



Gauge integrals and selections of weakly compact valued multifunctions



Domenico Candeloro^{a,*}, Luisa Di Piazza^b, Kazimierz Musiał^c,
Anna Rita Sambucini^a

^a Department of Mathematics and Computer Sciences, 06123 Perugia, Italy

^b Department of Mathematics, University of Palermo, Via Archirafi 34, 90123 Palermo, Italy

^c Institut of Mathematics, Wrocław University, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

ARTICLE INFO

Article history:

Received 22 August 2015

Available online 7 April 2016

Submitted by B. Cascales

Keywords:

Multifunction

Set-valued Pettis, Henstock and

McShane integrals

Selection

ABSTRACT

In the paper Henstock, McShane, Birkhoff and variationally multivalued integrals are studied for multifunctions taking values in the hyperspace of convex and weakly compact subsets of a general Banach space X . In particular the existence of selections integrable in the same sense of the corresponding multifunctions has been considered.

© 2016 Published by Elsevier Inc.

1. Introduction

After the pioneering works of Aumann and Debreu, the theory of multivalued functions has been intensively studied and several notions of integral have been developed using different techniques, see for example [48,40,41,12,3,24,14–16,25,43,4,5,26,27,20,6,10,37]. These notions have shown to be useful when modeling some theories in different fields as optimal control and mathematical economics, see for example [36,2,18,30,31]. The choice to deal with these types of integration is motivated by the fact that a very important tool in this framework is the Kuratowski and Ryll-Nardzewski theorem which guarantees the existence of measurable selectors, though this famous result requires the separability of the range space X .

The starting point of this research are the papers [14–16,43] in which this result was extended to the non-separable Banach spaces for the Pettis multivalued integral and the papers [27,25,26,24] where the existence of selections integrable in the same sense of the corresponding multifunctions has been considered for some gauge integrals in the hyperspace $cwk(X)$ ($ck(X)$) of convex and weakly compact (compact) subsets of a general Banach space X and [6,8,9] in which these arguments are studied in suitable Banach lattices.

* Corresponding author at: Department of Mathematics and Computer Sciences, 1, Via Vanvitelli, 06123 Perugia, Italy.

E-mail addresses: domenico.candeloro@unipg.it (D. Candeloro), luisa.dipiazza@unipa.it (L. Di Piazza), musial@math.uni.wroc.pl (K. Musiał), anna.sambucini@unipg.it (A.R. Sambucini).

In the present paper selection results are obtained for Henstock, McShane, Birkhoff and variational integrals of $ckw(X)$ -valued multifunctions. Also a number of examples are given when this is not possible.

The paper is organized as follows: in section 2 the Henstock, McShane, Birkhoff and variationally multi-valued integrals are considered for multifunctions taking values in $ckw(X)$, and an embedding result, which will be useful later, is recalled. Moreover some other properties of variational integrals are proved, as for example their Bochner measurability.

Section 3 is more specifically devoted to selections theorems: results concerning McShane (resp. Henstock) integrability of all selectors of a McShane (resp. Henstock) integrable multifunction are given, at least in the separable case, and examples have been given to show that in non-separable spaces the results can fail.

In case of a variationally Henstock (resp. McShane) integrable $ckw(X)$ -valued multifunction, the problem of existence of at least one variationally Henstock (resp. McShane) selection has been investigated. Full solution has been given for Banach spaces with the Radon–Nikodým Property. In case of a general space X we present a solution for the variational McShane integral. It remains an open problem if, in a general Banach space X , any $ckw(X)$ -valued variationally Henstock-integrable multifunction has a variationally Henstock-integrable selection. Results with positive answers are given in some particular cases.

In section 4, thanks to the structure of near vector space of $ckw(X)$, it is shown that the variationally Henstock integrability of F , with $0 \in F$, implies its Birkhoff integrability and this result is used in order to obtain a decomposition extending the one given in [27] for compact convex valued multifunctions. For scalarly measurable multifunctions some implications are also given, connecting Henstock, McShane and Pettis multivalued integrability to one another. Moreover an example of a variationally Henstock but not variationally McShane integrable multifunction is given.

2. Preliminary facts

Throughout $[0, 1]$ is the unit interval of the real line equipped with the usual topology and Lebesgue measure λ , \mathcal{L} denotes the family of all Lebesgue measurable subsets of $[0, 1]$, and \mathcal{I} is the collection of all closed subintervals of $[0, 1]$. If $I \in \mathcal{I}$ then $|I|$ denotes its length.

X is an arbitrary Banach space with its dual X^* . The closed unit ball of X^* is denoted by $B(X^*)$. $ckw(X)$ is the family of all non-empty convex weakly compact subsets of X and $ck(X)$ is the family of all compact members of $ckw(X)$. We consider on $ckw(X)$ the Minkowski addition ($A + B := \{a + b : a \in A, b \in B\}$) and the standard multiplication by scalars. $\|A\| := \sup\{\|x\| : x \in A\}$. d_H is the Hausdorff distance in $ckw(X)$. $ckw(X)$ with the Hausdorff distance is a complete metric space.

For every $C \in ckw(X)$ the *support function* of C is denoted by $s(\cdot, C)$ and defined on X^* by $s(x^*, C) = \sup\{\langle x^*, x \rangle : x \in C\}$, for each $x^* \in X^*$. For all unexplained definitions we refer to [17].

A map $\Gamma : [0, 1] \rightarrow 2^X \setminus \{\emptyset\}$ (= non-empty subsets of X) is called a *multifunction*. A multifunction $\Gamma : [0, 1] \rightarrow 2^X \setminus \{\emptyset\}$ is said to be a *simple multifunction* if there exists a finite collection $\{E_1, \dots, E_p\}$ of measurable subsets of $[0, 1]$, pairwise disjoint, such that Γ is constant on each E_j .

A multifunction $\Gamma : [0, 1] \rightarrow ckw(X)$ is said to be *Effros measurable* (or simply *measurable*) if for each open $O \subset X$, the set $\{t \in [0, 1] : \Gamma(t) \cap O \neq \emptyset\}$ is measurable.

A multifunction $\Gamma : [0, 1] \rightarrow ckw(X)$ is said to be *scalarly measurable* if for every $x^* \in X^*$, the map $s(x^*, \Gamma(\cdot))$ is measurable. Γ is said to be *Bochner measurable* if there exists a sequence of simple multifunctions $\Gamma_n : [0, 1] \rightarrow ckw(X)$ such that

$$\lim_{n \rightarrow \infty} d_H(\Gamma_n(t), \Gamma(t)) = 0$$

for almost all $t \in [0, 1]$.

It is well known that the measurability of a $ck(X)$ -valued multifunction yields its scalar measurability (if X is separable also the reverse implication is true). Moreover, each Bochner measurable multifunction is also measurable (see [36]). The reverse implication fails (see Example 3.8).

A measurable multifunction Γ is said to be *Aumann integrable* if it admits at least one Bochner integrable selection. A function $f : [0, 1] \rightarrow X$ is called a *selection of Γ* if $f(t) \in \Gamma(t)$, for every $t \in [0, 1]$.

A *partition \mathcal{P} in $[0, 1]$* is a collection $\{(I_1, t_1), \dots, (I_p, t_p)\}$, where I_1, \dots, I_p are nonoverlapping subintervals of $[0, 1]$, t_i is a point of $[0, 1]$, $i = 1, \dots, p$. If $\cup_{i=1}^p I_i = [0, 1]$, then \mathcal{P} is a *partition of $[0, 1]$* . If t_i is a point of I_i , $i = 1, \dots, p$, we say that \mathcal{P} is a *Perron partition of $[0, 1]$* .

A *gauge* on $[0, 1]$ is a positive function on $[0, 1]$. For a given gauge δ on $[0, 1]$, we say that a partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ is δ -*fine* if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1, \dots, p$.

