Lifting and some of its Applications to the Theory of Pettis Integral

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Summary. - *Basic facts concerning general liftings and liftings in products of finite measure spaces are presented. Then a few results on the Pettis integrability of Banach space valued functions and their liftings are proved.*

Introduction

During these few lectures I am going to present some of the basic properties of lifting and some of its applications to the theory of Pettis integration. Most of the lifting theory holds true for a large family of unbounded measures, but I will consider only finite measures, since in the theory of Pettis integral I assume the finiteness of the measure spaces. Most of basic theory of lifting is standard and taken from [4], [3] and [11]. The notion of an admissible density, admissibly generated lifting as well as the results of section 4 are new. More general results will be published in [5]. The theory of stable sets is based on [8]. The completeness considerations in sections 7 are new.

1. Preliminaries

Throughout all considered measure spaces are assumed to be probability spaces. Given measure space (Ω, Σ, μ) a set $N \in \Sigma$ with $\mu(N) = 0$ is called a *μ-null set* and for $A, B \in \Sigma$ we write $A \equiv B$

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iff $A \triangle B$, the symmetric difference of A and B, is a μ-null set. The family of all μ-null sets is denoted by Σ_0 and the family of all sets of positive μ−measure is denoted by Σ_μ^+. μ^* is the outer measure generated by μ. The Carathéodory completion of (Ω, Σ, μ) will be denoted by $(\Omega, \hat{\Sigma}, \hat{\mu})$. $\mathcal{M}(\mu)$ denotes the family of all real-valued μ-measurable functions on (Ω, Σ, μ). The space of all bounded functions from $\mathcal{M}(\mu)$ is denoted by $\mathcal{L}_\infty(\mu)$. It is endowed with the sup norm. Neither in $\mathcal{M}(\mu)$ nor in $\mathcal{L}_\infty(\mu)$ equivalent functions are identified. The space of equivalence classes in $\mathcal{L}_\infty(\mu)$ is denoted by $L_\infty(\mu)$ and the space of equivalence classes of μ-integrable functions is denoted by $L_1(\mu)$. The σ-algebra generated by a family $\mathcal{L}$ of sets is denoted by $\sigma(\mathcal{L})$. $\mathbf{N}$ and $\mathbf{R}$ stand for the natural numbers and the real numbers respectively. If $M \subseteq \Omega$, then $M^c := \Omega \setminus M$. If I is a nonempty set and $(\Omega_i, \Sigma_i, \mu_i)_{i \in I}$ is a family of probability spaces then, for each $\emptyset \neq J \subseteq I$ we denote by $(\Omega_J, \Sigma_J, \mu_J)$ the product measure space $\otimes_{i \in J}(\Omega_i, \Sigma_i, \mu_i)$ and by $(\Omega_J, \widehat{\Sigma}_J, \widehat{\mu}_J)$ its measure completion. If (Ω, Σ, μ) is a probability space and I is a non-empty set, we write μ^I for the product measure on Ω^I and Σ^I for its domain.

If $(\Omega_1, \Sigma_1, \mu_1), \dots, (\Omega_n, \Sigma_n, \mu_n)$ is a collection of measure spaces, then $\otimes_{i=1}^n (\Omega_i, \Sigma_i, \mu_i)$ is the direct product of the measure spaces and $\hat{\otimes}_{i=1}^n (\Omega_i, \Sigma_i, \mu_i)$ is the completion of that product space. In the case of idendical measure spaces we will be using also the notations $\otimes^n (\Omega, \Sigma, \mu)$ and $\hat{\otimes}^n (\Omega, \Sigma, \mu)$ or μ_n and $\hat{\mu}_n$. For any $\emptyset \neq J \subseteq I$ the canonical projection of Ω_I onto Ω_J is denoted by p_J and the σ-algebra $p_J^{-1}(\Sigma_J) \subseteq \Sigma_I$ is denoted by Σ_J^*. X is an arbitrary Banach space and $\tilde{B}$ is the set of all $x^{**} \in X^{**}$ which are weak*-cluster points of countable subsets of B_X, the closed unit ball of X.

2. Densities

DEFINITION 2.1. *A mapping $\tau : \Sigma \to \Sigma$ is called a lower density or density on Σ if the following properties hold true for all $A, B \in \Sigma$:*

(Da) $\tau(A) \equiv A$;

(Db) $A \equiv B \Longrightarrow \tau(A) = \tau(B)$;

(Dc) $\tau(A \cap B) = \tau(A) \cap \tau(B)$;

(Dd) $\tau(\emptyset) = \emptyset$ *and* $\tau(\Omega) = \Omega$.

EXAMPLE 2.2. Let μ be Lebesgue measure on $\mathbf{R}^n$, where $n \geq 1$. If Σ is the family of Lebesgue measurable sets we put for any $E \in \Sigma$

$$\tau(E) := \left\{ x \in \mathbf{R}^n : \lim_{\delta \searrow 0} \frac{\mu(E \cap B(x, \delta))}{\mu B(x, \delta)} = 1 \right\},$$

where $B(x, \delta)$ is the closed ball with radius δ and centre x.

LEMMA 2.3. *(see [3]) Let $\Xi \supset \Sigma_0$ be a sub$-\sigma-$algebra of Σ and let τ_0 be a lower density on Ξ. If $V \in \Sigma \setminus \Xi$, then there exists a lower density τ on $\sigma(\Xi \cup \{V\})$ such that $\tau|\Xi = \tau_0$.*

Proof. Let

$$\begin{aligned} M_1 &:= \text{ess inf}\{F \in \Xi : V \subseteq F\} \quad \text{and} \\ M_2 &:= \text{ess inf}\{F \in \Xi : V^c \subseteq F\}. \end{aligned}$$

If $Q \in \sigma(\Xi \cup \{V\})$, then there exist $A, B \in \Xi$ such that $Q = (A \cap V) \cup (B \cap V^c)$. We set

$$\tau[(A \cap V) \cup (B \cap V^c)] := $$
$$[V \cap \tau_0[(A \cap M_1) \cup (B \cap M_1^c)]] \cup [V^c \cap \tau_0[(B \cap M_2) \cup (A \cap M_2^c)]].$$

A direct calculation proves, that τ satisfies the required conditions. We shall verify only the uniqueness. So assume that

$$A_1 \cap V \cup B_1 \cap V^c = A_2 \cap V \cup B_2 \cap V^c,$$

with $A_1, A_2, B_1, B_2 \in \Xi$. Obviously, we have

$$A_1 \cap V = A_2 \cap V \quad \text{and} \quad B_1 \cap V^c = B_2 \cap V^c$$

and so
$$A_1 \triangle A_2 \subset V^c \quad \text{and} \quad B_1 \triangle B_2 \subset V.$$

It follows that

$$(A_1 \triangle A_2) \cap M_1 \subset M_1 \setminus V \quad \text{and} \quad (B_1 \triangle B_2) \cap M_1^c = \emptyset.$$

Since $(\mu \setminus \Xi)_*(M_1 \setminus V) = 0$ we have

$$\mu[(A_1 \cap M_1)\triangle(A_2 \cap M_1)] = 0 \quad \text{and} \quad B_1 \cap M_1^c = B_2 \cap M_1^c.$$

By simmetry, applying the equality $(\mu \setminus \Xi)_*(M_2 \setminus V^c) = 0$ we get also

$$\mu[(B_1 \cap M_2)\triangle(B_2 \cap M_2)] = 0 \quad \text{and} \quad A_1 \cap M_2^c = A_2 \cap M_2^c.$$

Since τ_0 is a density we get for the both representations the same value of τ. □

LEMMA 2.4. *Let* $(\Xi_n)_{n=1}^\infty$ *be an increasing sequence of sub$-\sigma-$alge-bras of* Σ *and let* $(\tau_n)_{n=1}^\infty$ *be a sequence of densities (i.e.* τ_n *is a lower density on* Ξ_n *). If* $\tau_{n+1}|\Xi_n = \tau_n$ *for every* $n \in \mathbf{N}$, *then there exists a lower density* τ *on* $\Xi := \sigma\left(\bigcup_{n=1}^\infty \Xi_n\right)$ *such that* $\tau|\Xi_n = \tau_n$ *for every* $n \in \mathbf{N}$.

Proof. For each $E \in \Xi$ we set

$$\tau(E) := \bigcap_{k=1}^\infty \bigcup_{n=1}^\infty \bigcap_{m=n}^\infty \tau_m\left(\left\{\omega \in \Omega : E_{\Xi_m}(\chi_E)(\omega) \geq 1 - 1/2^k\right\}\right).$$

We shall verify whether the conditions (Da), (Db), (Dc) and (Dd) are fullfilled.

(Db) and (Dd) follow directly from the definition of τ.

(Dc) Notice first that if $A \subset B$, then $\tau(A) \subseteq \tau(B)$. Indeed, we have then $E_{\Xi_m}(\chi_A) \leq E_{\Xi_m}(\chi_B)$ $\mu|\Xi_m-a.e.$, what yields the required result. Consequently, if $A, B \in \Xi$ are arbitrary, then $\tau(A \cap B) \subseteq \tau(A) \cap \tau(B)$. To prove the reverse inclusion, notice that

$$\chi_{A\cap B} = \chi_A + \chi_B - \chi_{A\cup B} \geq \chi_A + \chi_B - 1$$

and so

$$E_{\Xi_m}(\chi_{A\cap B}) \geq E_{\Xi_m}(\chi_A) + E_{\Xi_m}(\chi_B) - 1 \; \mu|\Xi_m - a.e.$$

Consequently, for $\mu|\Xi_m-$almost all $\omega \in \Omega$ the inequalities

$$E_{\Xi_m}(\chi_A)\omega \geq 1 - 1/2^{k+1} \quad \text{and} \quad E_{\Xi_m}(\chi_B)\omega \geq 1 - 1/2^{k+1}$$

yield

$$E_{\Xi_m}(\chi_{A \cap B})\omega \geq 1 - 1/2^k \, .$$

Thus,

$$\{E_{\Xi_m}(\chi_A) \geq 1 - 1/2^{k+1}\} \cap \{E_{\Xi_m}(\chi_B) \geq 1 - 1/2^{k+1}\}$$
$$\subseteq \mu|_{\Xi_m}\{E_{\Xi_m}(\chi_{A \cap B}) \geq 1 - 1/2^k\}$$

and so

$$\tau_m\Big\{E_{\Xi_m}(\chi_A) \geq 1 - 1/2^{k+1}\Big\} \cap \tau_m\Big\{E_{\Xi_m}(\chi_B) \geq 1 - 1/2^{k+1}\Big\}$$
$$\subseteq \tau_m\Big\{E_{\Xi_m}(\chi_{A \cap B}) \geq 1 - 1/2^k\Big\} \, .$$

Now, if $\omega \in \tau(A) \cap \tau(B)$, then for each $k \in \mathbf{N}$ there exist n_1, n_2 such that

$$\omega \in \bigcap_{m=n_1}^{\infty} \tau_m\Big(\Big\{E_{\Xi_m}(\chi_A) \geq 1 - 1/2^{k+1}\Big\}\Big) \text{ and}$$
$$\omega \in \bigcap_{m=n_2}^{\infty} \tau_m\Big(\Big\{E_{\Xi_m}(\chi_B) \geq 1 - 1/2^{k+1}\Big\}\Big) \, ,$$

what means that

$$\omega \in \bigcap_{m \geq \max(n_1, n_2)}^{\infty} \tau_m\Big(\Big\{E_{\Xi_m}(\chi_{A \cap B}) \geq 1 - 1/2^k\Big\}\Big) \, .$$

As $k \in \mathbf{N}$ was arbitrary, we get the required inclusion.

