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Category product densities and liftings

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Abstract

In this paper we investigate two main problems. One of them is the question on the existence of category liftings in the product of two topological spaces. We prove, that if $X \times Y$ is a Baire space, then, given (strong) category liftings ρ and σ on X and Y, respectively, there exists a (strong) category lifting π on the product space such that π is a product of ρ and σ and satisfies the following section property:

$$\big[\pi(E)\big]_x = \sigma\big(\big[\pi(E)\big]_x\big) \quad \text{for all } E \subseteq X \times Y$$

with Baire property and all $x \in X$. We give also an example, where some of the sections $[\pi(E)]^y$ must be without Baire property.

Then, we investigate the existence of densities respecting coordinates on products of topological spaces, provided these products are Baire spaces. The densities are defined on σ-algebras of sets with Baire property and select elements modulo the σ-ideal of all meager sets. In all the problems the situation in the "category case" turns out to be much better, than in case of products of measure spaces. In particular, in every product there exists a canonical strong density being a product of the

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canonical densities in the factors and there exist (strong) densities respecting coordinates with given a priori marginal (strong) densities.

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Introduction

In [20] the last three authors considered densities and liftings in products of two probability spaces with good section properties analogous to that for measures and measurable sets in the Fubini theorem. These properties have been then applied to prove the permanence of the measurability of stochastic processes under the modification by liftings [20]. In this paper we study the product situation for the σ-algebra $\mathfrak{B}_c(X)$ of all sets having the Baire property, selecting a representative element from each equivalence class of $\mathfrak{B}_c(X)$ modulo sets of the first category (see Graf [11], Maharam [18] and Oxtoby [22]). Following Oxtoby's [22], p. 74 remark that "the suggestion to look for a category analogue has very often proved to be a useful guide", we have attempted to check if this can be interesting in case of our investigations.

It has been already mentioned by Graf [11], Maharam [18], and Oxtoby [22] that the canonical density which selects from each equivalence its regular open representative is a category strong density, while even for a compact Radon measure space a measure-theoretic strong density may not exist (cf. Fremlin [9]). A different approach to category density in case of the real line was presented by Wilczyński [31], who defined it via density points. As the first result for categories we show that the canonical strong density on the product that is a Baire space is a product density of the canonical densities on the factors (see Proposition 3.1). It has been proven in [17] that for measure spaces such a result cannot be true in general (see also Remark 4.7). The formula defining the product-density from its marginals (see Proposition 3.1 and Proposition 4.1) makes clear the crucial point in the difference between the measure and the category case. A non-meager set with the Baire property in the product contains up to a meager set a rectangle with non-meager sides with the Baire property, while a famous result of Erdős and Oxtoby [5] exhibits an example of a measurable set in the product σ-algebra of quite arbitrary non-atomic *positive* measure spaces, containing up to a set of measure zero no rectangle of positive measure (compare also Remark 4.7). That fact makes it clear that in the category case we probably should apply completely different methods than in the case of measure product liftings. The latter is done, as a rule, by transfinite induction, relying crucially on the martingale theorem, not available in the category case.

There is now a question what is precisely the situation in case of category (strong) liftings. We prove that given arbitrary topological spaces X and Y such that the product space $X \times Y$ is Baire and given (strong) liftings ρ on $(X, \mathfrak{B}_c(X), \mathcal{M}(X))$ and σ on $(Y, \mathfrak{B}_c(Y), \mathcal{M}(Y))$ there always exists a (strong) lifting π_1 on $(X \times Y, \mathfrak{B}_c(X \times Y), \mathcal{M}(X \times Y))$ satisfying the product condition $\pi_1(A \times B) = \rho(A) \times \sigma(B)$ for all $A \in \mathfrak{B}_c(X)$,

$B \in \mathfrak{B}_c(Y)$ and such that for each $E \in \mathfrak{B}_c(X \times Y)$ and each $x \in X$ the section property $[\pi_1(E)]_x = \sigma([\pi_1(E)]_x)$ holds true (see Theorem 5.1). The above assumptions are immediately satisfied by Polish spaces X and Y. The latter answers affirmatively Question 10 from [26] in case of Polish spaces and shows that at least in case of products of Polish spaces the category (strong) liftings behaves better than the measure-theoretic ones. One should notice also that it is impossible (besides some trivial cases) to have also the relation $[\pi_1(E)]^y = \rho([\pi_1(E)]^y)$ for each $E \in \mathfrak{B}_c(X \times Y)$ and each $y \in Y$, even if $X = Y$. See [26].

