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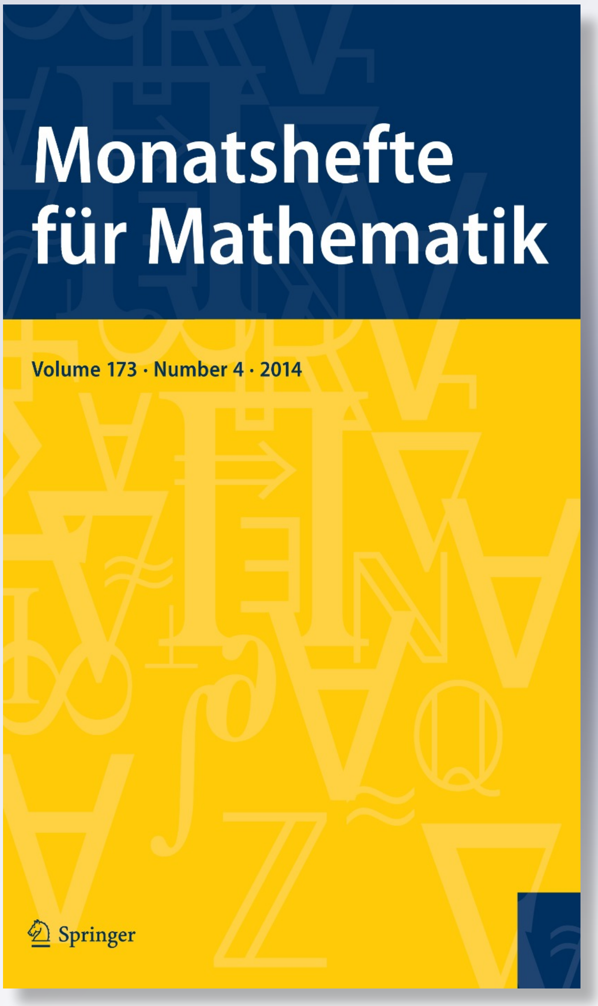

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Relations among Henstock, McShane and Pettis integrals for multifunctions with compact convex values

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Abstract Fremlin (Ill J Math 38:471–479, 1994) proved that a Banach space valued function is McShane integrable if and only if it is Henstock and Pettis integrable. In this paper we prove that the result remains valid also in case of multifunctions with compact convex values being subsets of an arbitrary Banach space (see Theorem 3.4). Di Piazza and Musiał (Monatsh Math 148:119–126, 2006) proved that if X is a separable Banach space, then each Henstock integrable multifunction which takes as its values convex compact subsets of X is a sum of a McShane integrable multifunction and a Henstock integrable function. Here we show that such a decomposition is true also in case of an arbitrary Banach space (see Theorem 3.3). We prove also that Henstock and McShane integrable multifunctions possess Henstock and McShane (respectively) integrable selections (see Theorem 3.1).

Keywords Multifunction · McShane integral · Henstock integral · Pettis integral · Henstock–Kurzweil–Pettis integral · Selection

Mathematics Subject Classification (2000) Primary 28B20 ; Secondary 26E25 · 26A39 · 28B05 · 46G10 · 54C60 · 54C65

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1 Introduction

In this paper we deal with relationship among Henstock, McShane and Pettis integrable multifunctions which take as their values compact convex subsets of a general Banach space, not necessarily separable. The Henstock and McShane integrals for multifunctions are, respectively, generalizations of the Henstock and McShane integrals for vector valued functions by means the notion of Hausdorff distance. In a remarkable paper [8, Theorem 8] Fremlin proved that a Banach space valued function in [0, 1] is McShane integrable if and only if it is both Henstock and Pettis integrable. Here we show that the same characterization holds true also in case of multifunctions with compact convex values being subsets of an arbitrary Banach space (see Theorem 3.4). An important step in our investigations is a proof of the existence of a Henstock (resp. McShane) integrable selection for a Henstock (resp. McShane) integrable multifunction with weakly compact convex values (see Theorem 3.1). It completes an earlier result from [1], where the existence of Pettis integrable selections of a Pettis integrable multifunction had been proven. Then, we use the existence of Henstock integrable selections to decompose each Henstock integrable multifunction as the sum of a McShane integrable multifunction and a Henstock integrable function (see Theorem 3.3). In this way we strengthen a previous decomposition for Henstock–Kurzweil–Pettis integrable multifunction taking values in a general Banach space: each Henstock–Kurzweil–Pettis integrable multifunction is the sum of a Pettis integrable multifunction and a Henstock–Kurzweil–Pettis integrable function [6].