Given a function $g : [0, 1] \rightarrow X$ and a partition $\mathcal{P} = \{(I_1, t_1), \dots, (I_p, t_p)\}$ in $[0, 1]$ we set

$$\sigma(g, \mathcal{P}) = \sum_{i=1}^p |I_i|g(t_i).$$

Definition 2.1. A multifunction $\Gamma : [0, 1] \rightarrow ck(X)$ is said to be *Henstock* (resp. *McShane*) integrable on $[0, 1]$, if there exists a non-empty closed convex set $\Phi_\Gamma([0, 1]) \subset X$ such that for every $\varepsilon > 0$ there exists a gauge δ on $[0, 1]$ such that for each δ -fine Perron partition (resp. partition) $\{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[0, 1]$, we have

$$d_H \left(\Phi_\Gamma([0, 1]), \sum_{i=1}^p \Gamma(t_i)|I_i| \right) < \varepsilon. \tag{1}$$

A multifunction $\Gamma : [0, 1] \rightarrow ck(X)$ is said to be *Birkhoff* (resp. *abs-Birkhoff*) integrable on $[0, 1]$, if there exists a non-empty closed convex set $\Phi_\Gamma([0, 1]) \in ck(X)$ with the following property: for every $\varepsilon > 0$ there is a countable partition Π_0 of $[0, 1]$ in \mathcal{L} such that for every countable partition $\Pi = (A_n)_n$ of $[0, 1]$ in \mathcal{L} finer than Π_0 and any choice $T = (t_n)_n$ in A_n , the series $\sum_n \lambda(A_n)\Gamma(t_n)$ is unconditionally convergent (resp. absolutely convergent) and

$$d_H \left(\Phi_\Gamma([0, 1]), \sum_n \Gamma(t_n)\lambda(A_n) \right) < \varepsilon. \tag{2}$$

A multifunction $\Gamma : [0, 1] \rightarrow ck(X)$ is said to be *Henstock* (resp. *McShane or Birkhoff*) integrable on $I \in \mathcal{I}$ if $\Gamma|_I$ is respectively integrable on $[0, 1]$. We write then $(H) \int_I \Gamma dt := \Phi_{\Gamma|_I}([0, 1])$ (resp. $(MS) \int_I \Gamma dt := \Phi_{\Gamma|_I}([0, 1])$ or $(Bi) \int_I \Gamma dt := \Phi_{\Gamma|_I}([0, 1])$). It is known that a multifunction that is Henstock (McShane, Birkhoff) integrable on $[0, 1]$ is in the same manner integrable on each $I \in \mathcal{I}$ (see e.g. [27]).

It is easily seen from the definition and the completeness of the Hausdorff metric that $ck(X)$ ($ck(X)$)-valued integrable multifunctions have compact (weakly compact) values of their integrals.

Moreover we would like to recall that a vector function $g : [0, 1] \rightarrow X$ is Birkhoff-integrable if it is McShane integrable, but just measurable gauges are involved in the notion of McShane integrability. More precisely

Definition 2.2. $g : [0, 1] \rightarrow X$ is Birkhoff-integrable if and only if there exists an element $y \in X$ such that for each $\varepsilon > 0$ a *measurable* gauge δ can be found on $[0, 1]$, such that, as soon as $\mathcal{P} = \{(t_j, I_j) : j = 1, \dots, n\}$ is any δ -fine partition of $[0, 1]$, it holds $\|\sigma(g, \mathcal{P}) - y\| \leq \varepsilon$.

(For the equivalence of this definition with the more common notion of Birkhoff integrability see [46,10, 13].)

Definition 2.3. A multifunction $\Gamma : [0, 1] \rightarrow cwk(X)$ is said to be *variationally Henstock (variationally McShane) integrable*, if there exists a finitely additive multifunction $\Phi_\Gamma : \mathcal{I} \rightarrow cwk(X)$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[0, 1]$ such that for each δ -fine Perron partition (partition) $\{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[0, 1]$, we have

$$\sum_{j=1}^p d_H(\Phi_\Gamma(I_j), \Gamma(t_j)|I_j|) < \varepsilon. \tag{3}$$

We write then $(vH) \int_0^1 \Gamma dt := \Phi_\Gamma([0, 1])$ ($(vMS) \int_0^1 \Gamma dt := \Phi_\Gamma([0, 1])$). We call the set multifunction Φ_Γ the *variational Henstock (McShane) primitive* of Γ . The variational integrals on $I \in \mathcal{I}$ are defined in the same way as the ordinary ones. The integrals are uniquely determined.

We say that a multifunction Γ is *scalarly Henstock integrable* if, for every $x^* \in X^*$, the function $s(x^*, \Gamma(\cdot))$ is *Henstock integrable*.

It follows from the definitions that if Γ is McShane (variationally McShane) integrable, then it is also Henstock (variationally Henstock) integrable (with the same values of the integrals). There is a McShane integrable function $f : [0, 1] \rightarrow B(L_\infty[0, 1])$ that is not variationally McShane integrable [33, 3D].

When a multifunction is a function $f : [0, 1] \rightarrow X$, then the set $\Phi_f([0, 1])$ is reduced to a vector of X and the above definitions coincide with those corresponding for vector valued functions. For vector valued functions on $[0, 1]$ the variational McShane integrability is equivalent to the Bochner integrability [19]. This result has been generalized in [22] to the case of vector valued functions defined in a compact finite Radon measure space.

For the definitions of Pettis and of Henstock–Kurzweil–Pettis integral for multifunctions we refer the reader to [42,45,43,44,14,23,25].

In [38] a Rådström embedding theorem is extended to the space $cwk(X)$ as follows.

Theorem 2.4. (See [38, Theorem 5.6].) *There exist a compact Hausdorff space Ω and a map $i : cwk(X) \rightarrow C(\Omega)$ such that*

- 2.4.1) $i(\alpha A + \beta C) = \alpha i(A) + \beta i(C)$ for every $A, C \in cwk(X), \alpha, \beta \in \mathbb{R}^+$;
- 2.4.2) $d_H(A, C) = \|i(A) - i(C)\|_\infty, A, C \in cwk(X)$;
- 2.4.3) $i(cwk(X)) = \overline{i(cwk(X))}$ (norm closure);
- 2.4.4) $i(\overline{\text{co}}(A \cup C)) = \max\{i(A), i(C)\}$ for all A, C in $cwk(X)$;
- 2.4.5) If $0 \in A$, then $i(A) \geq 0$.

The embedding i allows to reduce the Henstock (resp. McShane) integrability of multifunctions to the Henstock (resp. McShane) integrability of functions by embedding the family $cwk(X)$ into the Banach space $C(\Omega)$. Since $i(cwk(X))$ is a closed cone of $C(\Omega)$, a multifunction $\Gamma : [0, 1] \rightarrow cwk(X)$ is Henstock or variationally Henstock (resp. McShane or variationally McShane) integrable if and only if the single valued function $i \circ \Gamma : [0, 1] \rightarrow C(\Omega)$ is Henstock or variationally Henstock (resp. McShane or variationally McShane) integrable in the usual sense. The key point is that $i(cwk(X))$ is a closed cone. Consequently, if $z \in C(\Omega)$ is the value of the integral of $i \circ \Gamma$, then there exists a set $K \in cwk(X)$ with $i(K) = z$.

Observe that it follows directly from the definitions that if $i : cwk(X) \rightarrow Y$ is the Rådström embedding into a Banach space, then a multifunction $\Gamma : [0, 1] \rightarrow cwk(X)$ is G-integrable if and only if $i(\Gamma)$ is G-integrable (G stands for any of the gauge integrals). In [12, Proposition 2.6, Corollary 2.7] the authors proved it for Birkhoff. Thanks to the embedding, the following result, similar to that for single-valued function, is now obvious:

Proposition 2.5. *Let $\Gamma : [0, 1] \rightarrow cwk(X)$ be a Birkhoff integrable multifunction. Then Γ is McShane integrable.*

Proof. It is enough to observe that Γ is McShane integrable if and only if $i(\Gamma)$ is McShane integrable. \square

Finally $\mathcal{S}_H(\Gamma)$ [$\mathcal{S}_{MS}(\Gamma), \mathcal{S}_P(\Gamma), \mathcal{S}_{HKP}(\Gamma), \mathcal{S}_B(\Gamma), \mathcal{S}_{vH}(\Gamma), \mathcal{S}_{vMS}(\Gamma)$] denotes the family of all scalarly measurable selections of Γ that are Henstock [McShane, Pettis, Henstock–Kurzweil–Pettis, Birkhoff, variationally Henstock, variationally McShane] integrable.