As a negative result we provide an example of Polish spaces for which do not exist liftings $\sigma \in \Lambda(\mathcal{M}(Y))$ and $\pi \in \Lambda(\mathcal{M}(X \times Y))$ with the properties such that for each $\pi(E)$ all sections $[\pi(E)]^y$ have the Baire property in X and all sections $[\pi(E)]_x$ are invariant with respect to σ (Theorem 6.8).

The second problem considered in this paper concerns the existence of a density and a lifting θ_I on a Baire product $\prod_{i \in I} X_i$ of topological spaces such that if $\emptyset \neq J \subseteq I$ and $A \in \mathfrak{B}_c(X_J)$, then there is a $B \in \mathfrak{B}_c(X_J)$ such that $\theta_I(A \times X_{J^c}) = B \times X_{J^c}$. This is an obvious generalization of the two factor case. We say that such a density respects coordinates. The terminology is taken from measure products case, where it has been proposed by Fremlin [8]. There is a weaker version of respecting coordinates in which the set B is not required to have the property of Baire. In the last section of the paper we give an example of a lifting respecting coordinates in this weaker sense. This pathology cannot occur if each pair of subproducts $(\prod_{i \in J} X_i, \prod_{i \notin J} X_i)$ satisfies the Kuratowski–Ulam property. It also cannot occur if all factors are weakly α-favorable (see Section 8 for more details). In Theorem 7.2 (this is the basic result in case of arbitrary products) we prove that for an arbitrary non-empty collection $\{X_i\colon\ i \in I\}$ of topological spaces such that their product X_I is a Baire space, for any given a priori collection of (strong) densities δ_i for $i \in I$, on $(X_i, \mathfrak{B}_c(X_i), \mathcal{M}(X_i))$ there exists a (strong) density ξ_I on $(X_I, \mathfrak{B}_c(X_I), \mathcal{M}(X_I))$ respecting coordinates, being separately Baire additive and having the densities δ_i $(i \in I)$ as its marginals. A corresponding result for measure theoretic densities can be found in [8, 346B] or [15, Theorem 2.5].

The best known result in case of liftings on finite measure products is from [3], where it is shown that liftings respecting coordinates exist (no coordinate liftings are fixed in advance). In case of infinite product, Fremlin [8] proved the existence of liftings respecting coordinates if all the coordinate measure spaces are Maharam homogeneous. The general problem remains open. Also in the category products of more than two factors the existence of liftings respecting coordinates remains open.

1. Preliminaries

Throughout we assume that all topological spaces under consideration are non-empty. Let X be a topological space. The weight of X is denoted by $w(X)$. A family $\mathcal{U}_X$ of non-empty open sets in a topological space X will be called a *pseudo-basis* (π-basis for short) if every non-empty open set in X contains an element $U \in \mathcal{U}_X$. The minimal cardinality of a π-basis will be denoted by $\pi(X)$. For each subset A of X we denote by $\operatorname{cl} A$ (or by $\overline{A}$) and by $\operatorname{int}(A)$ the topological closure and interior of A, respectively. A set $A \subseteq X$ is *nowhere* *dense* if $\operatorname{int}(\operatorname{cl} A) = \emptyset$. A set $M \subseteq X$ is *meager* or of the *first category* if it is expressible as a union of a sequence of nowhere dense sets. A set $A \subseteq X$ is of the *second category* if it is not meager. We recall the standard observation (see, e.g., [21]) that when Y is a dense subspace of X, for subsets A of Y we have that A is nowhere dense in Y if and only if A is nowhere dense in X, and A is meager in Y if and only if A is meager in X.

An open set $A \subseteq X$ is said to be *regular open* in X if it coincides with the interior of its closure. A set $A \subseteq X$ has the *Baire property* if it can be represented in the form $A = G \triangle N$, where G is open and N is meager. A topological space X is called a *Baire space* if every non-empty open set in X is non-meager. $\mathcal{M}(X)$ denotes the collection of all meager subsets of the topological space X and $\mathfrak{B}_c(X)$ denotes the σ-algebra of sets possessing the Baire property. $\operatorname{add}(\mathcal{M}(X)) := \min\{\operatorname{card}\mathfrak{J}\colon \mathfrak{J} \subset \mathcal{M}(X) \,\&\, \bigcup \mathfrak{J} \notin \mathcal{M}(X)\}$. We write $A \subseteq B$ a.e. $(\mathcal{M}(X))$ or $A \subseteq_{\mathcal{M}} B$ if $A \setminus B \in \mathcal{M}(X)$ and similarly for equality in place of the inclusion.