Throughout [0, 1] is the unit interval of the real line equipped with the usual topology and the Lebesgue measure and $\mathcal{I}$ is the collection of all closed subintervals of [0, 1]. $\mathcal{L}$ denotes the family of all Lebesgue measurable subsets of [0, 1] and if $E \in \mathcal{L}$, then $|E|$ denotes its Lebesgue measure. A *partition* $\mathcal{P}$ *in* [0, 1] is a collection $\{(I_1, t_1), \ldots, (I_p, t_p)\}$, where $I_1, \ldots, I_p$ are nonoverlapping subintervals of [0, 1], t_i is a point of [0, 1], $i = 1, \ldots, p$. If $\cup_{i=1}^{p} I_i = [0, 1]$, then $\mathcal{P}$ is *a partition of* [0, 1]. If t_i is a point of I_i, $i = 1, \ldots, p$, we say that P is a *Perron partition of* [0, 1]. A *gauge* on [0, 1] is a positive function on [0, 1]. For a given gauge δ on [0, 1], we say that a partition $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ is δ-*fine* if $I_i \subset (t_i - \delta(x_i), t_i + \delta(x_i))$, $i = 1, \ldots, p$.

Given a function $g\colon [0, 1] \to X$ and a partition $\mathcal{P} = \{(I_1, t_1), \ldots, (I_p, t_p)\}$ in [0, 1] we set

$$\sigma(g, \mathcal{P}) = \sum_{i=1}^{p} |I_i| g(t_i).$$

Definition 1.1 (see [8]) Let X be a Banach space. A function $g : [0, 1] \to X$ is said to be *Henstock* (resp. *McShane*) *integrable* on [0, 1] if there exists a point $\Phi_g[0, 1] \in X$ with the following property: for every $\epsilon > 0$ there exists a gauge δ on [0, 1] such that

$$\left|\left|\sigma(g, \mathcal{P}) - \Phi_f[0, 1]\right|\right| < \varepsilon.$$

for each δ-fine Perron partition (resp. partition) $\mathcal{P}$ of [0, 1]. We set $(H) \int_0^1 g\,dt := \Phi_g[0, 1]$ (resp. $(MS) \int_0^1 g\,dt := \Phi_g[0, 1]$). □

We denote the set of all Henstock (resp. McShane) integrable functions on [0, 1], taking their values in X, by $\mathcal{H}([0,1],X)$ (resp. $\mathcal{MS}([0,1],X)$). In case when X is the real line, g is called *Henstock–Kurzweil integrable*, or simply *HK-integrable*, (resp. integrable, because for real functions McShane integrability coincides with Lebesgue integrability). Moreover the space of all HK-integrable scalar functions (resp. integrable functions) is denoted by $\mathcal{HK}[0,1]$ (resp. $\mathcal{MS}[0,1]$).

It is well known that if g is Henstock integrable on [0, 1], then g is Henstock integrable over all $I \in \mathcal{I}$ (see [8] or [13, Theorem 3.3.4.]). We write then $(H)\int_I f(t)\,dt := (H)\int_0^1 f(t)\chi_I(t)\,dt$ and in case of scalar functions: $(HK)\int_I f(t)\,dt$. Integrals without prefixes (HK) or (MS) are the ordinary Lebesgue integrals.

Throughout this paper X is a Banach space with its dual X^*. The closed unit ball of X^* is denoted by $B(X^*)$. $cb(X)$ is the family of all non-empty closed convex and bounded subsets of X. $ck(X)$ denotes the family of all compact members of $cb(X)$ and $cwk(X)$ is the collection of all weakly compact elements of $cb(X)$. For every $C \in cb(X)$ the *support function of* C is denoted by $s(\cdot, C)$ and defined on X^* by $s(x^*, C) = \sup\{\langle x^*, x\rangle : x \in C\}$, for each $x^* \in X^*$.

A map $\Gamma : [0,1] \to 2^X \setminus \{\emptyset\}$ (=non-empty subsets of X) is called a *multifunction*. Γ is said to be *scalarly measurable* if for every $x^* \in X^*$, the map $s(x^*, \Gamma(\cdot))$ is measurable. Γ is said to be *scalarly Henstock–Kurzweil* (resp. *scalarly*) *integrable* if, for every $x^* \in X^*$, the function $s(x^*, \Gamma(\cdot))$ is *Henstock–Kurzweil integrable* (resp.*integrable*). A function $f : [0,1] \to X$ is called a *selection of* Γ if, for every $t \in [0,1]$, one has $f(t) \in \Gamma(t)$.

