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Some new results on integration for multifunction

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Abstract It has been proven in Di Piazza and Musiał (Set Valued Anal 13:167–179, 2005, Vector measures, integration and related topics, Birkhauser Verlag, Basel, vol 201, pp 171–182, 2010) that each Henstock–Kurzweil–Pettis integrable multifunction with weakly compact values can be represented as a sum of one of its selections and a Pettis integrable multifunction. We prove here that if the initial multifunction is also Bochner measurable and has absolutely continuous variational measure of its integral, then it is a sum of a strongly measurable selection and of a variationally Henstock integrable multifunction that is also Birkhoff integrable (Theorem 3.4). Moreover,

Dedicated to Prof. Hans Weber on the occasion of his 70th birthday with deep esteem.

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in case of strongly measurable (multi)functions, a characterization of the Birkhoff integrability is given using a kind of Birkhoff strong property.

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1 Introduction

Integration of vector valued functions is strongly motivated by general problems of modern analysis including control theory, economics and differential inclusions. In many situations it is enough to use the well-known Bochner and Pettis integrals involved with the classical Lebesgue theory. In the last decades however gauge (non–absolute) integrals have been also considered [2–4,7,8,10–14,18,19,22,24,25,27,29] after the pioneering studies of G. Birkhoff, J. Kurzweil, R. Henstock and E. J. McShane. The Birkhoff integral was introduced in [1] and recently investigated in [9,15,16,31,39,40]. B. Cascales, V. M. Kadets, M. Potyrala, J. Rodriguez, and other authors considered the unconditional Riemann–Lebesgue multivalued integral.

This article is organized in the following manner. In Sect. 2 we give some preliminaries and the definitions. In Sect. 3 a new decomposition theorem for Bochner measurable and weakly compact valued Henstock–Kurzweil–Pettis integrable multifunction Γ with absolutely continuous variational measure of its integral is obtained, involving Birkhoff and variationally Henstock integrability of the multifunction G such that $\Gamma = f + G$. It is also shown that the conditions in Theorem 3.4 do not imply, in general, the variational integrability of the multifunction Γ and an example is given (Example 3.6).

Finally, in case of strongly measurable (multi)functions, a characterization of the Birkhoff integrability is given using a kind of Birkhoff strong property (see Definition 3.11).

2 Preliminary facts

Throughout [0, 1] is the unit interval of the real line equipped with the usual topology and Lebesgue measure λ, $\mathcal{L}$ denotes the family of all Lebesgue measurable subsets of [0, 1], and $\mathcal{I}$ is the collection of all closed subintervals of [0, 1]: if $I \in \mathcal{I}$ then its Lebesgue measure will be denoted by $|I|$.

X is an arbitrary Banach space with its dual X^*. The closed unit ball of X^* is denoted by $B(X^*)$. Following the notation in [17], $cwk(X)$ is the family of all nonempty convex weakly compact subsets of X and $ck(X)$ is the family of all compact members of $cwk(X)$. We consider on $cwk(X)$ the Minkowski addition ($A + B :=$ $\{a+b : a \in A, \ b \in B\}$) and the standard multiplication by scalars. d_H is the Hausdorff distance in $cwk(X)$ and $cwk(X)$ with this metric is a complete metric space. For every $A \in cwk(X)$, $\|A\| := d_H(A, \{0\}) = \sup\{\|x\| : x \in A\}$.

For every $C \in cwk(X)$ the *support function of C* is denoted by $s(\cdot, C)$ and defined on X^* by $s(x^*, C) := \sup\{\langle x^*, x \rangle : x \in C\}$, for each $x^* \in X^*$.

A map $\Gamma : [0, 1] \to 2^X \setminus \{\emptyset\}$ (= non-empty subsets of X) is called a *multifunction*.

A multifunction $\Gamma : [0, 1] \to cwk(X)$ is said to be *scalarly measurable* if for every $x^* \in X^*$, the map $s(x^*, \Gamma(\cdot))$ is measurable.
Γ is said to be *Bochner measurable* if there exists a sequence of simple multifunctions $\Gamma_n : [0, 1] \to cwk(X)$ such that

$$\lim_{n \to \infty} d_H(\Gamma_n(t), \Gamma(t)) = 0$$

for almost all $t \in [0, 1]$.