A useful tool to study the integrability of a single-valued function or of a multifunction is the variational measure associated to the primitive.

Definition 2.6. Given a finitely additive interval multimeasure $\Phi : \mathcal{I} \rightarrow cwk(X)$, a gauge δ and a set $E \subset [0, 1]$, we define

$$Var(\Phi, \delta, E) = \sup \sum_{j=1}^p \|\Phi(I_j)\|,$$

where the supremum is taken over all the δ -fine Perron partitions $\{(I_j, t_j)\}_{j=1}^p$ with $t_j \in E$ for $j = 1, \dots, p$. The set function

$$V_\Phi(E) := \inf_{\delta} \{Var(\Phi, \delta, E) : \delta \text{ is a gauge on } E\}$$

is called *the variational measure generated by Φ* . Moreover, we say that V_Φ is *absolutely continuous* with respect to λ and we write $V_\Phi \ll \lambda$ if for every $E \in \mathcal{L}$ with $\lambda(E) = 0$ we have $V_\Phi(E) = 0$.

In order to deduce more information from the integral of a multifunction, we state the following simple result.

Proposition 2.7. *Let $F : [0, 1] \rightarrow cwk(X)$ be any Henstock integrable mapping and f be a Henstock integrable selection of F . Then, for every interval $I \in \mathcal{I}$, one has*

$$(H) \int_I f dt \in (H) \int_I F dt \quad \text{and} \quad V_{\Phi_f}(I) \leq V_{\Phi_F}(I).$$

Proof. If $x^* \in X^*$, then $x^*f \leq s(x^*, F)$. Hence $(H) \int_I x^*f(t) dt \leq (H) \int_I s(x^*, F(t)) dt$. The Hahn–Banach theorem yields the required result. \square

Observe that the previous result holds also for McShane and Birkhoff integrable multifunctions: in these cases, moreover, the conclusion is valid not only for intervals, but also for arbitrary measurable subsets.

It is well known [11] that a variationally Henstock integrable vector valued function is Bochner measurable. Using the embedding of $cwk(X)$ into $C(\Omega)$ it is possible to prove a similar result for a multifunction.

Proposition 2.8. *Let $\Gamma : [0, 1] \rightarrow cwk(X)$ be variationally Henstock integrable. Then Γ is Bochner measurable.*

Proof. Since Γ is variationally-Henstock integrable, then also the single valued function $\gamma := i \circ \Gamma$ is so in $i(cwk(X))$. By [11, Theorem 9] $i \circ \Gamma$ is Bochner measurable.

We may assume that γ is separably valued. For each $n \in \mathbb{N}$ let $\{y_{n,k} : k \in \mathbb{N}\} \subset \gamma([0, 1])$ be a countable $1/n$ -net of $\gamma([0, 1])$. For each n, k let

$$\gamma'_n(t) = y_{n,k} \text{ if } t \in \gamma^{-1}(B(y_{n,k}, 1/n)).$$

Then $(\gamma'_n)_n$ is a sequence of countably valued functions that is uniformly converging to γ .

For each n let k_n be such that $\lambda\left(\bigcup_{i=1}^{k_n} \gamma^{-1}(B(y_{n,i}, 1/n))\right) > 1 - 1/n$. We define

$$\gamma_n(t) = \begin{cases} \gamma'_n(t) & t \in \bigcup_{i=1}^{k_n} \gamma^{-1}(B(y_{n,i}, 1/n)) \\ \text{a fixed point } y_n \in i(\text{ck}(X)) & \text{otherwise.} \end{cases}$$

Since the values of γ_n belong to $\gamma([0, 1]) = i \circ \Gamma([0, 1])$ and the embedding is one to one then each function γ_n defines in a unique way a simple multifunction $\Gamma_n : [0, 1] \rightarrow \text{ck}(X)$ such that $i \circ \Gamma_n(t) = \gamma_n(t)$ for every $t \in [0, 1]$. Clearly $d_H(\Gamma_n(t), \Gamma(t)) = \|\gamma_n(t) - \gamma(t)\| \rightarrow 0$ in λ -measure. Hence, there is a subsequence converging a.e. to Γ . In conclusion, Γ is the a.e. limit of a sequence of simple multifunctions, and therefore it is Bochner measurable. \square

3. Selections of $\text{ck}(X)$ -valued multifunctions

When X is an arbitrary Banach space, then it is well known that each scalarly measurable selection of a Pettis (resp. Henstock–Kurzweil–Pettis) integrable multifunction $\Gamma : [0, 1] \rightarrow \text{ck}(X)$ is also Pettis (resp. Henstock–Kurzweil–Pettis) integrable (see [15,43,25]).

For the Henstock integral the behavior is different. If X is separable and we consider $\text{ck}(X)$ -valued multifunctions, every measurable selection of a Henstock (resp. McShane) integrable multifunction is Henstock (resp. McShane) integrable (see [24]). This essentially depends on the fact that for functions taking values in a separable Banach space the Pettis and the McShane integrability coincide.

The answer is also affirmative if we consider $\text{ck}(X)$ -valued multifunctions taking values in any Banach space X with the property that the Pettis and the McShane integrability coincide, as the following proposition shows (see [29,51,1] and the bibliography inside, for the Banach spaces with such a property).

Proposition 3.1. *Let X be a Banach space with the property that the Pettis and the McShane integrability coincide. Then for any Henstock (resp. McShane) integrable multifunction $\Gamma : [0, 1] \rightarrow \text{ck}(X)$, every scalarly measurable selection of Γ is Henstock (resp. McShane) integrable.*

Proof. Let $\Gamma : [0, 1] \rightarrow \text{ck}(X)$ be Henstock integrable. According to [27, Theorem 3.1] there exists $f \in \mathcal{S}_H(\Gamma)$ and so by [25, Theorem 1] $\Gamma(t) = G(t) + f(t)$, where $G : [0, 1] \rightarrow \text{ck}(X)$ is a Pettis integrable multifunction. Now let $h : [0, 1] \rightarrow X$ be any scalarly measurable selection of Γ . So $h = g + f$, where g is a scalarly measurable selection of G . Since the multifunction G is Pettis integrable, g is also Pettis integrable (see [15] or [43]). Then by the hypothesis g is also McShane (and then Henstock) integrable. This gives that h as a sum of two Henstock integrable functions, is Henstock integrable. In case the multifunction Γ is McShane integrable, then it is enough to consider the decomposition with a McShane integrable selection of Γ (again, by [27, Theorem 3.1] one has $\mathcal{S}_{MS}(\Gamma) \neq \emptyset$). \square

If X is a general Banach space, and the multifunction is $\text{ck}(X)$ valued, the previous assertion is false as we are showing in the next proposition. To do it we use an example given in [1, Theorem 3.7] under ZFC, of a scalarly negligible function, which is not McShane integrable. We recall that a family \mathcal{F} of finite subsets of $[0, 1]$ is said to be *MC-filling* on $[0, 1]$ if it is hereditary (i.e. if $G \in \mathcal{F}$ whenever $G \subset F \in \mathcal{F}$) and there exists $\varepsilon > 0$ such that for every countable family (A_i) of disjoint sets whose union is $[0, 1]$, there is $F \in \mathcal{F}$ such that $\lambda^*(\cup\{A_i : F \cap A_i \neq \emptyset\}) > \varepsilon$, where λ^* is the outer Lebesgue measure.