We associate with each multifunction $\Gamma : [0,1] \to cb(X)$ the set

$$\mathcal{Z}_\Gamma := \{s(x^*, \Gamma) : \|x^*\| \le 1\},$$

where we consider functions, not equivalence classes of a.e. equal functions. Identifying equivalent functions we obtain the set $\mathbb{Z}_\Gamma$.

Let A and B be nonempty subsets of X. The *excess* of A over B is defined as $e(A,B) = \sup\{d(x,B) : x \in A\}$. The *Hausdorff distance of* A and B is defined by $d_H(A,B) := \max\{e(A,B), e(B,A)\}$. $cb(X)$ with the Hausdorff distance is a complete metric space. We consider on $cb(X)$ the Minkowski addition ($A \oplus B := \{a+b : a \in A,\ b \in B\}$) and the standard multiplication by scalars.

Definition 1.2 A multifunction $\Gamma : [0,1] \to cb(X)$ is said to be *Henstock–Kurzweil–Pettis integrable* (see [4]) *in* $cb(X)$ $[cwk(X), ck(X)]$ (resp. *Pettis integrable* (see [12]) *in* $cb(X)[cwk(X), ck(X)]$) if Γ is scalarly Henstock–Kurzweil (resp. scalarly) integrable on [0, 1] and for each subinterval $I \in \mathcal{I}$ (resp. $A \in \mathcal{L}$), there exists a set $\Phi_\Gamma(I) \in cb(X)[cwk(X), ck(X)]$ (resp. $M_\Gamma(A) \in cb(X)[cwk(X), ck(X)]$) such that for each $x^* \in X^*$, we have

$$s(x^*, \Phi_\Gamma(I)) = (HK)\int_I s(x^*, \Gamma(t))\,dt$$
$$\left(resp.\ s(x^*, M_\Gamma(A)) = \int_A s(x^*, \Gamma(t))\,dt\right). \tag{1}$$

We set $(HKP)\int_I \Gamma(t)\,dt := \Phi_\Gamma(I)$ and $(P)\int_A \Gamma(t)\,dt := M_\Gamma(A)$. □

Definition 1.3 A multifunction $\Gamma : [0,1] \to cb(X)$ is said to be *Henstock* (resp. *McShane*) integrable, if there exists a set $\Phi_\Gamma[0,1] \in cb(X)$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that for each δ-fine Perron partition (resp. partition) $\{(I_1,t_1),\ldots,(I_p,t_p)\}$ of $[0,1]$, we have

$$d_H\left(\Phi_\Gamma[0,1], \bigoplus_{i=1}^{p} \Gamma(t_i)|I_i|\right) < \varepsilon. \tag{2}$$

We write then $(H)\int_0^1 \Gamma(t)\,dt := \Phi_\Gamma[0,1]$ (resp. $(MS)\int_0^1 \Gamma(t)\,dt := \Phi_\Gamma[0,1]$).

If $\Phi_\Gamma[0,1] \in cwk(X)$ or $\Phi_\Gamma[0,1] \in ck(X)$, then sometimes we say that Γ is Henstock integrable in $cwk(X)$ or $ck(X)$, respectively. It is easily seen from the definition and the completeness of the Hausdorff metric that $cwk(X)[ck(X)]$-valued Henstock integrable multifunctions are integrable in $cwk(X)[ck(X)]$.

$\mathcal{S}_H(\Gamma)$ $[\mathcal{S}_{MS}(\Gamma)$, $\mathcal{S}_P(\Gamma)]$ denotes the family of all scalarly measurable selections of Γ that are Henstock [McShane, Pettis] integrable. □

We note that when a multifunction is a function $f : [0,1] \to X$, then the set $\Phi_f[0,1]$ is reduced to a vector of X and the above definitions coincide with those of Henstock and McShane integrability for vector valued functions.

It follows from the definitions that if Γ is McShane integrable, then it is also Henstock integrable (with the same values of the integrals).

According to Hörmander's equality (cf. [11, p. 9])

$$d_H\left(K, \bigoplus_{i=1}^{p} \Gamma(t_i)|I_i|\right) = \sup_{\|x^*\|\le 1} \left| s(x^*,K) - \sum_{i=1}^{p} s(x^*,\Gamma(t_i))|I_i| \right|. \tag{3}$$

for each δ-fine Perron partition (resp. partition) $\{(I_1,t_1),\ldots,(I_p,t_p)\}$ of $[0,1]$ and for each set $K \in cb(X)$. Hence, it follows from (2) and (3) that for each $x^* \in X^*$ the function $s(x^*,\Gamma(\cdot))$ is HK-integrable (or McShane integrable) and $s(x^*,\Phi_\Gamma[0,1]) = (HK)\int_0^1 s(x^*,\Gamma(t))\,dt$.