(Da) If $E \in \Xi$ is arbitrary, then

$$E_{\Xi_m}(\chi_E) \to \chi_E \quad \mu|\Xi - \text{a.e.} \, ,$$

so if we set

$$V_E \; := \; \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \Big\{E_{\Xi_m}(\chi_E) \geq 1 - 1/2^k\Big\}$$
$$= \; \{\omega : \liminf E_{\Xi_m}(\chi_E)(\omega) \geq 1\}$$

then $E \triangle V_E \in \Xi_0$. Since

$$\tau(E) \triangle V_E \subseteq$$
$$\bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \tau_m\Big(\Big\{E_{\Xi_m}(\chi_E) \geq 1 - \frac{1}{2^k}\Big\}\Big) \triangle \Big\{E_{\Xi_m}(\chi_E) \geq 1 - \frac{1}{2^k}\Big\} \in \Xi_0$$

we get $E \equiv \tau(E)$. The last assertion of the Lemma is clear.   □

THEOREM 2.5. *For an arbitrary (Ω, Σ, μ) there exists a lower density τ on Σ.*

Proof. Let $\mathcal{D}$ be the smallest cardinal with the property that there exists a collection $\mathcal{M} \subset \Sigma$ such that $\sigma(\mathcal{M})$ is dense in Σ in the pseudometric generated by μ. Let $\mathcal{M} = (M_\alpha)_{\alpha < \kappa}$ be numbered by ordinals less than κ, where κ is the first ordinal of the cardinality $\mathcal{D}$. Denote by η_0 the σ - algebra $\sigma(\Sigma_0)$ and for each $1 \le \alpha \le \kappa$ denote by η_α the σ - algebra generated by the family $\{M_\gamma : \gamma < \alpha\} \cup \eta_0$. We assume that $M_\alpha \notin \eta_\alpha$ for each α. It is clear that without loss of generality, we may do so.

We shall be constructing the final density inductively. τ_0 will be the only existing density on $(\Omega, \eta_0, \mu|\eta_0)$, i.e.

$$\tau_0(B) = \begin{cases} \emptyset, & \text{if } B \in \Sigma_0 \, ; \\ \Omega, & \text{if } B \notin \Sigma_0 \, . \end{cases}$$

Assume that for each $\alpha < \gamma < \kappa$ a density τ_α on η_α is already constructed. We assume, that $\alpha < \beta < \gamma$ yields $\tau_\beta|\eta_\alpha = \tau_\alpha$. We have to separate three cases.

A) $\gamma \le \kappa$ is a limit ordinal of uncountable cofinality. Then $\eta_\gamma = \bigcup_{\alpha < \gamma} \eta_\alpha$ and we define $\tau_\gamma \in \vartheta(\mu|\eta_\gamma)$ by setting

$$\tau_\gamma(B) := \tau_\alpha(B) \quad \text{if} \quad B \in \eta_\alpha \quad \text{and} \quad \alpha < \gamma \, .$$

B) There exists an increasing sequence (γ_n^γ) of ordinals that is cofinal to $\gamma \le \kappa$.

For simplicity put $\tau_n := \tau_{\gamma_n^\gamma}$ and $\eta_n := \eta_{\gamma_n^\gamma}$ for all $n \in \mathbf{N}$. Then $\eta_\gamma = \sigma(\cup_{n \in \mathbf{N}} \eta_n)$ and we can define τ_γ by setting

$$\tau_\gamma(B) := \bigcap_{k \in \mathbf{N}} \bigcup_{n \in \mathbf{N}} \bigcap_{m \ge n} \tau_m(\{E_{\eta_m}(\chi_B) > 1 - 1/2^k\}) \quad \text{for} \quad B \in \eta_\gamma \, .$$

It follows from Lemma 2.4 that τ_γ is a density on η_γ and $\tau_\gamma|\eta_n = \tau_n$ for each $n \in \mathbf{N}$.

C) If $\gamma = \beta + 1$ then, τ_γ is constructed with the help of Lemma 2.3.

Finally, we define τ just by setting $\tau = \tau_\kappa$. $\qquad \square$

DEFINITION 2.6. *Each density constructed in the way described in the proof of Theorem 2.5 will be called an admissible density. The family of all admissible densities on (Ω, Σ, μ) will be denoted by $A\vartheta(\mu)$.*

3. Liftings of functions and sets

DEFINITION 3.1. *A mapping $\rho : \mathcal{L}_\infty(\mu) \to \mathcal{L}_\infty(\mu)$ is called a lifting on $\mathcal{L}_\infty(\mu)$ if it possessess the following properties:*

(L1) $\rho(f) \equiv f$ for every $f \in \mathcal{L}_\infty(\mu)$;

(L2) ρ is linear and multiplicative;

(L3) $\rho(1) = 1$;

(L4) if $f \equiv g$, then $\rho(f) = \rho(g)$.

PROPOSITION 3.2. *If $\rho : \mathcal{L}_\infty(\mu) \to \mathcal{L}_\infty(\mu)$ is a lifting, then*

(L5) *If $f \geq 0$ $\mu-$a.e., then $\rho(f) \geq 0$ everywhere.*

(L6) *If $f \leq g$ $\mu-$a.e., then $\rho(f) \leq \rho(g)$ everywhere.*

(L7) *$|\rho(f)| = \rho(|f|)$.*

(L8) *$\sup(\rho(f), \rho(g)) = \rho(\sup(f, g))$ and $\inf(\rho(f), \rho(g)) = \rho(\inf(f, g))$.*

Proof.

(L5) If $f \geq 0$ $\mu-$a.e., then $f = (\sqrt{f})^2$ and so $\rho(f) = [\rho(\sqrt{f})]^2 \geq 0$.

(L6) Follows directly from $(L5)$ and the linearity of ρ.

(L7) We have $[\rho(|f|)]^2 = \rho(|f|^2) = \rho(f^2) = [\rho(f)]^2 = |\rho(f)|^2$. Since $\rho(|f|) \geq 0$ everywhere, we get the required equality $|\rho(f)| = \rho(|f|)$.

(L8) We use the equalities

$\square$

ρ can be aso considered as a mapping from $L_\infty(\mu)$ into $\mathcal{L}_\infty(\mu)$. With this terminology we get

PROPOSITION 3.3. *ρ is an isometry from $L_\infty(\mu)$ into $\mathcal{L}_\infty(\mu)$.*

Proof. Since $\pm f \leq \|f\|_\infty$ $\mu-$a.e., we have by $(L6), (L3)$ and the linearity of ρ

$$\pm\rho(f) = \rho(\pm f) \leq \rho(\|f\|_\infty) = \|f\|_\infty \rho(1) = \|f\|_\infty \text{ everywhere}.$$

Hence $\|\rho(f)\|_{\mathcal{L}_\infty} \leq \|f\|_\infty$.

To prove the reverse inequality notice, that since $\rho(f) \equiv f$ we have $\|f\|_\infty = \|\rho(f)\|_\infty \leq \|\rho(f)\|_{\mathcal{L}_\infty}$. $\qquad\square$

DEFINITION 3.4. *A mapping $\rho' : \Sigma \to \Sigma$ is called a lifting on Σ (or – more precisely – on (Ω, Σ, μ)) if besides the properties (Da), (Db), (Dc) and (Dd) the following property holds true for all $A, B \in \Sigma$:*

(De) $\rho'(A^c) = [\rho'(A)]^c$.

The collection of all liftings on (Ω, Σ, μ) is denoted by $\Lambda(\mu)$.

PROPOSITION 3.5. *If $\rho' : \Sigma \to \Sigma$ is a lifting on Σ then for every $A, B \in \Sigma$ the following property holds true:*

(Df) $$\rho'(A \cup B) = \rho'(A) \cup \rho'(B).$$

Proof.

$$\begin{aligned}
\rho'(A \cup B) &= \rho'[(A^c \cap B^c)^c] = [\rho'(A^c \cap B^c)]^c \\
&= [\rho'(A^c) \cap \rho'(B^c)]^c = [\rho'(A)^c \cap \rho'(B)^c]^c = \rho'(A) \cup \rho'(B).
\end{aligned}$$

$\qquad\square$

PROPOSITION 3.6. *Each lifting $\rho : \mathcal{L}_\infty(\mu) \to \mathcal{L}_\infty(\mu)$ uniquely determines a lifting $\rho' : \Sigma \to \Sigma$ satisfying for each $A \in \Sigma$ the equality $\rho(\chi_A) = \chi_{\rho'(A)}$. And conversely, if $\rho' : \Sigma \to \Sigma$ is a lifting on Σ, then there is a unique lifting $\rho : \mathcal{L}_\infty(\mu) \to \mathcal{L}_\infty(\mu)$ such that $\rho(\chi_A) = \chi_{\rho'(A)}$, for every $A \in \Sigma$.*

Proof. Let ρ be a lifting on $\mathcal{L}_\infty(\mu)$. If $A \in \Sigma$, then $\rho(\chi_A) = \rho[(\chi_A)^2] = [\rho(\chi_A)]^2$ is a $\{0, 1\}-$valued measurable function. Consequently, there exists a unique set $\rho'(A) \in \Sigma$ such that $\rho(\chi_A) = \chi_{\rho'(A)}$.

Let ρ' be a lifting on Σ. If $f = \sum_{i=1}^{k} \alpha_i \chi_{A_i} \in \mathcal{L}_\infty(\mu)$, then we set $\rho(f) := \sum_{i=1}^{k} \alpha_i \chi_{\rho'(A_i)}$. One can easily check that the definition of $\rho(f)$ does not depend on the representation of f. One can easily see, that on the space of simple functions ρ satisfies the conditions $(L1) - (L4)$ Moreover, ρ is on the space of simple functions uniquely determined. Now, if $\lim_{n \to \infty} \|f - f_n\|_{\sup} = 0$ and all f_n are simple, then we set $\rho(f) := \lim_{n \to \infty} \rho(f_n)$, where the convergence is pointwise. It is quite clear that the conditions $(L1) - (L4)$ are fulfilled. $\square$

Starting from this place we will use the same letters for both kinds of liftings and, we will assume that they are connected in the way described by Proposition 3.6.

The utility of liftings lies in excelent measurability properties of lifted sets and functions. When one considers an uncountable collection of measurable sets $E_t \in \Sigma$, then often the set $\bigcup_t E_t$ is non-measurable. It cannot happen so in the case of lifted sets. More precisely, the following result holds true:

THEOREM 3.7. *If $\{E_t : t \in T\}$ is an arbitrary family of sets such that $E_t \subseteq \rho(E_t)$ for each $t \in T$, then $\bigcup_{t \in T} E_t \in \Sigma$ and $\bigcup_{t \in T} E_t \subseteq \rho\left(\bigcup_{t \in T} E_t\right)$. More generally, if $\{f_t : t \in T\} \subseteq \mathcal{L}_\infty(\mu)$ is a uniformly bounded family of functions, such that $f_t \leq \rho(f_t)$ for every $t \in T$, then $\sup_{t \in T} f_t$ is a measurable function and $\sup_{t \in T} f_t \leq \rho(\sup_{t \in T} f_t)$.*

Proof. Let $\{E_t : t \in T\}$ be a family of sets satisfying for each $t \in T$ the inclusion $E_t \subseteq \rho(E_t)$. Moreover let Ξ be the collection of all at most countable subsets of T. Since μ is bounded there is a real number a such that $a = \sup_{\alpha \in \Xi} \mu(\bigcup_{t \in \alpha} E_t)$. Let $E_{t_n}, n \in \mathbf{N}$ be such that $a = \mu\left(\bigcup_n E_{t_n}\right)$. Put $E := \bigcup_n E_{t_n}$. Notice now that for arbitrary $t \in T$ the inclusions $E_t \subseteq \rho(E_t) \subseteq \rho(E)$ hold true. Consequently,

$$E \subseteq \bigcup_{t \in T} E_t \subseteq \rho(E).$$

This proves the measurability of the set $\bigcup_{t \in T} E_t$ and the required inclusion.