Proposition 3.2. *There exist a reflexive Banach space Y and a variationally McShane integrable multifunction $\Gamma : [0, 1] \rightarrow \text{cwk}(Y)$ such that $0 \in \Gamma(t)$ for each $t \in [0, 1]$ and Γ possesses a scalarly measurable selection that fails to be Henstock integrable.*

Proof. Let \mathcal{F} be a compact MC-filling family on $[0, 1]$, existing by [1, Theorem 3.5] and let X be equal to $C(\mathcal{F})$ (the space of continuous functions on \mathcal{F}). The function $f : [0, 1] \rightarrow C(\mathcal{F})$ defined by

$$f(t)(F) = 1_F(t), \text{ for } t \in [0, 1] \text{ and } F \in \mathcal{F}$$

is scalarly negligible and not McShane integrable. Moreover, there exists a reflexive Banach space Y , a one-to-one linear continuous mapping $T : Y \rightarrow C(\mathcal{F})$ and a function $g : [0, 1] \rightarrow Y$ such that: $f([0, 1]) \subset T(B(Y))$ and $T \circ g = f$. So also the function g is scalarly negligible and not McShane (and then not Henstock, due to [32]) integrable and $g([0, 1]) \subset B(Y)$. Since Y is a reflexive Banach space, the unit ball $B(Y)$ is a convex weakly compact set of Y , not norm compact. Now let us consider the constant multifunction $\Gamma : [0, 1] \rightarrow \text{cwk}(Y)$ defined by

$$\Gamma(t) := B(Y), \text{ for every } t \in [0, 1].$$

Clearly Γ is variationally McShane integrable and g is a scalarly measurable selection of Γ which fails to be Henstock integrable. \square

Proposition 3.3. *Each variationally Henstock integrable multifunction $\Gamma : [0, 1] \rightarrow \text{cwk}(X)$ possesses a strongly measurable selection.*

Proof. By Proposition 2.8 we know that Γ is Bochner measurable. We get the thesis applying [35, Theorem 2.9] which states that: in a metric space any Bochner measurable multifunction taking convex closed bounded values has strongly measurable selections. In [35] the metric is supposed to be bounded but the result is available without this assumption (see [34, Remark 3.7]). \square

A similar selection theorem can be stated for Birkhoff integrable mappings.

Theorem 3.4. *Let $\Gamma : [0, 1] \rightarrow \text{cwk}(X)$ be any Birkhoff integrable multifunction. Then, there exists at least one Birkhoff integrable selection.*

Proof. Since $\gamma := i \circ \Gamma$ is Birkhoff integrable, by [46, Remark 1] then γ is McShane integrable with a measurable gauge, namely there exists an element $H \in \text{cwk}(X)$ such that, for each $\varepsilon > 0$ a measurable gauge δ can be found, such that, as soon as $\mathcal{P} = \{(t_j, I_j) : j = 1, \dots, n\}$ is any δ -fine partition of $[0, 1]$, it holds

$$\|\sigma(\gamma, \mathcal{P}) - i(H)\|_\infty \leq \varepsilon$$

Now, since Γ is McShane integrable, by [27, Theorem 3.1], it admits a McShane-integrable selection g . It only remains to prove that, in the condition of McShane integrability of g , measurable gauges are involved. To this aim it is enough to repeat almost verbatim the proof of [27, Theorem 3.1], taking into account that, for each x^* in the dual space, the real-valued map x^*g is Lebesgue integrable, hence also Birkhoff integrable again using [46, Remark 1]. \square

Remark 3.5. The example given in Proposition 3.2 shows that in general not all scalarly measurable selections of a Birkhoff integrable multifunction are Birkhoff integrable.

If X is separable, then thanks to Pettis measurability Theorem, the answer is contained in [12, Proposition 3.1(ii)]. In fact in this case Γ admits strongly measurable selections each of them being Birkhoff integrable, and for every measurable set A it is

$$(B) \int_A \Gamma dt = \left\{ \int_A f dt, \quad f \in \mathcal{S}_B(\Gamma) \right\}.$$

In non-separable case we have that if $\Gamma : [0, 1] \rightarrow cwk(X)$ is Birkhoff integrable, then Γ is Pettis integrable by Proposition 2.5. So each scalarly measurable selection of Γ is Pettis integrable. Birkhoff integrability can be obtained if the range of Γ is separable, or if the selection is strongly measurable, thanks to [47, Corollary 5.11].

We recall that a multifunction $\Gamma : [0, 1] \rightarrow cwk(X)$ is said to be *integrably bounded* if there is a scalar valued function $h \in L_1[0, 1]$ such that $\|\Gamma(t)\| \leq |h(t)|$ for almost all $t \in [0, 1]$.

Proposition 3.6. *Let $\Gamma : [0, 1] \rightarrow cwk(X)$ be a scalarly measurable multifunction. Then the following conditions are equivalent:*

- (i) Γ is variationally McShane integrable;
- (ii) $i(\Gamma) \in L_1(\lambda, Y)$;
- (iii) Γ is Bochner measurable and integrably bounded.

Proof. The equivalence of (i) and (ii) follows from [22], where it has been proven that a Banach space valued function is variationally McShane integrable if and only if it is Bochner integrable. Consequently, Γ is variationally McShane integrable if and only if $i(\Gamma) \in L_1(\lambda, Y)$. The equality $\|i \circ \Gamma(t)\| = \|\Gamma(t)\|$ and the fact that $i(\Gamma)$ is Bochner measurable if and only if Γ is such, yield the equivalence of (ii) and (iii). \square

Theorem 3.7. *Each strongly measurable selection of a variationally McShane integrable multifunction $\Gamma : [0, 1] \rightarrow cwk(X)$ is vMS-integrable (= Bochner integrable). In particular, if X is separable, then each scalarly measurable selection of Γ is Bochner integrable. Consequently, Γ is Aumann integrable and the integrals coincide.*

Proof. That follows at once from Propositions 3.6 and 3.3 and the inequality $\|f(t)\| \leq \|\Gamma(t)\|$ that is valid for any selection f of Γ . \square

We would like to observe that in general the Aumann integrability of a measurable $cwk(X)$ valued multifunction does not imply its variational McShane integrability. In fact it is enough to consider Example 4.7: the multifunction G is not variationally McShane integrable, but is Aumann integrable, since the null function is a selection. Moreover in [3, Example 1] another example is given which is useful for this purpose. The example is reported here for the sake of completeness and compared with the variational McShane integrability:

Example 3.8. Let $X = l^2(\mathbb{N})$; for every $A \subset \mathbb{N}$ we consider

$$U_A = \{x \in X : \|x\| \leq 1, \text{ and } x_n = 0 \text{ if } n \notin A\} = \{1_A x : \|x\| \leq 1\},$$

where $(1_A x)_n = 1_A(n)x_n$. If $A \neq B$ then $d_H(U_A, U_B) \geq 1$ and so the set $\{U_A, A \subset \mathbb{N}\}$ is not separable.

Let $\Omega = [0, 1[$ and for every $\omega \in \Omega$ let $0, \omega_1 \cdots \omega_n \cdots$ be its dyadic representation, namely $\omega_1 = 1$ iff $\omega \in [1/2, 1[$, $\omega_2 = 1$ iff $\omega \in [1/4, 1/2[\cup [3/4, 1[$, etc. We set $B_1 = [1/2, 1[$, $B_2 = [1/4, 1/2[\cup [3/4, 1[$, etc.

Let $F(\omega) = U_{A(\omega)}$ where $A(\omega) = \{n \in \mathbb{N} : \omega_n = 1\}$. F is integrably bounded, takes weakly compact and convex values and its support function $s(y, F(\omega))$ is measurable since it is the limit of simple functions. In particular all scalarly measurable selections are Bochner integrable.

From [36, Proposition II.2.39] F is Effros measurable, but for every λ -null set N , the set $F(\Omega \setminus N)$ is not separable in the d_H -metric topology. Then immediately it follows that F is not Bochner measurable and then it cannot be variationally McShane integrable by Proposition 3.6.

Theorem 3.9. *If $\Gamma : [0, 1] \rightarrow ck(X)$ is Bochner measurable and Henstock integrable ($\Gamma : [0, 1] \rightarrow cwk(X)$ is variationally McShane integrable), then each scalarly measurable selection of Γ is Henstock (variationally McShane) integrable. Moreover in the second case Γ is Aumann integrable and the two integrals coincide.*

Proof. By [31, Theorem 3.6] the range of Γ is almost separably valued in X . Assume for simplicity that \tilde{X} is a separable subspace of X containing the range of Γ . It follows from [24, Theorem 2.(iv)] that all scalarly measurable selections of Γ are Henstock integrable. In case of variationally McShane integrable Γ , we apply Theorem 3.7 to \tilde{X} . \square

Proposition 3.10. *If $\Gamma : [0, 1] \rightarrow cwk(X)$ is Birkhoff integrable and its range is almost separably valued then each scalarly measurable selection of Γ is Birkhoff integrable, and then also McShane and Henstock integrable.*

Proof. Obviously Γ is Pettis integrable and then, by [15, Corollary 2.3], each scalarly measurable selection f of Γ is Pettis integrable. Since the range of Γ is almost separably valued the same occurs for f , so f is McShane integrable. Since f is scalarly measurable and its range is separable, by [46, Corollary 3], it is Bochner measurable. But then f satisfies the hypotheses of [46, Theorem 7] and so it is Birkhoff integrable (see also the Remark 3.5). \square

The previous results, and in particular Theorem 3.7, give raise to the following problem.

