

KURZWEIL-HENSTOCK AND KURZWEIL-HENSTOCK-PETTIS
INTEGRABILITY OF STRONGLY MEASURABLE FUNCTIONS

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(Received November 16, 2005)

Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday

Abstract. We study the integrability of Banach valued strongly measurable functions defined on $[0, 1]$. In case of functions f given by $\sum_{n=1}^{\infty} x_n \chi_{E_n}$, where x_n belong to a Banach space and the sets E_n are Lebesgue measurable and pairwise disjoint subsets of $[0, 1]$, there are well known characterizations for the Bochner and for the Pettis integrability of f (cf Musiał (1991)). In this paper we give some conditions for the Kurzweil-Henstock and the Kurzweil-Henstock-Pettis integrability of such functions.

Keywords: Kurzweil-Henstock integral, Kurzweil-Henstock-Pettis integral, Pettis integral

MSC 2000: 26A42, 26A39, 26A45

1. INTRODUCTION

In this paper we study the Kurzweil-Henstock and the Kurzweil-Henstock-Pettis integrability of strongly measurable functions. It is well known (cf [7, Lemma 5.1]) that each strongly measurable Banach valued function, defined on a measurable space, can be written as $f = g + \sum_{n=1}^{\infty} x_n \chi_{E_n}$, where g is a bounded strongly measurable function, x_n are vectors of the given Banach space and E_n are measurable and pairwise disjoint sets. As each bounded strongly measurable function is Bochner integrable, it is enough to study the integrability only for functions of the form $\sum_{n=1}^{\infty} x_n \chi_{E_n}$. In the case of the Bochner and Pettis integrals, a necessary and sufficient condition for the integrability of a function given by $\sum_{n=1}^{\infty} x_n \chi_{E_n}$ is, respectively, the

absolute and the unconditional convergence of the series $\sum_{n=1}^{\infty} x_n |E_n|$ (see Theorem A). In the case of the Kurzweil-Henstock or of the Kurzweil-Henstock-Pettis integrability, in general the series $\sum_{n=1}^{\infty} x_n |E_n|$ is only conditionally convergent. So the conditions for the integrability depend on the order of the terms $x_n |E_n|$. We present one sufficient condition for the Kurzweil-Henstock and the Kurzweil-Henstock-Pettis integrability of such functions.

2. BASIC FACTS

Let $[0, 1]$ be the unit interval of the real line equipped with the usual topology and the Lebesgue measure. If a set $E \subset [0, 1]$ is Lebesgue measurable, then $|E|$ denotes its Lebesgue measure. \mathcal{I} denotes the family of all closed subintervals of $[0, 1]$. A *partition in* $[0, 1]$ is a finite collection of pairs $\mathcal{P} = \{(I_1, t_1), \dots, (I_p, t_p)\}$, where I_1, \dots, I_p are nonoverlapping subintervals of $[0, 1]$ and $t_i \in I_i$, $i = 1, \dots, p$. If $\bigcup_{i=1}^p I_i = [0, 1]$ we say that \mathcal{P} is a *partition of* $[0, 1]$. Given a subset E of $[0, 1]$, we say that the partition \mathcal{P} is *anchored on* E if $t_i \in E$ for each $i = 1, \dots, p$. A *gauge* on $E \subset [0, 1]$ is a positive function on E . For a given gauge δ , we say that a partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ is δ -*fine* if $I_i \subset (t_i - \delta(x_i), t_i + \delta(x_i))$, $i = 1, \dots, p$.

Throughout this paper X is a Banach space with the dual X^* . The closed unit ball of X^* is denoted by $\mathcal{B}(X^*)$.

Definition 1. A function $f: [0, 1] \rightarrow X$ is said to be *Kurzweil-Henstock integrable*, or simply *KH-integrable*, on $[0, 1]$ if there exists $w \in X$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[0, 1]$ such that

$$\left\| \sum_{i=1}^p f(t_i) |I_i| - w \right\| < \varepsilon$$

for each δ -fine partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[0, 1]$. We set $w =: (\text{KH}) \int_0^1 f$.

We denote the set of all KH-integrable functions $f: [0, 1] \rightarrow X$ by $\text{KH}([0, 1], X)$.

