J Theor Probab (2007) 20: 545–560
DOI 10.1007/s10959-007-0075-0

Fubini Type Products for Densities and Liftings

K. Musiał · W. Strauss · N.D. Macheras

Received: 6 October 2005 / Revised: 31 May 2006 /
Published online: 10 May 2007

Abstract In our former paper (Fund. Math. 166, 281–303, 2000) we discussed densities and liftings in the product of two probability spaces with good section properties analogous to that for measures and measurable sets in the Fubini Theorem. In the present paper we investigate the following more delicate problem: Let (Ω, Σ, μ) and (Θ, T, ν) be two probability spaces endowed with densities υ and τ, respectively. Can we define a density on the product space by means of a Fubini type formula $(\upsilon \odot \tau)(E) = \{(\omega, \theta) : \omega \in \upsilon(\{\bar{\omega} : \theta \in \tau(E_{\bar{\omega}}\})\}$, for E measurable in the product, and the same for liftings instead of densities? We single out classes of marginal densities υ and τ which admit a positive solution in case of densities, where we have sometimes to replace the Fubini type product by its upper hull, which we call box product. For liftings the answer is in general negative, but our analysis of the above problem leads to a new method, which allows us to find a positive solution. In this way we solved one of the main problems of Musiał, Strauss and Macheras (Fund. Math. 166, 281–303, 2000).

The authors are indebted to the Mathematische Forschungsinstitut Oberwolfach for its support through the program Research in Pairs, during the preparation of this paper.

K. Musiał (✉)
Department of Mathematics, Wrocław University, Institute of Mathematics, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
e-mail: musial@math.uni.wroc.pl

W. Strauss
Universität Stuttgart, Fachbereich Mathematik Institut für Stochastik und Anwendungen Abteilung für Finanz- und Versicherungsmathematik, Postfach 80 11 40, 70511 Stuttgart, Germany
e-mail: strauss.w@arcor.de

N.D. Macheras
Department of Statistics and Insurance Science, University of Piraeus, 80 Karaoli and Dimitriou street, 185 34 Piraeus, Greece
e-mail: macheras@unipi.gr

Keywords Product densities · Product liftings · Product probability spaces

1 Introduction

The study of liftings compatible with the product structure of probability spaces started with a paper of M. Talagrand [15] and was carried on from different points of view by other authors (see [1, 5–11, 13]). In [10] we considered products of two probability spaces endowed with densities and liftings possessing good section properties analogous to that for measures and measurable sets in the Fubini Theorem. These properties were then applied by us to show the permanence of the measurability of stochastic processes under lifting modification (see [10] and [14]). As to be expected many convenient properties known from the Fubini Theorem for product measures fail for product liftings and densities. The main positive result proved in [10] was the following one: Given complete probability spaces (Ω, Σ, μ) and (Θ, T, ν) and liftings ρ for μ and σ for ν there exists a lifting π for the completed product measure $\mu \widehat{\otimes} \nu$ such that $[\pi(E)]_\omega = \sigma([\pi(E)]_\omega)$ for each $E \in \Sigma \widehat{\otimes} T$ and $\omega \in \Omega$, provided that σ is smooth enough.

In this paper we construct products of densities and liftings via sections, in a way paralleling Fubini formulae for product measures (see Definition 4.1), where measures are replaced by densities an liftings, respectively. For this reason we call it a Fubini type product. As often in lifting theory, in this process we are enfaced with problems of measurability and as well with problems of the validity of the basic lifting properties for the product. Moreover it turns out that even if we start with densities on the factors, we often can only achieve subdensity structure for the product. This convinced us that we have to consider so called primitive liftings, i.e. homorphisms with structural properties weaker than those of densities (see Preliminaries). For restoring the density property we define a second product, the box product, by taking the monotone hull of our former Fubini type product (see Definition 4.3). We single out primitive liftings in the factors (see Definition 3.2) being crucial for a positive solution of the above mentioned problems. To get the existence of the product we apply results from [10]. As one of the main positive results we obtain the following one: Given complete probability spaces (Ω, Σ, μ) and (Θ, T, ν) and liftings ρ for μ and σ for ν there exists a lifting π for the completed product measure $\mu \widehat{\otimes} \nu$ such that $[\pi(E)]^\theta = \rho([\pi(E)]^\theta)$ for each $E \in \Sigma \widehat{\otimes} T$ and $\theta \in \Theta$ provided that σ is smooth enough and the pair (ρ, σ) preserves measurability (Theorem 4.3). This solves one of the problems left open in [10]. This is completely different result from that mentioned above from [10]. We prove that without the additional assumption about preserving measurability, Ω-sections cannot be replaced by Θ-sections in [10].

Section 3 is of preparatory nature and serves for singling out primitive liftings which can appear as marginals in those products, which are again primitive liftings (see Theorem 3.1). In Sect. 4 we give the basic existence theorems for Fubini type and box products (see Theorems 4.1, 4.2, and 4.3). Section 5 is devoted to non-existence results (see Theorems 5.1, 5.2 and 5.4).

2 Preliminaries

For a given probability space (Ω, Σ, μ) a set $N \in \Sigma$ with $\mu(N) = 0$ is called a *μ-null set* and for $A, B \in \Sigma$ we write $A = B$ a.e. (μ) or $A =_\mu B$ iff $A \triangle B$, the symmetric difference of A and B, is a μ-null set. The family of all μ-null sets is denoted by Σ_0. The (Carathéodory) completion of (Ω, Σ, μ) will be denoted by $(\Omega, \widehat{\Sigma}, \widehat{\mu})$. $\mathcal{P}(\Omega)$ denotes the family of all subsets of Ω. The σ-algebra generated by a family $\mathcal{L}$ of sets is denoted by $\sigma(\mathcal{L})$. A filter $\mathcal{F}$ in Σ is said to be *μ-stable* if $A \in \mathcal{F}$ and $\mu(A \triangle B) = 0$ yields $B \in \mathcal{F}$. If $M \subseteq \Omega$, then $M^c := \Omega \setminus M$.

For given probability space (Ω, Σ, μ) and a map $\varpi : \Sigma \longrightarrow \mathcal{P}(\Sigma)$ we consider the following properties for $A, B \in \Sigma$

(L1) $\varpi(A) = A$ a.e.(μ);
(L2) $A = B$ a.e.(μ) implies $\varpi(A) = \varpi(B)$;
(N) $\varpi(\emptyset) = \emptyset$ and $\varpi(\Omega) = \Omega$;
(O) $A \subseteq B$ implies $\varpi(A) \subseteq \varpi(B)$;
(F) $\varpi(A) \cap \varpi(B) \subseteq \varpi(A \cap B)$.
(ϑ) $\varpi(A \cap B) = \varpi(A) \cap \varpi(B)$.
(U) $\varpi(A^c) = [\varpi(A)]^c$.

We call a $\varpi \in \Sigma^\Sigma$ satisfying (L1), (L2), and (N) a *primitive lifting* for μ and we denote by $P(\mu)$ the class of all primitive liftings. Due to (L1) each $\varpi \in P(\mu)$ can be uniquely extended to $\overline{\varpi} \in P(\widehat{\mu})$. Indeed, it is enough to set $\overline{\varpi}(A) = \varpi(B)$, where $A \in \widehat{\Sigma},\ B \in \Sigma$ and $\widehat{\mu}(A \triangle B) = 0$. A primitive lifting for μ will be called a *monotone lifting* for μ if it satisfies in addition the condition (O). If a primitive lifting satisfies also (F), we call it a *subdensity* for μ. We denote by $O(\mu)$ and $F(\mu)$ the class of all monotone liftings and subdensities, respectively. Monotone subdensities are called densities and denoted by $\vartheta(\mu)$ (notice that $\vartheta(\mu) = O(\mu) \cap F(\mu)$). Densities preserving complements are called liftings and denoted by $\Lambda(\mu)$. Compare with [4, Chap. III, Sects. 2 and 3] (see also [12]). We write $A\vartheta(\nu)$ and $AG\Lambda(\nu)$ for the classes of all *admissible densities* and all *admissibly generated liftings*, respectively (see [5] and [10] for the definitions). The space $\mathcal{P}(\Omega)^\Sigma$ will always be considered under pointwise order, i.e. $\varpi_1 \le \varpi_2$, if $\varpi_1(E) \subseteq \varpi_2(E)$ for every $E \in \Sigma$.

We denote by $(\Omega \times \Theta, \Sigma \otimes T, \mu \otimes \nu)$ the product probability space of the probability spaces (Ω, Σ, μ) and (Θ, T, ν) and by $(\Omega \times \Theta, \Sigma \widehat{\otimes} T, \mu \widehat{\otimes} \nu)$ its (Carathéodory) completion. For given $\upsilon \in P(\mu)$ and $\tau \in P(\nu)$ we denote by $\upsilon \otimes \tau$ the set of all $\varphi \in \vartheta(\mu \widehat{\otimes} \nu)$ satisfying $\varphi(A \times B) = \upsilon(A) \times \tau(B)$ for all $A \in \Sigma$ and all $B \in T$.

