valued strongly measurable scalarly integrable function is also Pettis integrable. In case of the Henstock–Kurzweil integrability a similar property is fulfilled by weakly sequentially complete Banach spaces. In fact in [25, Theorem 40] Gordon proved that a separable Banach space is weakly sequentially complete if and only if each X-valued Henstock–Kurzweil scalarly integrable function is also HKP integrable. Here we extend such a result to not necessarily separable Banach spaces by means of the notion of function (or multifunctions) determined by a WCG subspace of X (Theorem 3.3). Moreover by means of the same notion a characterization of the spaces possessing the Schur property is also given (Theorem 3.4).

1. Basic facts

$\mathcal{L}$ and $\mathcal{L}^+$ denote the family of all Lebesgue measurable subsets of $[0, 1]$ and that of positive measure, respectively. If $A \in \mathcal{L}$, then $|A|$ denotes its Lebesgue measure.

X^* is the dual space of X. The closed unit ball of X is denoted by $B(X)$. $c(X)$ denotes the collection of all nonempty closed convex subsets of X and $cb(X)$ is the collection of all bounded members of $c(X)$. $cwk(X)$ is the family of all nonempty convex weakly compact subsets of X and $ck(X)$ is the collection of all norm compact elements of $cwk(X)$.

We consider on $cb(X)$ the Minkowski addition ($A \oplus B := \overline{\{a + b : a \in A,\ b \in B\}}$) and the standard multiplication by scalars. For every $C \in c(X)$ the *support function of* C is denoted by $s(\cdot, C)$ and defined on X^* by $s(x^*, C) = \sup\{\langle x^*, x\rangle : x \in C\}$, for each $x^* \in X^*$. If $\emptyset \neq W \subset X$, then $|W| := \sup\{\|x\| : x \in W\}$.

$\tau(X^*, X)$ denotes the topology of uniform convergence on elements of $cwk(X)$ and $\tau_c(X^*, X)$ is the topology of uniform convergence on convex compact subsets of X. The weak*-topology of X^* will be denoted by $\sigma(X^*, X)$. It is known that $\tau_c(X^*, X)$ coincides with $\sigma(X^*, X)$ on $B(X^*)$ and $s(\cdot, C)$ is $\tau_c(X^*, X)$-continuous if and only if it is weak*-continuous on $B(X^*)$.

A map $\Gamma : [0, 1] \to c(X)$ is called a *multifunction*. A multifunction $\tilde{\Gamma} : [0, 1] \to c(X)$ is *dominated* by Γ if $\tilde{\Gamma}(t) \subseteq \Gamma(t)$ for every $t \in [0, 1]$. We associate with each Γ the set

$$\mathcal{Z}_\Gamma := \{s(x^*, \Gamma) : \|x^*\| \le 1\}.$$

If the multifunction is a function f, then we have

$$\mathcal{Z}_f := \{x^* f : \|x^*\| \le 1\}.$$

In both cases we consider functions, not equivalence classes of a.e. equal functions. Identifying scalarly equivalent functions we obtain respectively the sets $\mathbb{Z}_\Gamma$ and $\mathbb{Z}_f$.

We say that a space $Y \subset X$ *determines a multifunction* $\Gamma : [0, 1] \to ck(X)$ if $s(x^*, \Gamma) = 0$ a.e. for each $x^* \in Y^\perp := \{y^* \in X^* : \forall\, y \in Y\ y^*(y) = 0\}$ (the exceptional sets depend on x^*).

A function $f : [0, 1] \to X$ is called a *selector of* Γ if $f(t) \in \Gamma(t)$, for every $t \in [0, 1]$.

A family $\emptyset \neq W \subset L_1[0, 1]$ is said to be uniformly integrable if and only if it is uniformly absolutely continuous, that is if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $|A| < \delta$, then $\sup_{f \in W} \int_A |f(t)|\, dt < \varepsilon$. One should notice that in case of an arbitrary probability space a family of integrable functions is called uniformly integrable if it is bounded and uniformly absolutely continuous. However for an atomless probability the boundedness is a consequence of the uniform absolute continuity. It is a well known fact that $\emptyset \neq W \subset L_1[0, 1]$ is weakly relatively compact in $L_1[0, 1]$ if and only if W is uniformly integrable (cf. [20, Corollary IV.8.11]).

A *partition P in* $[0, 1]$ is a collection $\{(I_1, t_1), \ldots, (I_p, t_p)\}$, where $I_1, \ldots, I_p$ are nonoverlapping subintervals of $[0, 1]$ and t_i is a point of I_i, $i = 1, \ldots, p$. If $\cup_{i=1}^p I_i = [0, 1]$, we say that P is a *partition of* $[0, 1]$. A *gauge* on $[0, 1]$ is a positive function on $[0, 1]$. For a given gauge δ on $[0, 1]$, we say that a partition $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ is δ-*fine* if $I_i \subset (t_i - \delta(x_i), t_i + \delta(x_i))$, $i = 1, \ldots, p$. By $\mathcal{I}$ we denote the family of all nontrivial closed subintervals of $[0, 1]$. We recall that a function $h : [0, 1] \to \mathbb{R}$ is said to be *Henstock–Kurzweil-integrable*, or simply *HK-integrable*, on $[0, 1]$ if there exists $a \in \mathbb{R}$ with the following property: for every $\epsilon > 0$ there exists a gauge δ on $[0, 1]$ such that

$$\left| \sum_{i=1}^{p} h(t_i)|I_i| - a \right| < \varepsilon \tag{1}$$

for each δ-fine partition $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ of $[0, 1]$. We set $(HK)\int_0^1 h\,dt := a$.

A function h is said to be *HK-integrable on* $I \in \mathcal{I}$ if $h\chi_I$ is HK-integrable on $[0, 1]$.

It is known that if h is an HK-integrable function on $[0, 1]$, then for each $I \in \mathcal{I}$ the function $h\chi_I$ is also HK-integrable. We say then that h is HK-integrable on I and write $\Phi_h(I) = (HK)\int_I h(t)\,dt := (HK)\int_0^1 h(t)\chi_I(t)\,dt$. We denote by $HK[0, 1]$ the linear space of all real valued Henstock–Kurzweil-integrable functions on $[0, 1]$ (we identify functions that are equal a.e.). The space $HK[0, 1]$ is endowed with a norm that is equivalent to the original Alexiewicz norm (cf. [1])

$$\|g\|_A = \sup_{I \in \mathcal{I}} \left| (HK) \int_I g(t)\,dt \right|.$$

The completion $\widehat{HK}[0, 1]$ of $HK[0, 1]$ is isomorphic to the space of all distributions, each one of which is the distributional derivative of a continuous function (cf. [4]). The conjugate space $HK^*[0, 1]$ is linearly isometric to the space $BV[0, 1]$ of functions of bounded variation.

The weak topology of $HK[0,1]$ will be denoted by $\sigma(HK, BV)$. We will denote by τ_m and τ_p the topology of convergence in measure in the space of Lebesgue measurable functions and the topology of the pointwise convergence respectively.

If $f : [0,1] \to X$ is an X valued function, f is called *scalarly Henstock–Kurzweil* integrable (or simply *scalarly HK-integrable*), if, for each $x^* \in X^*$, the function x^*f is Henstock–Kurzweil integrable. By a result [24, Theorem 3] we know that f is scalarly Henstock–Kurzweil integrable if and only if f is *Henstock–Kurzweil–Dunford integrable*, or simply *HKD-integrable*, (i.e. if for each interval $I \in \mathcal{I}$, there exists a vector $w_I \in X^{**}$ such that for every $x^* \in X^*$, $\langle x^*, w_I\rangle = (HK)\int_I x^*f(t)\,dt$).

Definition 1.1. A multifunction $\Gamma : [0,1] \to c(X)$ is said to be *scalarly measurable* if for every $x^* \in X^*$, the functional $s(x^*, \Gamma(\cdot))$ is measurable. A multifunction $\Gamma : [0,1] \to c(X)$ is said to be *scalarly integrable* (resp. *scalarly HK-integrable*) if $s(x^*, \Gamma)$ is integrable (resp. HK-integrable) for every $x^* \in X^*$.

If $\Delta : [0,1] \to c(X)$ is another multifunction, then Γ and Δ are called *scalarly equivalent* if for each $x^* \in X^*$ the equality $s(x^*, \Gamma) = s(x^*, \Delta)$ holds true a.e.