The space $\text{KH}([0, 1], X)$ is endowed with the Alexiewicz norm (cf. [1])

$$\|f\|_A = \sup_{0 < \alpha \leq 1} \left\| (\text{KH}) \int_0^\alpha f(t) dt \right\|.$$

A family $\mathcal{A} \subset \text{KH}([0, 1], X)$ is said to be *Kurzweil-Henstock equiintegrable*, or simply *KH-equiintegrable*, on $[0, 1]$ if in Definition 1, for every $\varepsilon > 0$ there exists a gauge δ on $[0, 1]$ which works for all the functions in \mathcal{A} .

A function $f: [0, 1] \rightarrow X$ is said to be *scalarly Kurzweil-Henstock integrable*, or simply *scalarly KH-integrable*, if for each $x^* \in X^*$, the function x^*f is Kurzweil-Henstock integrable on $[0, 1]$.

Definition 2. A scalarly KH-integrable function $f: [0, 1] \rightarrow X$ is said to be *Kurzweil-Henstock-Dunford integrable* or simply *KHD-integrable*, if, for each non-empty interval $[a, b] \subset [0, 1]$, there exists a vector $w_{ab} \in X^{**}$ such that for every $x^* \in X^*$

$$(1) \quad \langle x^*, w_{ab} \rangle = (\text{KH}) \int_a^b x^* f(t) dt.$$

It follows from [5, Theorem 3] that a function $f: [0, 1] \rightarrow X$ is KHD-integrable if and only if f is scalarly KH-integrable.

The generalization of the Pettis integral obtained by replacing the Lebesgue integrability of the functions by the Kurzweil-Henstock integrability produces the Kurzweil-Henstock-Pettis integral (for the definition of the Pettis integral see [3]).

Definition 3. If a function $f: [0, 1] \rightarrow X$ is scalarly KH-integrable and for each subinterval $[a, b]$ of $[0, 1]$ and for each $x^* \in X^*$ there exists a vector $w_{[a,b]} \in X$ such that $x^*w_{[a,b]} = (\text{KH}) \int_a^b \langle x^*, f \rangle$, then f is said to be *Kurzweil-Henstock-Pettis integrable*, or simply *KHP-integrable*, on $[0, 1]$ and we set $w_{[a,b]} =: (\text{KHP}) \int_a^b f$.

We recall that a function $f: [0, 1] \rightarrow X$ is said to be *strongly measurable* if there is a sequence of simple functions f_n with $\lim_n \|f_n(t) - f(t)\| = 0$ for almost all $t \in [0, 1]$.

3. INTEGRATION OF STRONGLY MEASURABLE FUNCTION

The aim of this section is to give conditions for the Kurzweil-Henstock or the Kurzweil-Henstock-Pettis integrability of strongly measurable functions.

We start by recalling the following simple lemma (cf. Lemma 5.1 of [7]).

Lemma 1. *If $f: [0, 1] \rightarrow X$ is strongly measurable, then there exists a bounded strongly measurable function $g: [0, 1] \rightarrow X$ such that $f = g + \sum_{n=1}^{\infty} x_n \chi_{E_n}$ where $x_n \in X$ and the sets E_n are Lebesgue measurable and pairwise disjoint.*

Since each bounded strongly measurable function is Bochner integrable, and then Kurzweil-Henstock (and Kurzweil-Henstock-Pettis) integrable (see e.g. [4]), it is enough to give criteria of integrability only for functions of the form $\sum_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$ and the sets E_n are measurable and pairwise disjoint.

For the Bochner and Pettis integrals we have the following classical result:

Theorem A (cf [7]). Let $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$ and the sets E_n are Lebesgue measurable and pairwise disjoint subsets of $[0, 1]$. Then

- (i) f is Pettis integrable if and only if the series $\sum_{n=1}^{\infty} x_n |E_n|$ is unconditionally convergent;
- (ii) f is Bochner integrable if and only if the series $\sum_{n=1}^{\infty} x_n |E_n|$ is absolutely convergent.

In both cases $\int_E f = \sum_{n=1}^{\infty} x_n |E_n \cap E|$ for every measurable set E .

Here we would like to give similar conditions for the Kurzweil-Henstock and Kurzweil-Henstock-Pettis integrals.