If E is a subset of $\Omega \times \Theta$ and $(\omega, \theta) \in \Omega \times \Theta$ is fixed, then we use the ordinary notation E_ω, E^θ for the sections $\{\theta \in \Theta : (\omega, \theta) \in E\}$, $\{\omega \in \Omega : (\omega, \theta) \in E\}$, respectively.

It follows from our definitions that if $\varphi \in \vartheta(\mu \otimes \nu)$ and $E \in \Sigma \otimes T$, then $[\varphi(E)]_\omega \in T$ for every $\omega \in \Omega$ and $[\varphi(E)]^\theta \in \Sigma$ for every $\theta \in \Theta$.

Throughout the paper (Ω, Σ, μ) and (Θ, T, ν) are arbitrary but fixed complete probability spaces.

3 Lifting of Sections

Definition 3.1 For given $\tau \in \mathcal{P}(\Theta)^T$ we define the set

$$\tau_\bullet(E) := \{(\omega,\theta) \in \Omega \times \Theta : E_\omega \in T \wedge \theta \in \tau(E_\omega)\}$$

for all $E \in \Sigma \widehat{\otimes} T$. Similarly for $\upsilon \in \mathcal{P}(\Omega)^\Sigma$ we define

$$\upsilon^\bullet(E) := \{(\omega,\theta) \in \Omega \times \Theta : E^\theta \in \Sigma \wedge \omega \in \upsilon(E^\theta)\}$$

for all $E \in \Sigma \widehat{\otimes} T$.

Below we will discuss only $\tau_\bullet$ since the corresponding properties for $\upsilon^\bullet$ are easy to derive from those of $\tau_\bullet$.

Remark 3.1 For $\tau \in P(\nu)$ the mapping $\tau_\bullet : E \in \Sigma \widehat{\otimes} T \longrightarrow \mathcal{P}(\Omega \times \Theta)$ has the following properties.

(i) If $E, F \in \Sigma \widehat{\otimes} T$ then $E = F$ a.e.$(\mu \widehat{\otimes} \nu)$ implies for all $\theta \in \Theta$ the equality $[\tau_\bullet(E)]^\theta = [\tau_\bullet(F)]^\theta$ a.e.(μ).
In fact by Fubini's theorem there exists a set $N \in \Sigma_0$ such that $E_\omega = F_\omega$ a.e.(ν) for all $\omega \notin N$ what yields the equality $[\tau_\bullet(E)]^\theta \setminus N = [\tau_\bullet(F)]^\theta \setminus N$, for all $\theta \in \Theta$.
(ii) For $A \in \Sigma$ and $B \in T$ we get $\tau_\bullet(A \times B) = A \times \tau(B)$. In particular this makes clear, that (L2) is false for the operation $\tau_\bullet$.
(iii) $\tau_\bullet(\emptyset) = \emptyset$ and $\tau_\bullet(\Omega \times \Theta) = \Omega \times \Theta$.
(iv) If τ satisfies condition (F), so does $\tau_\bullet$.
(v) For $\tau \in \Lambda(\nu)$ we get $\tau_\bullet(E^c) = [\tau_\bullet(E)]^c$ for all $E \in \Sigma \otimes T$.
Indeed, we have $\tau(E_\omega) \cup \tau[(E^c)_\omega] = \Theta$, for every ω and the sets on the left hand side are disjoint.

Now the following critical problems arise.

1^0 When do we have $\tau_\bullet(E) \in \Sigma \widehat{\otimes} T$ for all $E \in \Sigma \widehat{\otimes} T$?
2^0 When do we have $\tau_\bullet(E) = E$ a.e.$(\mu \widehat{\otimes} \nu)$ for all $E \in \Sigma \widehat{\otimes} T$?
Since we deal with sections the following questions will become important.
3^0 For which $\theta \in \Theta$ is $[\tau_\bullet(E)]^\theta \in \Sigma$ for all $E \in \Sigma \widehat{\otimes} T$?
4^0 For which $\theta \in \Theta$ holds $[\tau_\bullet(E)]^\theta = E^\theta$ a.e.(μ) true?

Note that due to the completeness of $(\Omega \times \Theta, \Sigma \widehat{\otimes} T, \mu \widehat{\otimes} \nu)$ condition (L1) for $\tau_\bullet$ always implies 1^0 from which follows by means of Fubini's theorem the existence of a set $M_E \in T_0$ such that $[\tau_\bullet(E)]^\theta \in \Sigma$, even $[\tau_\bullet(E)]^\theta = E^\theta$ a.e.(μ) for all $\theta \notin M_E$ if $E \in \Sigma \widehat{\otimes} T$. But there are also reverse implications by the next proposition.

Proposition 3.1 *For given $\tau \in P(\nu)$ the following conditions are equivalent.*

(i) $\tau_\bullet(E) \in \Sigma \widehat{\otimes} T$ *for every* $E \in \Sigma \widehat{\otimes} T$.
(ii) $\tau_\bullet(E) = E$ *a.e.* $(\mu \widehat{\otimes} \nu)$ *for every* $E \in \Sigma \widehat{\otimes} T$.
(iii) *There exists a* $\varphi \in P(\mu \widehat{\otimes} \nu)$ *such that for all* $E \in \Sigma \widehat{\otimes} T$ *there exists a set* $N_E \in \Sigma_0$ *with* $\tau([\varphi(E)]_\omega) = [\varphi(E)]_\omega$ *for all* $\omega \notin N_E$.

(iv) *There exists a* $\varphi \in P(\mu \widehat{\otimes} \nu)$ *such that for all* $E \in \Sigma \widehat{\otimes} T$ *and all* $\omega \in \Omega$ *we have* $\tau([\varphi(E)]_\omega) = [\varphi(E)]_\omega$.

Proof The implication (ii) $\Longrightarrow$ (i) is trivially true due to the completeness of the probability space $(\Omega \times \Theta, \Sigma \widehat{\otimes} T, \mu \widehat{\otimes} \nu)$.

Ad (i) $\Longrightarrow$ (ii). If $\tau_\bullet(E) \in \Sigma \widehat{\otimes} T$ for some $E \in \Sigma \widehat{\otimes} T$, then $H := \tau_\bullet(E) \Delta E \in \Sigma \widehat{\otimes} T$ and by the Fubini theorem there exists a set $N_E \in \Sigma_0$ such that $[\tau_\bullet(E)]_\omega = E_\omega$ a.e.(ν) for all $\omega \notin N_E$. By Cavalieri's principle this implies $(\mu \widehat{\otimes} \nu)(H) = \int \nu([\tau_\bullet(E)]_\omega \Delta E_\omega)\, d\mu(\omega) = 0$, i.e. (L1) $\tau_\bullet(E) = E$ a.e. $(\mu \widehat{\otimes} \nu)$.

Ad (i) $\Longrightarrow$ (iv). Choose an arbitrary $\overline{\varphi} \in \vartheta(\mu \otimes \nu)$ (according to [3] $\vartheta(\mu \otimes \nu) \neq \emptyset$) and treat it as an element of $\vartheta(\mu \widehat{\otimes} \nu)$ (notice that we have always $[\overline{\varphi}(E)]_\omega \in T$ since $\varphi(E) \in \Sigma \otimes T$). Then define $\varphi(E) := \tau_\bullet(\overline{\varphi}(E))$ for each $E \in \Sigma \widehat{\otimes} T$. It follows $\varphi(E) \in \Sigma \widehat{\otimes} T$ for each $E \in \Sigma \widehat{\otimes} T$ by assumption. Due to the implication (i) $\Longrightarrow$ (ii) proved already we have $\varphi(E) =_{\mu \widehat{\otimes} \nu} \overline{\varphi}(E) =_{\mu \widehat{\otimes} \nu} E$ for all $E \in \Sigma \widehat{\otimes} T$, i.e. φ satisfies (L1).

For $E, F \in \Sigma \widehat{\otimes} T$ with $E = F$ a.e.$(\mu \widehat{\otimes} \nu)$ we get $\overline{\varphi}(E) = \overline{\varphi}(F)$, hence $\varphi(E) = \varphi(F)$, i.e. (L2) for φ.

Since $\overline{\varphi}$ satisfies (N) also φ satisfies condition (N), hence $\varphi \in P(\mu \widehat{\otimes} \nu)$.

For all $E \in \Sigma \widehat{\otimes} T$ we have $\tau([\varphi(E)_\omega) = [\varphi(E)]_\omega$ since $[\varphi(E)]_\omega = \tau([\overline{\varphi}(E)]_\omega)$ for all $\omega \in \Omega$. Hence (iv) is satisfied.

(iii) is an obvious consequence of (iv).