Hörmander's equality allows us to reduce the Henstock integrability of multifunctions to the Henstock integrability of functions by embedding the families $ck(X)$, $cwk(X)$ and $cb(X)$ into the Banach space $l_\infty(B(X^*))$. More precisely, let $j : cb(X) \to l_\infty(B(X^*))$ be the mapping defined by $j(K)(x^*) = s(x^*,K)$. Then $j(cb(X))$, $j(ck(X))$ and $j(cwk(X))$ are closed cones of $l_\infty(B(X^*))$ (see [2, Theorem II.19]). Therefore a multifunction $\Gamma : [0,1] \to cb(X)$ is Henstock integrable if and only if the single valued function $j \circ \Gamma : [0,1] \to l_\infty(B(X^*))$ is Henstock integrable in the usual sense. The key point is that $j(cb(X))$ $[j(ck(X)),\ j(cwk(X))]$ are closed cones. Consequently, if $z \in l_\infty(B(X^*))$ is the value of the Henstock integral of $j \circ \Gamma$, then there exists a set $K \in cb(X)\,[ck(X),\ cwk(X)]$ with $j(K) = z$.

2 Equi-integrability

In the theory of Lebesgue integration uniform integrability plays an essential role. Its counterpart in the theory of gauge integrals is the notion of equi-integrability.

Definition 2.1 Let $\{g_\alpha : \alpha \in \mathbb{A}\}$ be a family of real valued functions in $\mathcal{HK}[0,1]$ (resp. $\mathcal{MS}[0,1]$). We say that $\{g_\alpha : \alpha \in \mathbb{A}\}$ is *Henstock* (resp. *McShane*) *equi-integrable* on $[0,1]$ whenever for every $\varepsilon > 0$ there is a gauge δ such that

$$\sup\left\{\left|\sigma(g_\alpha, \mathcal{P}) - (HK)\int_0^1 g_\alpha\, dt\right| : \alpha \in \mathbb{A}\right\} < \varepsilon.$$

for each δ-fine Perron partition (resp. partition) $\mathcal{P}$ of $[0,1]$.

The next result follows immediately from the equality (3) and it has been explicitly formulated in [3] for Henstock integrable functions.

Proposition 2.2 *If $\Gamma : [0,1] \to cb(X)$ is a scalarly HK-integrable multifunction, then the following conditions are equivalent:*

(i) *Γ is Henstock (resp. McShane) integrable;*
(ii) *the family $\mathcal{Z}_\Gamma$ is Henstock (resp. McShane) equi-integrable;*

Direct consequences of the above characterizations are hereditary properties of Henstock and McShane integrals.

Proposition 2.3 (We refer to [8, Proposition 4]) *If $\Gamma : \Omega \to cb(X)$ is a Henstock integrable multifunction, then for each $I \in \mathcal{I}$ the multifunction $\Gamma\chi_I$ is Henstock integrable.*

Proposition 2.4 (We refer to [9, Theorem 2E]) *If $\Gamma : \Omega \to cb(X)$ is a multifunction that is McShane integrable, then for each $E \in \mathcal{L}$ the multifunction $\Gamma\chi_E$ is also McShane integrable.*

As an immediate corollary, we obtain the following result:

Corollary 2.5 *If $\Gamma : [0,1] \to cb(X)$ is Henstock integrable in $cb(X)[cwk(X), ck(X)]$, then it is also Henstock-Kurzweil-Pettis integrable in $cb(X)[cwk(X), ck(X)]$.*

If $\Gamma : [0,1] \to cb(X)$ is McShane integrable in $cb(X)[cwk(X), ck(X)]$, then it is also Pettis integrable in $cb(X)[cwk(X), ck(X)]$.

In order to prove our main results we need to know a little bit more about relations between the two types of equi-integrability. The idea of our proof is taken from [8, Theorem 8].

Proposition 2.6 *Let $\mathbb{A} = \{g_\alpha : [0,1] \to [0,\infty) : \alpha \in S\}$ be a family of functions satisfying the following conditions:*

(i) *$\mathbb{A}$ is Henstock equi-integrable;*
(ii) *$\mathbb{A}$ is norm relatively compact in the L_1 norm;*
(iii) *$\mathbb{A}$ is pointwise bounded.*

Then the family $\mathbb{A}$ *is also McShane equi-integrable.*

Proof At first we observe that, since the functions g_α in $\mathbb{A}$ are non negative and Henstock integrable, they are also Lebesgue and then McShane integrable (see [10]). Let $\varepsilon > 0$. For each $k \in N$ set $\eta_k = 2^{-k}\varepsilon/(2\varepsilon + 12(k+1)) > 0$.