The proof of the function part of the theorem is based on a similar idea.   □

In the sequel we will need yet the following simple fact:

LEMMA 3.8. *Let ρ be a lifting and let $f = \rho(f)$ be a function. Then, for each real α the inclusions*

$$\{f < \alpha\} \subseteq \rho(\{f < \alpha\}) \subseteq \rho(\{f \le \alpha\}) \subseteq \{f \le \alpha\}$$

hold true. In particular, $\{f < \alpha\} = \emptyset$ or $\{f < \alpha\} \in \Sigma_\mu^+$.

Proof. Assume first that $f = \rho(f) \ge 0$ and take an arbitrary number $\alpha \ge 0$. If $A = \{\omega : f(\omega) \ge \alpha\}$, then we have also for each $\omega \in \Omega$

$$f(\omega) \ge \alpha \chi_A(\omega).$$

Lifting of this inequality gives us for each omega

$$f(\omega) \ge \alpha \chi_{\rho(A)}(\omega).$$

This means that $\rho(A) \subseteq A$.

If f is arbitrary, then take m such that $f \ge m$ everywhere. Applying the first part of the proof, we get for $\alpha \ge m$

$$\rho\{f \ge \alpha\} = \rho\{f - m \ge \alpha - m\} \subseteq \{f - m \ge \alpha - m\} = \{f \ge \alpha\}.$$

Now we can prove the first inclusion of the lemma.

$$\{f < \alpha\} = \{f \ge \alpha\}^c \subseteq [\rho\{f \ge \alpha\}]^c = \rho\{f < \alpha\}.$$

This completes the proof.   □

We shall present now a method of constructing a lifting on a complete measure space if a density is given.

THEOREM 3.9. *Let (Ω, Σ, μ) be a complete measure space and let $\tau : \Sigma \to \Sigma$ be a lower density on Σ. Then there exists a lifting ρ on Σ, such that*

$$\tau(E) \subseteq \rho(E) \subseteq [\tau(E^c)]^c \tag{1}$$

for all $E \in \Sigma$. Each such ρ will be said to be generated by τ.

Proof. For each $\omega \in \Omega$ let

$$\mathcal{F}(\omega) := \{E \in \Sigma : \omega \in \tau(E)\}\,.$$

An easy calculation shows that $\mathcal{F}(\omega)$ is a filter base. Let $\mathcal{U}(\omega)$ be an arbitrary ultrafilter in Ω containing $\mathcal{F}(\omega)$ and let

$$\rho(F) := \{\omega : F \in \mathcal{U}(\omega)\}$$

for each $F \in \Sigma$.
We are going to prove that ρ is a lifting on Σ. To do it, notice first, that

$$\omega \in \tau(E) \Longleftrightarrow E \in \mathcal{F}(\omega) \qquad \& \qquad \omega \in \rho(E) \Longleftrightarrow E \in \mathcal{U}(\omega)\,. \quad (2)$$

Now, if $F \in \Sigma$ then since $\mathcal{U}(\omega)$ is an ultrafilter, we have

$$\omega \in \rho(F) \Leftrightarrow F \in \mathcal{U}(\omega) \Leftrightarrow E^c \notin \mathcal{U}(\omega) \Leftrightarrow \omega \notin \rho(F^c)\,.$$

It follows that $\rho(F^c) = [\rho(F)]^c$.
Since $\mathcal{F}(\omega) \subset \mathcal{U}(\omega)$ the relations (2) immediately proves that $\tau(E) \subseteq \rho(E)$ for every $E \in \Sigma$. The second inclusion of (1) follows from the first one and the condition (De).
Since τ is a lower density on Σ we have for each $F \in \Sigma$ $\tau(F) \equiv F$ and $[\tau(F^c)]^c \equiv (F^c)^c = F$. Together with (1), this proves the condition $(Da) : \rho(F) \equiv F$.
Consider now sets $A, B \in \Sigma$ such that $A \equiv B$ and let $\omega \in \Omega$ be an arbitrary point. Assume that $\omega \in \rho(A)$. This means that $A \in \mathcal{U}(\omega)$. Hence either $A \cap B \in \mathcal{U}(\omega)$ or $A \setminus B \in \mathcal{U}(\omega)$. But $A \setminus B \in \Sigma_0$ and so $\tau(A \setminus B) = \emptyset$. Then (1) yields $\rho(A \setminus B) = \emptyset$ and so $A \setminus B \notin \mathcal{U}(\omega)$. Consequently, $A \cap B \in \mathcal{U}(\omega)$ and so $B \in \mathcal{U}(\omega)$. It follows, that $\omega \in \rho(B)$, what proves (Db).
The multiplicativity of ρ follows directly from the multiplicativity of ultrafilters:

$$\begin{aligned}
\omega \in \rho(A \cap B) &\Leftrightarrow A \cap B \in \mathcal{U}(\omega) \\
&\Leftrightarrow \quad A \in \mathcal{U}(\omega) \quad \& \quad B \in \mathcal{U}(\omega) \\
&\Leftrightarrow \quad \omega \in \rho(A) \quad \& \quad \omega \in \rho(B) \\
&\Leftrightarrow \quad \omega \in \rho(A) \cap \rho(B)\,.
\end{aligned}$$

Consequently, $\rho(A \cap B) = \rho(A) \cap \rho(B)$. $\qquad\qquad \square$

DEFINITION 3.10. *Each lifting constructed on a complete measure space (Ω, Σ, μ) in the manner described by Theorem 3.9 from an admissible density will be called an admissibly generated lifting. The collection of all admissibly generated liftings on (Ω, Σ, μ) will be denoted by $AG\Lambda(\mu)$.*

As a direct consequence of Theorems 2.5 and 3.9 we get the existence of a lifting on an arbitrary complete measure space.

THEOREM 3.11. *On an arbitrary complete (Ω, Σ, μ) there exists an admissibly generated lifting.*

4. Product densities and liftings

DEFINITION 4.1. *For a family $((\Omega_i, \Sigma_i, \mu_i))_{i \in I}$ of probability spaces and a probability space (Ω, Σ, μ) such that $\Omega = \Omega_I$, $\Sigma \supseteq \Sigma_I$, $\mu \mid \Sigma_I = \mu_I$ we call a lifting π for μ a* product-lifting *of the liftings ρ_i for μ_i $(i \in I)$ and we write $\pi \in \otimes_{i \in I} \rho_i$ if the equation*

$$(P) \qquad \pi([A_{i_1}, ..., A_{i_n}]) = [\rho_{i_1}(A_{i_1}), ..., \rho_{i_n}(A_{i_n})].$$

holds true for all $n \in \mathbf{N}$, $i_1, ..., i_n \in I$ and all $A_{i_k} \in \Sigma_{i_k}$ $(k = 1, ..., n)$ where $[A_{i_1}, ..., A_{i_n}]$ denotes the cylinder set $\prod_{i \in I} B_i$ for $B_{i_k} = A_{i_k}$ $(k = 1, ..., n)$ and $B_i = \Omega_i$, $i \in I \setminus \{i_1, ..., i_n\}$. We use a similar definition for densities instead of liftings.
If (Ω, Σ, μ) is complete and ρ is a lifting on Σ then, following Talagrand [10], we say that ρ is a consistent lifting if for every finite product $\hat{\otimes}^k(\Omega, \Sigma, \mu)$ there exists a lifting ρ_k on $\hat{\otimes}^k(\Omega, \Sigma, \mu)$, such that

$$\rho_k(A_1 \times \cdots \times A_k) = \rho(A_1) \times \cdots \times \rho(A_k),$$

for arbitrary $A_1, \ldots, A_k \in \Sigma$. If (Ω, Σ, μ) is quite arbitrary probability space, ρ is a density and ρ_k is a density on $\otimes^k(\Omega, \Sigma, \mu)$, then ρ is called a consistent density.

The original Talagrand's proof of the existence of a consistent lifting on an arbitrary complete probability space goes through linear liftings. I will prove first the existence of a consistent density, and then – using Theorem 3.9 – the existence of a consistent lifting.

Following Fremlin [2] we say that $\varphi \in \vartheta(\mu_I)$ satisfies the condition $(*)$, if for arbitrary disjoint $J, K \subseteq I$

$(*)$ $\varphi(E \cup F) = \varphi(E) \cup \varphi(F)$ for each $E \in \Sigma_J^*$ and $F \in \Sigma_K^*$.

THEOREM 4.2. *Let (Θ, T, ν) be an arbitrary probability space. If $\upsilon \in A\vartheta(\nu)$ then for each (Ω, Σ, μ) and each $\tau \in \vartheta(\mu)$ there exists $\varphi \in \vartheta(\mu \otimes \nu)$ such that*

$$\varphi(A \times B) = \tau(A) \times \upsilon(B) \qquad \text{for all } A \in \Sigma \text{ and } B \in T$$

and φ satisfies $(*)$. If $(\Omega, \Sigma, \mu) = \otimes_{i \in I}(\Omega_i, \Sigma_i, \mu_i)$ and $\tau \in \otimes_{i \in I}\tau_i$ satisfies the condition $(*)$, then $\varphi \in \tau \otimes \upsilon$ can be chosen to satisfy $(*)$ in the product $\otimes_{i \in I}(\Omega_i, \Sigma_i, \mu_i) \otimes (\Theta, T, \nu)$.

Proof. Let there be given a $\tau \in \vartheta(\mu)$ and $\upsilon \in A\vartheta(\nu)$ altogether with other elements involved into the construction of $\upsilon \in A\vartheta(\nu)$. In particular the family $\mathcal{M} = (M_\alpha)_{\alpha < \kappa}$, the $\sigma-$subalgebras $(\eta_\alpha)_{\alpha < \kappa}$ and the sequences (γ_n^γ) cofinal with limit ordinals are fixed.
Using the transfinite induction, we shall be constructing now a transfinite sequence $(\hat\varphi_\alpha)_{\alpha < \kappa}$ with $\hat\varphi_\alpha \in \vartheta(\mu \otimes \nu \mid \Sigma \otimes \eta_\alpha)$ such that:

$$\hat\varphi_\alpha(A \times B) = \tau(A) \times \upsilon_\alpha(B) \quad \text{for all} \quad A \in \Sigma, \, B \in \eta_\alpha \qquad (3)$$

and

$$\hat\varphi_\beta \mid \Sigma \otimes \eta_\alpha = \hat\varphi_\alpha \quad \text{for} \quad \alpha < \beta < \kappa. \qquad (4)$$

Moreover, we assume that if $(\Omega, \Sigma, \mu) = \otimes_{i \in I}(\Omega_i, \Sigma_i, \mu_i)$ and $\tau \in \otimes_{i \in I}\tau_i$ satisfies $(*)$, then each $\hat\varphi_\alpha$ satisfies the condition $(*)$.
For $E \in \Sigma \otimes \eta_0$ we have $(\mu \otimes \nu)(E) = \int \nu(E_\omega) d\mu(\omega)$. Then $\widetilde{E} := \{\omega \in \Omega : \nu(E_\omega) = 1\} \in \Sigma$ and $E = \widetilde{E} \times \Theta$ $(\mu \otimes \nu)-$a.e.. Hence if we define

$$\hat\varphi_0(E) = \tau(\widetilde{E}) \times \Theta \quad \text{for all} \quad E = \widetilde{E} \times \Theta \quad (\mu \otimes \nu) - a.e.$$

we have $\hat\varphi_0 \in \vartheta(\mu \otimes \nu) \mid \Sigma \otimes \eta_0)$ and $\hat\varphi_0(A \times B) = \tau(A) \times \upsilon_0(B)$ for $A \in \Sigma$ and $B \in \eta_0$.
Assume now that given $\gamma < \kappa$, a system $(\hat\varphi_\alpha)$ satisfying the required conditions (3) and (4) has been constructed for all $\alpha < \gamma$.

We have to distinguish three cases.

A) γ is a limit ordinal of uncountable cofinality.