A function $f : [0, 1] \to X$ is called a *selection of Γ* if $f(t) \in \Gamma(t)$, for every $t \in [0, 1]$.

A *partition $\mathcal{P}$ in* $[0, 1]$ is a collection $\{(I_1, t_1), \ldots, (I_p, t_p)\}$, where $I_1, \ldots, I_p$ are nonoverlapping subintervals of $[0, 1]$, t_i is a point of $[0, 1]$, $i = 1, \ldots, p$.

If $\cup_{i=1}^{p} I_i = [0, 1]$, then $\mathcal{P}$ is *a partition of* $[0, 1]$. If t_i is a point of I_i, $i = 1, \ldots, p$, we say that $\mathcal{P}$ is a *Perron partition of* $[0, 1]$.

A *gauge* on $[0, 1]$ is a positive function on $[0, 1]$. For a given gauge δ on $[0, 1]$, we say that a partition $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ is *δ-fine* if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1, \ldots, p$.

We recall that an interval multifunction $\Phi : \mathcal{I} \to cwk(X)$ is said to be *finitely additive* if for every non-overlapping intervals $I_1, I_2 \in \mathcal{I}$ such that $I_1 \cup I_2 \in \mathcal{I}$ we have $\Phi(I_1 \cup I_2) = \Phi(I_1) + \Phi(I_2)$. Φ is then called an interval multimeasure.

A multifunction $M : \mathcal{L} \to cwk(X)$ is said to be a *multimeasure* if for every $x^* \in X^*$, the map $A \mapsto s(x^*, M(A))$ is a real-valued measure (see [28, Theorem 8.4.10]). We say that the multimeasure $M : \mathcal{L} \to cwk(X)$ is *λ-continuous* and we write $M \ll \lambda$, if $\lambda(A) = 0$ yields $M(A) = \{0\}$.

For the definitions of *Birkhoff, Henstock, McShane* and *variationally Henstock* $cwk(X)$-valued multifunctions we refer to [11, Definitions 2.2., 2.3, 2.8].

If $X = \mathbb{R}$ and Γ is a function, we speak of Henstock-Kurzweil integrability instead of Henstock one. Each of these integrals turns out to be an additive mapping, on $\mathcal{I}$ or $\mathcal{L}$ respectively, and is called *primitive*, or also *integral measure* of the mapping Γ. From the definition and the completeness of the Hausdorff metric the $cwk(X)$-valued integrals defined before have weakly compact values.

Definition 2.1 A multifunction $\Gamma : [0, 1] \to cwk(X)$ is said to be *Pettis* (resp. *Henstock-Kurzweil-Pettis integrable* (in short *(HKP)-integrable*)) if $s(x^*, \Gamma)$ is integrable (resp. Henstock-Kurzweil integrable) for each $x^* \in X^*$, and for each $A \in \mathcal{L}$ (resp. $I \in \mathcal{I}$) there exists an element $w_A \in cwk(X)$ (resp. $w_I \in cwk(X)$) such that $x^*(w_A) = \int_A s(x^*, \Gamma(t))dt$ (resp. $x^*(w_I) = (H)\int_I s(x^*, \Gamma(t))dt$) holds, for each $x^* \in X^*$.

There is a large literature concerning Pettis and Henstock–Kurzweil–Pettis integral for functions and multifunctions; we refer the reader to [15,21,22,32–35,37]. It follows from the classical properties of the Henstock–Kurzweil integral that the primitives of Henstock or (HKP) integrable multifunctions are interval multimeasures,

while the primitives of Pettis or Birkoff integrable multifunctions are multimeasures on the σ-algebra $\mathcal{L}$ (see [16, Theorem 4.1(i)]).

Moreover, as observed in [11], the multivalued integral of a Pettis integrable $cwk(X)$-valued multifunction is absolutely continuous with respect to λ and its *embedded* measure is an $l_\infty(B(X^*))$-valued countably additive measure.

For the definition of the variational measure V_Φ associated to a finitely additive interval measure $\Phi : \mathcal{I} \to \mathbb{R}$ we refer the reader to [6,23]. In particular, we recall that the variational measure V_ϕ of the primitive ϕ of a variationally Henstock integrable mapping is a (possibly unbounded) λ-continuous measure on $\mathcal{L}$: see [38, Proposition 3.3.1].