According to the hypothesis (ii), choose $h_{k,0}, \ldots, h_{k,i(k)} \in \mathbb{A}$ such that for every function $g_\alpha \in \mathbb{A}$ there exists $j_\alpha \leq i(k)$ with

$$||g_\alpha - h_{k,j_\alpha}||_1 \leq \eta_k.$$

Moreover by hypothesis (i) and the McShane integrability of the functions in $\mathbb{A}$, choose a gauge δ_k in $[a,b]$ such that:

$$\sup\left\{\left|\sigma(g_\alpha, \mathcal{P}) - \int_0^1 g_\alpha(t)\,dt\right| : \alpha \in \mathbb{A}\right\} < \eta_k. \tag{4}$$

for every δ_k-fine Perron partition $\mathcal{P}$ of [0, 1], and

$$\left|\sigma(h_{k,j}, \mathcal{P}) - \int_0^1 h_{k,j}(t)\,dt\right| < \eta_k. \tag{5}$$

for every $j \leq i(k)$ and every δ_k-fine partition $\mathcal{P}$ of [0, 1]. Now by hypothesis (iii) let $g\colon [0,1] \to [0,\infty)$ be a function such that

$$0 \leq g_\alpha(t) \leq g(t), \tag{6}$$

for every $t \in [0,1]$ and every $g_\alpha \in \mathbb{A}$.

For each k set

$$A_k = \{t \in [a,b] : \ k \leq g(t) < k+1\}. \tag{7}$$

Define a gauge δ on [0, 1] setting $\delta(t) = \delta_k(t)$, if $t \in A_k$.

Now let $\mathcal{Q} = \{(J_i, t_i) : \ i = 1, \ldots, p\}$ be a δ-fine partition of [0, 1] and take any $g_\alpha \in \mathbb{A}$.

Let us fix k. Set

$$T_k = \{i : \ i \leq p, \quad t_i \in A_k\} \quad \text{and} \quad H_k = \bigcup_{i \in T_k} J_i.$$

We are going to evaluate $|\int_{H_k} g_\alpha(t)\,dt - \sum_{i\in T_k} |J_i| g_\alpha(t_i)|$.

Take $j_\alpha \leq i(k)$ such that $\int_0^1 |g_\alpha(t) - h_{k,j_\alpha(t)}|\,dt \leq \eta_k$. Then also

$$\left|\int_{H_k} [g_\alpha(t) - h_{k,j_\alpha}(t)]\,dt\right| \leq \eta_k. \tag{8}$$

Note that, if $\mathcal{S}$ is any δ_k-fine partition in [0, 1], by (5) and Lemma 3.5.6. of [13], we have

$$\left| \int_{\bigcup_{(J,t)\in\mathcal{S}} J} h_{k,j_\alpha}(t)\,dt - \sigma(h_{k,j_\alpha},\mathcal{S}) \right| \le \eta_k. \tag{9}$$

So, if $\mathcal{R}$ is any δ_k-fine Perron partition in [0, 1], by (4), (9), and by Lemma 3.5.6. of [13], we get

$$\left| \sigma(g_\alpha,\mathcal{R}) - \int_{\bigcup_{(I,t)\in\mathcal{R}} I} g_\alpha(t)\,dt \right| \le \eta_k$$

and

$$\left| \sigma((g_\alpha - h_{k,j_\alpha}),\mathcal{R}) - \int_{\bigcup_{(I,t)\in\mathcal{R}} I} [g_\alpha(t) - h_{k,j_\alpha}(t)]\,dt \right| \le 2\eta_k.$$

Consequently, applying (8), we have

$$\sum_{(I,t)\in\mathcal{R}} |I|(g_\alpha - h_{k,j_\alpha})(t) \le 3\eta_k\,, \tag{10}$$

for every δ_k-fine Perron partition $\mathcal{R}$ in [0, 1]

Now set

$$V = \bigcup\{(t-\delta_k(t), t+\delta_k(t)) :\ g_\alpha(t) - h_{k,j_\alpha}(t) \ge \varepsilon\}.$$