Then

$$\Sigma \otimes \eta_\gamma = \bigcup_{\alpha < \gamma} (\Sigma \otimes \eta_\alpha) \tag{5}$$

Setting

$$\hat{\varphi}_\gamma(E) = \hat{\varphi}_\alpha(E) \quad \text{if} \quad E \in \Sigma \otimes \eta_\alpha,$$

we get unambiguously defined densities $\hat{\varphi}_\gamma \in \vartheta(\mu \otimes \nu \mid \Sigma \otimes \eta_\gamma)$ such that $\hat{\varphi}_\gamma \mid \Sigma \otimes \eta_\alpha = \hat{\varphi}_\alpha$ for all $\alpha < \gamma$.

It is a direct consequence of the relation (5) that the condition (3) is satisfied.

Clearly, $\hat{\varphi}_\gamma$ is a product density on the space $\otimes_{i \in I}(\Omega_i, \Sigma_i, \mu_i) \otimes (\Theta, \eta_\gamma, \nu \mid \eta_\gamma)$ and satisfies condition $(*)$.

B) γ is of countable cofinality.

For simplicity put $\upsilon_n := \upsilon_{\gamma_n^\gamma}$, $\hat{\varphi}_n := \hat{\varphi}_{\gamma_n^\gamma}$ and $\eta_n := \eta_{\gamma_n^\gamma}$ for all $n \in \mathbf{N}$.

Then

$$\Sigma \otimes \eta_\gamma = \sigma(\cup_{n \in \mathbf{N}} \Sigma \otimes \eta_n).$$

Hence, we can define

$$\hat{\varphi}_\gamma(P) := \bigcap_{k \in \mathbf{N}} \bigcup_{n \in \mathbf{N}} \bigcap_{m \geq n} \hat{\varphi}_m(\{E_{\Sigma \otimes \eta_m}(\chi_P) > 1 - 1/k\})$$

$$\text{for} \quad P \in \Sigma \otimes \eta_\gamma.$$

It follows by Lemma 2.4 that $\hat{\varphi}_\gamma \in \vartheta(\mu \otimes \nu \mid \Sigma \otimes \eta_\gamma)$, and $\hat{\varphi}_\gamma \mid \Sigma \otimes \eta_n = \hat{\varphi}_n$ for each $n \in \mathbf{N}$. Now for $A \in \Sigma$ and $B \in \eta_\gamma$ we have

$$\hat{\varphi}_\gamma(A \times B) := \bigcap_{k \in \mathbf{N}} \bigcup_{n \in \mathbf{N}} \bigcap_{m \geq n} \hat{\varphi}_m(\{E_{\Sigma \otimes \eta_m}(\chi_{A \times B}) > 1 - 1/k\}).$$

But for each $m, k \in \mathbf{N}$

$$
\begin{aligned}
\{E_{\Sigma \otimes \eta_m}(\chi_{A \times B}) > 1 - 1/k\} &= \{E_{\Sigma \otimes \eta_m}(\chi_A \otimes \chi_B) > 1 - 1/k\} \\
&= \{(E_\Sigma(\chi_A) \otimes E_{\eta_m}(\chi_B)) > 1 - 1/k\} \\
&= \{(\chi_A \otimes E_{\eta_m}(\chi_B)) > 1 - 1/k\} \\
&= A \times \{E_{\eta_m}(\chi_B) > 1 - 1/k\},
\end{aligned}
$$

where the equalities hold true $(\mu \otimes \nu | \Sigma \otimes \eta_m)-$a.e.. This implies that

$$
\begin{aligned}
\hat{\varphi}_\gamma(A \times B) & = \bigcap_{k \in \mathbf{N}} \bigcup_{n \in \mathbf{N}} \bigcap_{m \geq n} \hat{\varphi}_m(A \times \{E_{\eta_m}(\chi_B) > 1 - 1/k\}) \\
& = \bigcap_{k \in \mathbf{N}} \bigcup_{n \in \mathbf{N}} \bigcap_{m \geq n} (\tau(A) \times \upsilon_m(\{E_{\eta_m}(\chi_B) > 1 - 1/k\}) \\
& = \tau(A) \times (\bigcap_{k \in \mathbf{N}} \bigcup_{n \in \mathbf{N}} \bigcap_{m \geq n} \upsilon_m(\{E_{\eta_m}(\chi_B) > 1 - 1/k\})) \\
& = \tau(A) \times \upsilon_\gamma(B),
\end{aligned}
$$

i.e. $\hat{\varphi}_\gamma(A \times B) = \tau(A) \times \upsilon_\gamma(B)$ for all $A \in \Sigma$, $B \in \eta_\gamma$. Clearly $\hat{\varphi}_\gamma$ is a product density.
To show that $\hat{\varphi}_\gamma$ satisfies condition $(*)$, we need the following

Claim 1. For arbitrary $J, K \subseteq I$ with $J \cap K = \emptyset$, $A \in \Sigma_J^* \otimes \eta_\gamma$, $B \in \Sigma_K^* \times \Theta$ and $m, k \in \mathbf{N}$ we have $(\mu \otimes \nu \mid \Sigma \otimes \eta_m)-$a.e. the condition

$$
\begin{aligned}
\{E_{\Sigma \otimes \eta_m}(\chi_{A \cup B}) > 1 - 1/k\} \\
= \{E_{\Sigma \otimes \eta_m}(\chi_A) > 1 - 1/k\} \cup \{E_{\Sigma \otimes \eta_m}(\chi_B) > 1 - 1/k\}.
\end{aligned}
$$

Proof. Let J, K, A, B, m, k be as in the claim. For $A = \emptyset$ or $B = \emptyset$ the claim is obvious. Suppose that $A \neq \emptyset$ and $B \neq \emptyset$. It is clear that $(\mu \otimes \nu \mid \Sigma \otimes \eta_m)-$a.e. we have

$$
\begin{aligned}
\{E_{\Sigma \otimes \eta_m}(\chi_{A \cup B}) > 1 - 1/k\} \\
\supseteq \{E_{\Sigma \otimes \eta_m}(\chi_A) > 1 - 1/k\} \cup \{E_{\Sigma \otimes \eta_m}(\chi_B) > 1 - 1/k\}.
\end{aligned}
$$

To prove the converse relation notice that if $B = B_K \times \Omega_{I \setminus K} \times \Theta$, then

$$
E_{\Sigma \otimes \eta_m}(\chi_B) = \chi_B \quad (\mu \otimes \nu \mid \Sigma \otimes \eta_m) - \text{ a.e.}.
$$

Consequently, we get $(\mu \otimes \nu \mid \Sigma \otimes \eta_m)-$a.e. that

$$
\begin{aligned}
\{E_{\Sigma \otimes \eta_m}(\chi_{A \cup B}) > 1 - 1/k\} \\
= \{E_{\Sigma \otimes \eta_m}(\chi_A) + \chi_B - E_{\Sigma \otimes \eta_m}(\chi_A)\chi_B > 1 - 1/k\}. \quad (6)
\end{aligned}
$$

If $(\omega, \theta) \in B$, then

$$
E_{\Sigma \otimes \eta_m}(\chi_A)(\omega, \theta) + \chi_B(\omega, \theta) - E_{\Sigma \otimes \eta_m}(\chi_A)(\omega, \theta)\chi_B(\omega, \theta) = \chi_B(\omega, \theta),
$$

hence we get $(\mu \otimes \nu \mid \Sigma \otimes \eta_m)$−a.e. that

$$
\begin{aligned}
\{E_{\Sigma \otimes \eta_m}(\chi_A) &+ \chi_B - E_{\Sigma \otimes \eta_m}(\chi_A)\chi_B > 1 - 1/k\} \cap B = \\
&= \{\chi_B > 1 - 1/k\} \cap B \\
&\subseteq \Big(\{E_{\Sigma \otimes \eta_m}(\chi_A) > 1 - 1/k\} \cap B\Big) \\
&\quad \cup\Big(\{E_{\Sigma \otimes \eta_m}(\chi_B) > 1 - 1/k\} \cap B\Big).
\end{aligned} \tag{7}
$$

If $(\omega, \theta) \notin B$, then

$$
\begin{aligned}
E_{\Sigma \otimes \eta_m}(\chi_A)(\omega, \theta) &+ \chi_B(\omega, \theta) - E_{\Sigma \otimes \eta_m}(\chi_A)(\omega, \theta)\chi_B(\omega) \\
&= E_{\Sigma \otimes \eta_m}(\chi_A)(\omega, \theta),
\end{aligned}
$$

hence we have $(\mu \otimes \nu \mid \Sigma \otimes \eta_m)$−a.e. that

$$
\begin{aligned}
\{E_{\Sigma \otimes \eta_m}(\chi_A) &+ \chi_B - E_{\Sigma \otimes \eta_m}(\chi_A)\chi_B > 1 - 1/k\} \cap B^c = \\
&= \{\chi_B > 1 - 1/k\} \cap B^c \\
&\subseteq \Big(\{E_{\Sigma \otimes \eta_m}(\chi_A) > 1 - 1/k\} \cap B^c\Big) \\
&\quad \cup\Big(\{E_{\Sigma \otimes \eta_m}(\chi_B) > 1 - 1/k\} \cap B^c\Big).
\end{aligned} \tag{8}
$$

$(\mu \otimes \nu \mid \Sigma \otimes \eta_m)$−a.e. we have

$$
\begin{aligned}
\{E_{\Sigma \otimes \eta_m}(\chi_{A \cup B}) &> 1 - 1/k\} \\
&\subseteq \{E_{\Sigma \otimes \eta_m}(\chi_A) > 1 - 1/k\} \cup \{E_{\Sigma \otimes \eta_m}(\chi_B) > 1 - 1/k\}.
\end{aligned}
$$

This completes the proof of the claim 1.   □

To finish the proof of $(*)$ for $\widehat{\varphi}_\gamma$ observe that if $D = D_J \times \Omega_{J^c}$ with $D_J \in \Sigma_J \otimes \eta_\gamma$, then

$$
\begin{aligned}
E_{\Sigma \otimes \eta_m}(\chi_{D_J \times \Omega_{J^c}}) &= E_{(\Sigma_J \otimes \eta_m) \times \Omega_{J^c}}(\chi_{D_J} \otimes \chi_{\Omega_{J^c}}) \\
&= E_{\Sigma_J \otimes \eta_m}(\chi_{D_J}) \otimes \chi_{\Omega_{J^c}}.
\end{aligned} \tag{9}
$$