A scalarly integrable multifunction $\Gamma : [0,1] \to c(X)$ is called *Pettis integrable* in $cb(X)$, $[ck(X),\ cwk(X)]$ if for each $A \in \Sigma$ there exists a set $M_\Gamma(A) \in cb(X)$, $[ck(X),\ cwk(X)$, respectively] such that

$$s(x^*, M_\Gamma(A)) = \int_A s(x^*, \Gamma(t))\,dt \quad \text{for every } x^* \in X^*. \tag{2}$$

We set $(P)\int_A \Gamma(t)\,dt := M_\Gamma(A)$ and call $M_\Gamma(A)$ the *Pettis integral of* Γ over A. □

Definition 1.2. A multifunction $\Gamma : [0,1] \to c(X)$ is said to be *scalarly HK-integrable* if $s(x^*, \Gamma)$ is HK-integrable for every $x^* \in X^*$. A scalarly HK-integrable multifunction $\Gamma : [0,1] \to c(X)$ is said to be *Henstock–Kurzweil–Pettis integrable* (or simply *HKP-integrable*) in $cb(X)$, $[ck(X),\ cwk(X)]$ if for each $I \in \mathcal{I}$ there exists a set $\Phi_\Gamma(I) \in cb(X)$ $[ck(X),\ cwk(X)$, respectively] such that

$$s(x^*, \Phi_\Gamma(I)) = (HK)\int_I s(x^*, \Gamma(t))\,dt \quad \text{for every } x^* \in X^*. \tag{3}$$

We set $(HKP)\int_I \Gamma(t)\,dt := \Phi_\Gamma(I)$ and call $\Phi_\Gamma(I)$ the *Henstock–Kurzweil–Pettis integral of* Γ over I. □

If $f = \Gamma$ is an X-valued function, then we have an HKP-integrable function. In such a case Φ_f, considered as an X-valued function on $[0,1]$, is often called the *primitive* of f.

The space of scalarly equivalent HKP-integrable multifunctions can be endowed with a metric defined by

$$d_A(\Gamma, \Delta) := \sup\{\|s(x^*, \Gamma) - s(x^*, \Delta)\|_A :\ \|x^*\| \le 1\}.$$

If a multifunction $\Gamma : [0,1] \to cwk(X)$ is HKP-integrable in $cwk(X)$, then there is an HKP-integrable selector of Γ (see [19, Proposition 3]).

By the symbol $\mathcal{S}_{HKP}(\Gamma)$ we denote the family of all selectors of Γ that are HKP-integrable.

Proposition 1.3 (*See [19, Proposition 2]*)**.**

(i) *Let $\Gamma : [0,1] \to cwk(X)$ be a scalarly HK-integrable multifunction. Then Γ is HKP-integrable in $cwk(X)$ if and only if for each $I \in \mathcal{I}$ the mapping $x^* \longrightarrow (HK)\int_I s(x^*, \Gamma(t))\,dt$ is $\tau(X^*, X)$-continuous.*
(ii) *Let $\Gamma : [0,1] \to ck(X)$ be a scalarly HK-integrable multifunction. Then Γ is HKP-integrable in $ck(X)$ if and only if for each $I \in \mathcal{I}$ the mapping $x^* \longrightarrow (HK)\int_I s(x^*, \Gamma(t))\,dt$ is $\tau_c(X^*, X)$-continuous.*

Lemma 1.4. *Let $\Gamma : [0,1] \to cwk(X)$ be a scalarly HK-integrable multifunction. Then there exists a scalarly HK-integrable selector f of Γ. Moreover each scalarly measurable selector f of Γ is scalarly HK-integrable.*

Proof. Since Γ is scalarly HK-integrable, it is scalarly measurable. So by [9, Theorem 3.8] we have the existence of a scalarly measurable selector f of Γ. Then, proceeding as in the first part of [15, Lemma 2], we get the scalar HK-integrability of f. □

Another proof of the existence of a scalarly measurable selector of a scalarly measurable $\Gamma : [0,1] \to cwk(X)$ can be found in [22].

Proposition 1.5. *If $\Gamma : [0,1] \to ck(X)$ (resp. $\Gamma : [0,1] \to cwk(X)$) is HKP-integrable in $ck(X)$ (resp. $cwk(X)$), then each scalarly measurable multifunction $\tilde{\Gamma}$ dominated by Γ is HKP-integrable in $ck(X)$ (resp. $cwk(X)$).*

Proof. If $\tilde{\Gamma}$ is a scalarly measurable multifunction dominated by Γ, then for each $x^* \in X^*$ and $t \in [0,1]$ we have the inequality

$$-s(-x^*, \Gamma(t)) \le -s(-x^*, \tilde{\Gamma}(t)) \le s(x^*, \tilde{\Gamma}(t)) \le s(x^*, \Gamma(t)). \tag{4}$$

So, if $\tilde{\Gamma}$ is scalarly measurable, then the Henstock–Kurzweil-integrability of the support function $s(x^*, \tilde{\Gamma})$ follows immediately by the Henstock–Kurzweil-integrability of $s(x^*, \Gamma(t))$, for each $x^* \in X^*$. Indeed, we get from the inequalities (4),

$$0 \le s(x^*, \tilde{\Gamma}(t)) + s(-x^*, \Gamma(t)) \le s(x^*, \Gamma(t)) + s(-x^*, \Gamma(t)).$$

The function $s(x^*, \Gamma(\cdot)) + s(-x^*, \Gamma(\cdot))$ is non-negative and HK-integrable, hence it is Lebesgue integrable (cf. [26, Theorem 9.13]). Consequently also $s(x^*, \tilde{\Gamma}(\cdot) + s(-x^*, \Gamma(\cdot)))$ is Lebesgue integrable. Finally $s(x^*, \tilde{\Gamma}(t)) = [s(x^*, \tilde{\Gamma}(t)) + s(-x^*, \Gamma(t))] - s(-x^*, \Gamma(t))$ and so $s(x^*, \tilde{\Gamma}) \in HK[0, 1]$. Hence for each $I \in \mathcal{I}$ we have

$$-s(-x^*, \Phi_\Gamma(I)) \le (HK)\int_I s(x^*, \tilde{\Gamma}(t))\,dt \le s(x^*, \Phi_\Gamma(I)). \tag{5}$$

Since $\Phi_\Gamma(I)$ is convex compact (resp. convex weakly compact), the function $x^* \to s(x^*, \Phi_\Gamma(I))$ is $\tau_c(X^*, X)$-continuous (resp. $\tau(X^*, X)$-continuous). Consequently, (5) yields the continuity of the functional $x^* \to (HK)\int_I s(x^*, \tilde{\Gamma}(t))\,dt$ at zero. But since the functional is sublinear it is $\tau_c(X^*, X)$-continuous (resp. $\tau(X^*, X)$-continuous) on X^*.

By Proposition 1.3 the multifunction $\tilde{\Gamma}$ is HKP-integrable in $ck(X)$ (resp. $cwk(X)$). □

An important tool for our investigation is the decomposition theorem proved in [19, Theorem 2]. Since we use it many times throughout the paper, for readers' convenience, we state it here.

Theorem 1.6. *A scalarly HK-integrable multifunction $\Gamma : [0, 1] \to ck(X)$ (resp. $\Gamma : [0, 1] \to cwk(X)$) is HKP-integrable in $ck(X)$ (resp. $cwk(X)$) if and only if there is a representation $\Gamma = G + f$, where $G : [0, 1] \to ck(X)$ (resp. $G : [0, 1] \to cwk(X)$) is Pettis integrable in $ck(X)$ (resp. $cwk(X)$) and $f \in \mathcal{S}_{HKP}(\Gamma)$.*

2. HKP-integrability of $ck(X)$-valued multifunctions

The following two results are essential for our investigation. The first one follows directly from [32] but was not formulated there explicitly. We formulate it in the context of perfect measure spaces (cf. [36, p. 771] for the definition), but we will apply it for the Lebesgue measure on [0, 1].

Proposition 2.1. *Let (Ω, Σ, μ) be a complete and perfect probability space and, let $G : \Omega \to ck(X)$ be scalarly measurable. Then $\mathbb{Z}_G$ is compact in τ_p and $\mathbb{Z}_G$ is compact in τ_m.*

If $\mathbb{Z}_G$ is weakly relatively compact in $L_1(\mu)$ (for instance if $G : \Omega \to ck(X)$ is Pettis integrable in $cwk(X)$), then $\mathbb{Z}_G$ is compact in $\sigma(L_1(\mu), L_\infty(\mu))$ and in the norm topology of $L_1(\mu)$, hence $\sigma(L_1(\mu), L_\infty(\mu))$, τ_m and the norm topology of $L_1(\mu)$ coincide on $\mathbb{Z}_G$.

Proof. The τ_p-compactness of $\mathbb{Z}_G$ is a consequence of the weak* compactness of $B(X^*)$ and of the weak* continuity of the support functions $s(\cdot, G(\omega))$, $\omega \in \Omega$. If $\{s(x_n^*, G) : n \in \mathbb{N}\} \subset \mathbb{Z}_G$ is arbitrary, then from Fremlin's subsequence theorem (see [21] or [38, Chapter 8]) follows the existence of a subsequence $\{s(x_{n_k}^*, G) :\in \mathbb{N}\}$ that is a.e. convergent to a measurable function h. If x^* is a weak* cluster point of $\langle x_{n_k}^*\rangle_n$, then, due to the compactness of the sets $G(\omega)$, we have $h = s(x^*, G)$ a.e. As a result, the set $\mathbb{Z}_G$ is τ_m-compact.