Then by [8, Lemma 6] applied to the function $g_\alpha - h_{k,j_\alpha}$ we have

$$|[0,1]\cap V| \le 3\eta_k/\varepsilon.$$

Since

$$\bigcup\{J_i :\ i\in T_k,\ g_\alpha(t_i) - h_{k,j_\alpha}(t_i) \ge \varepsilon\} \subset V\,,$$

we have also

$$\sum_{\{i\in T_k:\ g_\alpha(t_i)-h_{k,j_\alpha}(t_i)\ge\varepsilon\}} |J_i| \le 3\eta_k/\varepsilon.$$

In a similar way we obtain

$$\sum_{\{i\in T_k:\ h_{k,j_\alpha}(t_i)-g_\alpha(t_i)\ge\varepsilon\}} |J_i| \le 3\eta_k/\varepsilon.$$

Moreover, by (6) and (7) we have for every $t_i \in A_k$

$$|h_{k,j_\alpha}(t_i) - g_\alpha(t_i)| \leq 2g(t) < 2(k+1)$$

Hence,

$$\sum_{i \in T_k} |J_i||h_{k,j_\alpha}(t_i) - g_\alpha(t_i)| \leq \varepsilon|H_k| + 12\eta_k(k+1)/\varepsilon. \tag{11}$$

By (8), (9) and (11) we obtain

$$\left|\int_{H_k} g_\alpha(t)\,dt - \sum_{i \in T_k} |J_i| g_\alpha(t_i)\right| < 2\eta_k + \varepsilon|H_k| + 12\eta_k(k+1)/\varepsilon < \varepsilon(2^{-k} + |H_k|).$$

And summing over k we get

$$\left|\int_a^b g_\alpha(t)\,dt - \sigma(g_\alpha, \mathcal{Q})\right| < \varepsilon \sum_k (2^{-k} + |H_k|) = 3\varepsilon.$$

Since this is true for every function g_α in $\mathbb{A}$ and for every δ-fine partition $\mathcal{Q}$ of $[0, 1]$, the family $\mathbb{A}$ is McShane equi-integrable. □

3 Main results

The following theorem is fundamental for our characterization of McShane integrable multifunctions.

Theorem 3.1 *If $\Gamma : [0,1] \to cwk(X)$ is Henstock integrable, then $\mathcal{S}_H(\Gamma) \neq \emptyset$.*
If $\Gamma : [0,1] \to cwk(X)$ is Henstock and Pettis integrable in $cwk(X)$, then $\mathcal{S}_{MS}(\Gamma) \neq \emptyset$.

Proof The first part of our proof is similar to that of [1, Theorem 2.5]. Since $H := \int_0^1 \Gamma(t)\,dt \in cwk(X)$, there exists a strongly exposed point $x_0 \in H$. Assume that $x_0^* \in B(X^*)$ is such that $x_0^*(x_0) > x_0^*(x)$ for every $x \in H \setminus \{x_0\}$ and the sets $\{x \in H : x_0^*(x) > x_0^*(x_0) - \alpha\}$, $\alpha \in \mathbb{R}$, form a neighborhood basis of x_0 in the norm topology on H. Define $G : [0,1] \to cwk(X)$ by $G(t) := \{x \in \Gamma(t) : x_0^*(x) = s(x_0^*, \Gamma(t))\}$. If Γ is Henstock integrable in $cwk(X)$, then Γ is also HKP-integrable in $cwk(X)$ (see Corollary 2.5). It follows from the continuity characterization of HKP-integrable multifunctions (see [6, Proposition 2]) that also G is HKP-integrable in $cwk(X)$. If Γ is Pettis integrable, then also G is Pettis integrable (see [1] or [12]). Let $g : [0,1] \to X$ be a selection of G constructed in [1] for the strongly exposed point x_0. In case of Henstock integrability of Γ, the selection g is HKP-integrable [6, Proposition 3] and in case when Γ is Pettis integrable, g is Pettis integrable (see [1, Corollary 2.3] or [12, Corollary 1.5]). In the both cases $x_0^*(x_0) = \int_0^1 x_0^* g(t)\,dt$.