Theorem 1. Let $f: [0, 1] \rightarrow X$ be defined by $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$ and the sets E_n are Lebesgue measurable and pairwise disjoint. Assume that the following condition is satisfied:

- (A) for every $\varepsilon > 0$ there exist a gauge δ and $k_0 \in \mathbb{N}$ such that given a δ -fine partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[0, 1]$ and given $s > r > k_0$ we have

$$(2) \quad \left\| \sum_{k=r}^s x_k \left| \bigcup_{t_j \in E_k} I_j \right| \right\| < \varepsilon.$$

Then f is Kurzweil-Henstock integrable with

$$(3) \quad (\text{KH}) \int_I f(t) dt = \sum_{n=1}^{\infty} x_n |E_n \cap I|$$

for every interval $I \in \mathcal{I}$.

Proof. Let $\varepsilon > 0$ be arbitrary and let δ and k_0 be respectively a gauge and a natural number such that inequality (2) is satisfied. We are going to prove first that the series $\sum_{n=1}^{\infty} x_n |E_n \cap I|$ is convergent for every $I \in \mathcal{I}$. To this purpose we set $f_m = \sum_{n=1}^m x_n \chi_{E_n}$ for every $m \in \mathbb{N}$, and observe that each function f_m is Bochner, and hence also Kurzweil-Henstock integrable, with its integral equal to $\sum_{k=1}^m x_k |E_k|$. Now let $s > r > k_0$ and, according to Definition 1, let δ_{r-1} and δ_s be two gauges related to f_{r-1} and f_s respectively, such that

$$\left\| \sum_{k=1}^{r-1} x_k |E_k| - \sum_{i=1}^p f_{r-1}(t_i) |I_i| \right\| < \varepsilon$$

and

$$\left\| \sum_{k=1}^s x_k |E_k| - \sum_{i=1}^p f_s(t_i) |I_i| \right\| < \varepsilon$$

for each partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[0, 1]$ which is both δ_{r-1} -fine and δ_s -fine. Now define $\delta^*(t) = \min\{\delta(t), \delta_{r-1}(t), \delta_s(t)\}$ and take any δ^* -fine partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[0, 1]$.

Then

$$\begin{aligned} (4) \quad \left\| \sum_{k=r}^s x_k |E_k| \right\| &= \left\| \sum_{k=1}^s x_k |E_k| - \sum_{k=1}^{r-1} x_k |E_k| \right\| \\ &\leq \left\| \sum_{k=1}^s x_k |E_k| - \sum_{i=1}^p f_s(t_i) |I_i| \right\| \\ &\quad + \left\| \sum_{k=1}^{r-1} x_k |E_k| - \sum_{i=1}^p f_{r-1}(t_i) |I_i| \right\| + \left\| \sum_{k=r}^s x_k \left| \bigcup_{t_i \in E_k} I_i \right| \right\| < 3\varepsilon. \end{aligned}$$

This proves that the series $\sum_{n=1}^{\infty} x_n |E_n|$ is norm convergent.

Now for each $i \in \mathbb{N}$ let K_i be a closed set and U_i an open set such that:

- (s1) $K_i \subseteq E_i \subseteq U_i$;
- (s2) $|U_i \setminus K_i| < 2^{-i} \varepsilon / (\|x_i\| + 1)$;
- (s3) if $j < i$, then $U_i \cap K_j = \emptyset$;
- (s4) if $i \leq k_0$, then

$$U_i \cap \bigcup_{k_0 \geq j \neq i} K_j = \emptyset.$$

It follows from (s3)–(s4) that if $\{(I_1, t_1), \dots, (I_p, t_p)\}$ is a δ -fine partition of $[0, 1]$, then $K_n \subset \bigcup_{t_i \in E_n} I_i$ for every $n \leq k_0$.

The functions f_1, f_2, \dots, f_{k_0} are Bochner, and hence also Kurzweil-Henstock integrable on $[0, 1]$. Let $\gamma(t)$ be a gauge on $[0, 1]$ such that

$$\left\| \sum_{k=1}^m x_k |E_k| - \sum_{i=1}^p f_m(t_i) |I_i| \right\| < \varepsilon$$

for each γ -fine partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[0, 1]$ and for $m = 1, 2, \dots, k_0$.