Now let J, K, A, B be as in Claim 1. Assume that

$$
A = A_J \times \Omega_{I \setminus J} \qquad \text{and} \qquad B = B_K \times \Omega_{I \setminus K} \times \Theta,
$$

where $A_J \in \Sigma_J \otimes \eta_\gamma$ and $B_K \in \Sigma_K$.
Applying the inductive assumption, Claim 1 and (9) we get

$$
\begin{aligned}
&\hat{\varphi}_m(\{E_{\Sigma \otimes \eta_m}(\chi_{A \cup B}) > 1 - 1/k\}) \\
&= \hat{\varphi}_m\Big(\{E_{\Sigma \otimes \eta_m}(\chi_A) > 1 - 1/k\} \cup \{E_{\Sigma \otimes \eta_m}(\chi_B) > 1 - 1/k\}\Big) \\
&= \hat{\varphi}_m\Big(\{E_{\Sigma_J \otimes \eta_m}(\chi_{A_J}) \otimes \chi_{\Omega_{I \setminus J}} > 1 - 1/k\} \\
&\qquad \cup \{E_{\Sigma_K}(\chi_{B_K}) \otimes \chi_{\Omega_{I \setminus K}} \otimes \chi_\Theta > 1 - 1/k\}\Big) \\
&= \hat{\varphi}_m\Big(\{E_{\Sigma_J \otimes \eta_m}(\chi_{A_J}) \otimes \chi_{\Omega_{I \setminus J}} > 1 - 1/k\}\Big) \\
&\qquad \cup \hat{\varphi}_m\Big(\{E_{\Sigma_K}(\chi_{B_K}) \otimes \chi_{\Omega_{I \setminus K}} \otimes \chi_\Theta > 1 - 1/k\}\Big) \\
&= \hat{\varphi}_m\Big(\{E_{\Sigma \otimes \eta_m}(\chi_A) > 1 - 1/k\}\Big) \cup \hat{\varphi}_m(B).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\hat{\varphi}_\gamma(A \cup B) \\
&= \bigcap_{k \in \mathbf{N}} \bigcup_{n \in \mathbf{N}} \bigcap_{m \geq n} \hat{\varphi}_m(\{E_{\Sigma \otimes \eta_m}(\chi_{A \cup B}) > 1 - 1/k\}) \\
&= \bigcap_{k \in \mathbf{N}} \bigcup_{n \in \mathbf{N}} \bigcap_{m \geq n} \Big[\hat{\varphi}_m\Big(\{E_{\Sigma \otimes \eta_m}(\chi_A) > 1 - 1/k\}\Big) \cup \hat{\varphi}_m(B)\Big] \\
&= \bigcap_{k \in \mathbf{N}} \bigcup_{n \in \mathbf{N}} \bigcap_{m \geq n} \Big[\hat{\varphi}_m\Big(\{E_{\Sigma \otimes \eta_m}(\chi_A) > 1 - 1/k\}\Big) \cup \hat{\varphi}_\gamma(B)\Big] \\
&= \hat{\varphi}_\gamma(A) \cup \hat{\varphi}_\gamma(B),
\end{aligned}
$$

i.e. $\hat{\varphi}_\gamma$ satisfies $(*)$.
C) $\gamma = \beta + 1$
To simplify the notations let $M := M_\beta$. It is well known that

$$
\Sigma \otimes \eta_\gamma = \{(K \cap (\Omega \times M)) \cup (L \cap (\Omega \times M^c)) : K, L \in \Sigma \otimes \eta_\beta\}.
$$

Let $M_1 \supseteq M$ and $M_2 \supseteq M^c$ be η_β−envelopes of M and M^c respectively, used in the process of describing of $\upsilon_\gamma \in A\vartheta(\nu|\eta_\gamma)$ (see Lemma 2.3). An easy calculation shows that

$$
E_1 = \Omega \times M_1 \quad \text{and} \quad E_2 = \Omega \times M_2 \tag{10}
$$

are $\Sigma \otimes \eta_\beta-$envelopes of $\Omega \times M$ and $\Omega \times M^c$ respectively.
Define

$$\hat{\varphi}_\gamma \Big[\Big(K \cap (\Omega \times M) \Big) \cup \Big(L \cap (\Omega \times M^c) \Big) \Big] :=$$
$$\Big((\Omega \times M) \cap \hat{\varphi}_\beta \big((K \cap E_1) \cup (L \cap E_1^c) \big) \Big)$$
$$\cup \Big((\Omega \times M^c) \cap \hat{\varphi}_\beta \big((L \cap E_2) \cup (K \cap E_2^c) \big) \Big)$$

for $K, L \in \Sigma \otimes \eta_\beta$.
By Lemma 2.3 it follows that $\hat{\varphi}_\gamma \in \vartheta(\mu \otimes \nu \mid \Sigma \otimes \eta_\gamma)$ and $\hat{\varphi}_\gamma \mid \Sigma \otimes \eta_\beta = \hat{\varphi}_\beta$.
For $A \in \Sigma$ and $B \in \eta_\gamma$ write $B = ((G \cap M) \cup (H \cap M^c))$ for $G, H \in \eta_\beta$.
Then we have

$$\begin{aligned}
A \times B &= A \times [(G \cap M) \cup (H \cap M^c)] \\
&= [(A \times G) \cap (\Omega \times M)] \cup [(A \times H) \cap (\Omega \times M^c)]
\end{aligned}$$

together with $K := A \times G$, $L := A \times H \in \Sigma \otimes \eta_\beta$. For simplicity put $E_0 := \Omega \times M$. By definition we have

$$\begin{aligned}
\hat{\varphi}_\gamma(A \times B) &= (E_0 \cap \hat{\varphi}_\beta[(K \cap E_1) \cup (L \cap E_1^c)]) \\
&\quad \cup (E_0^c \cap \hat{\varphi}_\beta[(L \cap E_2) \cup (K \cap E_2^c)]).
\end{aligned}$$

By an application of (10) this can be rewritten as

$$\hat{\varphi}_\gamma(A \times B) = [E_0 \cap \hat{\varphi}_\beta(A \times R)] \cup [E_0^c \cap \hat{\varphi}_\beta((A \times S)]$$

if

$$R := (G \cap M_1) \cup (H \cap M_1^c) \qquad \text{and} \qquad S := (H \cap M_2) \cup (G \cap M_2^c).$$

Since $R, S \in \eta_\beta$ this implies

$$\hat{\varphi}_\gamma(A \times B) = (E_0 \cap [\tau(A) \times \upsilon_\beta(R)]) \cup (E_0^c \cap [\tau(A) \times \upsilon_\beta(S)]).$$

By means of $E_0 = \Omega \times M$ the latter formula can be transformed into

$$\hat{\varphi}_\gamma(A \times B) = \tau(A) \times \upsilon_\gamma(B) \quad \text{for all} \quad A \in \Sigma \quad \text{and} \quad B \in \eta_\gamma.$$

Therefore $\hat{\varphi}_\gamma$ satisfies the condition (3).

To show that $\hat{\varphi}_\gamma$ satisfies $(*)$, consider $J, H \subseteq I$ with $J \cap H = \emptyset$. For each

$$A = (K \cap E_0) \cup (L \cap E_0^c) \in \Sigma_J \otimes \eta_\gamma \times \Omega_{I \setminus J}$$

and

$$B \in \Sigma_H \times \Theta \times \Omega_{I \setminus H},$$

where $K, L \in \Sigma_J \otimes \eta_\beta$. Applying the inductive assumption and the equality $B = (B \cap E_0)\big) \cup (B \cap E_0^c)$ we get

$$
\begin{aligned}
\hat{\varphi}_\gamma(A \cup B) & \\
= & \ \hat{\varphi}_\gamma\Big((K \cap E_0) \cup (L \cap E_0^c) \cup (B \cap E_0) \cup (B \cap E_0^c)\Big) \\
= & \ \hat{\varphi}_\gamma\Big[[(K \cup B) \cap E_0] \cup [(L \cup B) \cap E_0^c]\Big] \\
= & \ \Big[E_0 \cap \hat{\varphi}_\beta\Big([(K \cup B) \cap E_1] \cup [(L \cup B) \cap E_1^c]\Big)\Big] \\
& \cup \Big[E_0^c \cap \hat{\varphi}_\beta\Big([(L \cup B) \cap E_2] \cup [(K \cup B) \cap E_2^c]\Big)\Big] \\
= & \ \Big[E_0 \cap \Big(\hat{\varphi}_\beta[(K \cap E_1) \cup (L \cap E_1^c)] \cup \hat{\varphi}_\beta[(B \cap E_1) \cup (B \cap E_1^c)]\Big)\Big] \\
& \cup \Big[E_0^c \cap \Big(\hat{\varphi}_\beta[(L \cap E_2) \cup (K \cap E_2^c)] \cup \hat{\varphi}_\beta[(B \cap E_2) \cup (B \cap E_2^c)]\Big)\Big] \\
= & \ \Big[E_0 \cap \hat{\varphi}_\beta\Big((K \cap E_1) \cup (L \cap E_1^c)\Big)\Big] \\
& \cup \Big[E_0^c \cap \hat{\varphi}_\beta\Big(L \cap E_2) \cup (K \cap E_2^c)\Big)\Big] \\
& \cup \Big[E_0 \cap \hat{\varphi}_\beta(B)\Big] \cup \Big[E_0^c \cap \hat{\varphi}_\beta(B)\Big] \\
= & \ \hat{\varphi}_\gamma\Big((K \cap E_0) \cup (L \cap E_0^c)\Big) \cup \hat{\varphi}_\beta(B) \\
= & \ \hat{\varphi}_\gamma(A) \cup \hat{\varphi}_\gamma(B),
\end{aligned}
$$

i.e. $\hat{\varphi}_\gamma$ satisfies $(*)$.

We can define now $\varphi \in \vartheta(\mu \otimes \nu)$ possessing the required properties just by setting $\varphi = \hat{\varphi}_\kappa$. The densities are properly defined, since each element of $\Sigma \otimes T$ is measurable with respect to some $\Sigma \otimes \eta_\alpha$, with $\alpha \leq \kappa$. $\square$

COROLLARY 4.3. *Let (Ω, Σ, μ) be a probability space. If $\rho \in A\vartheta(\mu)$, then for each $n \in \mathbf{N}$ there exists $\rho_n \in \vartheta(\mu^n)$ such that*

$$\rho_n(A_1 \times \cdots \times A_n) = \rho(A_1) \times \cdots \times \rho(A_n)$$

for arbitrary $A_1, \ldots, A_n \in \Sigma$ and $\rho_{n+1}(A \times \Omega) = \rho_n(A) \times \Omega$ for each $A \in \otimes^n \Sigma$. In particular each admissible density is consistent.

THEOREM 4.4. *Let (Ω, Σ, μ) and (Θ, T, ν) be complete probability spaces and let $\tau \in \vartheta(\mu), \upsilon \in \vartheta(\nu)$ and $\varphi \in \vartheta(\mu \otimes \nu)$, be densities such that the condition $\varphi \in \tau \otimes \upsilon$ holds true and φ satisfies condition $(*)$. Then, for each $\rho \in \Lambda(\mu)$ and each $\sigma \in \Lambda(\nu)$ generated by τ and υ respectively, there exists $\pi \in \Lambda(\mu \hat{\otimes} \nu)$ such that*

$(P) \qquad \pi(A \times B) = \rho(A) \times \sigma(B) \quad \text{for all } A \in \Sigma \quad \text{and} \quad B \in T \,;$

$$\varphi(E) \subseteq \pi(E) \subseteq [\varphi(E^c)]^c \quad \text{for all} \quad E \in \Sigma \otimes T \,.$$

Proof. For each $\omega \in \Omega$ and each $\theta \in \Theta$ let

$$\mathcal{F}(\omega) := \{A \in \Sigma : \omega \in \tau(A)\} \quad \text{and} \quad \mathcal{F}(\theta) := \{B \in T : \theta \in \upsilon(B)\}$$

be the filterbases generated by τ and υ respectively.
Then let

$$\mathcal{U}(\omega) := \{A \in \Sigma : \omega \in \rho(A)\} \quad \text{and} \quad \mathcal{U}(\theta) := \{B \in T : \theta \in \sigma(B)\} \,.$$

be the ultrafilters generated by ρ and σ, respectively so that

$$\mathcal{F}(\omega) \subseteq \mathcal{U}(\omega) \subset \Sigma \quad \text{and} \quad \mathcal{F}(\theta) \subseteq \mathcal{U}(\theta) \subset T.$$

For each $(\omega, \theta) \in \Omega \times \Theta$ define the filterbasis by

$$\mathcal{F}(\omega, \theta) := \{E \in \Sigma \otimes T : (\omega, \theta) \in \varphi(E)\}.$$

Claim 1. *For each $(\omega, \theta) \in \Omega \times \Theta$, $A \in \mathcal{U}(\omega)$, $B \in \mathcal{U}(\theta)$, and $E \in \mathcal{F}(\omega, \theta)$, we have*

$$E \cap (A \times B) \neq \emptyset.$$

Proof. Let $(\omega, \theta), E$ and A, B be as in Claim 1. Assume that

$$E \cap (A \times B) = \emptyset.$$

Then we get by using $(*)$

$$
\begin{aligned}
\varphi(E) \quad &\subseteq \quad \varphi(A^c \times \Theta \cup \Omega \times B^c) \\
&= \quad \varphi(A^c \times \Theta) \cup \varphi(\Omega \times B^c) \\
&= \quad (\upsilon(A^c) \times \Theta) \cup (\Omega \times \tau(B^c)) \\
&\subseteq \quad (\rho(A^c) \times \Theta) \cup (\Omega \times \sigma(B^c)) \\
&= \quad [\rho(A) \times \sigma(B)]^c,
\end{aligned}
$$

i.e. $\varphi(E) \cap [\rho(A) \times \sigma(B)] = \emptyset$, which contradicts to the assumption $(\omega, \theta) \in \varphi(E) \cap [\rho(A) \times \sigma(B)]$ of claim 1. $\square$

By the above claim there exists an ultrafilter $\mathcal{U}(\omega, \theta) \subseteq \Sigma \hat{\otimes} T$ finer than $\mathcal{F}(\omega, \theta)$ such that

$$A \times B \in \mathcal{U}(\omega, \theta) \quad \text{for all} \quad A \in \mathcal{U}(\omega) \quad \text{and} \quad B \in \mathcal{U}(\theta). \qquad (11)$$

For each $E \in \Sigma \hat{\otimes} T$ put

$$\pi(E) := \{(\omega, \theta) \in \Omega \times \Theta : E \in \mathcal{U}(\omega, \theta)\}.$$

By Theorem 3.9 $\pi \in \Lambda(\mu \hat{\otimes} \nu)$.