Ad (iii) $\Longrightarrow$ (i). Let be given $\varphi \in P(\mu \widehat{\otimes} \nu)$ such that for all $E \in \Sigma \widehat{\otimes} T$ there exists a set $N_E \in \Sigma_0$ with $\tau([\varphi(E)]_\omega) = [\varphi(E)]_\omega$ for all $\omega \notin N_E$ and note that there exists a set $L_E \in \Sigma_0$ such that $E_\omega =_\nu [\varphi(E)]_\omega \in T$ for all $\omega \notin L_E$. Put $P_E := N_E \cup L_E$. Then $P_E \in \Sigma_0$ and $\tau(E_\omega) = \tau([\varphi(E)]_\omega) = [\varphi(E)]_\omega$ for all $\omega \notin P_E$. It follows that $\tau_\bullet(E) \Delta \varphi(E) \subseteq P_E \times \Theta \in (\Sigma \widehat{\otimes} T)_0$, hence $\tau_\bullet(E) \in \Sigma \widehat{\otimes} T$. □

Definition 3.2 Once the basic complete probability spaces (Ω, Σ, μ) and (Θ, T, ν) are fixed, we say that $\tau \in P(\nu)$ is a *Θ-marginal* if it satisfies one of the equivalent conditions of Proposition 3.1.

We say that $\tau \in P(\nu)$ *generates μ-measurable sections*, if $[\tau_\bullet(E)]^\theta \in \Sigma$ for all $E \in \Sigma \widehat{\otimes} T$ and all $\theta \in \Theta$.

If $\tau \in P(\nu)$ is a Θ-marginal and generates μ-measurable sections, then it is called *μ-smooth*.

$\tau \in P(\nu)$ is a *weak Θ-marginal* if for all $E \in \Sigma \widehat{\otimes} T$ exists a set $M_E \in T_0$ such that $[\tau_\bullet(E)]^\theta = E^\theta$ a.e.(μ) for all $\theta \notin M_E$.

In an obvious way one may define $\upsilon \in P(\mu)$ to be a (weak) Ω-marginal or generate ν-measurable sections.

It follows from Proposition 3.1 that each Θ-marginal is a weak Θ-marginal. Since all $\tau \in A\vartheta(\nu)$ are Θ-marginals and generate μ-measurable sections by [10, Corollary 2.7], there is a large variety of μ-smooth densities. Notice also that if $\tau \in F(\nu)$ is a Θ-marginal and $\widetilde{\tau} \in F(\nu)$ dominates τ, then also $\widetilde{\tau}$ is a Θ-marginal. Similarly, if $\tau \in F(\nu)$ is a weak Θ-marginal, then also $\widetilde{\tau} \in F(\nu)$ is a weak Θ-marginal.

Lemma 3.1 *Assume that* $\tau \in P(\nu)$ *is a* Θ*-marginal. Then* τ *is* μ*-smooth if and only if the* $\varphi \in P(\mu \widehat{\otimes} \nu)$ *from Proposition* 3.1(iii) *can be taken such that* $[\varphi(E)]^\theta \in \Sigma$ *for every* $\theta \in \Theta$.

Proof Let be given $\varphi \in P(\mu \widehat{\otimes} \nu)$ such that for all $E \in \Sigma \widehat{\otimes} T$ there exists a set $N_E \in \Sigma_0$ with $\tau([\varphi(E)]_\omega) = [\varphi(E)]_\omega$ for all $\omega \notin N_E$ and note that there exists a set $L_E \in \Sigma_0$ such that $E_\omega \in T$ and $E_\omega =_\nu [\varphi(E)]_\omega$ for all $\omega \notin L_E$. Put $P_E := N_E \cup L_E$. Then $P_E \in \Sigma_0$. Now for every $\theta \in \Theta$ we get

$$\begin{aligned}[][\tau_\bullet(E)]^\theta \cap P_E^c &= \{\omega \in P_E^c : \theta \in \tau(E_\omega)\} = \{\omega \in P_E^c : \theta \in \tau([\varphi(E)]_\omega)\} \\ &= \{\omega \in P_E^c : \theta \in [\varphi(E)]_\omega\} = [\varphi(E)]^\theta \cap P_E^c.\end{aligned}$$

Hence $[\tau_\bullet(E)]^\theta \Delta [\varphi(E)]^\theta \in \Sigma_0$. Since (Ω, Σ, μ) is complete we have then $[\varphi(E)]^\theta \in \Sigma$ if and only if $[\tau_\bullet(E)]^\theta \in \Sigma$. □

4 Fubini-Type Products

Definition 4.1 For given $\upsilon \in \mathcal{P}(\Omega)^\Sigma$ and $\tau \in \mathcal{P}(\Theta)^T$ we define a mapping $\upsilon \odot \tau : \Sigma \widehat{\otimes} T \to \mathcal{P}(\Omega \times \Theta)$ by the formula

$$(\upsilon \odot \tau)(E) := \{(\omega, \theta) \in \Omega \times \Theta : [\tau_\bullet(E)]^\theta \in \Sigma \quad \text{and} \quad \omega \in \upsilon([\tau_\bullet(E)]^\theta)\} \tag{4.1}$$

for all $E \in \Sigma \widehat{\otimes} T$.

In a similar way we define a map $\tau \odot_t \upsilon : \Sigma \widehat{\otimes} T \to \mathcal{P}(\Omega \times \Theta)$ by

$$(\tau \odot_t \upsilon)(E) := \{(\omega, \theta) \in \Omega \times \Theta : [\upsilon^\bullet(E)]_\omega \in T \quad \text{and} \quad \theta \in \tau([\upsilon^\bullet(E)]_\omega)\}$$

for all $E \in \Sigma \widehat{\otimes} T$.

Again we will discuss below only the product $\upsilon \odot \tau$ since it is easy to derive from results about $\upsilon \odot \tau$ corresponding results for the product $\tau \odot_t \upsilon$. By [10, Theorem 3.5], we have $\upsilon \odot \tau \neq \tau \odot_t \upsilon$ if the spaces (Ω, Σ, μ) and (Θ, T, ν) are not purely atomic, i.e. practically in all interesting cases.

Lemma 4.1 *For given $\upsilon \in P(\mu)$ and $\tau \in P(\nu)$ the map $\upsilon \odot \tau : \Sigma \widehat{\otimes} T \to \mathcal{P}(\Omega \times \Theta)$ has the following properties*:

(i) $\upsilon([\tau_\bullet(E)]^\theta) = [(\upsilon \odot \tau)(E)]^\theta$ *for all* $\theta \in \Theta$ *with* $[\tau_\bullet(E)]^\theta \in \Sigma$.
(ii) $[(\upsilon \odot \tau)(E)]^\theta = \upsilon([(\upsilon \odot \tau)(E)]^\theta)$ *for all* $E \in \Sigma \hat{\otimes} T$ *and all* $\theta \in \Theta$;
(iii) *The conditions* (L2) *and* (N) *are satisfied for* $\upsilon \odot \tau$;
(iv) $\upsilon \odot \tau \in \upsilon \otimes \tau$;
(v) *If* υ *and* τ *are densities*, *then* $\upsilon \odot \tau$ *satisfies condition* (F).

Proof (i) follows directly from (4.1).

Ad (ii). If $E \in \Sigma \widehat{\otimes} T$ and $\theta \in \Theta$ then $[(\upsilon \odot \tau)(E)]^\theta = \upsilon([\tau_\bullet(E)]^\theta)$ for all $\theta \in \Theta$ with $[\tau_\bullet(E)]^\theta \in \Sigma$ and $[(\upsilon \odot \tau)(E)]^\theta = \emptyset$ for all $\theta \in \Theta$ with $[\tau(E_\bullet)]^\theta \notin \Sigma$. In both cases it follows that $[(\upsilon \odot \tau)(E)]^\theta = \upsilon([(\upsilon \odot \tau)(E)]^\theta)$ for all $E \in \Sigma \hat{\otimes} T$ and all $\theta \in \Theta$.

Ad (iii). According to Remark 3.1 condition (L2) follows from (i). Clearly $\upsilon \odot \tau$ satisfies also (N).

Ad (iv). If $A \in \Sigma$ and $B \in T$, then according to condition (i) we get

$$\begin{aligned}[]
[(\upsilon \odot \tau)(A \times B)]^\theta &= \upsilon([\tau_\bullet(A \times B)]^\theta) \\
&= \upsilon(\{\omega \in \Omega : \theta \in \tau([A \times B]_\omega)\}) = \upsilon(\{\omega \in A : \theta \in \tau(B)\}).
\end{aligned}$$

Ad (v). Let $E, F \in \Sigma \widehat{\otimes} T$ be arbitrary. According to (iii) we may assume, without loss of generality, that $E, F \in \Sigma \otimes T$. Then $\tau(E_\omega) \cap \tau(F_\omega) = \tau(E_\omega \cap F_\omega)$ for all $\omega \in \Omega$. Let us fix now $\theta \in \Theta$ such that $[\upsilon \odot \tau(E) \cap \upsilon \odot \tau(F)]^\theta = [\upsilon \odot \tau(E)]^\theta \cap [\upsilon \odot \tau(F)]^\theta \neq \emptyset$.