Define $\delta_0(t) = \min\{\text{dist}(t, U_i^c), \delta(t), \gamma(t)\}$ if $t \in E_i$ and let $\{(I_1, t_1), \dots, (I_p, t_p)\}$ be a δ_0 -fine partition of $[0, 1]$. For each $m > k_0$, by (2)–(4) we get

$$\begin{aligned} & \left\| \sum_{k=1}^m x_k |E_k| - \sum_{i=1}^p f_m(t_i) |I_i| \right\| \leq \left\| \sum_{k=1}^{k_0} x_k (|E_k| - \sum_{t_i \in E_k} |I_i|) \right\| \\ & \quad + \left\| \sum_{k=k_0+1}^m x_k \left| \bigcup_{t_i \in E_k} I_i \right| \right\| + \left\| \sum_{k=k_0+1}^m x_k |E_k| \right\| \\ & < \sum_{k=1}^{k_0} \|x_k\| \left| |E_k| - \left| \bigcup_{t_i \in E_k} I_i \right| \right| + 4\varepsilon \leq \sum_{k=1}^{k_0} \|x_k\| |U_k \setminus K_k| + 4\varepsilon \\ & < 2\varepsilon \sum_{k=1}^{k_0} \frac{\|x_k\|}{2^k(\|x_k\| + 1)} + 4\varepsilon < 6\varepsilon. \end{aligned}$$

Thus, the sequence (f_m) is Kurzweil-Henstock equiintegrable. Moreover, since $\lim_{m \rightarrow \infty} f_m(t) = f(t)$ in $[0, 1]$, by [8, Theorem 1] f is Kurzweil-Henstock integrable and (f_m) converges to f in the Alexiewicz topology. So, in particular, for each $I \in \mathcal{I}$ we have

$$(\text{KH}) \int_I f(t) dt = \lim_n (\text{KH}) \int_I f_n(t) dt,$$

and the assertion follows. \square

Remark 1. Within the proof of the previous theorem it is also showed that condition (A) implies the Kurzweil-Henstock equiintegrability of the sequence (f_m) . It is easy to check that the same proof can be used to prove the reverse implication. So we have also:

If the sequence $(f_m = \sum_{k=1}^m x_k \chi_{E_k})$ is Kurzweil-Henstock equiintegrable, then the function $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$ is Kurzweil-Henstock integrable and

$$(\text{KH}) \int_I f(t) dt = \sum_{n=1}^{\infty} x_n |E_n \cap I|$$

for every interval $I \in \mathcal{I}$.

Remark 2. There exist points $x_n \in X$ and pairwise disjoint Lebesgue measurable sets E_n , $n = 1, 2, \dots$, such that the series $\sum_{n=1}^{\infty} x_n |E_n \cap I|$ is convergent for every $I \in \mathcal{I}$, the function $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$ is Kurzweil-Henstock integrable and

$$(\text{KH}) \int_0^1 f(t) dt \neq \sum_{n=1}^{\infty} x_n |E_n|.$$

P r o o f. Let π be a permutation of \mathbb{N} such that the series

$$\sum_{n=1}^{\infty} (-1)^{\pi(n)} [\pi(n) + 1] \left(\frac{1}{\pi(n)} - \frac{1}{\pi(n) + 1} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{\pi(n)}}{\pi(n)}$$

is convergent but

$$\sum_{n=1}^{\infty} \frac{(-1)^{\pi(n)}}{\pi(n)} \neq \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Let

$$(5) \quad f = \sum_{n=1}^{\infty} (-1)^{\pi(n)} [\pi(n) + 1] \chi_{[1/(\pi(n)+1), 1/\pi(n))}.$$

Remark that the function f can be written also in the form

$$f = \sum_{n=1}^{\infty} (-1)^n (n+1) \chi_{[1/(n+1), 1/n)}.$$

Since the function f is Riemann improper integrable on $[0, 1]$, then it is Kurzweil-Henstock integrable, with

$$(KH) \int_0^1 f(t) dt = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Note that the series

$$\sum_{n=1}^{\infty} (-1)^{\pi(n)} [\pi(n) + 1] \left| I \cap \left[\frac{1}{\pi(n) + 1}, \frac{1}{\pi(n)} \right) \right|$$

is convergent, but

$$(KH) \int_0^1 f(t) dt \neq \sum_{n=1}^{\infty} \frac{(-1)^{\pi(n)}}{\pi(n)}.$$

□

R e m a r k 3. It follows from Remark 2 that condition (A) of Theorem 1 is not necessary for the KH-integrability of a function f given by $\sum_{n=1}^{\infty} x_n \chi_{E_n}$, with $x_n \in X$ and E_n pairwise disjoint Lebesgue measurable sets.