Question 3.11. Does there exist at least one variationally Henstock integrable selection of a $cwk(X)$ -valued variationally Henstock integrable multifunction?

If X is an arbitrary Banach space, we do not know the answer to the previous question. But for Banach spaces possessing the Radon–Nikodým property there exist variationally Henstock integrable selections of $cwk(X)$ -valued variationally Henstock integrable multifunctions, as it is stated in the next theorem.

To present a proof, we need the notion of the variational measure associated to the primitive given in Definition 2.6.

Theorem 3.12. *Let X be a Banach space with the Radon–Nikodým property and let $\Gamma : [0, 1] \rightarrow cwk(X)$ be a variationally Henstock integrable multifunction. Then every strongly measurable selection of Γ is variationally Henstock integrable.*

Proof. Thanks to Proposition 3.3 we know that there exist strongly measurable selections of Γ . Let f be one of such selections. Since Γ is variationally Henstock integrable, it is also Henstock–Kurzweil–Pettis integrable. Therefore by [26, Proposition 1.5] $f \in \mathcal{S}_{HKP}(\Gamma)$.

Let us denote by F its Henstock–Kurzweil–Pettis primitive and by Φ_Γ the Henstock primitive of Γ . By [49, Proposition 3.3.1] we have $V_{\Phi_\Gamma} \ll \lambda$. Hence also $V_F \ll \lambda$. Since X has the Radon–Nikodým property, by [7, Theorem 3.6] we infer that F is the variational Henstock primitive of f . Therefore f is variationally Henstock integrable. \square

Comparing the previous theorem with [Proposition 3.2](#), we see that the condition of strong measurability of the selection cannot be relaxed. Indeed, in the example given in [3.2](#) X is reflexive, and it is shown that a constant $cwk(X)$ -valued multifunction has a scalarly measurable selection that is not even Henstock-integrable.

Another partial answer is the following:

Theorem 3.13. *If $\Gamma : [0, 1] \rightarrow cwk(X)$ is a bounded multifunction which is variationally Henstock and Pettis integrable, then $\mathcal{S}_{vMS}(\Gamma) \neq \emptyset$.*

Proof. Thanks to [Proposition 3.3](#) the multifunction Γ admits a strongly measurable selection f which is Pettis integrable by [\[15, Corollary 2.3\]](#). Let μ_f be its indefinite Pettis integral. Then $R(\mu_f)$, the range of μ_f , is relatively norm compact (classical result) and the variation $|\mu_f|$ of μ_f is of σ -finite variation (cf. [\[42, Theorem 4.1\]](#)).

Now, since f is a selection of Γ which is bounded, then there exists $M > 0$ such that for every $E \in \mathcal{L}$ it is $|\mu_f|(E) \leq M\lambda(E)$, so μ_f is moderated. Then thanks to [\[22, Lemma 2\]](#) f is variationally McShane integrable. \square

This result could be extended to multifunctions F of the following type:

$$\Gamma : [0, 1] \rightarrow cwk(X) : \exists M_n \in \mathbb{R} \text{ and } O_n = O_n^\circ \text{ such that } V_\Gamma(\cap_n O_n^c) = 0 \text{ and } \|\Gamma(t) \cdot 1_{O_n}(t)\| \leq M_n.$$

These assumptions cannot be further weakened since in [\[50, Example 1\]](#) an example of a Birkhoff and strongly measurable function is given, whose variational indefinite integral is not moderated on any open interval.

It is well known that if Γ is a $cwk(X)$ valued multifunction defined on a complete probability space (resp. $[0, 1]$), and all its scalarly measurable selections are Pettis integrable (resp. Henstock–Kurzweil–Pettis integrable), then Γ is Pettis integrable (resp. Henstock–Kurzweil–Pettis integrable) ([\[15\]](#) or [\[43\]](#) (resp. [\[25\]](#))). A similar result is valid in case of a separable X and a $ck(X)$ -valued Henstock integrable multifunction ([\[24, Theorem 2\]](#)).

The following proposition shows that the above assertion remains true also for the McShane integral.

Proposition 3.14. *Let X be a separable Banach space and let $\Gamma : [0, 1] \rightarrow ck(X)$ be a Bochner measurable multifunction.*

- (i) *If all measurable selections of Γ are McShane integrable, then Γ is McShane integrable.*
- (ii) *If all measurable selections of Γ are Birkhoff integrable, then Γ is Birkhoff integrable.*

Proof. Since all measurable selections of Γ are McShane integrable, they are also Pettis and Henstock integrable. So by [\[15, Theorem 4.2\]](#) and [\[16, Theorem 3.8\]](#), Γ is Pettis integrable, and by [\[24, Theorem 2\]](#), Γ is Henstock integrable. An application of [\[27, Theorem 3.4\]](#) gives us the McShane integrability of Γ . The second statement follows from [3.14.i\)](#) and [\[4, Corollary 4.2\]](#). \square

Remark 3.15. We would like to point out that even in the case of separable Banach spaces, the assertion of [Proposition 3.14](#) is false if we consider $cwk(X)$ valued multifunctions. In fact Γ may fail to be even Henstock integrable. Indeed let X be a separable Banach space without the Schur property. Then, following the proof of [\[14, Theorem 2.1\]](#) it is possible to construct a Pettis integrable multifunction $\Gamma : [0, 1] \rightarrow cwk(X)$, such that $i \circ \Gamma$ is not scalarly measurable. Since Γ is Pettis integrable, then (see [\[15\]](#) or [\[43\]](#)), each measurable selection of Γ is Pettis, so McShane integrable and then Henstock integrable. But Γ cannot be Henstock

integrable (hence neither McShane). In fact in such a case also $i \circ \Gamma$ would be Henstock integrable, whereas $i \circ \Gamma$ fails to be scalarly measurable.

4. Variationally Henstock and McShane integrability of $ckw(X)$ -valued multifunctions

Some results concerning variationally Henstock and McShane integrable multifunctions are collected here.

Proposition 4.1. *Let $G : [0, 1] \rightarrow cwk(X)$ be variationally Henstock integrable. If $0 \in G(t)$ a.e., then G is Birkhoff integrable.*

Proof. Let i be the embedding of $ckw(X)$ into $C(\Omega)$ of Theorem 2.4. Then we just have to prove that $i(G)$ is Birkhoff integrable.

To this aim, we observe that $i(G)$ is variationally Henstock integrable, hence Bochner-measurable, thanks to Proposition 2.8. This implies that $i(G)$ is also Riemann-measurable, according with [46, Theorem 1]. So, in order to prove that $i(G)$ is Birkhoff integrable, it is enough to show that it is McShane-integrable, thanks to [46, Theorem 7]. Since $0 \in G(t)$ one has that $i(G(t))$ is non-negative for almost all $t \in [0, 1]$, and $i(G)$ is variationally Henstock integrable.

Then, thanks to [32, Corollary 9 (iii)], it will be sufficient to prove convergence in $C(\Omega)$ of all series of the type $\sum_n (vH) \int_{I_n} i(G)$, where $(I_n)_n$ is any sequence of pairwise non-overlapping subintervals of $[0, 1]$.