Finally, a useful tool to study the $cwk(X)$-valued multifunctions is the Rådström embedding (see for example [30]) given by $i(A) := s(\cdot, A)$. It satisfies the following properties:

(1) $i(\alpha A + \beta C) = \alpha i(A) + \beta i(C)$ for every $A, C \in cwk(X), \alpha, \beta \in \mathbb{R}^+$;
(2) $d_H(A, C) = \|i(A) - i(C)\|_\infty, \quad A, C \in cwk(X)$;
(3) $i(cwk(X)) = \overline{i(cwk(X))}$ (norm closure).

Observe that it follows directly from the definitions that a multifunction $\Gamma : [0, 1] \to cwk(X)$ is Birkhoff (resp. Henstock, McShane, variationally Henstock) integrable if and only if $i(\Gamma)$ is integrable in the same sense.

3 Variationally Henstock integrable selections

Concerning existence of selections of gauge integrable mappings with values in $cwk(X)$, several results have been obtained recently in [4,10,11,21,22,39]. The purpose of this paper is to revisit the last results, trying to generalize it to a more general case. We begin with some useful Lemmas.

Lemma 3.1 *Let $\Gamma : [0, 1] \to cwk(X)$ be any (HKP)-integrable multifunction, and let $f : [0, 1] \to X$ be any (HKP)-integrable selection of Γ. Then, if ϕ and Φ denote respectively the primitives of f and Γ, it is*

$$\phi(I) \in \Phi(I)$$

for every subinterval $I \subset [0, 1]$.

Proof It can be deduced as in the proof of [10, Proposition 2.7]. □

Lemma 3.2 *Let $\Gamma : [0, 1] \to cwk(X)$ be any (HKP)-integrable multifunction, and let f be any (HKP)-integrable selection of Γ. If the variational measure V_Φ associated to the primitive Φ of Γ is absolutely continuous with respect to the Lebesgue measure λ, the same holds for the variational measure of the primitive ϕ of f. In particular this holds whenever Γ is variationally Henstock integrable.*

Proof From the previous Lemma we have that $\phi(I) \in \Phi(I)$ for every interval $I \subset [0, 1]$. Then $V_\phi \leq V_\Phi$. Since V_Φ is λ-continuous the same holds for V_ϕ, of course. □

A crucial tool is the following.

Theorem 3.3 *Let $G : [0, 1] \to cwk(X)$ be any McShane integrable mapping. Then G is variationally Henstock integrable if and only if it is Bochner measurable and the variational measure of its integral is λ-continuous.*

Proof If G is McShane integrable, then also $i(G)$ is McShane, hence Pettis, integrable and we may apply [6, Lemma 4.1]. □

Here is the new decomposition theorem which extends [11, Theorem 5.1].

Theorem 3.4 *Let $\Gamma : [0, 1] \to cwk(X)$ satisfy the following conditions:*

(3.4.1) *Γ is Bochner measurable;*
(3.4.2) *Γ is HKP-integrable;*
(3.4.3) *the variational measure associated to the primitive of Γ is λ-continuous.*

Then Γ can be decomposed as the sum $\Gamma = f + G$, where f is any strongly measurable selection of Γ and G is a variationally Henstock and Birkhoff integrable multifunction.

Proof Let f be any Bochner measurable selection of Γ. Then f is HKP-integrable and the mapping G defined by $\Gamma = G + f$ is Pettis integrable (see [21, Theorem 1]). Moreover $i(G)$ and G are strongly measurable. By [16, Proposition 4.5] $i(G)$ is Pettis integrable, since the range of $i(G)$ is separable. So, by [37, Corollary 5.11] $i(G)$ is McShane and so G is McShane integrable.

By (3.4.3) and by Lemma 3.2 the variational measure V_ϕ associated to f is λ-continuous and so the variational measure associated to the primitive of G is λ-continuous too. So, the mapping G is variationally Henstock integrable thanks to Theorem 3.3. Birkhoff integrability of G follows from [10, Proposition 4.1] since the support functionals of G are non-negative. □

Remark 3.5 We observe that the conditions (3.4.1), (3.4.2), (3.4.3) do not imply in general variational Henstock integrability of Γ. But the implication holds true, as already observed, if (3.4.2) is replaced by the stronger request that Γ is Pettis integrable, by Theorem 3.3.