However, there are cases in which the convergence of the series $\sum_{n=1}^{\infty} x_n |E_n|$ implies the Kurzweil-Henstock integrability of the function $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$ and the equality (3) holds.

Proposition 1. Let (a_n) be a decreasing sequence converging to zero such that $a_1 = 1$, and let $f: [0, 1] \rightarrow X$ be defined by $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$ and $E_n = [a_{n+1}, a_n]$. If the series $\sum_{n=1}^{\infty} x_n |E_n|$ is convergent, then condition (A) of Theorem 1 is satisfied.

Proof. Let $\varepsilon > 0$ be given, and let k_0 be a natural number such that

$$(6) \quad \left\| \sum_{n=r}^{\infty} x_n |E_n| \right\| < \frac{\varepsilon}{5} \quad \text{and} \quad \|x_r\| |E_r| < \frac{\varepsilon}{5},$$

for every $r > k_0$. Now we define a gauge δ on $[0, 1]$ in the following way:

$$\delta(0) = a_{k_0},$$

$$\delta(x) = \text{dist}(x, E_n^c) \text{ if } x \in E_n^0 \text{ (the interior of } E_n),$$

$$\delta(a_n) = \frac{1}{5} \min\{2^{-n} \varepsilon / (\|x_{n-1}\| + \|x_n\| + 1), \text{dist}(a_n, 0)\} \text{ for } n = 1, 2, \dots$$

Now let $\{(I_1, t_1), \dots, (I_p, t_p)\}$ be any δ -fine partition of $[0, 1]$. By the definition of δ , the point 0 has to be the tag of one of the pairs of the partition; we may assume that $t_1 = 0$. Let \bar{n} be the first natural number such that $a_{\bar{n}+1} \in I_1$. By the definition of $\delta(0)$ it follows that $a_{\bar{n}+1} < a_{k_0}$, hence $\bar{n} \geq k_0$. Then

$$(7) \quad \left| \bigcup_{t_j \in E_{\bar{n}}} I_j \right| \leq |E_{\bar{n}}|$$

and

$$(8) \quad \left| \bigcup_{t_j \in E_n} I_j \right| = 0 \quad \text{for } n > \bar{n}.$$

Hence

$$\left\| \sum_{n>r} x_n \left| \bigcup_{t_j \in E_n} I_j \right| \right\| = 0$$

for each $r \geq \bar{n}$.

Besides, for $1 < n < \bar{n}$ we have

$$(9) \quad \left| \bigcup_{t_j \in E_n} I_j \right| = |E_n| + \varepsilon'_n \delta(a_{n+1}) - \varepsilon''_n \delta(a_n)$$

for suitable $\varepsilon'_n, \varepsilon''_n \in [0, 1]$.

So, for $\bar{n} > r > k_0$, by (6), (7), (8) and (9) we obtain

$$\begin{aligned} \left\| \sum_{n>r} x_n \left| \bigcup_{t_j \in E_n} I_j \right| \right\| &= \left\| x_{\bar{n}} \left| \bigcup_{t_j \in E_{\bar{n}}} I_j \right| + \sum_{\bar{n}>n>r} x_n (|E_n| + \varepsilon'_n \delta(a_{n+1}) - \varepsilon''_n \delta(a_n)) \right\| \\ &\leq \|x_{\bar{n}}\| |E_{\bar{n}}| + \left\| \sum_{\bar{n}>n>r} x_n |E_n| \right\| + \sum_{\bar{n}>n>r} \|x_n\| \delta(a_{n-1}) + \sum_{\bar{n}>n>r} \|x_n\| \delta(a_n) \\ &\leq \frac{\varepsilon}{5} + 2\frac{\varepsilon}{5} + \frac{1}{5} \sum_{n>r} \frac{\varepsilon}{2^n} + \frac{1}{5} \sum_{n>r} \frac{\varepsilon}{2^n} < \varepsilon. \end{aligned}$$

This completes the proof. \square

Theorem 1 and Proposition 1 yield

Theorem 2. Let $f: [0, 1] \rightarrow X$ be defined by $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$, $E_n = [a_{n+1}, a_n)$ and (a_n) is a decreasing sequence converging to zero such that $a_1 = 1$. If the series $\sum_{n=1}^{\infty} x_n |E_n|$ is convergent, then f is Kurzweil-Henstock integrable and

$$(KH) \int_I f(t) dt = \sum_{n=1}^{\infty} x_n |E_n \cap I|$$

for every interval $I \in \mathcal{I}$.