The map $\Psi(E) := (vH) \int_E G$ is defined, nonnegative and finitely additive on the algebra \mathcal{H} generated by all intervals. By [7, Corollary 3.1] $V_\Psi \ll \lambda$. Since $0 \in G(t)$ then $s(x^*, \Psi(E)) \geq 0$ for every $x^* \in X^*$ and every $E \in \mathcal{H}$, and so by [28, Theorem 4.6] the map Ψ can be extended to \mathcal{L} in a σ -additive way; let $\tilde{\Psi} : \mathcal{L} \rightarrow cwk(X)$ be its extension. So, fixed any sequence of pairwise non-overlapping subintervals $(I_n)_n$ of $[0, 1]$, let $E = \cup_n I_n$. Then $i(\tilde{\Psi})(E) := \sum_n (vH) \int_{I_n} i(G) \in C(\Omega)$. \square

Theorem 4.2. *Let $\Gamma : [0, 1] \rightarrow cwk(X)$ be a variationally Henstock integrable multifunction. If $\mathcal{S}_{vH}(\Gamma) \neq \emptyset$ (this is fulfilled in case of X possessing RNP, by Proposition 3.3 and Theorem 3.12), then for every $f \in \mathcal{S}_{vH}(\Gamma)$ the multifunction $G : [0, 1] \rightarrow cwk(X)$ defined by $\Gamma(t) = G(t) + f(t)$ is Birkhoff integrable.*

Proof. Let $f \in \mathcal{S}_{vH}(\Gamma)$ be fixed. Define $G : [0, 1] \rightarrow cwk(X)$ by $G(t) := \Gamma(t) - f(t)$. Then G is also variationally Henstock integrable (in $ckw(X)$) and $0 \in G(t)$ for every $t \in [0, 1]$. By Proposition 4.1 the multifunction G is Birkhoff integrable. \square

The next two results generalize [27, Theorem 3.4], proved there for $ckw(X)$ -valued multifunctions with compact valued integrals.

Theorem 4.3. *Let $\Gamma : [0, 1] \rightarrow cwk(X)$ be a vH -integrable multifunction. If $\mathcal{S}_{vH}(\Gamma) \neq \emptyset$ (this is fulfilled in case of X possessing RNP, by Proposition 3.3 and Theorem 3.12), then the following conditions are equivalent:*

- (a) $\mathcal{S}_{vH}(\Gamma) \subset \mathcal{S}_{MS}(\Gamma)$;
- (b) $\mathcal{S}_{vH}(\Gamma) \subset \mathcal{S}_P(\Gamma)$;
- (c) $\mathcal{S}_P(\Gamma) \neq \emptyset$;
- (d) Γ is Pettis integrable;
- (e) Γ is McShane integrable.

Proof. (a) \Rightarrow (b) is valid, because each McShane integrable function is also Pettis integrable ([32, Theorem 8]).

(b) \Rightarrow (c) is obvious.

(c) \Rightarrow (d) Take $f \in \mathcal{S}_P(\Gamma)$. Since Γ is Henstock integrable too, it is also HKP-integrable and so applying [25, Theorem 2] we obtain a representation $\Gamma = G + f$, where $G : [0, 1] \rightarrow \text{cwk}(X)$ is Pettis integrable in $\text{cwk}(X)$. Consequently, Γ is also Pettis integrable in $\text{cwk}(X)$ and so (d) is fulfilled.

(d) \Rightarrow (e) In virtue of [27, Theorem 3.1] Γ has a McShane integrable selection f . It follows from Theorem 4.2 that the multifunction $G : [0, 1] \rightarrow \text{cwk}(X)$ defined by $\Gamma(t) = G(t) + f(t)$ is McShane integrable.

(e) \Rightarrow (a) It is a consequence of Proposition 3.6 and [15, Corollary 2.3], in fact, if $f \in \mathcal{S}_{vH}(\Gamma)$, then $f \in \mathcal{S}_H(\Gamma) \cap \mathcal{S}_P(\Gamma)$ and so f is McShane integrable. \square

Theorem 4.4. *Let $\Gamma : [0, 1] \rightarrow \text{cwk}(X)$ be an integrably bounded multifunction satisfying Theorem 4.3. Then all the statements given in Theorem 4.3 are equivalent to*

(f) Γ is variationally McShane integrable.

Proof. The equivalence of the four first conditions can be proved as in Theorem 4.3.

(e) \Rightarrow (f) Assume that Γ is integrably bounded by $h \in L_1[0, 1]$ and let i be the Rådström embedding of $\text{cwk}(X)$ into a Banach space Z . Since Γ is variationally Henstock integrable, also $i(\Gamma)$ is variationally Henstock integrable. If M_Γ is the indefinite Pettis integral of Γ , then it is a measure in the Hausdorff metric, hence also $i(M_\Gamma)$ is countably additive in Z . Moreover, if $I \in \mathcal{I}$, then

$$\langle z^*, i(M_\Gamma(I)) \rangle = (HK) \int_I \langle z^*, i(\Gamma)(I) \rangle d\lambda \quad \text{for every } z^* \in Z^*. \quad (4)$$

But $i(\Gamma)$ is integrably bounded and so every $\langle z^*, i(\Gamma)(I) \rangle$ is Lebesgue integrable. Consequently, both sides of the equality (4) may be extended to scalar measures on \mathcal{L} . By the integral boundedness assumption we have also $|i(M_\Gamma)(E)| \leq \int_E |h| d\lambda$ for all $E \in \mathcal{L}$. Thus, $i(M_\Gamma)$ is moderated (in fact finite). Then, thanks to [22, Lemma 2], $i(\Gamma)$ is variationally McShane integrable. This proves the required result.

(f) \Rightarrow (e) is obvious. \square

A comparison with the Birkhoff integrability is the following

Proposition 4.5. *If Γ is Bochner measurable and abs(Bi)-integrable, then Γ is vMS-integrable.*

Proof. Thanks to Theorem 2.4 the function $i \circ \Gamma$ is Bochner measurable and abs(Birkhoff)-integrable then, applying [50, Corollary 1], $i \circ \Gamma$ is vMS-integrable. \square

While for the converse implication only Birkhoff-integrability could be obtained, see [50, Corollary 3].

Corollary 4.6. *If X has RNP and Γ is variationally Henstock integrable, then every strongly measurable selection of Γ is variationally Henstock and Birkhoff integrable. Moreover, Γ turns out to be Birkhoff integrable too.*

Proof. The first part is an easy consequence of Theorem 3.12 and [15, Corollary 2.3], since by [47, Corollary 5.11], for finite measure spaces Birkhoff integrability of strongly measurable functions is equivalent to their Pettis integrability. The final assertion about Γ follows from Theorem 4.2 since both the selection f and the translated mapping G are Birkhoff integrable. \square

We would like to observe that in general the equivalent conditions in Theorem 4.3 do not imply the variational McShane integrability, even in case of single valued functions (see [21]). Moreover, also Proposition 4.1 cannot be obtained, if we consider variational McShane integrability instead of Birkhoff integrability, as the

following example shows. Observe first that if Γ satisfies assumptions of Proposition 4.1, then by Proposition 2.5 is Pettis integrable since it is McShane integrable. The multifunction of the following Example 4.7 satisfies Proposition 4.1, so it will be vH-integrable, Pettis integrable but not vMS-integrable.

Example 4.7. Assume that $\sum_n x_n$ is unconditionally but not absolutely convergent in X . We assume that $\|x_n\| \leq 1$, for all $n \in \mathbb{N}$. Moreover, let $I_n =: (2^{-n}, 2^{-n+1})$, $n \in \mathbb{N}$. We define $f : [0, 1] \rightarrow X$ by

$$f(t) = \sum_{n=1}^{\infty} 2^n x_n 1_{I_n}(t).$$

It has been proven in [21,39] that f is variationally Henstock and Pettis integrable but is not variationally McShane integrable. Consider now the multifunction $G : [0, 1] \rightarrow ck(X)$ defined by $G(t) = \text{conv}\{f(t), 0\}$.

Claim 1. G is variationally Henstock integrable.