So, we shall now give an example of a mapping $f : [0, 1] \to c_0$, satisfying (3.4.1), (3.4.2), (3.4.3), but not Henstock integrable (and so, a fortiori, not variationally Henstock integrable). The example is the same as [5, Example 2], and [26, Example].

Example 3.6 Fix any disjoint sequence of closed subintervals $(J_n)_n := ([a_n, b_n])_n$ in $[0, 1]$, such that $0 = a_1 < b_1 < a_2 < b_2 < \cdots$ and $\lim_n b_n = 1$.

Now the function f is defined as follows:

$$f_n(t) := \frac{1}{2}|J_{2n-1}|\chi_{J_{2n-1}}(t) - \frac{1}{2}|J_{2n}|\chi_{J_{2n}}(t),$$

for each $t \in [0, 1]$, where f_n denotes the nth component of f. Proceeding as in the quoted papers [5,26], it is possible to see that f is (HKP)-integrable, and, for every interval $I \subset [0, 1]$:

$$\int_I f \, dt = \left(\frac{\lambda(I \cap J_{2n-1})}{2|J_{2n-1}|} - \frac{\lambda(I \cap J_{2n})}{2|J_{2n}|} \right)_n.$$

However, as observed in [5, Example 2], the set $\{ \int_I f(t) dt : I \in \mathcal{I} \}$ is not relatively norm compact in c_0, hence f is not Henstock integrable, thanks to [5, Proposition 1].

Since c_0 is separable, f is strongly measurable; then, we only have to prove that the variational measure associated with the integral Γ of f is λ-continuous.

To this aim, let us fix any null set $A \in \mathcal{L}$, and fix any $\varepsilon > 0$. Then clearly $\lambda(A \cap J_n) = 0$ for every integer n. So, for every n there exists an open set U_n such that $A \cap J_n \subset U_n$ and $\lambda(U_n) \leq \frac{\varepsilon}{2^n} |J_n|$.

Now, for each element $t \in A$, we define $\delta(t)$ in such a way that $[t - \delta(t), t + \delta(t)] \subset U_n$ whenever $t \in A \cap J_n^0$, and $[t - \delta(t), t + \delta(t)] \subset]b_{n(t)}, a_{n(t)+1}[$ as soon as $t \notin \cup_n J_n$ and $n(t)$ is such that $t \in]b_{n(t)}, a_{n(t)+1}[$. Finally, if $t \in A$ is one of the points a_n or b_n, we choose $\delta(t)$ in such a way that $[t - \delta(t), t + \delta(t)] \subset U_n$ and $[t - \delta(t), t + \delta(t)]$ intersects just one of the intervals J_n.

So, if $\{(I_k, t_k), k = 1, \ldots, K\}$ is any δ-fine Henstock partition in $[0, 1]$, with tags in A, for each index k the interval I_k intersects at most one of the intervals J_n, and therefore $F(I_k) := \int_{I_k} f(t) dt$ has no more than one component different from 0 (say n), and

$$\left\| \int_{I_k} f(t) dt \right\| \leq \frac{\lambda(I_k \cap J_n)}{2|J_n|}.$$

Hence, summing as $k = 1, \ldots, K$, we get

$$\sum_k \|F(I_k)\| = \sum_n \sum_{t_k \in J_n} \left\| \int_{I_k} f(t) dt \right\| \leq \sum_n \sum_{t_k \in J_n} \frac{\lambda(I_k \cap J_n)}{2|J_n|} \leq$$
$$\leq \sum_n \frac{\lambda(U_n)}{2|J_n|} \leq \sum_n \frac{\varepsilon}{2^n} = \varepsilon.$$

This proves that f satisfies condition (3.4.3) as required.