Open Problem. Suppose that f is defined like in Theorem 1 and it is KH-integrable. Does there exist a permutation π of \mathbb{N} such that

$$(KH) \int_I f(t) dt = \sum_{n=1}^{\infty} x_{\pi(n)} |E_{\pi(n)} \cap I|$$

for each $I \in \mathcal{I}$?

By applying Theorem 1 the following two results follow:

Theorem 3. Let $f: [0, 1] \rightarrow X$ be defined by $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$ and the sets E_n are Lebesgue measurable and pairwise disjoint. We assume also that the following conditions are satisfied:

- (a) $\sum_{n=1}^{\infty} x_n |E_n|$ is weakly convergent;
- (b) for every $\varepsilon > 0$ and every $x^* \in X^*$ there exist a gauge δ and $k_0 \in \mathbb{N}$ such that for each δ -fine partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[0, 1]$ and each $n > m > k_0$ we have

$$\left| \sum_{k=m}^n x^*(x_k) \left| \bigcup_{t_j \in E_k} I_j \right| \right| < \varepsilon.$$

Then f is Kurzweil-Henstock-Pettis integrable and for every $I \in \mathcal{I}$ and $x^* \in X^*$ we have

$$(10) \quad (\text{KH}) \int_I \langle x^*, f(t) \rangle dt = \sum_{n=1}^{\infty} x^*(x_n) |E_n \cap I|.$$

Theorem 4. Let $f: [0, 1] \rightarrow X$ be defined by $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$ and the sets E_n are Lebesgue measurable and pairwise disjoint. We assume also that the following conditions are satisfied:

- (a) $\sum_{n=1}^{\infty} x_n |E_n|$ is weakly Cauchy;
- (b) for every $\varepsilon > 0$ and every $x^* \in X^*$ there exist a gauge δ and a natural number k_0 such that

$$\left| \sum_{k=m}^n x^*(x_k) \right| \bigg| \bigcup_{t_j \in E_k} I_j \bigg| < \varepsilon$$

for each δ -fine partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[0, 1]$ and each $n > m > k_0$.

Then f is Kurzweil-Henstock-Dunford integrable and for every $I \in \mathcal{I}$ and $x^* \in X^*$ we have

$$(11) \quad (\text{KH}) \int_I \langle x^*, f(t) \rangle dt = \sum_{n=1}^{\infty} x^*(x_n) |E_n \cap I|.$$

Theorem 5. Let $f: [0, 1] \rightarrow X$ be defined by $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$, $E_n = [a_{n+1}, a_n)$ and (a_n) is a decreasing sequence converging to zero such that $a_1 = 1$. If the series $\sum_{n=1}^{\infty} x_n |E_n|$ is weakly convergent, then f is Kurzweil-Henstock-Pettis integrable and

$$(\text{KH}) \int_I x^* f(t) dt = \sum_{n=1}^{\infty} x^*(x_n) |E_n \cap I|$$

for every interval $I \in \mathcal{I}$ and every $x^* \in X^*$.

If the series $\sum_{n=1}^{\infty} x_n |E_n|$ is weak*-convergent, then f is Kurzweil-Henstock-Dunford integrable and

$$(\text{KH}) \int_I x^* f(t) dt = \sum_{n=1}^{\infty} x^*(x_n) |E_n \cap I|$$

for every interval $I \in \mathcal{I}$ and every $x^* \in X^*$.

We recall that a function $G: [0, 1] \rightarrow \mathbb{R}$ is a KH-primitive if and only if for each $N \subset [0, 1]$ with $|N| = 0$ and for each $\varepsilon > 0$ there exists a gauge δ in N such that

$$\sum_{i=1}^p |G(I_i)| < \varepsilon$$

for each δ -fine partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ anchored on N . Here $G([a, b]) = G(b) - G(a)$. In this case the derivative of G exists almost everywhere in $[0, 1]$ and G is the KH-primitive of G' (see e.g. [2] and the bibliography there).