Proof. For each x^* we have $s(x^*, G(t)) = (x^* f)^+(t)$ and so if $I \in \mathcal{I}$, then

$$\int_I s(x^*, G(t)) dt = \int_I (x^* f)^+(t) dt = \int_I \left(\sum_{n=1}^{\infty} 2^n \langle x^*, x_n \rangle^+ 1_{I_n}(t) \right) dt = \sum_{n=1}^{\infty} 2^n |I_n \cap I| \langle x^*, x_n \rangle^+$$

where $\langle x^*, x_n \rangle^+ = \langle x^*, x_n \rangle$ if $\langle x^*, x_n \rangle \geq 0$ and 0 otherwise. Let $\varepsilon > 0$ be fixed and $k \in \mathbb{N}$ be such that $\sup_{\|x^*\| \leq 1} \sum_{i=k}^{\infty} |\langle x^*, x_i \rangle| < \varepsilon/2$. We define a gauge setting

$$\delta(t) = \begin{cases} \min\{|t - 2^{-n}|, |t - 2^{-n+1}|\} & \text{if } t \in I_n \\ \varepsilon 2^{-2n-2} & \text{if } t = 2^{-n-1} \\ \varepsilon 2^{-k} & t = 0 \end{cases}$$

Let $\{(J_1, t_1), \dots, (J_p, t_p)\}$ be a δ -fine Perron partition of $[0, 1]$. We have to evaluate the expression

$$\begin{aligned} & \sum_{i=1}^p \sup_{\|x^*\| \leq 1} \left| \int_{J_i} s(x^*, G(t)) dt - s(x^*, G(t_i)) |J_i| \right| \\ &= \sum_{i=1}^p \sup_{\|x^*\| \leq 1} \left| \sum_{n=1}^{\infty} 2^n |I_n \cap J_i| \langle x^*, x_n \rangle^+ - \sum_{n=1}^{\infty} 2^n \langle x^*, x_n \rangle^+ 1_{I_n}(t_i) |J_i| \right|. \end{aligned}$$

If $t_i \in I_{n_i}$, then $J_i \subset I_{n_i}$ and we have

$$\begin{aligned} & \left| \sum_{n=1}^{\infty} 2^n |I_n \cap J_i| \langle x^*, x_n \rangle^+ - \sum_{n=1}^{\infty} 2^n \langle x^*, x_n \rangle^+ 1_{I_n}(t_i) |J_i| \right| \\ &= |2^{n_i} \langle x^*, x_{n_i} \rangle^+ |J_i| - 2^{n_i} \langle x^*, x_{n_i} \rangle^+ |J_i| = 0. \end{aligned}$$

If $t_i = 2^{-n_i}$, then

$$\begin{aligned} & \left| \sum_{n=1}^{\infty} 2^n |I_n \cap J_i| \langle x^*, x_n \rangle^+ - \sum_{n=1}^{\infty} 2^n \langle x^*, x_n \rangle^+ 1_{I_n}(t_i) |J_i| \right| = \left| \sum_{n=1}^{\infty} 2^n |I_n \cap J_i| \langle x^*, x_n \rangle^+ \right| \\ &= 2^{n_i-1} |I_{n_i-1} \cap J_i| \langle x^*, x_{n_i-1} \rangle^+ + 2^{n_i} |I_{n_i} \cap J_i| \langle x^*, x_{n_i} \rangle^+ \end{aligned}$$

$$\begin{aligned} &\leq 2^{n_i-1} \varepsilon 2^{2n_i-2} \langle x^*, x_{n_i-1} \rangle^+ + 2^{n_i} \varepsilon 2^{2n_i-2} \langle x^*, x_{n_i} \rangle^+ \\ &= \varepsilon 2^{n_i-3} \langle x^*, x_{n_i-1} \rangle^+ + \varepsilon 2^{n_i-2} \langle x^*, x_{n_i} \rangle^+ < \varepsilon 2^{-n_i}. \end{aligned}$$

If $t_i = 0$, then

$$\begin{aligned} &\left| \sum_{n=1}^{\infty} 2^n |I_n \cap J_i| \langle x^*, x_n \rangle^+ - \sum_{n=1}^{\infty} 2^n \langle x^*, x_n \rangle^+ 1_{I_n}(t_i) |J_i| \right| \\ &= \sum_{n=1}^{\infty} 2^n |I_n \cap J_i| \langle x^*, x_n \rangle^+ \leq \sum_{n=k}^{\infty} 2^n |I_n| \langle x^*, x_n \rangle^+ \leq \sum_{n=k}^{\infty} |\langle x^*, x_n \rangle| \leq \varepsilon / 2^{-k}. \end{aligned}$$

So we can conclude that

$$\sum_{i=1}^p \sup_{\|x^*\| \leq 1} \left| \int_{J_i} s(x^*, G(t)) dt - s(x^*, G(t_i)) |J_i| \right| < 2\varepsilon. \quad \square$$

Claim 2. *G is not variationally McShane integrable and not abs-Birkhoff.*

Proof. Let $\delta : [0, 1] \rightarrow (0, \infty)$ be an arbitrary gauge. Let $m \in \mathbb{N}$ be the smallest number satisfying the inequality $2^{-m} \leq \delta(0)$. For each $p > m$ we take a particular δ -fine partition of $[0, 1]$: $\mathcal{P} := \{([0, 2^{-p}], 0), (I_p, 0), \dots, (I_{m+1}, 0), (J_1, s_1), \dots, (J_q, s_q)\}$, where we require each (J_i, s_i) to be only $\delta(s_i)$ -small. We have then

$$\begin{aligned} &\sum_{(J,t) \in \mathcal{P}} \sup_{\|x^*\| \leq 1} \left| \sum_{n=1}^{\infty} 2^n |I_n \cap J| \langle x^*, x_n \rangle^+ - \sum_{n=1}^{\infty} 2^n \langle x^*, x_n \rangle^+ 1_{I_n}(t) |J| \right| \\ &\geq \sum_{i=m+1}^p \sup_{\|x^*\| \leq 1} \left| \sum_{n=1}^{\infty} 2^n |I_n \cap I_i| \langle x^*, x_n \rangle^+ \right| = \sum_{i=m+1}^p \sup_{\|x^*\| \leq 1} 2^i |I_i| \langle x^*, x_i \rangle^+ = \sum_{i=m+1}^p \|x_i\| \rightarrow \infty \end{aligned}$$

when $p \rightarrow \infty$. For the last part, since G is Bochner measurable if it were abs-Birkhoff, then by [Proposition 4.5](#) then G would be variationally McShane-integrable. \square

Acknowledgments

The paper was written while the third named author was visiting the Departments of Mathematics and Computer Sciences of the University of Palermo (Italy) thanks to the Grant Prot. N. U 2014-001415 of GNAMPA – INDAM (Italy).

The first and the last authors have been supported by University of Perugia – Department of Mathematics and Computer Sciences – Grant Nr. 2010.011.0403 and, respectively, by Prin “Metodi logici per il trattamento dell’informazione” and “Descartes”. The second author was supported by F.F.R. 2013 – University of Palermo (Italy).

References

[1] A. Avilés, G. Plebanek, J. Rodriguez, The McShane integral in weakly compactly generated spaces, *J. Funct. Anal.* 259 (11) (2010) 2776–2792.
 [2] E.J. Balder, A.R. Sambucini, Fatou’s lemma for multifunctions with unbounded values in a dual space, *J. Convex Anal.* 12 (2) (2005) 383–395.
 [3] A. Boccuto, A.R. Sambucini, A McShane integral for multifunctions, *J. Concr. Appl. Math.* 2 (4) (2004) 307–325.