Then we have $[\tau_\bullet(E)]^\theta \in \Sigma$ and $[\tau_\bullet(F)]^\theta \in \Sigma$. Hence,

$$\begin{aligned}[]
[\tau_\bullet(E \cap F)]^\theta &= \{\omega : \theta \in \tau(E_\omega \cap F_\omega)\} = \{\omega : \theta \in \tau(E_\omega) \cap \tau(F_\omega)\} \\
&= \{\omega : \theta \in \tau(E_\omega)\} \cap \{\omega : \theta \in \tau(F_\omega)\} = [\tau_\bullet(E)]^\theta \cap [\tau_\bullet(F)]^\theta \in \Sigma.
\end{aligned}$$

Consequently,

$$\upsilon([\tau_\bullet(E \cap F)]^\theta) = \upsilon([\tau_\bullet(E)]^\theta \cap [\tau_\bullet(F)]^\theta)$$

and then

$$\begin{aligned}
&[(\upsilon \odot \tau)(E)]^\theta \cap [(\upsilon \odot \tau)(F)]^\theta = \upsilon([\tau_\bullet(E)]^\theta) \cap \upsilon([\tau_\bullet(F)]^\theta) \\
&= \upsilon([\tau_\bullet(E)]^\theta \cap [\tau_\bullet(F)]^\theta) = \upsilon([\tau_\bullet(E \cap F)]^\theta) = [(\upsilon \odot \tau)(E \cap F)]^\theta,
\end{aligned}$$

i.e. $[(\upsilon \odot \tau)(E) \cap (\upsilon \odot \tau)(F)]^\theta \subseteq [(\upsilon \odot \tau)(E \cap F)]^\theta$. If $[\upsilon \odot \tau(E)]^\theta \cap [\upsilon \odot \tau(F)]^\theta = \emptyset$ then, the last inclusion also holds true and so we obtain the condition (F) for $\upsilon \odot \tau$. □

Remark 4.1 Since $\upsilon \odot \tau$ satisfies (L2) by Lemma 4.1 and since for any $G \in \Sigma \widehat{\otimes} T$ there exists $E \in \Sigma \otimes T$ with $E = G$ a.e.$(\mu \widehat{\otimes} \nu)$, we may restrict ourselves to work with $(\upsilon \odot \tau)(E)$ for $E \in \Sigma \otimes T$ only, what we will do below without any further comments. Note that for $E \in \Sigma \otimes T$ we have $E_\omega \in T$ for all $\omega \in \Omega$ and this simplifies the definition of $(\upsilon \odot \tau)(E)$.

Lemma 4.2 *If υ satisfies condition* (O) *and $\tau \in O(\nu)$ generates μ-measurable sections then it follows that $\upsilon \odot \tau$ satisfies condition* (O).

Proof If $E, F \in \Sigma \otimes T$ and $E \subseteq F$, we get $E_\omega \subseteq F_\omega$ for all $\omega \in \Omega$, hence $\tau(E_\omega) \subseteq \tau(F_\omega)$ for all $\omega \in \Omega$. The latter implies $[\tau_\bullet(E)]^\theta \subseteq [\tau_\bullet(F)]^\theta$ for all $\theta \in \Theta$. Since by assumption $[\tau_\bullet(E)]^\theta, [\tau_\bullet(F)]^\theta \in \Sigma$ for all $\theta \in \Theta$, we get $\upsilon([\tau_\bullet(E)]^\theta) \subseteq \upsilon([\tau_\bullet(F)]^\theta)$ for all $\theta \in \Theta$, hence $\upsilon \odot \tau$ satisfies condition (O). □

Remark 4.2 There is now an obvious question, whether for given $\upsilon \in P(\mu)$ and $\tau \in P(\nu)$, the set function $\upsilon \odot \tau$ is in $P(\mu \widehat{\otimes} \nu)$. As it will be shown in Remark 5.1, even if $\upsilon \in \vartheta(\mu)$ and $\tau \in \vartheta(\nu)$, then it may happen that $\upsilon \odot \tau$ does not satisfy (L1).

Definition 4.2 The pair $(\upsilon, \tau) \in P(\mu) \times P(\nu)$ *preserves measurability* if it satisfies the condition

$$(\upsilon \odot \tau)(E) \in \Sigma \widehat{\otimes} T \quad \text{for all } E \in \Sigma \widehat{\otimes} T.$$

Proposition 4.1 *If $\upsilon \in P(\mu)$ is an Ω-marginal and $\tau \in P(\nu)$ is a Θ-marginal, then (υ, τ) preserves measurability.*

Proof Let $E \in \Sigma \widehat{\otimes} T$ be arbitrary. Then $\tau_\bullet(E) \in \Sigma \widehat{\otimes} T$ by the marginality assumption applied for τ and E. Then, $\upsilon^\bullet(\tau_\bullet(E)) \in \Sigma \widehat{\otimes} T$ again by the marginality assumption applied for υ and the set $\tau_\bullet(E)$. But $\upsilon^\bullet(\tau_\bullet(E)) = \{(\omega, \theta) : [\tau_\bullet(E)]^\theta \in \Sigma \;\&\; \omega \in \upsilon([\tau_\bullet(E)]^\theta)\} = (\upsilon \odot \tau)(E)$. □

Proposition 4.2 *Assume that $\upsilon \in P(\mu)$ is arbitrary, $\tau \in F(\nu)$ is a weak Θ-marginal and (υ, τ) preserves measurability. Let $\widetilde{\upsilon} \in \vartheta(\mu)$ and $\widetilde{\tau} \in F(\nu)$ dominate υ and τ, respectively. Then, $(\widetilde{\upsilon}, \widetilde{\tau})$ preserves measurability.*

Proof Since $\widetilde{\tau}$ is also a weak Θ-marginal, for every $E \in \Sigma \widehat{\otimes} T$ there is $M_E \in T_0$ such that $[\tau_\bullet(E)]^\theta \in \Sigma$ and $[\widetilde{\tau}_\bullet(E)]^\theta \in \Sigma$, for every $\theta \notin M_E$. Consequently, if $\theta \notin M_E$, then

$$\upsilon([\tau_\bullet(E)]^\theta) \subseteq \widetilde{\upsilon}([\widetilde{\tau}_\bullet(E)]^\theta),$$

what yields

$$[(\upsilon \odot \tau)(E)]^\theta \subseteq [(\widetilde{\upsilon} \odot \widetilde{\tau})(E)]^\theta \quad \text{for every } \theta \notin M_E.$$

This proves that $(\widetilde{\upsilon}, \widetilde{\tau})$ preserves measurability. □

Proposition 4.3 *For $\upsilon \in P(\mu)$ and $\tau \in P(\nu)$ the product $\upsilon \odot \tau$ satisfies condition* (L1) *if and only if the pair $(\upsilon, \tau) \in P(\mu) \times P(\nu)$ preserves measurability and τ is a weak Θ-marginal.*

Proof Let the pair $(\upsilon, \tau) \in P(\mu) \times P(\nu)$ preserve measurability and write $\varphi := \upsilon \odot \tau$. Then $\varphi(E) \in \Sigma \widehat{\otimes} T$ for $E \in \Sigma \widehat{\otimes} T$ and it follows from Cavalieri's principle that $(\mu \widehat{\otimes} \nu)(\varphi(E) \Delta E) = \int \mu([\varphi(E)]^\theta \Delta E^\theta)\, d\nu(\theta) = 0$, since for ν-almost all θ we have $[\varphi(E)]^\theta = \upsilon([\tau_\bullet(E)]^\theta) = \upsilon(E^\theta) =_\mu E^\theta$, where we use the assumption that τ is a weak Θ-marginal. Thus, $\varphi(E) = E$ a.e.$(\mu \widehat{\otimes} \nu)$.

Conversely it follows from (L1) for $\varphi = \upsilon \odot \tau$ that the pair (υ, τ) preserves measurability by the completeness of the space $(\Omega \times \Theta, \Sigma \widehat{\otimes} T, \mu \widehat{\otimes} \nu)$. From Fubini's theorem we infer from (L1) that for all $E \in \Sigma \widehat{\otimes} T$ there exists a set $M_E \in T_0$, such that $[\tau_\bullet(E)]^\theta =_\mu \upsilon([\tau_\bullet(E)]^\theta = [\varphi(E)]^\theta =_\mu E^\theta$ for all $\theta \notin M_E$. □

Proposition 4.4 *Assume that the pair $(\upsilon, \tau) \in P(\mu) \times P(\nu)$ preserves measurability. If $\tau \in P(\nu)$ is a weak Θ-marginal, then $\upsilon \odot \tau \in P(\mu \widehat{\otimes} \nu)$. If in addition $\upsilon \in \vartheta(\mu)$ and $\tau \in \vartheta(\nu)$ then $\upsilon \odot \tau \in F(\mu \widehat{\otimes} \nu)$.*

Proof The product $\upsilon \odot \tau$ satisfies condition (L2) and (N) by Lemma 4.1(iii), so we need only to show condition (L1). But (L1) follows from Proposition 4.3.