It is known that in $L_1(\mu)$ a sequence that is weakly convergent and convergent in measure is convergent in the norm of $L_1(\mu)$. Thus, if $\mathbb{Z}_G$ is also weakly relatively compact in $L_1(\mu)$ then, being τ_m-compact, it is norm compact and all the three topologies coincide on $\mathbb{Z}_G$.

The weak relative compactness of $\mathbb{Z}_G$ for a Pettis integrable $G : \Omega \to ck(X)$ has been proven in [8, Theorem 4.1] and [32, Theorem 2.5]. □

Proposition 2.2 (*See [16, Proposition 2]*). *If $f : [0, 1] \to X$ is scalarly measurable, then $\mathbb{Z}_f$ is τ_m-compact. If $f : [0, 1] \to X$ is HKP-integrable, then $\mathbb{Z}_f$ is $\sigma(HK, BV)$-compact and $\sigma(HK, BV)$ coincides on $\mathbb{Z}_f$ with τ_m.*

Proposition 2.3. *If $\Gamma : [0, 1] \to ck(X)$ is scalarly measurable, then $\mathbb{Z}_\Gamma$ is compact in τ_p and $\mathbb{Z}_\Gamma$ is compact in τ_m. If Γ is HKP-integrable in $cwk(X)$, then $\mathbb{Z}_\Gamma$ is compact also in $\sigma(HK, BV)$ and the two topologies coincide on $\mathbb{Z}_\Gamma$.*

Proof. τ_p-compactness of $\mathbb{Z}_\Gamma$ and τ_m-compactness of $\mathbb{Z}_\Gamma$ follow exactly in the same way as that of $\mathbb{Z}_G$ and $\mathbb{Z}_G$ in Proposition 2.1. Assume $\Gamma : [0, 1] \to ck(X)$ is HKP-integrable in $cwk(X)$. In virtue of [19, Theorem 1] $\Gamma = G + f$ where G is Pettis integrable in $cwk(X)$ and $f \in \mathcal{S}_{HKP}(\Gamma)$. In virtue of Proposition 2.1 the set $\mathbb{Z}_G$ is weakly compact in $L_1[0, 1]$ and so it is also $\sigma(HK, BV)$-compact in $L_1[0, 1]$. $\mathbb{Z}_f$ is $\sigma(HK, BV)$-compact and τ_m-compact by Proposition 2.2. Since the norm topology on $\mathbb{Z}_G$ coincides with τ_m, taking into account the decomposition, we see that $\sigma(HK, BV)$ and τ_m coincide on the set $\mathbb{Z}_\Gamma$ and $\mathbb{Z}_\Gamma$ is compact in the both topologies. □

To prove a characterization of HKP-integrable $ck(X)$-valued multifunctions, we need a supplementary fact concerning real functions.

Lemma 2.4 (*See [35, p. 157]*). *Let $\langle f_n(t)\rangle$ be a sequence of non-negative integrable functions converging in measure to a function $f(t)$ on $[0,1]$. If $\lim_n \int_0^1 f_n(t)dt = \int_0^1 f(t)dt$, then $\lim_n \int_E f_n(t)dt = \int_E f(t)dt$ for all $E \in \mathcal{L}$.*

The following theorem extends [16, Theorem 3] to compact valued multifunctions.

Theorem 2.5. *A scalarly HK-integrable multifunction $\Gamma\colon [0,1] \to ck(X)$ is HKP integrable in $cwk(X)$ if and only if it satisfies the following conditions*

(S) $\mathcal{S}_{HKP}(\Gamma) \neq \emptyset$;
(TC) *on the set $\mathbb{Z}_\Gamma \subset HK[0,1]$ the topology $\sigma(HK, BV)$ coincides with τ_m;*
(D) *Γ is determined by a WCG space $Y \subseteq X$.*

Proof. $\Rightarrow$ Assume $\Gamma : [0,1] \to ck(X)$ is HKP-integrable in $cwk(X)$. In virtue of [19, Theorem 1] $\Gamma = G + f$ where G is Pettis integrable in $cwk(X)$ and $f \in \mathcal{S}_{HKP}(\Gamma)$. In virtue of Proposition 2.3 the condition (TC) is fulfilled and the set $\mathbb{Z}_\Gamma$ is compact in the both topologies. Concerning condition (D) it follows at once from the fact that both the Pettis integrable multifunction G and the *HKP*-integrable multifunction f are determined by a WCG space $Y \subseteq X$ (see [32, Proposition 2.2] and [16, Theorem 3], respectively).

$\Leftarrow$ Assume now that the conditions (S), (TC) and (D) are fulfilled. Take $f \in \mathcal{S}_{HKP}(\Gamma)$. Define $G : [0,1] \to ck(X)$ by $\Gamma = G + f$. Since $s(x^*, \Gamma) = s(x^*, G) + x^*f$ and $s(x^*, G) \geq 0$ everywhere and for every $x^* \in X^*$, G is scalarly integrable. Moreover, it follows from the inequalities

$$-s(-x^*, \Gamma(t)) \leq x^*f(t) \leq s(x^*, \Gamma(t)) \quad \text{for every } x^* \in X^* \text{ and } t \in [0,1] \tag{6}$$

that f, and then G are determined by Y.

In order to prove the HKP-integrability of Γ we need only to show that G is Pettis integrable. According to Propositions 2.1 and 2.3, respectively, the sets $\mathcal{Z}_G$ and $\mathcal{Z}_\Gamma$ are τ_m compact. Take an arbitrary sequence $\langle x_n^*\rangle_n$ with $\{x_n^* : n \in \mathbb{N}\} \subset B(X^*)$ and assume for simplicity that $s(x_n^*, \Gamma) \longrightarrow s(x_0^*, \Gamma)$ and $s(x_n^*, G) \longrightarrow s(x_0^*, G)$ a.e. Due to (TC) and Proposition 2.2 the sequence $\langle s(x_n^*, G)\rangle_n$ is $\sigma(HK, BV)$ convergent to $s(x_0^*, G)$. Now we apply Lemma 2.4 to get the weak convergence $s(x_n^*, G) \longrightarrow s(x_0^*, G)$. Thus, we have proven that $\mathbb{Z}_G$ is weakly compact in $L_1[0,1]$. Being weakly compact and compact in measure in $L_1[0,1]$, the set $\mathbb{Z}_G$ is compact in the norm topology of $L_1[0,1]$. In virtue of [32, Theorem 3.3] G is Pettis integrable in $ck(X)$. □

As a direct consequence of the proof of the above theorem, we obtain the following interesting description of $ck(X)$-valued multifunction that are HKP-integrable in $cwk(X)$. The result can be considered as a partial completion of [19, Theorem 1] and of [17].

Theorem 2.6. *If $\Gamma\colon [0,1] \to ck(X)$ is HKP-integrable in $cwk(X)$ and $f \in \mathcal{S}_{HKP}(\Gamma)$, then $\Gamma = G + f$ and $G\colon [0,1] \to ck(X)$ is Pettis integrable in $ck(X)$.*

As a particular case of the above result, we obtain a partial but essential generalization of [32, Theorem 3.6] in case when $\Gamma\colon [0,1] \to ck(X)$ is Pettis-integrable in $cwk(X)$ with respect to the Lebesgue measure. But as the result holds true in case of an arbitrary finite complete perfect measure space (Ω, Σ, μ), we sketch its general proof also.

Theorem 2.7. *If $\Gamma\colon \Omega \to ck(X)$ is Pettis-integrable in $cwk(X)$ and μ is complete and perfect, then Γ is Pettis integrable in $ck(X)$ and $\bigcup_{E\in\Sigma} M_\Gamma(E)$ is a norm relatively compact subset of X.*

Proof. According to [8, Theorem 2.5], Γ possesses a Pettis integrable selector f. Define $G : [0,1] \to ck(X)$ by $\Gamma = G + f$. Since Γ is determined by a WCG space (see [32, Proposition 2.2]), it follows from (6) that also G and f are determined by a WCG space. According to Proposition 2.1 the set $\mathbb{Z}_G$ is norm relatively compact in $L_1(\mu)$ and so — in virtue of [32, Theorem 3.3] — G is Pettis integrable in $ck(X)$. According to [23, Proposition 3J] the range of the integral of f is norm relatively compact. It follows that Γ is Pettis integrable in $ck(X)$. □

Remark 2.8. Theorem 2.7 fails for non-perfect measures. In [32, Remark 3.5] is described an example of a $ck(l_\infty)$-valued multifunction, defined on a space with non-perfect measure, that is Pettis integrable in $cwk(l_\infty)$ but not in $ck(l_\infty)$.