Claim 2. *For each $A \in \Sigma$ and $B \in T$, we have*

$$\rho(A) \times \sigma(B) = \pi(A \times B).$$

Proof. For $A \in \Sigma$ and $B \in T$ we get from (11) that

$$
\begin{aligned}
(\omega, \theta) \in \rho(A) \times \sigma(B) \quad &\Longleftrightarrow \quad \omega \in \rho(A) \& \theta \in \sigma(B) \\
&\Longleftrightarrow \quad A \in \mathcal{U}(\omega) \& B \in \mathcal{U}(\theta) \\
&\Longrightarrow \quad A \times B \in \mathcal{U}(\omega, \theta) \\
&\Longleftrightarrow \quad (\omega, \theta) \in \pi(A \times B),
\end{aligned}
$$

i.e.

$$\rho(A) \times \sigma(B) \subseteq \pi(A \times B).$$

It remains to show that

$$\pi(A \times B) \subseteq \rho(A) \times \sigma(B) \,.$$

Applying the first part of the proof we get for each $A \in \Sigma$ and $B \in T$ that

$$
\begin{aligned}
(\rho(A) \times \sigma(B))^c &= \left[\rho(A^c) \times \Theta\right] \cup \left[\Omega \times \sigma(B^c)\right] \\
&\subseteq \pi(A^c \times \Theta) \cup \pi(\Omega \times B^c) \\
&= [\pi(A \times B)]^c.
\end{aligned}
$$

$\square$

This completes the proof of the claim and hence of the whole theorem. $\square$

As a consequence of the Theoreme 4.2 and 4.4 we get Talagrand's result on the existence of consistent liftings:

THEOREM 4.5. *Let (Ω, Σ, μ) be complete. If $\rho \in AG\Lambda(\mu)$, then for each $n \in \mathbf{N}$ there exists $\rho_n \in \Lambda(\hat{\mu}^n)$, such that*

$$\rho_n(A_1 \times \cdots \times A_n) = \rho(A_1) \times \cdots \times \rho(A_n)$$

for all $A_1, \ldots, A_n \in \Sigma$ and $\rho_{n+1}(A \times \Omega) = \rho_n(A) \times \Omega$ for each $A \in \widehat{\bigotimes}^n \Omega$. In particular each admissibly generated lifting is consistent.

5. Stable sets

Throughout this section (Ω, Σ, μ) is a complete probability space, τ_p is the topology of pointwise convergence in $\mathcal{M}(\mu)$ and τ_m is the topology of convergence si measure.

LEMMA 5.1. *If $f : \Omega \to \mathbf{R}$ is non-measurable, then there exist numbers $\alpha < \beta$ and a set $A \in \Sigma_\mu^+$ such that*

$$\mu^*(A \cap \{f < \alpha\}) = \mu^*(A \cap \{f > \beta\}) = \mu(A) \,.$$

DEFINITION 5.2. *Let $\mathcal{Z} \subset \mathcal{M}(\mu)$ be τ_p-bounded. A set $A \in \Sigma_\mu^+$ for which there exist numbers $\alpha < \beta$ such that*

$$\forall k, l \in \mathbf{N}\, \mu_{k+l}^* \Big[\bigcup_{f \in \mathcal{Z}} \Big((\{f < \alpha\}^k \times \{f > \beta\}^l) \cap A^{k+l} \Big) \Big] = \mu(A)^{k+l}$$

is called a critical set for $\mathcal{Z}$. $\mathcal{Z}$ is said to be $\mu - stable$ if there is no critical set for $\mathcal{Z}$, i.e. if for all $A \in \Sigma_\mu^+$ and all $\alpha < \beta$ there exist $k, l \in \mathbf{N}$ such that

$$\mu_{k+l}^* \Big[\bigcup_{f \in \mathcal{Z}} \Big((\{f < \alpha\}^k \times \{f > \beta\}^l) \cap A^{k+l} \Big) \Big] < \mu(A)^{k+l}$$

The existence of a non-measurable τ_p-cluster point of $\mathcal{Z}$ yields the existence of a critical set (see [8] or [7]). It follows that stable sets $\mathcal{Z} \subset \mathcal{M}(\mu)$ are pointwise relatively compact in $\mathcal{M}(\mu)$. It is an easy – but important – consequence of the definition that τ_p-closure of a stable set $\mathcal{Z} \subset \mathcal{M}(\mu)$ is again stable.

Another important property is presented by the following

LEMMA 5.3. *Let ρ be a consistent lifting and let $\mathcal{Z} \subset \mathcal{L}_\infty(\mu)$ be a family of $\rho-$invariant functions. If each countable subset of $\mathcal{Z}$ is stable, then $\mathcal{Z}$ itself is also stable.*

Proof. Assume that $\mathcal{Z}$ is non-stable. Then there exist $\alpha < \beta$ and a critical set $A \in \Sigma_\mu^+$. Without loss of generality, we may assume that $\rho(A) = A$. Thus, we have for each $k, l \in \mathbf{N}$

$$\mu_{k+l}^* \Big(\bigcup_{f \in \mathcal{Z}} \{f < \alpha\}^k \times \{f > \beta\}^l \cap A^{k+l} \Big) = [\mu(A)]^{k+l}. \qquad (12)$$

According to Lemma 3.8 we have

$$\{f < \alpha\} \subseteq \rho(\{f < \alpha\}) \quad \text{and} \quad \{f > \beta\} \subseteq \rho(\{f > \beta\})$$

for each $f \in \mathcal{Z}$. Moreover, the consistency of ρ implies the relation

$$\rho_{k+l} \Big(\{f < \alpha\}^k \times \{f > \beta\}^l \cap A^{k+l} \Big) =$$
$$[\rho(\{f < \alpha\})]^k \times [\rho(\{f > \beta\})]^l \cap [\rho(A)]^{k+l}$$
$$\supseteq \{f < \alpha\}^k \times \{f > \beta\}^l \cap A^{k+l}.$$

Consequently, it follows from Theorem 3.7 that the set $\bigcup_{f \in \mathcal{Z}} \{f < \alpha\}^k \times \{f > \beta\}^l$ is measurable and (12) may be replaced by

$$\mu_{k+l}\Big(\bigcup_{f \in \mathcal{Z}} \{f < \alpha\}^k \times \{f > \beta\}^l \cap A^{k+l} \Big) = [\mu(A)]^{k+l} . \qquad (13)$$

Exactly as in the proof of Theorem 3.7 one can find such countable set $\mathcal{Z}' \subset \mathcal{Z}$ that

$$\mu_{k+l}\Big(\bigcup_{f \in \mathcal{Z}'} \{f < \alpha\}^k \times \{f > \beta\}^l \cap A^{k+l} \Big) = [\mu(A)]^{k+l} .$$

This proves the non-stability of $\mathcal{Z}'$ and completes the proof. $\square$

The next result follows directly from Lemma 5.3.

PROPOSITION 5.4. *If $\mathcal{Z} \subset \mathcal{L}_\infty(\mu)$ is stable and $\rho \in \Lambda(\mu)$ is consistent, then $\rho(\mathcal{Z})$ is also stable.*

LEMMA 5.5. *Let $\mathcal{A}$ be a subset of Σ. If f is a weak cluster point in $L_2(\mu)$ of a family $\{\chi_A : A \in \mathcal{A}\}$ and if $B := \{f > 0\}$, then*

$$\forall l \in \mathbf{N} \; \mu_l^*\Big(\bigcup_{A \in \mathcal{A}} A^l \cap B^l \Big) = [\mu(B)]^l .$$

Proof. Assume first that $l = 1$ and take an arbitrary $C \subseteq B$ of positive measure. Since $\int_C f \, d\mu > 0$ and $\int_C f \, d\mu$ belongs to the closure of $\{\int_\Omega \chi_A \cdot \chi_C \, d\mu : A \in \mathcal{A}\}$, we have $\mu(A \cap C) > 0$ for some A. In particular, the set $\bigcup_{A \in \mathcal{A}}(A \cap B)$ intersects every $C \subseteq B$ of positive measure.

In the case of an arbitrary $l \in \mathbf{N}$ notice, that the collection of all functions of the form $\prod_{i=1}^{l} h_i(\omega_i), \omega_1, \ldots, \omega_l \in \Omega$, with $g_1, \ldots, g_l \in L_2(\mu)$ is total in $L_2(\mu_l)$. Hence the function $h_l : \Omega^l \to \mathbf{R}$ defined by

$$f_l(\omega_1, \ldots, \omega_l) := \prod_{i=1}^{l} f(\omega_i)$$

is a weak cluster point in $L_2(\mu)$ of the set $\{\chi_{A^l} : A \in \mathcal{A}\}$. Since $\{f_l > 0\} = B^l$, the result follows from the first part of the proof. $\square$

THEOREM 5.6. *[8] If $\mathcal{Z} \subset \mathcal{M}(\mu)$ is $\mu-$stable, then the identity map $(\mathcal{Z}, \tau_p) \to (\mathcal{Z}, \tau_m)$ is continuous. In particular $\mathcal{Z}$ is relatively compact in the topology of convergence in measure.*

Proof. Suppose that there exists a net $(f_s)_{s \in S} \subset \mathcal{Z}$ converging pointwise to some $g \in \mathcal{M}(\mu)$ but not convergent in measure to g. Without loss of generality, we may assume that for some $\delta > 0$

$$\int_\Omega |f_s - g| \wedge 1 \, d\mu \geq \delta \qquad \text{for all } s \in S \,.$$

Passing to a subnet, we may assume that either

$$\int_\Omega (f_s - g)^+ \wedge 1 \, d\mu \geq \delta/2 \quad \text{or} \quad \int_\Omega (f_s - g)^- \wedge 1 \, d\mu \geq \delta/2$$

for all $s \in S$. We assume that the first inequality takes place. Without loss of generality, we may now assume that the net $[(f_s - g)^+ \wedge 1]_{s \in S}$ converges weakly in $L_2(\mu)$ to a function $h \in L_2(\mu)$. Then clearly

$$\int_\Omega h \, d\mu \geq \delta/2 > 0 \,.$$

Let us fix now $a > 0$ so that $\mu\{h > 3a\} > 0$ and the choose $A \in \Sigma_\mu^+$ and $c \in \mathbf{R}$ such that

$$A \subseteq \{h > 3a\} \quad \text{and} \quad A \subseteq \{c - a \leq g < c\} \,. \tag{14}$$

We are going to prove now that A is a critical set for $\mathcal{Z}$. To do it , notice first that if $B \subseteq A$ of positive measure is arbitrary, then since $\int_B h \, d\mu > 3a\mu(B)$, there exists $s_0 \in S$ such that

$$\int_B (f_s - g)^+ \wedge 1 \, d\mu > 3a\mu(B) \quad \text{for all } s \geq s_0 \,.$$

Then we have

$$3a\mu(B) < \int_B (f_s - g)^+ \wedge 1 \, d\mu$$
$$= \int_{\{f_s \geq g + 2a\} \cap B} (f_s - g)^+ \wedge 1 \, d\mu + \int_{\{f_s < g + 2a\} \cap B} (f_s - g)^+ \wedge 1 \, d\mu$$
$$\leq \mu[\{f_s \geq g + 2a\} \cap B] + 2a\mu(B) \,.$$

Consequently,

$$\mu[\{f_s \geq g + 2a\} \cap B] \geq a\mu(B) \quad \text{for all } s \geq s_0 \,. \tag{15}$$

Since $c - a \leq g$ on B, we have for all $s \geq s_0$

$$\int_B \chi_{\{f_s \geq c+a\}} \, d\mu = \mu[\{f_s \geq c+a\} \cap B] \geq \mu[\{f_s \geq g+2a\} \cap B] \geq a\mu(B) \,.$$

Since B was an arbitrary subset of A it follows from the above inequality that every weak cluster point of $\left(\chi_{\{f_s \geq c+a\}}\right)_{s \in S}$ in $L_2(\mu)$ is a.e. positive on A.