Lemma 4.1(v) yields $\upsilon \odot \tau \in F(\mu \widehat{\otimes} \nu)$ for $\upsilon \in \vartheta(\mu)$ and $\tau \in \vartheta(\nu)$. □

Corollary 4.1 *If $\upsilon \in \vartheta(\mu)$ is an Ω-marginal and $\tau \in \vartheta(\nu)$ is a Θ-marginal, then $\upsilon \odot \tau \in F(\mu \widehat{\otimes} \nu)$.*

Proof By Proposition 4.1 the pair (υ,τ) preserves measurability. Since $\tau\in\vartheta(\nu)$ is a weak Θ-marginal, we have $\upsilon\odot\tau\in F(\mu\widehat{\otimes}\nu)$ by Proposition 4.4. □

Theorem 4.1 *If the pair $(\upsilon,\tau)\in\vartheta(\mu)\times\vartheta(\nu)$ preserves measurability and τ is a weak Θ-marginal and generates μ-measurable sections, then $\upsilon\odot\tau\in\vartheta(\mu\widehat{\otimes}\nu)$.*

The same is true if we replace $\vartheta(\mu)$, $\vartheta(\nu)$, and $\vartheta(\mu\widehat{\otimes}\nu)$ by $O(\mu)$, $O(\nu)$, and $O(\mu\widehat{\otimes}\nu)$, respectively.

Proof Since τ is a weak Θ-marginal we obtain $\upsilon\odot\tau\in F(\mu\widehat{\otimes}\nu)$ by Proposition 4.4. Then Lemma 4.2 implies $\upsilon\odot\tau\in O(\mu\widehat{\otimes}\nu)$. Hence $\upsilon\odot\tau\in F(\mu\widehat{\otimes}\nu)\cap O(\mu\widehat{\otimes}\nu)=\vartheta(\mu\widehat{\otimes}\nu)$.

The proof for monotone liftings is contained in the above. □

Corollary 4.2 *If $\upsilon\in\vartheta(\mu)$ is an Ω-marginal and $\tau\in\vartheta(\nu)$ is μ-smooth (in particular if $\tau\in A\vartheta(\nu)$), then $\upsilon\odot\tau\in\vartheta(\mu\widehat{\otimes}\nu)$.*

Proof By Proposition 4.1 the pair (υ,τ) preserves measurability. Now apply Theorem 4.1. □

Corollary 4.3 *If $\tau\in A\vartheta(\nu)$ and $\upsilon\in A\vartheta(\mu)$ then $\upsilon\odot\tau\in\vartheta(\mu\widehat{\otimes}\nu)$.*

Proof It follows from [10, Corollary 2.7], that υ is ν-smooth and τ is μ-smooth, hence according to Corollary 4.2 $\upsilon\odot\tau\in\vartheta(\mu\widehat{\otimes}\nu)$. □

Before the next theorem we need an additional step. The notion of the upper hull appears in a paper of J. Gapaillard [2]. For a map $\varpi:\Sigma\longrightarrow\mathcal{P}(\Omega)$ we define the *upper hull of* ϖ by means of

$$(\varpi^m)(A):=\bigcup_{A\supseteq B\in\Sigma}\varpi(B).$$

For given $\xi\in F(\mu)$ we denote by $\Lambda_\xi(\mu):=\{\rho\in\Lambda(\mu):\xi\le\rho\}$ the set of all liftings dominating ξ.

Proposition 4.5 *For given map $\varpi:\Sigma\longrightarrow\mathcal{P}(\Omega)$ the map $\varpi^m:\Sigma\longrightarrow\mathcal{P}(\Omega)$ has the following properties.*

(i) *ϖ^m satisfies condition* (O), *$\varpi\le\varpi^m$, and for any map $\xi:\Sigma\longrightarrow\mathcal{P}(\Omega)$ satisfying condition* (O) *and such that $\varpi\le\xi$, the relation $\varpi^m\le\xi$ holds true.*
(ii) *If $\varpi\in F(\mu)$, then $\varpi^m\in\vartheta(\mu)$.*
(iii) *If $\varpi\in F(\mu)$ satisfies condition* (U) *in addition, then $\varpi^m\in\Lambda(\mu)$.*
(iv) *If $\varpi\in F(\mu)$, then $\Lambda_\varpi(\mu)=\Lambda_{\varpi^m}(\mu)$.*

Proof (i) is obvious and ad (ii) note that ϖ^m is a monotone lifting by [2], hence $\varpi^m(A\cap B)\subseteq\varpi^m(A)\cap\varpi^m(B)$ for all $A,B\in\Sigma$, and it is sufficient to show that $\varpi^m\in F(\mu)$. Indeed let $A,B,C,D\in\Sigma$ with $A\supseteq C$ and $B\supseteq D$ be given. It follows $A\cap B\supseteq C\cap D$, hence $\varpi^m(A\cap B)\supseteq\varpi(C\cap D)\supseteq\varpi(C)\cap\varpi(D)$ and for fixed D we have $\varpi^m(A\cap B)\supseteq(\bigcup_{A\supseteq C\in\Sigma}\varpi(C))\cap\varpi(D)$ hence $\varpi^m(A\cap B)\supseteq\varpi^m(A)\cap\varpi(D)$

for all D with $B \supseteq D \in \Sigma$, consequently $\varpi^m(A \cap B) \supseteq \varpi^m(A) \cap \bigcup_{B \supseteq D \in \Sigma} \varpi(D)$, i.e. $\varpi^m(A \cap B) \supseteq \varpi^m(A) \cap \varpi^m(B)$.

Ad (iii). By (ii) we have only to show that ϖ^m satisfies condition (U) too. But for all $A \in \Sigma$ we get $\Omega = \varpi(A) \cup [\varpi(A)]^c = \varpi(A) \cup \varpi(A^c) \subseteq \varpi^m(A) \cup \varpi^m(A^c) \subseteq \Omega$, hence $\varpi^m(A) \cup \varpi^m(A^c) = \Omega$.

If $A \cap B = \emptyset$ for some $A, B \in \Sigma$ and if $C \subseteq A$, $D \subseteq B$ for other $C, D \in \Sigma$ we get $\varpi(C) \cap \varpi(D) \subseteq \varpi(C \cap D) = \emptyset$, and this implies $\varpi^m(A) \cap \varpi^m(B) = \emptyset$, in particular $\varpi^m(A) \cap \varpi^m(A^c) = \emptyset$. Together with $\varpi^m(A) \cup \varpi^m(A^c) = \Omega$ we infer $\varpi^m(A^c) = [\varpi^m(A)]^c$ for all $A \in \Sigma$.

Item (iv) follows from the minimality condition satisfied by ϖ^m according to (i). □

Definition 4.3 For $\upsilon \in \mathcal{P}(\Omega)^\Sigma$ and $\tau \in \mathcal{P}(\Theta)^T$ we define the $\boxdot$-*product* $\upsilon \boxdot \tau : \Sigma \widehat{\otimes} T \longrightarrow \mathcal{P}(\Omega \times \Theta)$ by $\upsilon \boxdot \tau = (\upsilon \odot \tau)^m$.

Lemma 4.3 *Assume that $\upsilon \in \vartheta(\mu)$ and $\tau \in \vartheta(\nu)$ are arbitrary. Then, $\upsilon \boxdot \tau \geq \upsilon \odot \tau$, and $(\upsilon \boxdot \tau)(A \times B) = \upsilon(A) \times \tau(B)$ for all $A \in \Sigma$ and all $B \in T$.*

Proof The first inequality is a consequence of the construction of the $\boxdot$-product.

To prove the product property, let us fix an arbitrary $A \in \Sigma$ and $B \in T$. If $F \in \Sigma \otimes T$ and $F \subseteq A \times B$, then $\tau(F_\omega) \subseteq \tau[(A \times B)_\omega]$ for every $\omega \in \Omega$ and, consequently $\tau_\bullet(F) \subseteq A \times \tau(B)$ by Remark 3.1. It follows that if $\theta \in \tau(B)$ is arbitrary and $[\tau_\bullet(E)]^\theta \in \Sigma$, then $[(\upsilon \odot \tau)(F)]^\theta = \upsilon([\tau_\bullet(F)]^\theta) \subseteq \upsilon([A \times \tau(B)]^\theta) \subseteq \upsilon(A)$. Otherwise $[(\upsilon \odot \tau)(F)]^\theta = \emptyset$. Hence, $\upsilon \odot \tau(F) \subseteq \upsilon(A) \times \tau(B) = \upsilon \odot \tau(A \times B)$. And so $(\upsilon \boxdot \tau)(A \times B) \subseteq (\upsilon \odot \tau)(A \times B)$. Conversely, by Lemma 4.1 we get $\upsilon(A) \times \tau(B) = (\upsilon \odot \tau)(A \times B) \subseteq (\upsilon \boxdot \tau)(A \times B)$.