For each $E \in \mathfrak{B}_c(X)$ we denote by $\varphi_X(E)$ the regular open set equivalent to E. $\varphi_X : \mathfrak{B}_c(X) \to \mathfrak{B}_c(X)$ defined in that way is a strong density (see [11, Section 9], [18, Section 4], or [22, p. 88]). φ_X will be called the *canonical density* on $(X, \mathfrak{B}_c(X), \mathcal{M}(X))$.

A set $A \in \mathfrak{B}_c(X) \setminus \mathcal{M}(X)$ is an $\mathcal{M}(X)$-*atom* of $\mathfrak{B}_c(X)$ if A cannot be decomposed into two disjoint elements of $\mathfrak{B}_c(X) \setminus \mathcal{M}(X)$. Notice that φ_X is a lifting precisely when every regular open set in X is clopen, i.e., precisely when X is extremally disconnected.

Lower densities and liftings on $(X, \mathfrak{B}_c(X), \mathcal{M}(X))$ are defined exactly in the same way as densities and liftings for measure spaces (cf. [12], [25, Chapter 28]). We call them *category lower densities* and *category liftings*, while we call the densities and liftings for measure spaces *measure-theoretic densities* and *measure-theoretic liftings*. If no confusion arises we say “density” instead of “category lower density” and “measure-theoretic lower density” and “lifting” instead of “category lifting” and “measure-theoretic lifting”. The family of all (lower) densities on $(X, \mathfrak{B}_c(X), \mathcal{M}(X))$ is denoted by $\vartheta(\mathcal{M}(X))$, and the family of lifting, by $\Lambda(\mathcal{M}(X))$. Each density $\delta \in \vartheta(\mathcal{M}(X))$ generates a collection of filters $\{\mathcal{F}(x)\colon x \in X\}$ containing no elements of $\mathcal{M}(X)$: $\mathcal{F}(x) = \{A \in \mathfrak{B}_c(X)\colon x \in \delta(A)\}$.

For the densities $\delta \in \vartheta(\mathcal{M}(X))$, $\upsilon \in \vartheta(\mathcal{M}(Y))$ and $\xi \in \vartheta(\mathcal{M}(X \times Y))$ we say that ξ is a product of δ and υ, and we write it as $\xi \in \delta \otimes \upsilon$ if

$$\xi(A \times B) = \delta(A) \times \upsilon(B) \quad \text{for all } A \in \mathfrak{B}_c(X) \text{ and } B \in \mathfrak{B}_c(Y).$$

We use similar notation for a density ξ for the category algebra of an infinite product $\prod_{i\in I} X_i$ with densities δ_i in X_i, writing $\xi \in \bigotimes_{i\in I} \delta_i$ if $\xi(\prod_{i\in I} A_i) = \prod_{i\in I} \delta_i(A_i)$ for each product set $\prod_{i\in I} A_i$ where $A_i \in \mathfrak{B}_c(X_i)$ and $A_i = X_i$ for all but finite collection of $i \in I$.

The collection of all strong densities and of all strong liftings on $(X, \mathfrak{B}_c(X), \mathcal{M}(X))$ will be denoted by $\vartheta_s(\mathcal{M}(X))$ and by $\Lambda_s(\mathcal{M}(X))$, respectively.

Each time we consider strong densities on a topological space X, we assume that X is a Baire space. The assumption is necessary for the existence of a strong density in $\vartheta_s(\mathcal{M}(X))$. In fact, assume that X is a topological space admitting a strong density φ. Then for each non-empty open set G we have $G \subseteq \varphi(G)$, from which it follows that $\varphi(G) \neq \emptyset$ and hence G is not meager.

Given a probability space (Ω, Σ, μ) the family of all μ-null sets is denoted by Σ_0. The family of all elements of Σ of positive μ-measure is denoted by Σ_+. The (Carathéodory) completion of (Ω, Σ, μ) will be denoted by $(\Omega, \widehat{\Sigma}, \hat{\mu})$.

Given probability spaces (Ω, Σ, μ) and (Θ, T, ν), we denote by $\Sigma \otimes T$ the product σ-algebra generated by Σ and T. $(\Omega \times \Theta, \Sigma \otimes T, \mu \otimes \nu)$ is the corresponding product probability space and $(\Omega \times \Theta, \Sigma \widehat{\otimes} T, \mu \widehat{\otimes} \nu)$ denotes its (Carathéodory) completion.