The function from the above example has a trivial decomposition: $f(t) = \{0\} + f(t)$. To have an example of a multifunction $\Gamma : [0, 1] \to cwk(c_0)$ satisfying (3.4.1), (3.4.2), (3.4.3) and possessing a non-trivial decomposition, it is sufficient to take any variationally Henstock integrable multifunction $\Gamma_0 : [0, 1] \to cwk(c_0)$, and define: $\Gamma(t) = \Gamma_0(t) + f(t)$, for $t \in [0, 1]$, where f is the function defined above. Since $i(\Gamma) = i(\Gamma_0) + i(\{f(t)\})$, we easily see that $i(\Gamma)$ (and so Γ) satisfies (3.4.1), (3.4.2), (3.4.3), but it cannot be Henstock integrable, otherwise $i(\{f\})$ (and therefore f) would be, by difference.

As a consequence of Theorem 3.4 we can obtain the following corollary which appears also in [11, Thorem 5.3] with a different proof.

Corollary 3.7 *Let* $\Gamma : [0, 1] \to cwk(X)$ *be any variationally Henstock integrable multifunction. Then every strongly measurable selection* f *of* Γ *is variationally Henstock integrable and* $\Gamma = G + f$, *where* G *is Birkhoff and variationally Henstock integrable.*

Proof Since Γ is variationally Henstock integrable, it satisfies (3.4.1), (3.4.2) and (3.4.3), so from Theorem 3.4 we deduce that, for every strongly measurable selection f the mapping G defined by $\Gamma = G + f$ is variationally Henstock and Birkhoff integrable. Finally, since $i(\{f\}) = i(\Gamma) - i(G)$, it follows easily that $i(\{f\})$ (and therefore also f) is variationally Henstock integrable. □

We have cited before [37, Corollary 5.11] which ensures that for strongly measurable functions, the Pettis and the Birkhoff integrability are the same. We shall now present one more characterization of the Birkhoff integrability for strongly measurable integrands.

Proposition 3.8 *Assume that* $f : [0, 1] \to X$ *is strongly measurable. Then for every* $\varepsilon > 0$ *there exists a countable partition of* $[0, 1]$, *made of measurable sets* $(A_h)_h$, *such that* $f|_{A_h}$ *is Bochner integrable for every* $h \in \mathbb{N}$ *and*

$$\sum_k \left\| f(t_k)\lambda(A'_k) - \int_{A'_k} f \right\| \le \varepsilon \tag{1}$$

holds true, for any partition $\{A'_k\}_k$ *refining* $(A_n)_n$, *and any choice of points* t_k *in* A'_k, $k \in \mathbb{N}$.

Proof Since f is Bochner measurable, then f is also Lusin measurable, see [36, Section 3]. Now, in order to prove (1), fix arbitrarily $\varepsilon > 0$. Thanks to Lusin measurability of f there exists a sequence $(B_n)_n$ of pairwise disjoint closed subsets of $[0, 1]$, such that $\sum_{n=1}^{+\infty} \lambda(B_n) = 1$, and a corresponding sequence $(\delta_n)_n$ of positive real numbers, such that

$$\|f(t) - f(t')\| \le \varepsilon$$

holds for every n, as soon as t, t' are in B_n and $|t - t'| \le \delta_n$. Since $f_{|B_n}$ is continuous, it is obviously Bochner integrable in B_n, hence we can deduce also that

$$\left\| f(t) - \frac{\int_{B_n \cap I} f}{\lambda(B_n \cap I)} \right\| \le \varepsilon \tag{2}$$

holds true, for every n, every interval $I \subset [0, 1]$ with $|I| \le \delta_n$ and every point $t \in B_n \cap I$ (provided the latter is non-negligible).

Now, in order to construct the announced partition $(A_n)_n$, for every integer n fix a finite partition $\{I_1^n, \ldots, I_{N(n)}^n\}$ of $[0, 1]$ consisting of half-open intervals of the same length, with $|I_j^n| < \delta_n$ for all j, and set $A_j^{n+1} := B_n \cap I_j^n$ whenever the intersection is non-negligible. From (2), we see that

$$\left\| f(t_j^{n+1}) - \frac{\int_{A_j^{n+1}} f}{\lambda(A_j^{n+1})} \right\| \leq \varepsilon \tag{3}$$

holds true, for every n and every choice of points $t_j^{n+1} \in A_j^{n+1}$. Finally, since the sequence $(A_j^{n+1})_{\{n \in \mathbb{N}, j \in \{1,\dots,N(n)\}\}}$ is composed by pairwise disjoint sets, we can rearrange them in a sequence $(A_h)_{h=2}^{\infty}$. Adding also the null set $A_1 := [0,1] \setminus (\cup_{h=2}^{\infty} A_h)$, we claim that the partition $(A_h)_h$ obtained in this way is the requested one.