Hence $\upsilon \boxdot \tau(A \times B) = \upsilon(A) \times \tau(B)$. □

Lemma 4.4 *For $\varphi \in F(\mu \widehat{\otimes} \nu)$ and $\upsilon \in O(\mu)$ the following holds true.*

(i) *From $\upsilon([\varphi(E)]^\theta) \supseteq [\varphi(E)]^\theta$ for all $E \in \Sigma \widehat{\otimes} T$ and all $\theta \in \Theta$ follows $\upsilon([\varphi^m(E)]^\theta) \supseteq [\varphi^m(E)]^\theta$ for all $E \in \Sigma \widehat{\otimes} T$ and all $\theta \in \Theta$.*

(ii) *If for each $E \in \Sigma \widehat{\otimes} T$ exists a set $M_E \in T_0$ such that $\upsilon([\varphi(E)]^\theta) \subseteq [\varphi(E)]^\theta$ for every $\theta \notin M_E$ then $\upsilon([\varphi^m(E)]^\theta) \subseteq [\varphi^m(E)]^\theta$ for every $\theta \notin N_E$, where $N_E \in T_0$ and $N_E \supseteq M_E$.*

(iii) *If for each $E \in \Sigma \widehat{\otimes} T$ exists a set $M_E \in T_0$ such that $\upsilon([\varphi(E)]^\theta) = [\varphi(E)]^\theta$ for every $\theta \notin M_E$ then $\upsilon([\varphi^m(E)]^\theta) = [\varphi^m(E)]^\theta$ for every $\theta \notin N_E$, where $N_E \in T_0$ and $N_E \supseteq M_E$.*

Proof Ad (i). For all $E, F \in \Sigma \widehat{\otimes} T$ with $F \subseteq E$ we get $[\varphi^m(E)]^\theta \supseteq [\varphi(F)]^\theta$, hence $\upsilon([\varphi^m(E)]^\theta) \supseteq \upsilon([\varphi(F)]^\theta) \supseteq [\varphi(F)]^\theta$ for every $\theta \in \Theta$. This implies $\upsilon([\varphi^m(E)]^\theta) \supseteq [\varphi^m(E)]^\theta$ for all $\theta \in \Theta$.

Ad (ii). For $E \in \Sigma \widehat{\otimes} T$ we have $\varphi^m(E) = \varphi(E)$ a.e. $(\mu \widehat{\otimes} \nu)$. By Fubini's theorem there exists a $M_E \in T_0$ with $[\varphi^m(E)]^\theta = [\varphi(E)]^\theta$ a.e.(μ) for every $\theta \notin M_E$. This implies the existence of $N_E \in T_0$ such that $N_E \supseteq M_E$ and $\upsilon([\varphi^m(E)]^\theta) = \upsilon([\varphi(E)]^\theta) \subseteq [\varphi(E)]^\theta \subseteq [\varphi^m(E)]^\theta$ for every $\theta \notin N_E$.

Ad (iii). For all $E, F \in \Sigma \widehat{\otimes} T$ with $F \subseteq E$ and all $\theta \notin M_E$ we have $\upsilon([\varphi^m(F)]^\theta) \supseteq \upsilon([\varphi(F)]^\theta) = [\varphi(F)]^\theta$ and this implies the existence of $N_E \in T_0$ such that $N_E \supseteq M_E$

and $\upsilon([\varphi^m(E)]^\theta) \supseteq [\varphi^m(E)]^\theta$ for every $\theta \notin N_E$. The inverse equation holds true by (ii). □

Theorem 4.2 *Let the pair $(\upsilon, \tau) \in \vartheta(\mu) \times \vartheta(\nu)$ preserve measurability and let τ be a weak Θ-marginal, Then*

(i) $\upsilon \boxdot \tau \in \vartheta(\mu \widehat{\otimes} \nu)$, *and* $\varphi \geq \upsilon \boxdot \tau$ *for all* $\varphi \in \vartheta(\mu \widehat{\otimes} \nu)$ *with* $\varphi \geq \upsilon \odot \tau$.
(ii) $[(\upsilon \boxdot \tau)(E)]^\theta \subseteq \upsilon([(\upsilon \boxdot \tau)(E)]^\theta)$ *for all* $E \in \Sigma \hat{\otimes} T$ *and all* $\theta \in \Theta$.
(iii) *For every* $E \in \Sigma \hat{\otimes} T$ *there exists a set* $M_E \in T_0$ *such that* $[(\upsilon \boxdot \tau)(E)]^\theta = \upsilon([(\upsilon \boxdot \tau)(E)]^\theta)$ *for all* $\theta \notin M_E$.
(iv) *If in addition* $\upsilon \in \Lambda(\mu)$ *and* $\tau \in \Lambda(\nu)$, *then for every* $E \in \Sigma \hat{\otimes} T$ *there exists a set* $K_E \in T_0$ *such that* $[(\upsilon \boxdot \tau)(E^c)]^\theta = ([(\upsilon \boxdot \tau)(E)]^c)^\theta)$ *for all* $\theta \notin K_E$.

Proof (i) follows from Proposition 4.4 and Proposition 4.5. (ii) follows from Lemma 4.4(i), and (iii) from Lemma 4.4(iii).

To prove (iv), notice first that $\tau_\bullet(E^c) = [\tau_\bullet(E)]^c$ for all $E \in \Sigma \otimes T$, according to Remark 3.1(v). Then, let $K_E := \{\theta : [\tau_\bullet(E)]^\theta \notin \Sigma\}$. Then $(\omega, \theta) \in (\upsilon \odot \tau)(E^c) \Longleftrightarrow \omega \in \upsilon[([\tau_\bullet(E)]^\theta)^c] \Longleftrightarrow \omega \notin \upsilon([\tau_\bullet(E)]^\theta) \Longleftrightarrow (\omega, \theta) \in [(\upsilon \odot \tau)(E)]^c$ for all $(\omega, \theta) \in \Omega \times K_E^c$. □

The assumptions of the last theorem are in particular satisfied if υ is a Ω-marginal and τ is a Θ-marginal (apply Proposition 4.1 and the comments following Definition 3.2).

Theorem 4.3 *For each $\rho \in \Lambda(\mu)$ and each weak Θ-marginal $\tau \in \vartheta(\nu)$ such that (ρ, τ) preserves measurability, there exist $\sigma \in \Lambda(\nu)$ dominating τ and $\pi_2 \in \Lambda(\mu \widehat{\otimes} \nu)$ dominating $\rho \boxdot \sigma$ such that*

(i) $\pi_2 \in \rho \otimes \sigma$
(ii) $[\pi_2(E)]^\theta = \rho([\pi_2(E)]^\theta)$ *for all* $\theta \in \Theta$ *and* $E \in \Sigma \widehat{\otimes} T$.

Proof Let

$$\begin{aligned}\Phi := \{&\varphi \in \vartheta(\mu \widehat{\otimes} \nu) : \forall \theta \in \Theta \forall E \in \Sigma \widehat{\otimes} T [\varphi(E)]^\theta \subseteq \rho([\varphi(E)]^\theta) \\ &\& \forall E \in \Sigma \widehat{\otimes} T \, \upsilon \boxdot \tau(E) \subseteq \varphi(E) \, \& \, \forall B \in T \forall \theta \in \Theta [\varphi(\Omega \times B)]^\theta \in \{\Omega, \emptyset\}\}.\end{aligned}$$

Notice first that $\rho \boxdot \tau \in \Phi$, since according to Theorem 4.2 and Lemma 4.3 we have $\rho \boxdot \tau \in \vartheta(\mu \widehat{\otimes} \nu)$, $[\rho \boxdot \tau(E)]^\theta \subseteq \rho([\rho \boxdot \tau(E)]^\theta)$ and $\rho \boxdot \tau(\Omega \times B) = \Omega \times \tau(B)$.

We consider Φ with inclusion as the partial order: $\varphi \leq \widetilde{\varphi}$ if $\varphi(E) \subseteq \widetilde{\varphi}(E)$ for each $E \in \Sigma \widehat{\otimes} T$. Following arguments analogous to those in the proof of Lemma 2.8 from [10] we can find a maximal element in Φ, which we denote by π_2 satisfying the following properties:

$$\mu([\pi_2(E)]^\theta \cup [\pi_2(E^c)]^\theta) = 1 \quad \text{for all } \theta \in \Theta \text{ and } E \in \Sigma \widehat{\otimes} T$$

and

$$[\pi_2(E)]^\theta = \rho([\pi_2(E)]^\theta) \quad \text{for all } \theta \in \Theta \text{ and } E \in \Sigma \widehat{\otimes} T.$$

Setting $\sigma(B) := \{\theta : [\pi_2(\Omega \times B)]^\theta = \Omega\}$ for each $B \in T$, we get $\sigma \in \Lambda(\nu)$ satisfying condition (i). □

The next result is complementary to Theorem 2.13 of [10].

Corollary 4.4 *Let $\rho \in \Lambda(\mu)$ be arbitrary and let $\sigma \in AG\Lambda(\nu)$ be generated by an admissible density $\tau \in A\vartheta(\nu)$. If (ρ, τ) preserves measurability, then there exists $\pi_2 \in \Lambda(\mu \widehat{\otimes} \nu)$ dominating $\rho \boxdot \sigma$ and such that*

(i) $\pi_2 \in \rho \otimes \sigma$
(ii) $[\pi_2(E)]^\theta = \rho([\pi_2(E)]^\theta)$ *for all $\theta \in \Theta$ and $E \in \Sigma \widehat{\otimes} T$.*

Proof By assumption, σ dominates τ, that is $\tau(B) \subseteq \sigma(B)$ for every $B \in T$. Since $\tau \in A\vartheta(\nu)$, it is μ-smooth, and so it follows from Proposition 4.2 that (ρ, σ) preserves measurability. The rest follows from Theorem 4.3, since lifting can be dominated only by itself. □

5 Non-Existence Results

In case $(\Omega, \Sigma, \mu) = (\Theta, T, \nu)$ we consider the measurable, self-inverse bijection $s : (\omega, \theta) \in \Omega \times \Omega \longrightarrow (\theta, \omega) \in \Omega \times \Omega$.