Let $\varepsilon > 0$ and $0 < \varepsilon' < \varepsilon/2$ be arbitrary. Then, let $0 < \eta < \varepsilon'$ be such that

$$\forall\, x \in H \quad [|x_0^*(x) - x_0^*(x_0)| < \eta \Rightarrow \|x - x_0\| < \varepsilon']. \tag{12}$$

Since Γ is Henstock integrable and $x_0^* g$ is HK-integrable we can find a gauge δ on $[0, 1]$ such that for each δ-fine Perron partition $\mathcal{P} := \{(I_1, t_1), \ldots, (I_p, t_p)\}$ of $[0, 1]$

$$d_H\left(H, \bigoplus_{i=1}^{p} \Gamma(t_i)|I_i|\right) < \eta/2 \tag{13}$$

and

$$\left|\int_0^1 x_0^* g(t)\, dt - \sum_{i=1}^{p} x_0^* g(t_i)|I_i|\right| < \eta/2. \tag{14}$$

It is a consequence of (13) that there exists a point $x_{\mathcal{P}} \in H$ with

$$\left\|\sum_{i=1}^{p} g(t_i)|I_i| - x_{\mathcal{P}}\right\| < \eta/2$$

and so, taking into account also (14) and the equality $x_0^*(x_0) = \int_0^1 x_0^* g(t)\, dt$, we have

$$|x_0^*(x_{\mathcal{P}}) - x_0^*(x_0)| \leq \left|x_0^*(x_{\mathcal{P}}) - x_0^*\left(\sum_{i=1}^{p} g(t_i)|I_i|\right)\right| + \left|\sum_{i=1}^{p} x_0^* g(t_i)|I_i| - x_0^*(x_0)\right| < \eta.$$

Now, (12) yields $\|x_{\mathcal{P}} - x_0\| < \varepsilon'$ and finally

$$\left\|\sum_{i=1}^{p} g(t_i)|I_i| - x_0\right\| \leq \left\|\sum_{i=1}^{p} g(t_i)|I_i| - x_{\mathcal{P}}\right\| + \|x_{\mathcal{P}} - x_0\| < \varepsilon.$$

The above inequality proves the Henstock integrability of g. If g is also Pettis integrable, then it follows from [8] that g is McShane integrable. □

The following lemma is related to [6, Lemma 1].

Corollary 3.2 *Let $\Gamma : [0, 1] \to cwk(X)$ be a multifunction that is Henstock-integrable in $ck(X)$. If $s(x^*, \Gamma) \geq 0$ almost everywhere for each $x^* \in X^*$ separately, then Γ is McShane integrable in $ck(X)$.*

Proof According to [6, Lemma 1] Γ is Pettis integrable in $cwk(X)$. But if $E \in \mathcal{L}$, then

$$(P)\int_E \Gamma(t)\, dt \subset (P)\int_0^1 \Gamma(t)\, dt = (H)\int_0^1 \Gamma(t)\, dt \in ck(X).$$

Hence Γ is Pettis integrable in $ck(X)$ and all $f \in \mathcal{S}_P(\Gamma)$ have norm relatively compact ranges of their Pettis integrals. Hence, it follows from [12, Theorem 3.7] and its proof that $\mathbb{Z}_\Gamma$ is relatively compact in $L_1[0,1]$. In view of Propositions 2.2 and 2.6 the multifunction Γ is McShane integrable in $ck(X)$. □

The next theorem is a strong generalization of [5, Theorem 2], where an identical result has been proven for compact valued multifunctions taking their values in a separable Banach space.

Theorem 3.3 *Let $\Gamma : [0,1] \to cwk(X)$ be a scalarly Henstock–Kurzweil integrable multifunction. Then the following conditions are equivalent:*

(i) *Γ is Henstock integrable in $ck(X)$;*
(ii) *$\mathcal{S}_H(\Gamma) \neq \emptyset$ and for every $f \in \mathcal{S}_H(\Gamma)$ the multifunction $G : [0,1] \to cwk(X)$ defined by $\Gamma(t) = G(t) + f(t)$ is McShane integrable in $ck(X)$;*
(iii) *there exists $f \in \mathcal{S}_H(\Gamma)$ such that the multifunction $G : [0,1] \to cwk(X)$ defined by $\Gamma(t) = G(t) + f(t)$ is McShane integrable in $ck(X)$.*

If $\Gamma : [0,1] \to ck(X)$, then also the multifunctions G are $ck(X)$-valued.

Proof (i) ⇒ (ii) According to Theorem 3.1 $\mathcal{S}_H(\Gamma) \neq \emptyset$. Let $f \in \mathcal{S}_H(\Gamma)$ be fixed. Define $G : [0,1] \to ck(X)$ by $G(t) := \Gamma(t) - f(t)$. Then G is also Henstock integrable in $ck(X)$ and $s(x^*, G(t)) \geq 0$ for every $x^* \in X^*$ and every $t \in [0,1]$. So $s(x^*, G(t))$ is integrable. By Corollary 3.2 the multifunction G is McShane integrable in $ck(X)$. Obviously (ii) ⇒ (iii). The implication (iii) ⇒ (i) follows at once from Proposition 2.2. □

The next theorem generalizes [8, Theorem 8].