The next theorem gives a complete description of those $ck(X)$-integrable compact valued multifunctions Γ for which the total range $\Phi_\Gamma(\mathcal{I}) := \bigcup_{I\in\mathcal{I}} \Phi_\Gamma(I)$ of the integral is norm relatively compact. The theorem solves also the problem formulated in [19, Question 1]. First we present a general situation (the result should be compared with [27, Theorem 8.5.10]).

Proposition 2.9. *If a multifunction $\Gamma : [0,1] \to cwk(X)$ is HKP-integrable in $cwk(X)$, then the set $\Phi_\Gamma(\mathcal{I}) := \bigcup_{I\in\mathcal{I}} \Phi_\Gamma(I)$ is weakly relatively compact.*

Proof. Assume that Γ is HKP-integrable in $cwk(X)$. Then according to [19, Theorem 1], there exists $f \in \mathcal{S}_{HKP}(\Gamma)$ such that the multifunction $G : [0,1] \to cwk(X)$ defined by $\Gamma(t) = G(t) + f(t)$ is Pettis integrable in $cwk(X)$. Then, for each $I \in \mathcal{I}$, $\Phi_\Gamma(I) = M_G(I) + (HKP)\int_I f$. In virtue of [27, Theorem 8.5.10] the set $\bigcup M_G(\Sigma) := \bigcup_{E\in\Sigma} M_\Gamma(E)$ is weakly relatively compact. (In fact it is weakly compact since $M_G(I) \subset M_G[0,1]$, for each $I \in \mathcal{I}$.) Moreover also the set $\{(HKP)\int_I f : I \in \mathcal{I}\}$ is weakly relatively compact (see [18, Theorem 1]). Therefore the set $\Phi_\Gamma(\mathcal{I})$ is weakly relatively compact. □

One should notice however that even in case of HKP-integrable functions the range of its HKP-integral may be non relatively compact (see [18, Example 2]). Thus, even for $ck(X)$-valued multifunctions Γ the set $\Phi_\Gamma(\mathcal{I})$ is not always relatively compact.

Theorem 2.10. *If $\Gamma: [0, 1] \to ck(X)$ is scalarly HK-integrable, then the following conditions are equivalent:*

(i) *Γ is HKP-integrable in $ck(X)$ and $\Phi(\mathcal{I})$ is relatively compact;*
(ii) *Each scalarly measurable selector of Γ is HKP-integrable and has norm relatively compact range of its integral;*
(iii) *Each scalarly measurable selector of Γ is HKP-integrable and has continuous primitive.*

Proof. (i) $\Rightarrow$ (ii) If f is a scalarly measurable selector of Γ, then its HKP-integrability follows from Proposition 1.5. As $f \in \mathcal{S}_{HKP}(\Gamma)$, we have $\Phi_f(I) \in \Phi_\Gamma(I) \subset \Phi(\mathcal{I})$, for each $I \in \mathcal{I}$.

(ii) $\Rightarrow$ (i) In virtue of [19, Theorem 1] Γ is HKP-integrable in $cwk(X)$. Applying Theorem 2.6 we have a representation $\Gamma = G + f$ with the properties mentioned there. Theorem 2.7 yields then the norm relative compactness of the set $M_G(\mathcal{I})$. Since $\Phi_\Gamma(\mathcal{I}) \subset M_G(\mathcal{I}) + \bigcup_{I\in\mathcal{I}} \Phi_f(I)$ and the last set is in norm relatively compact, we have the required property of the range of Φ_Γ.

The equivalence (ii) $\Leftrightarrow$ (iii) is a result of Naralenkov [34]. □

Our next result reverts to [32, Theorem 4.6], where Pettis integrability has been characterized in the language of core.

Definition 2.11 (*[32, Definition 4.3]*)**.** Let $\Gamma: \Omega \to c(X)$ be a multifunction. For each $E \in \Sigma$ we define the *core of Γ on E* by the formula

$$cor_\Gamma(E) := \bigcap_{\mu(N)=0} \overline{conv}\, \Gamma(E \setminus N) = \bigcap_{\mu(N)=0} \overline{conv}\left(\bigcup_{\omega\in E\setminus N} \Gamma(\omega)\right).$$

One can immediately see that $E \subset F$ yields $cor_\Gamma(E) \subset cor_\Gamma(F)$. Moreover, if Γ dominates $\Theta : \Omega \to c(X)$, then $cor_\Theta(E) \subset cor_\Gamma(E)$, for each $E \in \Sigma$.

Theorem 2.12. *A scalarly HK-integrable multifunction $\Gamma: [0, 1] \to ck(X)$ is HKP integrable in $cwk(X)$ if and only if it satisfies the following conditions*

(S) *$\mathcal{S}_{HKP}(\Gamma) \neq \emptyset$;*
(TC) *on the set $\mathbb{Z}_\Gamma \subset HK[0, 1]$ the topology $\sigma(HK, BV)$ coincides with τ_m;*
(CC) *If $\Delta : [0, 1] \to ck(X)$ is scalarly measurable and dominated by Γ, then $cor_\Delta(E) \neq \emptyset$, for every $E \in \mathcal{L}^+$.*

The condition (CC) *may be replaced by a formally weaker*

(CS) *If $f : [0, 1] \to X$ is a scalarly measurable selector of Γ, then $cor_f(E) \neq \emptyset$, for every $E \in \mathcal{L}^+$.*

Proof. $\Rightarrow$ If Γ is HKP-integrable in $cwk(X)$, then the conditions (S) and (TC) follow from Theorem 2.5. Let $\Delta : [0, 1] \to ck(X)$ be scalarly measurable and dominated by Γ. By Proposition 1.5 Δ is HKP-integrable. Hence, by [19, Theorem 1] there exists $f \in \mathcal{S}_{HKP}(\Delta)$. Since for every $E \in \mathcal{L}^+$ we have $cor_f(E) \subset cor_\Delta(E)$ and $cor_f(E) \neq \emptyset$ by [16, Theorem 3], the (CC) condition is fulfilled.

$\Leftarrow$ Assume that (S), (TC) and (CS) are fulfilled. Define $G : [0, 1] \to ck(X)$ by $\Gamma = G + f$, where $f \in \mathcal{S}_{HKP}(\Gamma)$. In order to prove the HKP-integrability of Γ we need only to show that G is Pettis integrable in $cwk(X)$.

The weak compactness of $\mathbb{Z}_G$ can be proved exactly as in the proof of Theorem 2.5.

According to [8] or [32, Theorem 4.8], in order to prove Pettis integrability of G in $cwk(X)$, we should prove that each scalarly measurable selector g of G is Pettis integrable. Let us fix one such g. In virtue of the condition (CS) we have $cor_{g+f}(E) \neq \emptyset$ for every $E \in \mathcal{L}^+$.

According to [30, Corollary 3.1], one can decompose [0, 1] into pairwise disjoint sets $S_k \in \mathcal{L}^+$ in such a way that $g + f$ is scalarly bounded on each S_k, $k \in \mathbb{N}$. Hence, for each $k \in \mathbb{N}$ the collection $\mathcal{Z}_{g+f}$ restricted to S_k is weakly relatively compact. Talagrand's characterization of Pettis integrability (see [38, Theorem 5-2-2], [30, Theorem 6.1] or [31, Theorem 4.10]) yields Pettis integrability of $g + f$ on each S_k.

Invoking again to [38, Theorem 5-2-2] (either to [30, Theorem 6.2] or to [31, Theorem 4.5]), we see that there exists a set $B(X) \supset W_k \in cwk(X)$ that spans a Banach space which determines and $(g + f)\chi_{S_k}$. If $W := \bigcup_{k=1}^{\infty} \frac{1}{2^k} W_k$, then $W \in cwk(X)$ and it spans a WCG space that determines $g + f$.

Let us notice now that f is also determined by a WCG Banach space, since we have assumed HKP-integrability of f (see [16, Theorem 3]). Thus, f as well as $g + f$ are determined by a common WCG Banach space, what immediately yields determination of g by the same WCG space.

We already know that $\mathbb{Z}_G$ is weakly relatively compact in $L_1[0, 1]$. So by the Dunford characterization of weakly relatively compact subsets of $L_1[0, 1]$, the family $\mathbb{Z}_G$ is (bounded in $L_1[0, 1]$ and) uniformly absolutely continuous. Since $-s(-x^*, G) \leq x^*g \leq s(x^*, G)$ everywhere, for each x^*, also $\mathbb{Z}_g$ is (bounded in $L_1[0, 1]$ and) uniformly absolutely continuous. Applying once again the Dunford characterization, we infer that $\mathbb{Z}_g$ is weakly relatively compact in $L_1[0, 1]$.