We are ready to prove that A is a critical set for $\mathscr{Z}$. Fix $t_1, \ldots, t_k \in A^k$ and set

$$\tilde{\mathscr{Z}} := \{f \in \mathscr{Z} : f(t_i) \leq c \text{ for } i = 1, \ldots, k\} \,.$$

Since by (14) we have $g(t_i) < c$ for $i = 1, \ldots, k$, the set $\tilde{\mathscr{Z}}$ is a neighbourhood of g in the topology of pointwise convergence. Consequently, $f_s \in \tilde{\mathscr{Z}}$ for sufficiently large $s \in S$. It follows now from Lemma 5.5 and from (15) that

$$\mu_l^*\left(\bigcup_{f \in \tilde{\mathscr{Z}}} \{f_s \geq c + a\}^l \cap A^l\right) = [\mu(A)]^l \,. \tag{16}$$

Hence,

$$\mu_{k+l}^*\left(\bigcup_{f \in \mathscr{Z}} [\{f \leq c\}^k \times \{f \geq c + a\}^l] \cap A^{k+l}\right) = [\mu(A)]^{k+l} \,. \tag{17}$$

Indeed, if $Q := \bigcup_{f \in \mathscr{Z}} [\{f \leq c\}^k \times \{f \geq c+a\}^l] \cap A^{k+l}$, then for each $t_1, \ldots, t_k \in A^k$ we have $\mu_l^*(Q_{t_1, \ldots, t_k}) = [\mu(A)]^l$, where

$$Q_{t_1, \ldots, t_k} := \{(s_1, \ldots, s_l) \in A^l : (t_1, \ldots, t_k, s_1, \ldots, s_l) \in Q\} \,.$$

This proves (17) and the whole theorem. $\qquad\square$

6. Integration of lifted functions

A function $f : \Omega \to X$ is Pettis integrable with respect to μ if it is weakly measurable and for each $E \in \Sigma$ there exists $\nu_f(E) \in X$ such that for each functional $x^* \in X^*$ we have $x^* \nu_f(E) = \int_E x^* f \, d\mu$. Identifying weakly equivalent Pettis integrable functions we get a linear space which we denote by $P(\mu, X)$. It is well known that the space can be normalized by setting

$$\|f\|_1 := \sup_{\|x^*\| \leq 1} \int_\Omega |x^* f| \, d\mu \ .$$

Whether function $f : \Omega \to X$ is Pettis integrable or not depends on the behaviour of the set

$$\mathscr{Z}_f := \{x^* f : \|x^*\| \leq 1\}$$

It is well known (cf [6]) that if $f \in P(\mu, X)$, then $\mathscr{Z}_f$ is weakly compact in $L_1(\mu)$ and τ_p−compact. These two conditions however do not guarantee the integrability of f. It turns out that the stability of $\mathscr{Z}_f$ is sufficient.

DEFINITION 6.1. *[8] A function $f : \Omega \to X$ is said to be properly measurable if the set $\mathscr{Z}_f$ is stable.*

THEOREM 6.2. *[8] Let $f : \Omega \to X$ be a scalarly bounded function. If f is properly measurable, then f is Pettis integrable and the range of its integral is a norm relatively compact subset of X.*

Proof. We have to prove only that the map $x^* \to x^* f$ from the unit ball of X^* into $L_1(\mu)$ is weak*−norm continuous. Let $(x_s^* f)_{s \in S}$ be a net that is pointwise convergent to $x_0^* f$ and $\|x_s^*\| \leq 1$ for all $s \in S$. According to Theorem 5.6 the net $(x_s^* f)_{s \in S}$ is convergent to $x_0^* f$ in μ−measure. Because of the boundedness of the net in $L_\infty(\mu)$ the Lebesque dominated convergence theorem may be applied, yielding the convergence in the norm of $L_1(\mu)$. $\qquad \square$

If $f : \Omega \to X^*$ is a weak*-measurable and weak*-bounded function, and $\rho : \mathcal{L}_\infty(\mu) \to \mathcal{L}_\infty(\mu)$ is a lifting, then $\rho_0(f) : \Omega \to X^*$ is the unique function satisfying for each $x \in X$ the equality

$$\langle x, \rho_0(f) \rangle = \rho(\langle x, f \rangle)$$

(see [4]). Similarly, if $f : \Omega \rightarrow X^{***}$ is a weak*-measurable and weak*-bounded function (or $f : \Omega \rightarrow X^*$ is a weakly-measurable and weakly-bounded) then $\rho_2(f) : \Omega \rightarrow X^{***}$ is the unique function satisfying for each $x^{**} \in X^{**}$ the equality

$$\langle x^{**}, \rho_2(f) \rangle = \rho(\langle x^{**}, f \rangle). \tag{18}$$

If $f : \Omega \rightarrow X$ is weakly measurable and weakly bounded, then $\rho_1(f) : \Omega \rightarrow X^{**}$ is uniquely determined by

$$\langle x^*, \rho_1(f) \rangle = \rho(\langle x^*, f \rangle). \tag{19}$$

It is a consequence of Theorem 3.7 that

$$\|\rho_0(f)\| := \sup\{|\rho(\langle x, f \rangle)| : \|x\| \leq 1\},$$
$$\|\rho_1(f)\| := \sup\{|\rho(\langle x^*, f \rangle)| : \|x^*\| \leq 1\}$$

and

$$\|\rho_2(f)\| := \sup\{|\rho(\langle x^{**}, f \rangle)| : \|x^{**}\| \leq 1\}$$

are measurable functions.

It is known (cf [8]) that $\rho_1(f) : \Omega \rightarrow X^{**}$ and $\rho_0(f) : \Omega \rightarrow X^*$ are weak*-Borel measurable and the measures $\xi_0 := \mu\rho_0(f)^{-1}$ and $\xi_1 := \mu\rho_1(f)^{-1}$ are Radon measures on the completions Ξ_f^0 and Ξ_f^1 of the σ-algebras of weak*-Borel subsets of X^* and X^{**} respectively. We denote by K_f^0 and K_f^1 the weak*- closures of $\rho_0(f)(\Omega)$ and $\rho_1(f)(\Omega)$ respectively.

There are known examples of scalarly bounded and weakly measurable functions, that are not Pettis integrable (cf [8]). It is however expected that liftings of such functions should be often Pettis integrable. It turns out that in general this is true. The result we are presenting here refers to Theorem 7-3-7 of Talagrand [8], where the Pettis integrability of each $\rho_0(f)$ taking its values in a convex Pettis set (see [8] for the definition) was proved and, to Theorem 6-2-1 of [8], where the Pettis integrability of $\rho_0(f)$ was proved under the assumption of the validity of Axiom L and the perfectness of μ. The following theorem is a generalization of Talagrand's theorem 6-2-1 from [8].

THEOREM 6.3. *Let $\rho : \mathcal{L}_\infty(\mu) \rightarrow \mathcal{L}_\infty(\mu)$ be a consistent lifting. Then*

(i) If $f : \Omega \to X$ is weakly measurable, weakly bounded and each countable subset of $\mathcal{Z}_f$ is stable, then $\rho_1(f) \in P(\mu, X^{**})$ and is properly measurable;

(ii) If $f : \Omega \to X^*$ is weak*-measurable, weak*-bounded and each countable subset of the set $\{\langle x, f \rangle : \|x\| \leq 1\}$ is stable, then $\rho_0(f) \in P(\mu, X^*)$ and is properly measurable;

(iii) If $f : \Omega \to X^*$ is weakly measurable, weakly bounded and each countable subset of the set $\mathcal{Z}_f$ is stable, then $\rho_2(f) \in P(\mu, X^{***})$ and is properly measurable.

In all the above cases the Pettis integrals have norm relatively compact ranges.

Proof. Since all proofs are based on the same idea, we shall prove (ii). According to Lemma 5.3 the set $\{\langle x, \rho_0(f) \rangle : \|x\| \leq 1\}$ is stable and hence also its pointwise closure, which coincides with $\mathcal{Z}_{\rho_0(f)}$, is stable. Now the assertion follows from Theorem 6.2.   □

LEMMA 6.4. *If $f : \Omega \to X^*$ is a scalarly bounded Pettis integrable function, then*

$$\langle x^{**}, \nu_f(E) \rangle = \int_E \langle x^{**}, \rho_0(f) \rangle \, d\mu$$

for each $x^{**} \in \tilde{B}$ and each $E \in \Sigma$.

Proof. Let (x_α) be a countable net $\sigma(X^{**}, X^*)$ - convergent to a functional $x^{**} \in \tilde{B}$. We have for each $\omega \in \Omega$

$$\lim_\alpha \langle x_\alpha, f(\omega) \rangle = \langle x^{**}, f(\omega) \rangle \quad \text{and}$$

$$\lim_\alpha \langle x_\alpha, \rho_0(f)(\omega) \rangle = \langle x^{**}, \rho_0(f)(\omega) \rangle \, .$$

Since $\langle x_\alpha, f \rangle = \langle x_\alpha, \rho_0(f) \rangle \quad \mu - $ a.e. and there are only countably many different functionals x_α, we have $\langle x^{**}, f \rangle = \langle x^{**}, \rho_0(f) \rangle$ μ-a.e. The required equality of the integrals is now a direct consequence of the Pettis integrability of f.   □

THEOREM 6.5. *Let $f : \Omega \to X^*$ be a scalarly bounded Pettis integrable function. If each $x^{**} \in X^{**}$ is ξ_0-measurable (in particular not containing l_1 by X is sufficient), then $\rho_0(f) \in P(\mu, X^*)$.*

Proof. In order to prove the Pettis integrability of $\rho_0(f)$, we have to show only – according to Theorem 6.2 from [6] – that for an arbitrary $z \in \nu_f(\Sigma)^\perp$ (=the anihilator of $\nu_f(\Sigma)$) the equality $\langle z, \rho_0(f) \rangle = 0$ holds true μ-a.e.. Suppose that there exists $z \in \nu_f(\Sigma)^\perp$ such that $\mu\{\omega \in \Omega : \langle z, \rho_0(f)(\omega) \rangle > 0\} > 0$ and $\|z\| \leq 1$. Then, since ξ_0 is a Radon measure and $\xi_0(K_f^0) = 1$, there exist a weak*-compact set $L \subseteq \{x^* \in K_f^0 : \langle z, x^* \rangle > 0\}$ and a positive real number a such that $\xi_0(L) > 0$, z is continuous on L and $\langle z, x^* \rangle > a$ for each $x^* \in L$. Take now an arbitrary net $(x_\alpha)_{\alpha \in A}$ in B_X that is $\sigma(X^{**}, X^*)$-convergent to z. It is known, that there exists a net of convex combinations of the elements of $(x_\alpha)_{\alpha \in A}$ that is convergent to z in the Mackey topology $\tau(X^{**}, X^*)$. In particular the convergence to z is uniform on $\nu_f(\Sigma)$. To avoid unnecessary complications, we assume at once that the initial net $(x_\alpha)_{\alpha \in A}$ is Mackey convergent to z. Then, for each $n \in \mathbf{N}$ there exists $\alpha_n \in A$ such that

$$|\langle x_\alpha, \nu_f(E) \rangle| \leq 1/n \quad \text{for all} \quad E \in \Sigma \quad \text{and all} \quad \alpha \geq \alpha_n \,.$$