Theorem 3.4 *Let $\Gamma : [0,1] \to cwk(X)$ be a scalarly measurable multifunction. Then the following conditions are equivalent:*

(i) *Γ is McShane integrable in $ck(X)$;*
(ii) *Γ is Pettis and Henstock integrable in $ck(X)$;*
(iii) *Γ is Henstock integrable in $ck(X)$ and $\mathcal{S}_H(\Gamma) \subset \mathcal{S}_{MS}(\Gamma)$.*
(iv) *Γ is Henstock integrable in $ck(X)$ and $\mathcal{S}_H(\Gamma) \subset \mathcal{S}_P(\Gamma)$;*
(v) *Γ is Henstock integrable in $ck(X)$ and $\mathcal{S}_P(\Gamma) \neq \emptyset$.*

Proof (i) ⇒ (ii) [This implication holds true for an arbitrary $\Gamma : [0,1] \to cb(X)$ that is McShane integrable in $cb(X)$]. As we have already mentioned in Corollary 2.5, if Γ is McShane integrable in $ck(X)$, it is also Henstock and Pettis integrable in $ck(X)$.

(ii) ⇒ (i) In virtue of Theorem 3.1 Γ has a McShane integrable selection f. It follows from Theorem 3.3 that there exists a multifunction $G : [0,1] \to cwk(X)$ that is McShane integrable in $ck(X)$ such that $\Gamma = G + f$. It follows that Γ is also McShane integrable in $ck(X)$.

(i) ⇒ (iii) If $f \in \mathcal{S}_H(\Gamma)$ then, according to Theorem 3.3, $\Gamma = G + f$ for a McShane integrable G. But as Γ is Pettis integrable, also f is Pettis integrable (cf. [12, Corollary 1.5]). In view of [8, Theorem 8] f is McShane integrable.

(iii) ⇒ (iv) is valid, because each McShane integrable function is also Pettis integrable [9, Theorem 2C].

(iv) ⇒ (v) In view of Theorem 3.1 $\mathcal{S}_H(\Gamma) \neq \emptyset$ and so (iv) implies $\mathcal{S}_P(\Gamma) \neq \emptyset$.

(v) ⇒ (ii) Take $f \in \mathcal{S}_P(\Gamma)$. Since Γ is Henstock integrable, it is also HKP-integrable and so applying [6, Theorem 2] we obtain a representation $\Gamma = G + f$, where $G : [0, 1] \to cwk(X)$ is Pettis integrable in $ck(X)$. Consequently, Γ is also Pettis integrable in $ck(X)$ and so (ii) is fulfilled. □

Remark 3.5 We observe that one cannot add to Theorem 3.4 the condition: "*Γ is Pettis integrable in $ck(X)$ and $\mathcal{S}_H(\Gamma) \neq \emptyset$*". In fact let $f : [0, 1] \to X$ be a scalarly negligible function, Pettis but not McShane integrable (it is well known that under (CH) there are such functions [7]). We define a multifunction $\Gamma : [0, 1] \to ck(X)$ by setting $\Gamma(t) = conv\{0, f(t)\}$. Since for every $x^* \in X^*$ we have $s(x^*, \Gamma(t)) = [x^* f(t)]^+$, the multifunction Γ is scalarly integrable and measurable. Moreover, since f is scalarly negligible, then $s(x^*, \Gamma(t)) = [x^* f(t)]^+ = 0$ a.e. So Γ itself is Pettis integrable in $ck(X)$ with integral equal to $\{0\}$. Since f is not McShane integrable by [3, Lemma 1] there exists $\varepsilon_0 > 0$ such that for every gauge δ we can find a δ-fine partition $\{(I_i, t_i) : i \in I\}$ satisfying the inequality

$$\left\| \sum_{i \in I} f(t_i)|I_i| \right\| > \varepsilon_0.$$

So there exists $x_0^* \in B(X^*)$ such that

$$\sum_{i \in I} x_0^* f(t_i)|I_i| > \varepsilon_0,$$

and then

$$\sum_{i \in I} s(x_0^*, \Gamma(t_i))|I_i| = \sum_{i \in I} [x_0^* f(t_i)]^+ |I_i| > \varepsilon_0.$$

By Proposition 2.2 it follows that Γ is not McShane integrable, and by Theorem 3.4(v) Γ is not Henstock integrable. On the other hand Γ has a McShane integrable zero selection.

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