For $\varphi \in P(\mu \widehat{\otimes} \mu)$ put $\varphi^s(E) := [\varphi(E^s)]^s$ for all $E \in \Sigma \hat{\otimes} \Sigma$.

In order to distinguish Ω-marginals from Θ-marginals despite of $\Omega = \Theta$, we write 1-marginal instead of Ω-marginal and 2-marginal instead of Θ-marginal.

Lemma 5.1 *The map $\varphi \in P(\mu \widehat{\otimes} \mu) \longrightarrow \varphi^s \in P(\mu \widehat{\otimes} \mu)$ is a self-inverse bijection with $[\varphi^s]^s = \varphi$.*

(i) $\varphi \in \vartheta(\mu \widehat{\otimes} \mu)$ *if and only if* $\varphi^s \in \vartheta(\mu \widehat{\otimes} \mu)$.
(ii) $[\varphi^s(E)]_\omega = [\varphi(E^s)]^\omega$ *and* $[\varphi^s(E)]^\omega = [\varphi(E^s)]_\omega$ *for all $E \in \Sigma \hat{\otimes} \Sigma$ and all $\omega \in \Omega$.*
(iii) *$\sigma \in P(\mu)$ is a 1-marginal if and only if it is a 2-marginal, and the same for μ-smooth and weak i-marginal, $i = 1, 2$. For this reason we speak of (μ-smooth and weak) marginal, respectively, in this situation.*

Proof (i) and (ii) are obvious.

Ad (iii). For a 1-marginal $\sigma \in P(\mu)$ choose $\varphi \in P(\mu \widehat{\otimes} \mu)$ such that for all $E \in \Sigma \widehat{\otimes} \Sigma$ exists $N_E \in \Sigma_0$ with $\sigma([\varphi(E)]_\omega) = [\varphi(E)]_\omega$ for all $\omega \notin N_E$. This implies

$$\sigma([\varphi^s(E)]^\omega) = \sigma([\varphi(E^s)]_\omega) = [\varphi(E^s)]_\omega = [\varphi^s(E)]^\omega \quad \text{for all } \omega \notin N_E.$$

In case of μ-smooth σ note that $[\varphi(E)]^\omega \in \Sigma$ implies $[\varphi^s(E)]_\omega = [\varphi(E^s)]^\omega \in \Sigma$ for all $\omega \in \Omega$.

For weak i-marginals apply (ii). □

Theorem 5.1 *Denote by $(\Omega, \mathcal{S}, \Sigma, \mu)$ the hyperstonian space of the Lebesgue probability space on $[0, 1]$ and consider the product $(\Omega \times \Omega, \mathcal{S} \widehat{\otimes} \mathcal{S}, \mu \widehat{\otimes} \mu)$. Let σ be* *the canonical strong lifting on* $(\Omega, \mathcal{S}, \Sigma, \mu)$. *Then* σ *is neither* 1- *nor* 2-*marginal and the condition* (L1) *fails for* $\sigma_\bullet$ *and* $\sigma^\bullet$. *Moreover, neither* $\sigma_\bullet(E) \in \Sigma \widehat{\otimes} \Sigma$ *for all* $E \in \Sigma \widehat{\otimes} \Sigma$ *nor* $\sigma^\bullet(E) \in \Sigma \widehat{\otimes} \Sigma$ *for all* $E \in \Sigma \widehat{\otimes} \Sigma$.

Proof Suppose if possible that σ is a 1-marginal. Then, by Lemma 5.1, it is also a 2-marginal and so, according to Proposition 4.1 the pair (σ, σ) preserves measurability. So applying Theorem 4.3 one can find $\pi \in \Lambda(\mu \widehat{\otimes} \mu) \cap (\sigma \otimes \sigma)$. The latter implies for the lifting topologies $\tau_\sigma := \{A \in \Sigma : A \subseteq \sigma(A)\}$ and $\tau_\pi := \{E \in \Sigma \widehat{\otimes} \Sigma : E \subseteq \pi(E)\}$ of σ and π, respectively, that $\tau_\pi \supseteq \tau_\sigma \times \tau_\sigma$, since for $A, B \in \tau_\sigma$ follows $A \times B \subseteq \sigma(A) \times \sigma(B) = \pi(A \times B)$ (compare [9]), hence $\mathcal{S} \times \mathcal{S} \subseteq \tau_\sigma \times \tau_\sigma \subseteq \tau_\pi \subseteq \Sigma \widehat{\otimes} \Sigma$, so $\mathcal{S} \times \mathcal{S} \subseteq \Sigma \widehat{\otimes} \Sigma$, a contradiction according to Fremlin [1]. The remaining facts follow from Proposition 3.1. □

Remark 5.1 (a) If we assume that the pair (σ, τ) preserves measurability where σ is the canonical strong lifting on the hyperstonian space $(\Omega, \mathcal{S}, \Sigma, \mu)$ of the Lebesgue measure space over $[0, 1]$ and $\tau \in P(\mu)$ is a weak marginal, then by Proposition 4.4 we get $\varphi = \sigma \odot \tau \in P(\mu \widehat{\otimes} \mu)$ and σ is a marginal of φ by Lemma 4.1, (ii), contradicting Theorem 5.1.

This means that the pair (σ, τ) does not preserves measurability. By Lemma 4.3 the latter implies that $\upsilon \odot \tau$ does not satisfy condition (L1).

(b) In [10, Question 4.7], we ask if for given complete probability spaces (Ω, Σ, μ) and (Θ, T, ν) there exist $\rho \in \Lambda(\mu)$, $\sigma \in \Lambda(\nu)$ and $\pi \in \Lambda(\mu \widehat{\otimes} \nu)$ such that ($\pi \in \rho \otimes \sigma$ and) for each $E \in \Sigma \widehat{\otimes} T$ there exist sets $N_E \in \Sigma_0$ and $M_E \in T_0$ with the property that whenever $\omega \notin N_E$ and $\theta \notin M_E$ then

$$\rho([\pi(E)]^\theta) = [\pi(E)]^\theta \quad \text{and} \quad \sigma([\pi(E)]_\omega) = [\pi(E)]_\omega.$$

It follows from Theorem 5.1 and Proposition 3.1 that if we consider strong liftings on hyperstonian spaces, then the answer to above question is to the negative.

Question 5.1 What is the situation in case of the Radon product of hyperstonian spaces?

The following theorem improves Theorem $4.3^\perp$ from [10] and has a similar proof, which we enclose for completeness.

Theorem 5.2 *Let* (Ω, Σ, μ) *be non-atomic and let* (Θ, T, ν) *be non-atomic and perfect. There exists no lifting* $\rho \in \Lambda(\mu)$ *and no* $\varphi \in P(\mu \widehat{\otimes} \nu)$ *satisfying the following two conditions*:

(j) *There exists* $\overline{\omega} \in \Omega$ *such that for each* $E \in \Sigma \widehat{\otimes} T$

$$[\varphi(E)]_{\overline{\omega}} \in T;$$

(jj) *For each* $E \in \Sigma \widehat{\otimes} T$ *there exists a set* $M_E \in T_0$ *such that*

$$[\varphi(E)]^\theta = \rho([\varphi(E)]^\theta) \quad \textit{for each } \theta \notin M_E.$$

Proof Suppose that a lifting ρ and a $\varphi \in P(\mu \widehat{\otimes} \nu)$ satisfying all the above assumptions exist. If $E \in \Sigma \widehat{\otimes} T$, then by the Fubini Theorem there exists a set $K_E \in T_0$ such that

$$[\varphi(E)]^\theta = E^\theta \quad \text{a.e.}(\mu), \text{ for each } \theta \notin K_E.$$

Set $L_E = K_E \cup M_E$. Then for each $\omega \in \Omega$

$$\{\theta \in \Theta : E^\theta \in \mathcal{U}(\omega)\} \cap (\Theta \setminus L_E) = [\varphi(E)]_\omega \cap L_E^c.$$

Since (Θ, T, ν) is complete and $[\varphi(E)]_{\overline{\omega}} \in T$, we have

$$\{\theta \in \Theta : E^\theta \in \mathcal{U}(\overline{\omega})\} \in T$$

for every $E \in \Sigma \widehat{\otimes} T$, if $\mathcal{U}(\overline{\omega}) := \{A \in \Sigma : \overline{\omega} \in \rho(A)\}$ for all $\overline{\omega} \in \Omega$. This however contradicts Theorem $4.2^\perp$ from [10], since each $\mathcal{U}(\omega)$ is a measure stable ultrafilter. □

The last theorem remains true if we replace the σ-algebra $\Sigma \otimes T$ by a σ-algebra $\Xi \supseteq \Sigma \otimes T$ and the measure $\mu \otimes \nu$ by a probability measure ξ on Ξ such that $\xi | \Sigma \otimes T = \mu \otimes \nu$, and such that the Fubini Theorem is satisfied for $(\Omega \times \Theta, \Xi, \xi)$.