Theorem 6. *Let $f: [0, 1] \rightarrow X$ be defined by $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$ and the sets E_n are pairwise disjoint intervals. Then f is Kurzweil-Henstock-Pettis integrable and*

$$(12) \quad (\text{KHP}) \quad \int_I f(t) dt = \sum_{n=1}^{\infty} x_n |E_n \cap I| \quad \text{for every } I \in \mathcal{I}$$

if and only if the following conditions are satisfied:

- (a) $\sum_{n=1}^{\infty} x_n |E_n \cap I|$ is weakly convergent for each $I \in \mathcal{I}$;
- (b) for every $\varepsilon > 0$, every $x^* \in X^*$ and every $N \subset [0, 1]$ with $|N| = 0$, there exists a gauge δ in N such that

$$(13) \quad \left| \sum_{n=1}^{\infty} x^*(x_n) |E_n \cap \bigcup_{i=1}^p I_i| \right| < \varepsilon$$

for each δ -fine partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ anchored on N .

Proof. If f is KH-integrable and condition (12) is satisfied, then condition (a) follows directly from (12). Now we prove condition (b). By hypothesis, for every $x^* \in X^*$ the scalar function x^*f is KH-integrable. Set

$$\alpha(t) = (\text{KH}) \int_0^t \langle x^*, f(s) \rangle ds = \sum_{n=1}^{\infty} x^*(x_n) |E_n \cap [0, t]|.$$

Since $\alpha(t)$ is the KH-primitive of x^*f , hence for each $N \subset [0, 1]$ with $|N| = 0$ and for each $\varepsilon > 0$ there exists a gauge δ on N such that

$$\sum_{i=1}^p |\alpha(I_i)| < \varepsilon$$

for each δ -fine partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ anchored on N .

Therefore

$$\left| \sum_{n=1}^{\infty} x^*(x_n) \left| E_n \cap \bigcup_{i=1}^p I_i \right| \right| \leq \sum_{i=1}^p \left| \sum_{n=1}^{\infty} x^*(x_n) |E_n \cap I_i| \right| = \sum_{i=1}^p |\alpha(I_i)| < \varepsilon.$$

Assume now that conditions (a) and (b) are satisfied, and fix $x^* \in X^*$. By condition (b), for every $\varepsilon > 0$ and every $N \subset [0, 1]$ with $|N| = 0$, there exists a gauge δ on N such that inequality (13) holds for each δ -fine partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ anchored on N . Set once again $\alpha(t) = \sum_{n=1}^{\infty} x^*(x_n) |E_n \cap [0, t]|$. Then, by (a) and (13), we infer

$$\begin{aligned} \sum_{i=1}^p |\alpha(I_i)| &= \left| \sum_{I^+} \alpha(I_i) \right| + \left| \sum_{I^-} \alpha(I_i) \right| \\ &= \left| \sum_{n=1}^{\infty} x^*(x_n) \left| E_n \cap \bigcup_{I^+} I_i \right| \right| + \left| \sum_{n=1}^{\infty} x^*(x_n) \left| E_n \cap \bigcup_{I^-} I_i \right| \right| < 2\varepsilon, \end{aligned}$$

where I^+ and I^- denote the set of all indices $i = 1, \dots, p$ such that $\alpha(I_i)$ is positive or negative, respectively. Therefore $\alpha(t)$ is a KH-primitive. As the sets E_n are intervals, it follows easily that $\alpha'(t) = \sum_{n=1}^{\infty} x^*(x_n) \chi_{E_n}$ almost everywhere in $[0, 1]$.

Then $\alpha(t)$ is the KH-primitive of the function $x^*f = \sum_{n=1}^{\infty} x^*(x_n) \chi_{E_n}$. Consequently

$$(\text{KH}) \int_0^1 \langle x^*, f(t) \rangle dt = \sum_{n=1}^{\infty} x^*(x_n) |E_n| = \left\langle x^*, \sum_{n=1}^{\infty} x_n |E_n| \right\rangle.$$

Hence the function f is KHP-integrable and (12) holds. \square

Open problem. Is Theorem 6 still true if the sets E_n are arbitrary pairwise disjoint Lebesgue measurable sets?

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