- [4] A. Boccuto, A.R. Sambucini, A note on comparison between Birkhoff and McShane-type integrals for multifunctions, *Real Anal. Exchange* 37 (2) (2012) 315–324.
- [5] A. Boccuto, A.M. Minotti, A.R. Sambucini, Set-valued Kurzweil–Henstock integral in Riesz space setting, *Panamer. Math. J.* 23 (1) (2013) 57–74.
- [6] A. Boccuto, D. Candeloro, A.R. Sambucini, Henstock multivalued integrability in Banach lattices with respect to pointwise non atomic measures, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* 26 (4) (2015) 363–383, <http://dx.doi.org/10.4171/RLM/710>, arXiv:1503.08285v1.
- [7] B. Bongiorno, L. Di Piazza, K. Musiał, A variational Henstock integral characterization of the Radon–Nikodym Property, *Illinois J. Math.* 53 (1) (2009) 87–99.
- [8] D. Candeloro, A.R. Sambucini, Order-type Henstock and McShane integrals in Banach lattice setting, in: *Proceedings of the SISY 2014 – IEEE 12th International Symposium on Intelligent Systems and Informatics*, ISBN 978-1-4799-5995-2, 2014, pp. 55–59.
- [9] D. Candeloro, A.R. Sambucini, Comparison between some norm and order gauge integrals in Banach lattices, *Panamer. Math. J.* 25 (3) (2015) 1–16, arXiv:1503.04968 [math.FA].
- [10] D. Candeloro, A. Croitoru, A. Gavrilut, A.R. Sambucini, An extension of the Birkhoff integrability for multifunctions, *Mediterr. J. Math.* (2015), <http://dx.doi.org/10.1007/s00009-015-0639-7>, arXiv:1507.06444 [math.FA].
- [11] S. Cao, The Henstock integral for Banach-valued functions, *SEA Bull. Math.* 16 (1) (1992) 35–40.
- [12] B. Cascales, J. Rodríguez, Birkhoff integral for multi-valued functions, *J. Math. Anal. Appl.* 297 (2) (2004) 540–560.
- [13] B. Cascales, J. Rodríguez, The Birkhoff integral and the property of Bourgain, *Math. Ann.* 331 (2) (2005) 259–279.
- [14] C. Cascales, V. Kadets, J. Rodríguez, The Pettis integral for multi-valued functions via single-valued ones, *J. Math. Anal. Appl.* 332 (1) (2007) 1–10.
- [15] C. Cascales, V. Kadets, J. Rodríguez, Measurable selectors and set-valued Pettis integral in non-separable Banach spaces, *J. Funct. Anal.* 256 (2009) 673–699.
- [16] C. Cascales, V. Kadets, J. Rodríguez, Measurability and selections of multifunctions in Banach spaces, *J. Convex Anal.* 17 (2010) 229–240.
- [17] C. Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes Math., vol. 580, Springer-Verlag, Berlin–New York, 1977.
- [18] K. Cichoń, M. Cichoń, Some applications of nonabsolute integrals in the theory of differential inclusions in Banach spaces, in: G.P. Curbera, G. Mockenhaupt, W.J. Ricker (Eds.), *Vector Measures, Integration and Related Topics*, in: *Operator Theory: Advances and Applications*, vol. 201, Birkhauser-Verlag, ISBN 978-3-0346-0210-5, 2010, pp. 115–124.
- [19] W. Congxin, Y. Xiaobo, A Riemann-type definition of the Bochner integral, *J. Math. Study* 27 (1994) 32–36.
- [20] A. Croitoru, A. Gavrilut, A.E. Iosif, Birkhoff weak integrability of multifunctions, *Int. J. Pure Math.* 2 (2015) 47–54.
- [21] L. Di Piazza, V. Marraffa, The McShane, PU and Henstock integrals of Banach valued functions, *Czechoslovak Math. J.* (ISSN 0011-4642) 52 (127) (2002) 609–633.
- [22] L. Di Piazza, K. Musiał, A characterization of variationally McShane integrable Banach-space valued functions, *Illinois J. Math.* 45 (1) (2001) 279–289.
- [23] L. Di Piazza, K. Musiał, Characterizations of Kurzweil–Henstock–Pettis integrable functions, *Studia Math.* (ISSN 0039-3223) 176 (2) (2006) 159–176.
- [24] L. Di Piazza, K. Musiał, A decomposition theorem for compact-valued Henstock integral, *Monatsh. Math.* (ISSN 0026-9255) 148 (2) (2006) 119–126.
- [25] L. Di Piazza, K. Musiał, A decomposition of Denjoy–Khinchine–Pettis and Henstock–Kurzweil–Pettis integrable multifunctions, in: G.P. Curbera, G. Mockenhaupt, W.J. Ricker (Eds.), *Vector Measures, Integration and Related Topics*, in: *Operator Theory: Advances and Applications*, vol. 201, Birkhauser Verlag, 2010, pp. 171–182.
- [26] L. Di Piazza, K. Musiał, Henstock–Kurzweil–Pettis integrability of compact valued multifunctions with values in an arbitrary Banach space, *J. Math. Anal. Appl.* (ISSN 0022-247X) 408 (2013) 452–464.
- [27] L. Di Piazza, K. Musiał, Relations among Henstock, McShane and Pettis integrals for multifunctions with compact convex values, *Monatsh. Math.* 173 (4) (2014) 459–470.
- [28] L. Di Piazza, G. Porcello, Radon–Nikodym theorems for finitely additive multimeasures, *Z. Anal. Anwend.* 34 (4) (2015) 373–389, <http://dx.doi.org/10.4171/ZAA/1545>.
- [29] L. Di Piazza, D. Preiss, When do McShane and Pettis integrals coincide?, *Illinois J. Math.* (ISSN 0019-2082) 47 (4) (2003) 1177–1187.
- [30] L. Di Piazza, B. Satco, A new result on impulsive differential equations involving non-absolutely convergent integrals, *J. Math. Anal. Appl.* 352 (2009) 954–963.
- [31] L. Di Piazza, V. Marraffa, B. Satco, Set valued integrability in non separable Fréchet spaces and applications, *Math. Slovaca* (2015), in press, arXiv:1507.07349v1 [Math:FA].
- [32] D.H. Fremlin, The Henstock and McShane integrals of vector-valued functions, *Illinois J. Math.* 38 (3) (1994) 471–479.
- [33] D.H. Fremlin, J. Mendoza, On the integration of vector-valued functions, *Illinois J. Math.* 38 (1994) 127–147.
- [34] R.W. Hansell, Extended Bochner measurable selectors, *Math. Ann.* 277 (1987) 79–94.
- [35] C.J. Himmelberg, F.S. Van Vleck, K. Prikry, The Hausdorff metric and measurable selections, *Topology Appl.* 20 (1985) 121–133.
- [36] S. Hu, N.S. Papageorgiou, *Handbook of Multivalued Analysis I and II*, Math. Appl., vol. 419, Kluwer Academic Publisher, Dordrecht, 1997.
- [37] S.B. Kaliaj, A variational McShane integral characterization of the weak Radon–Nikodym property, *Bull. Aust. Math. Soc.* 85 (2012) 456–462, <http://dx.doi.org/10.1017/S0004972712000093>.
- [38] C.C.A. Labuschagne, A.L. Pinchuck, C.J. van Alten, A vector lattice version of Rådström’s embedding theorem, *Quaest. Math.* 30 (3) (2007) 285–308.
- [39] V. Marraffa, Strongly measurable Kurzweil–Henstock type integrable functions and series, *Quaest. Math.* 31 (2008) 379–386.

- [40] A. Martellotti, A.R. Sambucini, On the comparison between Aumann and Bochner integral, *J. Math. Anal. Appl.* 260 (1) (2001) 6–17.
- [41] A. Martellotti, A.R. Sambucini, The finitely additive integral of multifunctions with closed and convex values, *Z. Anal. Anwend.* 21 (4) (2002) 851–864.
- [42] K. Musiał, Topics in the theory of Pettis integration, *Rend. Istit. Mat. Univ. Trieste* 23 (1991) 177–262.
- [43] K. Musiał, Pettis integrability of multifunctions with values in arbitrary Banach spaces, *J. Convex Anal.* 18 (3) (2011) 769–810.
- [44] K. Musiał, Approximation of Pettis integrable multifunctions with values in arbitrary Banach spaces, *J. Convex Anal.* 20 (2013).
- [45] K. Musiał, Pettis integral, in: *Handbook of Measure Theory I*, Elsevier Science B.V., 2002, pp. 532–586.
- [46] K.M. Naralencov, A Lusin type measurability property for vector-valued functions, *J. Math. Anal. Appl.* 417 (1) (2014) 293–307, <http://dx.doi.org/10.1016/j.jmaa.2014.03.029>.
- [47] B.J. Pettis, On integration in vector spaces, *Trans. Amer. Math. Soc.* 44 (2) (1938) 277–304.
- [48] R.S. Phillips, Integration in a convex linear topological space, *Trans. Amer. Math. Soc.* 47 (1940) 114–145.
- [49] G. Porcello, Multimeasures and integration of multifunctions in Banach spaces, Dottorato di Ricerca in Matematica e Informatica XXIV ciclo, University of Palermo, Italy, https://iris.unipa.it/retrieve/handle/10447/91026/99048/Tesi_Dottorato_Giovanni_Porcello.pdf.
- [50] M. Potyrala, The Birkhoff and variationally McShane integrals of vector valued functions, *Acta Univ. Lodz., Folia Math.* 13 (1) (2006) 31–39.
- [51] J. Rodriguez, On the equivalence of McShane and Pettis integrability in non-separable Banach spaces, *J. Math. Anal. Appl.* 341 (1) (2008) 80–90.