Finally we may apply [30, Theorem 6.2] or [31, Theorem 4.5] to obtain Pettis integrability of g. Thus, G is Pettis integrable in $cwk(X)$ and that completes the whole proof. □

In case of multifunctions with relatively compact total range $\Phi(\mathcal{I})$ of the integral the above theorem has the following form:

Theorem 2.13. *A scalarly HK-integrable multifunction Γ: [0, 1] $\to ck(X)$ is HKP integrable in $ck(X)$ and $\Phi(\mathcal{I})$ is norm relatively compact if and only if it satisfies the following conditions*

(S) *There exists $f \in \mathcal{S}_{HKP}(\Gamma)$ with norm relatively compact range of its HKP-integral;*
(TC) *on the set $\mathbb{Z}_\Gamma \subset HK[0, 1]$ the topology $\sigma(HK, BV)$ coincides with τ_m;*
(CC) *If $\Delta : [0, 1] \to ck(X)$ is scalarly measurable and dominated by Γ, then $cor_\Delta(E) \neq \emptyset$, for every $E \in \mathcal{L}^+$.*

The condition (CC) *may be replaced by* (CS).

Proof. Only the implication $\Leftarrow$ requires an explanation. We keep the notation of the proof of Theorem 2.12 which yields Pettis integrability of G in $cwk(X)$. But then, it follows from Theorem 2.7 that G is Pettis integrable in $ck(X)$ and $\bigcup_{E\in\Sigma} M_\Gamma(E)$ is a norm relatively compact subset of X. This fact composed with (S) forces HKP-integrability of Γ in $ck(X)$ and completes the whole proof. □

3. Integration in weakly sequentially complete Banach spaces and in Banach spaces possessing the Schur property

When one investigates Pettis integrable functions, then the space c_0 is that space which makes problems. If $c_0 \subset X$ isomorphically, then there are X-valued scalarly integrable functions that are not Pettis integrable. Gordon proved in [25, Theorem 40] that in case of Denjoy–Pettis integral a similar role to spaces not containing c_0 is played by weakly sequentially complete separable Banach spaces.

Let us recall that X is called weakly sequentially complete if each weakly Cauchy sequence in X is weakly convergent. It is known that no weakly sequentially complete Banach space can contain an isomorphic copy of c_0.

Gordon [25, Theorem 40 + Example 41] proved that a separable Banach space X is weakly sequentially complete if and only if each X-valued scalarly Denjoy–Pettis-integrable function $f : [0, 1] \to X$ is Denjoy–Pettis integrable. His proof remains valid for the HKP-integral also and we will apply it here. It is our aim to extend that result to not necessarily separable Banach spaces.

To formulate our next result we will recall first Romanowski's lemma (see [37] or [26, Lemma 5.18]):

Lemma 3.1. *Let $\mathcal{F}$ be a family of open subintervals of (α, β) and assume that $\mathcal{F}$ has the following properties;*

(a) *If $(a, b) \in \mathcal{F}$ and $(b, c) \in \mathcal{F}$, then $(a, c) \in \mathcal{F}$;*
(b) *If $(a, b) \in \mathcal{F}$ and $(c, d) \subset (a, b)$, then $(c, d) \in \mathcal{F}$;*
(c) *If $(a, b) \in \mathcal{F}$ for every $[a, b] \subset (c, d)$, then $(c, d) \in \mathcal{F}$;*
(d) *If $H \subset [\alpha, \beta]$ is perfect and all open intervals contiguous to H are in $\mathcal{F}$, then there exists $I \in \mathcal{F}$ such that $I \cap H \neq \emptyset$.*

Then $(\alpha, \beta) \in \mathcal{F}$. □

Theorem 3.2. *Let X be a weakly sequentially complete Banach space. A scalarly HK-integrable function $f : [0, 1] \to X$ is HKP-integrable if and only if it is determined by a WCG space.*

Proof. $\Leftarrow$ Let $\Phi : \mathcal{I} \to X^{**}$ be the HKD-integral of f and let $\mathcal{F}$ be the collection of all open intervals $I \subset (0, 1)$ such that $\Phi(J) \in X$ for every open $J \subset I$. We are going to prove that $(0, 1) \in \mathcal{F}$ and so we have to verify condition (a)–(d) of Lemma 3.1 in case $(\alpha, \beta) = (0, 1)$.

It is obvious that conditions (a) and (b) hold true. Assume now that $(a, b) \in \mathcal{F}$ for every $[a, b] \subset (c, d)$. We are going to prove that f is HKP-integrable on $[c, d]$. Since for each $x^* \in X^*$ the function x^*f is HK-integrable, we have for every $x^* \in X^*$

$$x^*\Phi[c, d] = \int_c^d x^*f(t)\,dt = \lim_n \int_{c+1/n}^{d-1/n} x^*f(t)\,dt = \lim_n x^*\Phi[c + 1/n, d - 1/n].$$

Thus, the sequence $\{\Phi[c + 1/n, d - 1/n]; n \in \mathbb{N}\}$ is weakly Cauchy and weak* convergent to $\Phi[c, d]$. As the space X is assumed to be weakly sequentially complete and weak* convergent sequences may have only one limit point, we have $\Phi[c, d] \in X$ and this verifies condition (c).

We are proving that $(0, 1) \in \mathcal{F}$. Let H be a perfect subset of [0, 1] such that each interval in (0, 1) contiguous to H belongs to $\mathcal{F}$. Since f is HKD-integrable on [0, 1] it follows from [25, Theorem 33] that there exists an interval $[u, v]$ with $u, v \in H$ and $H \cap (u, v) \neq \emptyset$ such that f is Dunford integrable in $[u, v]$. Since $c_0 \not\subseteq X$ and f is determined by a WCG space, by [32, Theorem 2.14] f is Pettis integrable on $[u, v]$. Thus, (d) is fulfilled and so (0, 1) belongs to $\mathcal{F}$ what means (in virtue of (c)) that f is HKP-integrable in [0, 1].

$\Rightarrow$ If f is HKP-integrable, then according to [16, Theorem 3], it is determined by a WCG space. □

Taking into account the characterization of weakly sequentially complete Banach spaces by Gordon [25], we obtain the following result:

Theorem 3.3. *The following conditions are equivalent for an arbitrary Banach space X:*

(i) *X is weakly sequentially complete Banach space;*
(ii) *Each scalarly HK-integrable function $f : [0, 1] \to X$ that is determined by a WCG space is HKP-integrable;*

(iii) *Each scalarly HK-integrable multifunction* $\Gamma : [0, 1] \to cwk(X)[ck(X)]$ *that is determined by a WCG space, is HKP-integrable in* $cwk(X)$.

Proof. (i) ⇒ (ii) has been proven in Theorem 3.2.

(ii) ⇒ (iii) Let f be a scalarly measurable selector of Γ (Lemma 1.4). By (ii) $f \in \mathcal{S}_{HKP}(\Gamma)$. Hence, it follows from [19, Theorem 1] that Γ is HKP-integrable in $cwk(X)$.

(iii) ⇒ (i) Gordon [25, Example 41] proved that if X is a separable non weakly sequentially complete Banach space, then there exists an X-valued scalarly HK-integrable function that is not HKP-integrable. □

We recall that a Banach space X has the *Schur property* is each sequence weakly convergent to 0 is also norm convergent. It well known that each space with the Schur property is weakly sequentially complete. The next theorem generalizes one of the results from [34] and answers Question 3 from [18].

Theorem 3.4. *The following conditions are equivalent for an arbitrary Banach space X:*

(i) *X has the Schur property;*
(ii) *Each scalarly HK-integrable function* $f : [0, 1] \to X$ *that is determined by a WCG space, is HKP-integrable and the range of its integral is norm relatively compact;*
(iii) *Each scalarly HK-integrable multifunction* $\Gamma : [0, 1] \to ck(X)$ *that is determined by a WCG space, is HKP-integrable in* $ck(X)$.

Proof. (i) ⇒ (iii) According to Theorem 3.3, if $\Gamma : [0, 1] \to ck(X)$ is scalarly HK-integrable and determined by a WCG space, then it is HKP-integrable in $cwk(X)$. The Schur property of X forces the integrability in $ck(X)$.

(iii) ⇒ (ii) If $f : [0, 1] \to X$ is scalarly integrable and determined by a WCG space, then by (iii) it is HKP-integrable. Also if $\Gamma(t) := conv\{0, f(t)\}$, then Γ is HKP-integrable in $ck(X)$. But $\Phi_f(\mathcal{I}) \subset \Phi_\Gamma[0, 1]$, which is compact. This completes the proof.

(ii) ⇒ (i) Assume that X fails the Schur property. Then we can find a sequence $\{x_n : n \in \mathbb{N}\}$ that is weakly convergent to zero and all its terms are of norm 1. Moreover, due to Bessaga–Pełczyński selection principle (cf. [13, p. 42]), we may choose a subsequence that is a basis of the space $Y := \overline{span}\{x_n : n \in \mathbb{N}\}$. Assume that $\{x_n : n \in \mathbb{N}\}$ is already such. We are going to construct an Y-valued function that is HKP-integrable but the range of its integral is not relatively compact.