Note that following the same arguments as in the proof of Theorem 5.2 we can prove the following:

Theorem 5.3 *Denote by $(\Omega, \mathcal{S}, \Sigma, \mu)$ the hyperstonian space of the Lebesgue probability space on $[0, 1]$ and by σ its canonical strong lifting. Then there exists no $\varphi \in P(\mu \widehat{\otimes}_R \mu)$, where $\mu \widehat{\otimes}_R \mu$ denotes the completed Radon product of μ, satisfying the following two conditions*

(j) *There exists $\overline{\theta} \in \Omega$ such that for each $E \in \Sigma \widehat{\otimes}_R \Sigma$*

$$[\varphi(E)]^{\overline{\theta}} \in \Sigma;$$

(jj) *For each $E \in \Sigma \widehat{\otimes}_R \Sigma$ there exists a set $N_E \in \Sigma_0$ such that*

$$[\varphi(E)]_\omega = \sigma([\varphi(E)]_\omega) \quad \textit{for each } \omega \notin N_E.$$

Corollary 5.1 *Let (Ω, Σ, μ) and (Θ, T, ν) be as in Theorem 5.2 and let $\rho \in \Lambda(\mu), \sigma \in AG\Lambda(\nu)$, and $\pi_2 \in \Lambda(\mu \widehat{\otimes} \nu)$ and liftings satisfying the conclusion of Theorem 4.3, then for each $\omega \in \Omega$ there exists $E \in \Sigma \widehat{\otimes} T$ such that $[\pi_2(E)]_\omega$ is non-measurable.*

It follows from the above corollary, that Theorem 4.3 is the best possible result, at least when ν is perfect.

Corollary 5.2 *Let (Ω, Σ, μ) be non-atomic and perfect, and (Θ, T, ν) be non-atomic, and let $\rho \in \Lambda(\mu), \sigma \in AG\Lambda(\nu)$ and $\pi_2 \in \Lambda(\mu \widehat{\otimes} \nu)$ be liftings satisfying the conclusion of Theorem 4.3. Then there exists $E \in \Sigma \widehat{\otimes} T$ such that*

$$\mu^*\{\omega \in \Omega : [\pi_2(E)]_\omega \neq \sigma([\pi_2(E)]_\omega)\} > 0.$$

The following question remains open:

Question 5.2 Let $\upsilon \in \vartheta(\mu) \setminus \Lambda(\mu)$, $\tau \in A\vartheta(\nu)$ and $\varphi \in \vartheta(\mu \widehat{\otimes} \nu)$ be as in Theorem 4.2. Is the set $[\varphi(E)]_\omega$ measurable for each $\omega \in \Omega$?

Proposition 5.1 *Let the pair $(\upsilon, \tau) \in \Lambda(\mu) \times \Lambda(\nu)$ preserve measurability and let $\tau \in \Lambda(\nu)$ be a Θ-marginal. Then the following conditions are equivalent.*

(i) *τ is μ-smooth;*
(ii) *$\upsilon \odot \tau \in \Lambda(\mu \widehat{\otimes} \nu)$.*

Proof If τ is μ-smooth then for all $E \in \Sigma \widehat{\otimes} T$ and all $\theta \in \Theta$ we have $[\tau_\bullet(E)]^\theta \in \Sigma$. It follows from Proposition 4.4 that $\upsilon \odot \tau \in F(\mu \widehat{\otimes} \nu)$. It will suffice to show $(\upsilon \odot \tau)(E^c) = [(\upsilon \odot \tau)(E)]^c$ for all $E \in \Sigma \otimes T$. Then Proposition 4.5(iii) shows $\upsilon \odot \tau \in \Lambda(\mu \widehat{\otimes} \nu)$. We have

$$\begin{aligned}(\omega, \theta) \in (\upsilon \odot \tau)(E^c) &\Longleftrightarrow \omega \in \upsilon([\tau_\bullet(E^c)]^\theta) = \upsilon([[\tau_\bullet(E)^c]^\theta) = \upsilon([[\tau_\bullet(E)]^\theta])^c \\ &\Longleftrightarrow \omega \notin \upsilon([\tau_\bullet(E)]^\theta) \Longleftrightarrow (\omega, \theta) \in [(\upsilon \odot \tau)(E)]^c \\ &\qquad \text{for all } (\omega, \theta) \in \Omega \times \Theta.\end{aligned}$$

Hence (i) implies (ii).

For the converse implication note that $(\upsilon \odot \tau)(E^c) = [(\upsilon \odot \tau)(E)]^c$ for all $E \in \Sigma \otimes T$ yields for each θ either $[(\upsilon \odot \tau)(E)]^\theta \neq \emptyset$ or $[(\upsilon \odot \tau)(E^c)]^\theta \neq \emptyset$ and so $[\tau_\bullet(E)]^\theta \in \Sigma$ for every θ, i.e. τ generates μ-measurable sections. Being a Θ-marginal, τ is μ-smooth. □

The next result says that in practically all situations where the application of liftings makes sense, the Fubini type product as well as the box product of liftings is never a lifting.

Theorem 5.4 *Let (Ω, Σ, μ) be non-atomic and let (Θ, T, ν) be non-atomic and perfect. If the pair $(\upsilon, \tau) \in \Lambda(\mu) \times \Lambda(\nu)$ preserves measurability and $\tau \in \Lambda(\nu)$ is a Θ-marginal then $\upsilon \odot \tau \in F(\mu \widehat{\otimes} \nu)$ but $\upsilon \odot \tau \notin \Lambda(\mu \widehat{\otimes} \nu)$.*

Proof The existence of $\upsilon \odot \tau \in F(\mu \widehat{\otimes} \nu)$ follows from Proposition 4.4 and the rest from Theorem 5.2 in connection with Proposition 5.1. □

Contrary to the last theorem it was already remarked in [10] that under the assumption of real-valued measurable cardinals and taking μ to be a universal measure on Ω of measurable cardinality, we get an example of a lifting with all sections measurable. Applying Theorem 4.3, Proposition 4.2 and Proposition 5.1 we obtain the following result:

Theorem 5.5 *Let Ω be an uncountable set of measurable cardinality and let μ be a universal non-atomic probability on Ω. Let (Θ, T, ν) be arbitrary nonatomic and complete and, let $\rho \in AG\Lambda(\mu)$ and $\sigma \in AG\Lambda(\nu)$ be arbitrary. Then $\rho \boxdot \sigma \in \Lambda(\mu \widehat{\otimes} \nu)$.*

References

1. Fremlin, D.H.: Measure Theory, vol. 3. T. Fremlin (ed.) (2002)
2. Gapaillard, J.: Relevements monotones. Arch. Math. **24**, 169–178 (1973)
3. Graf, S., von Weizsäcker, H.: On the existence of lower densities in non-complete measure spaces. In: Bellow, A., Kölzow, D. (eds.) Measure Theory (Oberwolfach 1975). Lecture Notes in Math., vol. 541, pp. 155–158. Springer, Berlin (1976)
4. Ionescu Tulcea, A., Ionescu Tulcea, C.: Topics in the Theory of Lifting. Springer, Berlin (1969)
5. Macheras, N.D., Musiał, K., Strauss, W.: On products of admissible liftings and densities. Z. Anal. Anw. **18**, 651–667 (1999)
6. Macheras, N.D., Musiał, K., Strauss, W.: Linear liftings respecting coordinates. Adv. Math. **153**, 403–416 (2000)
7. Macheras, N.D., Musiał, K., Strauss, W.: On strong liftings on projective limits. Glasgow Math. J. **45**, 503–525 (2003)
8. Macheras, N.D., Strauss, W.: Products of lower densities. Z. Anal. Anw. **14**, 25–32 (1995)
9. Macheras, N.D., Strauss, W.: On products of almost strong liftings. J. Austr. Math. Soc. (Ser. A) **60**, 311–333 (1996)
10. Musiał, K., Strauss, W., Macheras, N.D.: Product liftings and densities with lifting invariant and density invariant sections. Fund. Math. **166**, 281–303 (2000)
11. Musiał, K., Strauss, W., Macheras, N.D.: Liftings for topological products of measures. In: Atti Sem, Mat. Fis. Univ. Modena e Reggio Emilia, LII, pp. 69–72 (2004)
12. Strauss, W., Macheras, N.D., Musiał, K.: Liftings. In: Handbook of Measure Theory, pp. 1131–1184. Elsevier, Amsterdam (2002)
13. Strauss, W., Macheras, N.D., Musiał, K.: Non-existence of certain types of liftings and densities in product spaces with σ-ideals. Real Anal. Exch. **29**(1), 473–479 (2003/2004)
14. Strauss, W., Macheras, N.D., Musiał, K.: Splitting of liftings in probability spaces. Ann. Probab. **32**, 2389–2408 (2004)
15. Talagrand, M.: Closed convex hull of set of measurable functions, Riemann-measurable functions and measurability of translations. Ann. Inst. Fourier (Grenoble) **32**, 39–69 (1989)