Indeed, let $\{A_k'\}_k$ be any refinement of $(A_h)_h$. Since each non-negligible A_k' is contained in some set of the type $B_n \cap I_j$, we also have, similarly to (2)

$$\left\| f(t_k) - \frac{\int_{A_k'} f}{\lambda(A_k')} \right\| \leq \varepsilon \tag{4}$$

as soon as $t_k \in A_k'$. So

$$\sum_k \left\| f(t_k)\lambda(A_k') - \int_{A_k'} f \right\| \leq \sum_n \sum_j \sum_{A_k' \subset B_n \cap I_j^n} \left\| f(t_k)\lambda(A_k') - \int_{A_k'} f \right\|$$

$$\leq \sum_n \sum_j \sum_{A_k' \subset B_n \cap I_j^n} \varepsilon \lambda(A_k') = \sum_n \sum_j \varepsilon \lambda(B_n \cap I_j^n) = \sum_n \varepsilon \lambda(B_n) = \varepsilon,$$

as requested.   □

Remark 3.9 We observe that if f satisfies (1), in general f is not variationally Henstock integrable: indeed, in [6, Remark 4.3] it is shown that for every Banach space X there are Pettis integrable mappings $f : [0,1] \to X$ whose variational measure associated to the integral fails to be λ-continuous. In particular, if X is separable, such functions are also Birkhoff integrable and strongly measurable.

Corollary 3.10 *Assume that $f : [0,1] \to X$ is strongly measurable. Then f is Birkhoff integrable if and only if there exists a countable partition $(A_k)_k$ of $[0,1]$ such that the restriction $f|_{A_k}$ is bounded whenever $\lambda(A_k) > 0$ and $\left\{ \sum_k f(t_k)\lambda(A_k), t_k \in A_k \right\}$ is made up of unconditionally convergent series.*

Proof It is enough to apply Proposition 3.8 together with [14, Proposition 2.2].   □

By the properties of the Rådström embedding we can obtain also that a Bochner measurable multifunction $\Gamma : [0,1] \to cwk(X)$ is Birkhoff integrable if and only is there exists a countable partition of $[0,1]$: $(A_k)_k$ such that the restriction $F|_{A_k}$ is bounded whenever $\lambda(A_k) > 0$ and $\left\{ \sum_k \Gamma(t_k)\lambda(A_k), t_k \in A_k \right\}$ is made up of unconditionally convergent series.

Finally, we would like to remark that, for strongly measurable maps, Birkhoff integrability can be labeled in a form that recalls the notion of variational Henstock integrability (though there is no direct implication between the two notions, in general). We first give a definition.

Definition 3.11 Given a function $f : [0, 1] \to X$, we say that f is *strongly Birkhoff* integrable if there exists a λ-continuous σ-additive measure $\phi : \mathcal{L} \to X$ such that, for every $\varepsilon > 0$ it is possible to find a countable partition P of $[0, 1]$, made with measurable sets, such that, as soon as $\{A_n\}_n$ is a refinement of P, it holds

$$\sum_n \| f(t_n)\lambda(A_n) - \phi(A_n) \| \leq \varepsilon$$

for every choice of $t_n \in A_n, n \in \mathbb{N}$.

We now observe that, as soon as $f : [0, 1] \to X$ is strongly measurable and Birkhoff integrable, it turns out to be also *strongly Birkhoff* integrable, thanks to Proposition 3.8, and of course ϕ is the (Birkhoff) integral function of f.

Conversely, we have the following results.