If I is a non-empty set and $\langle X_i \rangle_{i \in I}$ is a family of arbitrary topological spaces then, for each $\emptyset \neq J \subseteq I$ we denote by X_J the product topological space $\prod_{i \in J} X_i$. If $J = \emptyset$, then for simplicity of notation we identify $X_J \times Y$ with Y.

We say that a $\varphi \in \vartheta(\mathcal{M}(X_I))$ is *separately Baire additive* if for any non-empty sets $J, K \subseteq I$ with $J \cap K = \emptyset$ we have

$$\varphi(E \cup F) = \varphi(E) \cup \varphi(F) \quad \text{for all } E \in \mathfrak{B}_c(X_J) \times X_{J^c} \text{ and } F \in \mathfrak{B}_c(X_K) \times X_{K^c}.$$

For measure theoretic densities this notion is due to Fremlin [8], where it is called the $(*)$ property.

We call a lifting $\pi \in \Lambda(\mathcal{M}(X_I))$ a *product-lifting* if there are liftings $\rho_i \in \Lambda(\mathcal{M}(X_i))$, for $i \in I$, such that the equation

$$\pi\big([A_{i_1}, \ldots, A_{i_n}]\big) = \big[\rho_{i_1}(A_{i_1}), \ldots, \rho_{i_n}(A_{i_n})\big], \tag{P}$$

holds true for all $n \in \mathbb{N}$, $i_1, \ldots, i_n \in I$ and all $A_{i_k} \in \mathfrak{B}_c(X_{i_k})$ $(k = 1, \ldots, n)$ where $[A_{i_1}, \ldots, A_{i_n}]$ denotes the cylinder set $\prod_{i \in I} B_i$ for $B_{i_k} = A_{i_k}$ $(k = 1, \ldots, n)$ and $B_i = X_i$, $i \in I \setminus \{i_1, \ldots, i_n\}$. We write then $\pi \in \bigotimes_{i \in I} \rho_i$. If $I := [n] := \{1, \ldots, n\}$ then we write $\pi \in \rho_1 \otimes \cdots \otimes \rho_n$.

We say that $\varphi_I \in \vartheta(\mathcal{M}(X_I))$ (or $\varphi_I \in \Lambda(\mathcal{M}(X_I))$) *respects coordinates* if for each proper $\emptyset \neq J \subseteq I$ the inclusion $\varphi_I(\mathfrak{B}_c(X_J) \times X_{J^c}) \subseteq \mathfrak{B}_c(X_J) \times X_{J^c}$ holds true.

It can be easily seen that if φ_I respects coordinates then, for each $\emptyset \neq J \subseteq I$ there is a uniquely determined density $\varphi_J \in \vartheta(\mathcal{M}(X_J))$ given by $\varphi_J(A) \times X_{J^c} = \varphi_I(A \times X_{J^c})$, for all $A \in \mathfrak{B}_c(X_J)$. And conversely, if for each $\emptyset \neq J \subseteq I$ there is a density φ_J on $\mathfrak{B}_c(X_J)$ such that $\varphi_I(A \times X_{J^c}) = \varphi_J(A) \times X_{J^c}$, whenever $A \in \mathfrak{B}_c(X_J)$, then φ_I respects coordinates. From this point of view one could speak about completely product density instead of density respecting coordinates.

Let X be a topological space. A density $\delta \in \vartheta(\mathcal{M}(X))$ is *consistent* if for every $n \in \mathbb{N}$ there exists a density $\delta^n \in \vartheta(\mathcal{M}(X^n))$, such that

$$\delta^n(A_1 \times \cdots \times A_n) = \delta(A_1) \times \cdots \times \delta(A_n)$$

for all $A_1, \ldots, A_n \in \mathfrak{B}_c(X)$ (see Talagrand [28], for the corresponding definition for measure-theoretic densities). We use a similar definition for liftings.