Let $A_n := \{\alpha \in A : \alpha \geq \alpha_n\}$. Since for each $n \in \mathbf{N}$ and each $x^* \in L$ the net $(\langle x_\alpha, x^* \rangle)_{\alpha \in A_n}$ is convergent to $\langle z, x^* \rangle$, we can find for each collection of points $x_1^*, \ldots, x_n^* \in L$ an index $\alpha_{x_1^*, \ldots, x_n^*} \in A_n$ such that

$$|\langle z, x_i^* \rangle - \langle x_{\alpha_{x_1^*, \ldots, x_n^*}}, x_i^* \rangle| < 1/n \quad \text{for each} \quad i \leq n \,.$$

Equivalently,

$$L^n \subseteq \bigcup_{\alpha \in A_n} \{x^* : |\langle z, x^* \rangle - \langle x_\alpha, x^* \rangle| < 1/n\}^n \,.$$

Now, as a consequence of the compactness of L and the continuity of $z|L$, there exists a finite set $B_n \subset A_n$ such that the inclusion

$$L^n \subseteq \bigcup_{\alpha \in B_n} \{x^* : |\langle z, x^* \rangle - \langle x_\alpha, x^* \rangle| < 1/n\}^n$$

holds true.

It follows that $z|L$ is a pointwise cluster point of the countable set $\{x_\alpha|L : x_\alpha \in \bigcup_{n=1}^\infty B_n\}$. Consequently, there exists $x_0^{**} \in X^{**}$ that is a weak*−cluster point of the set $\{x_\alpha : \alpha \in \bigcup_{n=1}^\infty B_n\}$ and $x_0^{**}|L =$

$z|L$. It follows from the construction of x_0^{**} that $x_0^{**} \in \nu_f(\Sigma)^{\perp}$ and so

$$\int_{\rho_0(f)^{-1}(L)} \langle x_0^{**}, \rho_0(f) \rangle \, d\mu \geq$$
$$\geq a\mu(\rho_0(f)^{-1}(L)) > 0 = \langle x_0^{**}, \nu_f(\rho_0(f)^{-1}(L)) \rangle .$$

Since, $x_0^{**} \in \tilde{B}$ and f is Pettis integrable, we get a contradiction with Lemma 6.4. $\qquad \square$

The next result is a direct consequence of the above theorem.

THEOREM 6.6. *Let $f : \Omega \to X$ be a scalarly bounded Pettis integrable function. If each $x^{***} \in X^{***}$ is ξ_1-measurable (in particular not containing l_1 by X^* is sufficient), then $\rho_1(f) \in P(\mu, X^{**})$.*

COROLLARY 6.7. *If $f \in P(\mu, X^*)$ is scalarly bounded, $K \subset X^*$ is a convex Pettis set and $\rho_0(f) : \Omega \to K$, then $\rho_0(f) \in P(\mu, X^*)$. If $f \in P(\mu, X)$ is scalarly bounded, $K \subset X^{**}$ is a convex Pettis set and $\rho_1(f) : \Omega \to K$, then $\rho_1(f) \in P(\mu, X^{**})$.*

7. Completeness of $LLN_{\infty}(\mu, X)$

We shall denote by $LLN(\mu, X)$ the space of all properly measurable functions $f : \Omega \to X$ with the property that the set $\mathcal{Z}_f$ is order bounded in $L_1(\mu)$. The space $LLN(\mu, X)$ will be considered with the Glivenko-Cantelli seminorm, defined in [9] for an arbitrary function $f : \Omega \to X$ by setting

$$\|f\|_{GC} = \limsup_n \int^* g_n \, d\mu^{\infty} ,$$

where

$$g_n(\omega) = \sup_{\|x^*\| \leq 1} \frac{1}{n} \sum_{i \leq n} |x^*(f(\omega_i))| .$$

Moreover, let

$$\|f\|_1^* := \int^* \|f\| \, d\mu .$$

One can easily see that

$$\|f\|_{GC} \leq \|f\|_1^* \, . \tag{20}$$

One should recall at this moment that according to [9], the GC-seminorm and the Pettis seminorm on $LLN(\mu, X)$ are equivalent. In particular functions from $LLN(\mu, X)$ that are weakly equivalent are not distinguishable by the GC-norm. This permits us to identify weakly equivalent elements of $LLN(\mu, X)$ and investigate the quotient space.

If $1 \leq r \leq \infty$ then - identifying weakly equivalent functions - we denote by $LLN_r(\mu, X)$ the linear space

$$\{f \in LLN(\mu, X) : \|f\|_{P_r} := \sup_{\|x^*\| \leq 1} \|x^* f\|_r < \infty\} \, ,$$

where $\|x^* f\|_r$ is the $L_r(\mu)$-norm of $x^* f$. One can easily check that $\| \cdot \|_{P_r}$ is a norm for each $r \in [1, \infty]$.

Dobric in [1] posed a question about completeness of $LLN_1(\mu, X)$. We are going to prove that it is almost never complete.

THEOREM 7.1. *If X is infinite dimensional and μ is not purely atomic, then $LLN_1(\mu, X)$ is non-complete.*

Proof. If $P_c(\mu, X) := \{f \in P(\mu, X) : \nu_f(\Sigma)$ is norm relatively compact $\}$ then $P_c(\mu, X)$ endowed with the Pettis norm is non-complete. Let (f_n) be a Cauchy sequence in $P_c(\mu, X)$ that is not convergent in $P_c(\mu, X)$. Since each set $\nu_{f_n}(\Sigma)$ is norm relatively compact, it follows from [6](Theorem 9.1) that there exist simple functions $h_n : \Omega \to X$ with $\|f_n - h_n\|_1 < 1/n$. It is easily seen that the sequence (h_n) is Cauchy in $P_c(\mu, X)$. Since simple functions are properly measurable and $LLN_1(\mu, X) \subseteq P_c(\mu, X)$ the sequence (h_n) is also Cauchy in $LLN_1(\mu, X)$. It follows that the sequence (h_n) is divergent in $LLN_1(\mu, X)$. This completes the proof. □

Question. Does there exist $1 < r < \infty$ and an infinite dimensional X such that $LLN_r(\mu, X^*)$ is complete ?

THEOREM 7.2. *The space $LLN_\infty(\mu, X^*)$ is complete.*

Proof. Let ρ be a consistent lifting on $L_\infty(\mu)$ and let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $LLN_\infty(\mu, X^*)$. We have then

$$
\begin{aligned}
\|f_n - f_m\|_{P_\infty} &= \sup_{\|x^{**}\| \leq 1} \|x^{**} f_n - x^{**} f_m\|_\infty \\
&= \sup_{\|x^{**}\| \leq 1} \|x^{**}\rho_2(f_n) - x^{**}\rho_2(f_m)\|_\infty \\
&= \sup_{\|x^{**}\| \leq 1} \sup_\omega |x^{**}\rho_2(f_n)(\omega) - x^{**}\rho_2(f_m)(\omega)| \\
&= \sup_\omega \|\rho_2(f_n)(\omega) - \rho_2(f_m)(\omega)\| .
\end{aligned}
$$

It follows that the sequence $\left(\rho_2(f_n)\right)_{n=1}^\infty$ is uniformly Cauchy in the norm topology of X^{***}. Let $h : \Omega \longrightarrow X^{***}$ be the pointwise limit of the sequence $\left(\rho_2(f_n)\right)_{n=1}^\infty$. The uniform convergence yields the equality $h = \rho_2(h)$. Moreover, since each $f_n \in LLN_\infty(\mu, X^*)$ and ρ is consistent, the function $\rho_2(f_n)$ is properly measurable. Clearly it is also pointwise bounded by the function $\|\rho_2(f_n)\| \in L_\infty(\mu)$. Consequently, $\rho_2(f_n) \in LLN_\infty(\mu, X^{***})$. Moreover, the uniform convergence of the sequence $(\rho_2(f_n))$ yields $h \in LLN_\infty(\mu, X^{***})$.
Since the norm of $LLN_\infty(\mu, X^*)$ is stronger than the norm generated by the variation of integrals, the convergence of $(\rho_2(f_n))$ to h yields the norm convergence $\lim_n \|\nu_{f_n}(E) - \nu_h(E)\| = 0$ for each $E \in \Sigma$. It follows that ν_h takes its values in X^*.
Let Q be the canonical injection of X into X^{**} and let $h = Q^*h + h^\perp$, where $h^\perp : \Omega \to X^\perp$ and $X^\perp$ is the annihilator of X in X^{***}. Notice that
$$
\{\langle x, Q^*h \rangle : \|x\| \leq 1\} \subseteq \{\langle x^{**}, h \rangle : \|x^{**}\| \leq 1\}
$$
and the set on the right is stable in virtue of the proper measurability of h. It follows that the set on the left is also stable, and so Q^*h is properly measurable. Since $\|Q^*h\| \leq \|h\| \in L_\infty(\mu)$ everywhere on Ω, we get $Q^*h \in LLN_\infty(\mu, X^*)$.
Then $h = \rho_2(Q^*h) + \rho_2(h^\perp)$ and all three functions are Pettis integrable. Moreover, the integrals of h and $\rho_2(Q^*h)$ are X^*-valued whereas the integral of $\rho_2(h^\perp)$ takes its values in $X^\perp$. Since $X^* \cap X^\perp = 0$, we have for each $E \in \Sigma$ $\int_E \rho_2(h^\perp) = 0$. This yields for each

$x^{**} \in X^{**}$ the equality $\langle x^{**}, \rho_2(h^{\perp}) \rangle = 0 \quad \mu-$a.e. But we have for each $x^{**} \in X^{**}$

$$
\begin{aligned}
\rho(\langle x^{**}, h \rangle) &= \rho(\langle x^{**}, Q^*h \rangle) + \rho(\langle x^{**}, h^{\perp} \rangle) \\
&= \rho(\langle x^{**}, Q^*h \rangle) + \langle x^{**}, \rho_2(h^{\perp}) \rangle
\end{aligned}
$$

and so

$$
\langle x^{**}, \rho_2(h) \rangle = \rho(\langle x^{**}, h \rangle) = \rho(\langle x^{**}, Q^*h \rangle) = \langle x^{**}, \rho_2(Q^*h) \rangle .
$$

Consequently, $\rho_2(h) = \rho_2(Q^*h)$.

Thus, we get the following equalities:

$$
\begin{aligned}
0 &= \limsup_n \sup_\omega \| \rho_2(f_n)(\omega) - \rho_2(Q^*h)(\omega) \| \\
&= \lim_n \sup_{\|x^{**}\| \le 1} \sup_\omega |x^{**}\rho_2(f_n)(\omega) - x^{**}\rho_2(Q^*h)(\omega)| \\
&= \lim_n \sup_{\|x^{**}\| \le 1} \| x^{**}\rho_2(f_n) - x^{**}\rho_2(Q^*h) \|_\infty \\
&= \lim_n \sup_{\|x^{**}\| \le 1} \| x^{**}f_n - x^{**}Q^*h \|_\infty \\
&= \lim_n \| f_n - Q^*h \|_{P_\infty} .
\end{aligned}
$$

This proves the completeness of $LLN_\infty(\mu, X^*)$. $\square$

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