Proposition 3.12 *Assume that $f : [0, 1] \to X$ is strongly Birkhoff integrable. Then it is also Birkhoff integrable.*

Proof Fix $\varepsilon > 0$, and let $P := \{E_j : j \in \mathbb{N}\}$ be the corresponding partition in the definition of strong Birkhoff integrability. Choose now any finer partition $\{A_k\}_k$, set $x := \sum_k \phi(E_k) = \phi([0, 1])$, and observe that there exists an integer N such that $\| x - \sum_{k \leq n} \phi(A_k) \| \leq \varepsilon$ for every $n \geq N$. So, for every $n \geq N$, and every choice of points $t_k \in A_k$, we have

$$\left\| \sum_{k=1}^n f(t_k)\lambda(A_k) - x \right\| \leq \left\| \sum_{k=1}^n f(t_k)\lambda(A_k) - \sum_{k=1}^n \phi(A_k) \right\| + \varepsilon$$

$$\leq \sum_{k=1}^n \| f(t_k)\lambda(A_k) - \phi(A_k) \| + \varepsilon \leq \sum_{k=1}^{\infty} \| f(t_k)\lambda(A_k) - \phi(A_k) \| + \varepsilon \leq 2\varepsilon.$$

This means that

$$\limsup_n \left\| \sum_{k=1}^n f(t_k)\lambda(A_k) - x \right\| \leq 2\varepsilon.$$

This property in [9] is called *simple-Birkhoff* integrability, and is proved in [9, Theorem 3.18] to be equivalent to Birkhoff integrability of f. $\square$

Proposition 3.13 *Assume that $f : [0, 1] \to X$ is strongly Birkhoff integrable. Then f is also strongly measurable.*

Proof We shall follow the technique of [19, Lemma 3]. First of all, let us denote by V the separable subspace of X generated by the range of ϕ. Next, for every $\varepsilon > 0$ let us set

$$T_\varepsilon := \{t \in [0, 1] : d_H(f(t), V) > \varepsilon\}.$$

Now, we shall prove that $\lambda(T_\varepsilon) = 0$ for all $\varepsilon > 0$, from which it will follow that f is separably-valued, and so also strongly measurable, thanks to the Pettis Theorem. So, assume by contradiction that there exists a $\varepsilon' > 0$ such that $\lambda^*(T_{\varepsilon'}) = a > 0$. Let A' denote any measurable set such that $T_{\varepsilon'} \subset A'$ and $\lambda(A') = \lambda^*(T_{\varepsilon'}) = a$.

Now, fix $0 < \varepsilon < \varepsilon'$ and take any partition $\Pi \equiv \{E_j\}_j$ of measurable sets satisfying the condition of strong Birkhoff integrability of f with respect to $a\varepsilon$. Moreover let $\{A_n\}_n$ be any finer partition, such that $A' \cap E_j$ and $E_j \setminus A'$ are among the sets A_n. Without loss of generality, we can assume that $\lambda(A_n) > 0$ for all n. Observe that, if A_n is of the type $A' \cap E_j$, then $A_n \cap T_{\varepsilon'}$ is non-empty, otherwise $A' \setminus A_n$ would contain $T_{\varepsilon'}$ but would have smaller measure than a. So, we can choose $t_n \in A_n \cap T_{\varepsilon'}$ as soon as A_n is of the type $A' \cap E_j$. Thus we have

$$\sum_n{}' \| f(t_n)\lambda(A_n) - \phi(A_n)\| \le a\varepsilon$$

where the $\sum'$ runs along the sets A_n of the type $A' \cap E_j$. Therefore, since A' is the union of all the sets A_n of the type above,

$$\sum_n{}' \| f(t_n)\lambda(A_n) - \phi(A_n)\| \le \varepsilon \sum_n{}' \lambda(A_n)$$

i.e.

$$\sum_n{}' \left\| f(t_n) - \frac{\phi(A_n)}{\lambda(A_n)} \right\| \lambda(A_n) \le \varepsilon \sum_n{}' \lambda(A_n).$$

This inequality implies that, for at least one integer n, it is

$$\left\| f(t_n) - \frac{\phi(A_n)}{\lambda(A_n)} \right\| \le \varepsilon < \varepsilon',$$

but this is impossible, since $\dfrac{\phi(A_n)}{\lambda(A_n)} \in H$ and $t_n \in T_{\varepsilon'}$. This proves that $\lambda(T_\varepsilon) = 0$ for every $\varepsilon > 0$, and so f is strongly measurable. $\qquad\square$

Summarizing, we get the following conclusion.

Theorem 3.14 *Assume that $f : [0, 1] \to X$ is any mapping. Then f is strongly Birkhoff integrable if and only if it is strongly measurable and Birkhoff integrable.*

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