We will use an idea of Gamez and Mendoza [24]. So consider a sequence of intervals $A_n = [a_n, b_n] \subseteq [0, 1]$ such that $a_1 = 0$, $b_n < a_{n+1}$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} b_n = 1$ and define $f : [0, 1] \to Y$ by

$$f(t) = \sum_{n=1}^{\infty} x_n \left(\frac{\chi_{A_{2n-1}}(t)}{|A_{2n-1}|} - \frac{\chi_{A_{2n}}(t)}{|A_{2n}|} \right).$$

One can easily see that the sequence $\left\langle \sum_{n=1}^{k} x_n \left(\frac{\chi_{A_{2n-1}}(t)}{|A_{2n-1}|} - \frac{\chi_{A_{2n}}(t)}{|A_{2n}|} \right) \right\rangle_k$ is eventually constant for each $t \in [0, 1]$, hence norm convergent.

We claim that f is HKP-integrable and for each $I \in \mathcal{I}$

$$(HKP) \int_I f(t)\,dt = \sum_{n=1}^{\infty} x_n \left(\frac{|A_{2n-1} \cap I|}{|A_{2n-1}|} - \frac{|A_{2n} \cap I|}{|A_{2n}|} \right). \tag{7}$$

One can easily see that the series is norm convergent because it has always only finitely many terms.

If $I = [a, b]$ and $b < 1$, then the sum in (7) is finite and the equality holds true. Consider now a sequence $c_k \nearrow 1$ and a functional $x^* \in B(Y^*)$. Since $\{x_n : n \in \mathbb{N}\}$ is a normalized basis there exists in Y^* a biorthogonal sequence $\{x_n^* : n \in \mathbb{N}\}$ being a weak*-basis of Y^*. In particular $x^* = \sum_{n=1}^{\infty} \alpha_n x_n^*$, where the series is weak*-convergent. We should notice that as $x_n \to 0$ weakly, we have $\alpha_n = x^*(x_n) \to 0$. We have now

$$(HK) \int_0^{c_k} x^* f(t)\,dt = \begin{cases} \alpha_n \dfrac{|A_{2n-1} \cap [0, c_k]|}{|A_{2n-1}|} & \text{if } a_{2n-1} < c_k < b_{2n-1} \\ \alpha_n & \text{if } b_{2n-1} \le c_k \le a_{2n} \\ \alpha_n \left(1 - \dfrac{|A_{2n} \cap [0, c_k]|}{|A_{2n}|} \right) & \text{if } a_{2n} < c_k < b_{2n} \\ 0 & \text{otherwise.} \end{cases}$$

Since $c_k \nearrow 1$ is arbitrary, it follows that $\lim_k (HK) \int_0^{c_k} x^* f(t)\,dt = 0$ and so we may apply [26, Theorem 9.21] to get the HK-integrability of $x^* f$ on $[0, 1]$ and the equality (7) for $I = [0, 1]$.

Since $(HKP) \int_{A_{2n-1}} f(t)\,dt = x_n$, $n \in \mathbb{N}$, the range of the integral is not relatively compact. □

4. Convergence theorem for the HKP-integral

Convergence theorem of Vitali type is fundamental in the theory of absolute integration. Several results of that type are well known in case of Bochner or Pettis integrable multifunctions with values in separable Banach spaces. Theorems of that type for non-separable Banach spaces have been proven for Pettis integrable functions in [29–31,2]. The corresponding results for Pettis integrable multifunctions can be found in [32]. We are going to present here and in the next section two convergence theorems for non-absolute integrals, the first for HKP-integrable multifunctions and the second for Henstock integrable multifunctions. Their formulation reverts to the corresponding form of the results in [12], [16, Theorems 5,6] and [32]. From our point of view [32, Theorem 3.16] is of particular interest.

To prove our convergence results, we need another form of Proposition 2.3 and another condition guaranteeing the HKP-integrability (Theorem 4.2).

We recall that a family $\{h_\alpha : \alpha \in \mathbb{A}\}$ of real valued functions in $HK[0,1]$ is said to be *Henstock–Kurzweil equi-integrable* (or *equi-HK-integrable*) on $[0,1]$ whenever for every $\varepsilon > 0$ there is a gauge δ such that

$$\sup\left\{\left|\sum_{i=1}^{p} h_\alpha(t_i)|I_i| - (HK)\int_0^1 h_\alpha(t)dt\right| : \alpha \in \mathbb{A}\right\} < \varepsilon$$

for each δ-fine partition $\{(I_i, t_i) : i \le p\}$ of $[0,1]$.

Lemma 4.1. *Let $\Gamma : [0,1] \to ck(X)$ be a scalarly HK-integrable multifunction such that each infinite subset of $\mathcal{Z}_\Gamma$ contains an infinite equi-HK-integrable subset. Then the set $\mathbb{Z}_\Gamma$ is $\sigma(HK, BV)$ and τ_m compact, and the two topologies coincide on $\mathbb{Z}_\Gamma$.*

Proof. According to Proposition 2.3 the set $\mathbb{Z}_\Gamma$ is τ_m-compact. Take now a sequence $\langle s(x_n^*, \Gamma)\rangle_n$ in $\mathbb{Z}_\Gamma$ converging in measure to $s(x^*, \Gamma) \in \mathbb{Z}_\Gamma$. If $(s(x_{n_k}^*, \Gamma))_k \to s(x^*, \Gamma)$ a.e., it is also pointwise bounded because Γ is compact valued. So applying the equi-integrability of a subsequence $\langle s(x_{n_{k_p}}^*, \Gamma)\rangle_p$, we see that $s(x_{n_{k_p}}^*, \Gamma)_p \to s(x^*, \Gamma)$ weakly in $HK[0,1]$ (see [5, Theorem 4]). It follows that each subsequence of $\langle s(x_n^*, \Gamma)\rangle_n$ contains a subsequence weakly convergent to $s(x^*, \Gamma)$ and so $s(x_n^*, \Gamma) \to s(x^*, \Gamma)$ weakly in $HK[0,1]$. Thus the identity mapping $(\mathbb{Z}_\Gamma, \tau_m) \longrightarrow (\mathbb{Z}_\Gamma, \sigma(HK, BV))$ is continuous and since $\mathbb{Z}_\Gamma$ is τ_m-compact, it is a homeomorphism. Consequently, $\sigma(HK, BV)$ and τ_m coincide on $\mathbb{Z}_\Gamma$. This completes the proof. □

Theorem 4.2. *Let $\Gamma : [0,1] \to ck(X)$ be a scalarly HK-integrable multifunction determined by a WCG space. If each infinite family of $\mathcal{Z}_\Gamma$ contains an infinite equi-HK-integrable subfamily, then Γ is HKP-integrable in $cwk(X)$.*

Proof. We are going to prove that each scalarly measurable selector of Γ is HKP-integrable. It will follow then from [19, Theorem 2] that Γ is HKP-integrable in $cwk(X)$. Since Γ is determined by a WCG space, the same holds true also for its selectors. Let f be a scalarly HK-integrable selector of Γ, existing by Lemma 1.4. In order to prove its HKP-integrability, we need to show (see [16, Theorem 3]) that $\sigma(HK, BV)$ and τ_m coincide on $\mathcal{Z}_f$. Due to τ_m-compactness of $\mathcal{Z}_f$ (see Proposition 2.2) it is enough to show that the identity mapping $(\mathcal{Z}_f, \tau_m) \longrightarrow (\mathcal{Z}_f, \sigma(HK, BV))$ is continuous. To achieve it, we will prove that each τ_m-convergent sequence $\langle x_n^* f\rangle_n$ contains a subsequence that is $\sigma(HK, BV)$-convergent to the same limit.

Let $G : [0,1] \to ck(X)$ be defined by $G(t) := \Gamma(t) - f(t),\ t \in [0,1]$. It is important to observe that $s(x^*, G(t)) \ge 0$ for every $x^* \in X^*$ and every $t \in [0,1]$. Moreover let $\{x_n^* : n \in \mathbb{N}\} \subset B(X^*)$ be an arbitrary sequence such that $\langle x_n^* f\rangle_n$ i s a.e. convergent to an element of $\mathcal{Z}_f$.

Let $\overline{x}^*$ be a weak*-cluster point of $\langle x_n^*\rangle_n$. According to Fremlin's subsequence theorem there exists a subsequence $\langle x_{n_k}^*\rangle$ such that we have a.e.

$$s(x_{n_k}^*, \Gamma) \to s(\overline{x}^*, \Gamma) \qquad s(-x_{n_k}^*, \Gamma) \to s(-\overline{x}^*, \Gamma) \quad \text{and} \quad x_{n_k}^* f \to \overline{x}^* f \text{ a.e.}$$

and hence also

$$s(x_{n_k}^*, G) \to s(\overline{x}^*, G) \qquad s(-x_{n_k}^*, G) \to s(-\overline{x}^*, G) \quad \text{a.e.}$$

Without loss of generality we may assume that the sequences $\langle s(x_{n_k}^*, \Gamma)\rangle_k$ and $\langle s(-x_{n_k}^*, \Gamma)\rangle_k$ are equi-HK-integrable. Since obviously the sequences are pointwise bounded, by [14, Theorem 4] we obtain for each $I \in \mathcal{I}$ the convergence

$$(HK)\int_I s(x_{n_k}^*, \Gamma(t))\,dt \longrightarrow (HK)\int_I s(\overline{x}^*, \Gamma(t))\,dt \tag{8}$$

and

$$(HK)\int_I s(-x_{n_k}^*, \Gamma(t))\,dt \longrightarrow (HK)\int_I s(-\overline{x}^*, \Gamma(t))\,dt. \tag{9}$$

Taking into account the definition of G and adding (8) and (9) we get the convergence

$$s(x_{n_k}^*, G) + s(-x_{n_k}^*, G) \longrightarrow s(\overline{x}^*, \Gamma) + s(-\overline{x}^*, \Gamma) \quad \text{a.e.}$$

and for each $I \in \mathcal{I}$

$$(L)\int_I [s(x_{n_k}^*, G(t)) + s(-x_{n_k}^*, G(t))]\,dt \longrightarrow (HK)\int_I [s(\overline{x}^*, \Gamma(t)) + s(-\overline{x}^*, \Gamma(t))]\,dt.$$

In view of Lemma 2.4 we have

$$s(x^*_{n_k}, G) + s(-x^*_{n_k}, G) \longrightarrow s(\overline{x}^*, \Gamma) + s(-\overline{x}^*, \Gamma) = s(\overline{x}^*, G) + s(-\overline{x}^*, G) \quad \text{weakly in } L_1[0,1],$$

and so the Vitali–Hahn–Saks Theorem yields the uniform integrability of the family

$$\{s(x^*_{n_k}, G(t)) + s(-x^*_{n_k}, G) : k \in \mathbb{N}\}.$$

Hence also the functions $\{s(x^*_{n_k}, G)\,dt : k \in \mathbb{N}\}$ are uniformly integrable. In particular, the Vitali convergence theorem yields the convergence

$$(L)\int_I s(x^*_{n_k}, G(t))\,dt \longrightarrow (L)\int_I s(\overline{x}^*, G(t))\,dt. \tag{10}$$

Thus, it follows from (8) and (10) that

$$(HK)\int_I x^*_{n_k} f(t)\,dt \longrightarrow (HK)\int_I \overline{x}^* f(t)\,dt \quad \text{for every } I \in \mathcal{I}.$$

Notice now that it follows from our assumptions, due to [14, Theorem 4], that $\sup_k \|s(\pm x^*_{n_k}, \Gamma)\|_A < \infty$. Since f is a selector, we have $\|x^* f\|_A \le \|s(x^*, \Gamma)\|_A + \|s(-x^*, \Gamma)\|_A$ for each $x^* \in X^*$ and so $\sup_k \|x^*_{n_k} f\|_A < \infty$. This means that $x^*_{n_k} f \longrightarrow \overline{x}^* f$ weakly in $HK[0,1]$ (cf. [6, Proposition 3.3]).

It follows from the Krein–Šmulian Theorem that $\mathbb{Z}_f$ is $\sigma(HK, BV)$-compact and $\sigma(HK, BV)$ is weaker than τ_m. As both are compact on $\mathbb{Z}_f$, they coincide. It follows from [16, Theorem 3] that $f \in \mathcal{S}_{HKP}(\Gamma)$. This completes the whole proof. □

As an immediate corollary of the above proposition we have the following generalization of [16, Proposition 4]:

Theorem 4.3. *Let $f : [0,1] \to X$ be scalarly HK-integrable function determined by a WCG. If each infinite subset of $\mathcal{Z}_f$ contains an infinite equi-HK-integrable subset, then f is HKP-integrable.*

Now we are in a position to prove a Vitali type convergence theorem for HKP-integrable functions.

Theorem 4.4. *Let $\Gamma_n : [0,1] \to ck(X)$, $n \in \mathbb{N}$, be a sequence of multifunctions that are HKP-integrable in $cwk(X)$ and let $\Gamma : [0,1] \to ck(X)$ be a multifunction. Assume that the following conditions are satisfied:*

(a) *$s(x^*, \Gamma_n) \to s(x^*, \Gamma)$ a.e. for each $x^* \in X^*$ (the exceptional sets depend on x^*);*
(b) *for each $x^* \in X^*$ the sequence $\langle s(x^*, \Gamma_n)\rangle_n$ is pointwise bounded;*
(c) *each countable subset of $\bigcup_n \mathcal{Z}_{\Gamma_n}$ is equi-HK-integrable.*

Then Γ is HKP-integrable in $cwk(X)$ and

$$\lim_n \|s(x^*, \Gamma_n) - s(x^*, \Gamma)\|_A = 0 \quad \text{for each } x^* \in X^*. \tag{11}$$

Due to the Banach–Steinhaus theorem the condition (b) *is equivalent to*

(b′) $\forall\, t \in [0,1]\ \sup_n |\Gamma_n(t)| < \infty$.

Proof. By condition (a), (b) and (c) and by [14, Theorem 4] it follows that Γ is scalarly HK-integrable. We are going to prove that each sequence in $\mathcal{Z}_\Gamma$ is equi-HK-integrable. So let $\{s(x^*_m, \Gamma) : x^*_m \in \mathcal{B}(X^*)\}$ be an arbitrary sequence. At first we observe that the sequence $\langle s(x^*_m, \Gamma_n(t))\rangle_{n,m}$ is pointwise bounded in $[0,1]$. In fact let us fix $t \in [0,1]$. By (b) we have $\sup_n |s(x^*, \Gamma_n(t))| < \infty$ and so the Banach–Steinhaus theorem gives $\sup_n |\Gamma_n(t)| < +\infty$ for every $t \in [0,1]$. So we have $\sup_{m,n} |s(x^*_m, \Gamma_n(t))| \le \sup_n |\Gamma_n(t)| < +\infty$.

Now set $N_m = \{t \in [0,1] : s(x^*_m, \Gamma_n(t)) \nrightarrow s(x^*_m, \Gamma(t))\}$ and $N = \bigcup_{m=1}^{\infty} N_m$. Let us fix an arbitrary $\varepsilon > 0$. According to (c) the sequence $\langle s(x^*_m, \Gamma_n)\rangle_{n,m}$ is equi-HK-integrable and so by [26, p. 361], there is a gauge δ' such that if $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ is a δ'-fine partition of $[0,1]$, then for every $n, m \in \mathbb{N}$

$$\left|\sum_{i=1}^{p} s(x^*_m, \Gamma_n(t_i))\chi_{N^c}(t_i)|I_i| - (HK)\int_0^1 s(x^*_m, \Gamma_n(t))\,dt\right| < \varepsilon.$$

Moreover, it follows from (a) that for each m there is $n_m \in \mathbb{N}$ such that for every $n \ge n_m$

$$\left|\sum_{i=1}^{p} s(x^*_m, \Gamma_n(t_i))\chi_{N^c}(t_i)|I_i| - \sum_{i=1}^{p} s(x^*_m, \Gamma(t_i))\chi_{N^c}(t_i)|I_i|\right| < \varepsilon.$$

By (c), for each $m \in \mathbb{N}$ the sequence $\langle s(x^*_m, \Gamma_n)\rangle_n$ is equi-HK-integrable and by (b) it is pointwise bounded. Then, by [14, Theorem 4], there is $k_m \ge n_m$ such that for every $n \ge k_m$

$$\left|(HK)\int_0^1 s(x^*_m, \Gamma_n(t))\,dt - (HK)\int_0^1 s(x^*_m, \Gamma(t))\,dt\right| < \varepsilon.$$

Consequently, for each m

$$\begin{aligned}\left|\sum_{i=1}^{p} s(x_m^*, \Gamma(t_i))\chi_{N^c}(t_i)|I_i| - (HK)\int_0^1 s(x_m^*, \Gamma(t))\,dt\right| &\le \left|\sum_{i=1}^{p} s(x_m^*, \Gamma(t_i))\chi_{N^c}(t_i)|I_i| - \sum_{i=1}^{p} s(x_m^*, \Gamma_{k_m}(t_i))\chi_{N^c}(t_i)|I_i|\right| \\ &+ \left|\sum_{i=1}^{p} s(x_m^*, \Gamma_{k_m}(t_i))\chi_{N^c}(t_i)|I_i| - (HK)\int_0^1 s(x_m^*, \Gamma_{k_m}(t))\,dt\right| \\ &+ \left|(HK)\int_0^1 s(x_m^*, \Gamma_{k_m}(t))\,dt - (HK)\int_0^1 s(x_m^*, \Gamma(t))\,dt\right| < 3\varepsilon.\end{aligned}$$

But according to [14, Lemma 1] there is a gauge δ'' on N such that if $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ is a δ''-fine partition anchored in N, then $\sup_m \sum_{i=1}^{p} |s(x_m^*, \Gamma(t_i))|\,|I_i| < \varepsilon$. So if $\delta(t) := \delta'(t)\chi_{N^c}(t) + \delta''(t)\chi_N(t)$, then for each partition $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ of $[0, 1]$ that is δ-fine, we have

$$\left|\sum_{i=1}^{p} s(x_m^*, \Gamma(t_i))|I_i| - (HK)\int_0^1 s(x_m^*, \Gamma(t))\,dt\right| < 4\varepsilon,$$

what proves the equi-integrability of the sequence $\langle s(x_m^*, \Gamma)\rangle_m$.

It order to prove the HKP-integrability of Γ we need to show yet the existence of a weakly compactly generated space $Y \subset X$ such that $s(x^*, \Gamma) = 0$ a.e., if $x^* \in Y^{\perp}$. But as each Γ_n is a $ck(X)$ valued multifunction HKP-integrable in $cwk(X)$, there is a weakly compact set $W_n \subset B(X)$ such that $s(x^*, \Gamma_n) = 0$ a.e. if $x^* \in Y_n^{\perp}$, where Y_n is the Banach space generated by W_n. Consequently, if $W = \bigcup_{n=1}^{\infty} 2^{-n} W_n$, then W is weakly compact and the Banach space Y generated by W has the required property.

Since each sequence $\langle s(x^*, \Gamma_n)\rangle_n$ is equi-integrable and pointwise bounded applying once again [14, Theorem 4] we get

$$\lim_n \|s(x^*, \Gamma_n) - s(x^*, \Gamma)\|_A = 0 \quad \text{for each } x^* \in X^*. \quad \square$$

Remark 4.5. A more careful analysis of the above proof shows that one can weaken the condition (c) to the following form: each infinite set $\{s(x_{m_k}^*, \Gamma_{m_k}) : k \in \mathbb{N}, \|x_{m_k}^*\| \le 1\}$ contains an infinite subset $\{s(x_{m_{k_l}}^*, \Gamma_{m_{k_l}}) : l \in \mathbb{N}, \|x_{m_{k_l}}^*\| \le 1\}$ that is equi-HK-integrable.

5. Convergence theorems for the Henstock integral

One may ask whether the convergence in the equality (11) may be uniform on $B(X^*)$. We will show now that it is possible but under stronger assumptions on the convergence and integrability.

We recall first that if A and B are nonempty subsets of X, the *Hausdorff distance of* A and B is defined by $d_H(A, B) := \max\{e(A, B), e(B, A)\}$ where $e(A, B) = \sup\{d(x, B) : x \in A\}$ and $d(x, B) = \inf\{\|x - y\| : y \in B\}$.

Definition 5.1. $\Gamma : [0, 1] \to cb(X)$ is said to be Henstock integrable if there exists a set $\Phi_\Gamma[0, 1] \in cb(X)$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[0, 1]$ such that for each δ-fine partition $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ of $[0, 1]$, we have

$$d_H\left(\Phi_\Gamma[0, 1], \sum_{i=1}^{p} \Gamma(t_i)|I_i|\right) < \varepsilon.$$

We say that Γ is Henstock integrable on $I \in \mathcal{I}$ if $\Gamma\chi_I$ is Henstock integrable. $\square$

It is easily seen from the definition and the completeness of the Hausdorff metric that $cwk(X)[ck(X)]$-valued Henstock integrable multifunctions are integrable in $cwk(X)[ck(X)]$.

According to Hörmander's equality (cf. [27, p. 9]) we have the equality

$$d_H\left(K, \sum_{i=1}^{p} \Gamma(t_i)|I_i|\right) = \sup_{\|x^*\| \le 1} \left|s(x^*, K) - \sum_{i=1}^{p} s(x^*, \Gamma(t_i))|I_i|\right| \tag{12}$$

for each δ-fine partition $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ of $[0, 1]$ and set $K \in cb(X)$.

Hörmander's equality allows us to reduce the Henstock integrability of multifunctions to the Henstock integrability of functions by embedding the families $ck(X)$, $cwk(X)$ and $cb(X)$ into the Banach space $l_\infty(B(X^*))$. This standard approach works properly for the Debreu, Birkhoff and McShane integrals (cf. [13,10,3], respectively). In case of the Pettis integral that method not always can be applied but in many cases it is also useful (cf. [33]).

More precisely, let $j : cb(X) \to l_\infty(B(X^*))$ be the mapping defined by $j(K)(x^*) = s(x^*, K)$. Then $j(cb(X)), j(ck(X))$ and $j(cwk(X))$ are closed cones of $l_\infty(B(X^*))$ (see [11, Theorem II.19]). Therefore a multifunction $\Gamma : [0, 1] \to cb(X)$ is Henstock integrable if and only if the single valued function $j \circ \Gamma : [0, 1] \to l_\infty(B(X^*))$ is Henstock integrable in the usual sense. The key point is that $j(cb(X)), j(ck(X))$ and $j(cwk(X))$ are closed cones. Consequently, if $z \in l_\infty(B(X^*))$ is the value of the Henstock integral of $j \circ \Gamma$, then there exists a set $K \in cb(X)$ with $j(K) = z$.

As an immediate corollary we obtain

Theorem 5.2. *A scalarly HK-integrable multifunction $\Gamma : [0, 1] \to cb(X)$ is Henstock integrable if and only if $\mathcal{Z}_\Gamma$ is equi-HK-integrable.*

And as a direct consequence of the convergence theorem for Henstock integrable functions in [14, Theorem 4], we get the following convergence result.

Theorem 5.3. *Let $\Gamma_n : [0, 1] \to cb(X)$, $n \in \mathbb{N}$, be a sequence of multifunctions that are Henstock integrable and let $\Gamma : [0, 1] \to cb(X)$ be a multifunction. Assume that the following conditions are satisfied:*

(a) $\lim_n d_H(\Gamma_n, \Gamma) = 0$ *a.e.;*
(b) *for each $x^* \in X^*$ the sequence $\langle s(x^*, \Gamma_n)\rangle_n$ is pointwise bounded;*
(c) $\bigcup_n \mathcal{Z}_{\Gamma_n}$ *is equi-HK-integrable.*

Then Γ is Henstock integrable and

$$\lim_n d_A(\Gamma_n, \Gamma) = \lim_n \sup_{x^* \in B(X^*)} \|s(x^*, \Gamma_n) - s(x^*, \Gamma)\|_A = 0. \tag{13}$$

Due to the Banach–Steinhaus theorem the condition (b) *is equivalent to*

(b′) $\forall\, t \in [0, 1]\ \sup_n |\Gamma_n(t)| < \infty$.

Remark 5.4. It is worthwhile to recall that a function $f : [0, 1] \to \mathbb{R}$ is Denjoy–Perron integrable on [0, 1] if there exists an $ACG^*_{(s)}$ function $F : [0, 1] \to \mathbb{R}$ such that $F' = f$ a.e. We refer to [5] for the definition of $ACG^*_{(s)}$ function. By a classical result, a function $f : [0, 1] \to \mathbb{R}$ is Denjoy–Perron integrable on [0, 1] if and only if it is HK-integrable on [0, 1] (see [26, Theorem 11.3 and 11.4]). See also [5,7] for equivalent definitions of ACG conditions. So if in the definition of HKP-integral we replace the HK-integrability for real functions with the Denjoy–Perron integrability, we obtain an integral equivalent to the HKP one. For sake of simplicity we call HKP-integral also this Denjoy extension of the Pettis integral (see also the definition of HKP integral in [34]).

It is well known that a sequence of real valued pointwise convergent functions is equi-HK-integrable if and only if the sequence of their primitives is uniformly $ACG^*_{(s)}$ (see e.g. [28,5]). So taking into account such an equivalence, by Theorem 5.3 we can formulate also the following result:

Theorem 5.5. *Let $\Gamma_n : [0, 1] \to cb(X)$, $n \in \mathbb{N}$, be a sequence of multifunctions that are Henstock integrable and let $\Gamma : [0, 1] \to cb(X)$ be a multifunction. Assume that the following conditions are satisfied:*

(a) $\lim_n d_H(\Gamma_n, \Gamma) = 0$ *everywhere in* [0, 1]*;*
(b) *the set $\{s(x^*, \Phi_n) : \|x^*\| \le 1, n \in \mathbb{N}\}$ is uniformly $ACG^*_{(s)}$, where Φ_n's are H-primitives of Γ_n's.*

Then Γ is Henstock integrable and

$$\lim_n d_A(\Gamma_n, \Gamma) = \lim_n \sup_{x^* \in B(X^*)} \|s(x^*, \Gamma_n) - s(x^*, \Gamma)\|_A = 0. \tag{14}$$

Acknowledgments

The authors are grateful to the anonymous reviewers for their valuable remarks.

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