If X is a topological space with a complete finite measure μ on Σ then, (X, Σ, μ) is called a *category measure space* if and only if $\Sigma = \mathfrak{B}_c(X)$ and $\Sigma_0 = \mathcal{M}(X)$. μ is called then a *category measure*. For an arbitrary probability space (Ω, Σ, μ) we define its associated *hyperstonian space* $(X, \mathcal{T}, \mathfrak{B}_c(X), \nu)$ by means of: $X = \mathrm{Stone}(\Sigma/\mu)$, the Stone space of the measure algebra of (Ω, Σ, μ). $\mathcal{T}$ denotes the topology generated by $\{s(a)\colon a \in \Sigma/\mu\}$, where $s(a) \subseteq X$ is the open-and-closed set corresponding to a according to the Stone duality. $\nu = \tilde{\mu} \circ \pi : \mathfrak{B}_c(X) \to R$, where $\pi : \mathfrak{B}_c(X) \to \Sigma/\mu$ is the canonical epimorphism and $\tilde{\mu} : \Sigma/\mu \to R$ is defined by $\tilde{\mu}(a) := \mu(A)$ if $a = A^\bullet$ for $A \in \Sigma$

(where $A^\bullet$ denotes the class of all sets in Σ that are equivalent with A). We may say that "$(X, \mathcal{T}, \mathfrak{B}_c(X), \nu)$ is a hyperstonian space" instead of $(X, \mathcal{T}, \mathfrak{B}_c(X), \nu)$ is the hyperstonian space associated with the complete probability space (Ω, Σ, μ), if confusion is unlikely.

It is well known that if $(X, \mathcal{T}, \mathfrak{B}_c(X), \mu)$ is a hyperstonian space, then the elements of $\mathfrak{B}_c(X)$ are precisely those expressible in the form $E = s(a) \triangle N$ where $a \in \Sigma/\mu$, $s(a)$ is the corresponding open-and-closed subset of X, and N is meager. The system of the meager sets coincides with that of nowhere dense sets in X, and the system of the regular open sets coincides with that of the open-and-closed sets in X (see, e.g. [8, 321K]).

Each hyperstonian space is a category measure space, but there are category measure spaces which are not hyperstonian (see, e.g. [22, Section 22]).

Other unexplained notations and terminology come from [25].

2. Basic facts concerning Baire property

To begin a deeper investigation of densities on product spaces we need to prove or recall a few particular properties of regular open sets in product spaces and of sets having the property of Baire in Baire product spaces.

We recall that a topological space X is Baire if and only if player I does not have a winning strategy in the Banach–Mazur game for two players, I and II, in which, starting with player I, the players alternately play the terms of a decreasing sequence $U_1 \supseteq U_2 \supseteq \cdots$ of nonempty open sets and player I wins if the intersection of the sequence is empty. When the stronger condition that player II has a winning strategy holds, X is called *weakly α-favorable*. The standard proofs of the Baire category theorem for locally compact and for completely metrizable spaces show that these spaces are weakly α-favorable. (See [24] for more details. The characterization of Baire spaces is Theorem 2.1 of that paper.) If X and Y are Baire spaces and (X, Y) is a Kuratowski–Ulam pair, then $X \times Y$ is Baire. It is not hard to see (and well-known), using the game-theoretic characterizations, that if X is Baire and Y is weakly α-favorable, then $X \times Y$ is Baire. Also, it is easily seen that an arbitrary product $\prod_{i \in I} X_i$ of weakly α-favorable spaces is weakly α-favorable (see [30]). For examples of Baire spaces whose product is not Baire, see [21,6,19].

The following fact has been communicated to us by J. Pawlikowski.

Lemma 2.1. *Let X and Y be topological spaces such that $w(Y) < \operatorname{add}(\mathcal{M}(X))$. If $U \subseteq X \times Y$ is regular open, then there is a set $R \in \mathcal{M}(X)$ such that U_x is regular open for every $x \notin R$.*

Proof. We are going to prove that if $F \subseteq X \times Y$ is closed, then the set $\{x\colon (\operatorname{int} F)_x = \operatorname{int}(F_x)\}$ is comeager. To do it let us fix a base $\{V_\alpha\colon \alpha < w(Y)\}$ of the topology in Y. Note that $W_\alpha := \{x\colon V_\alpha \subseteq F_x\}$ is closed. Now, if $H = \bigcup_\alpha(\operatorname{int}(W_\alpha) \times V_\alpha)$, then $\operatorname{int}(F) = H$. Moreover, $\operatorname{int}(F_x) = \bigcup_\alpha\{V_\alpha\colon x \in W_\alpha\}$ and $H_x = \bigcup_\alpha\{V_\alpha\colon x \in \operatorname{int}(W_\alpha)\}$. So, for x outside the meager set $\bigcup_\alpha(W_\alpha \setminus \operatorname{int}(W_\alpha))$ we have $\operatorname{int}(F_x) = H_x$. Setting $F = \operatorname{cl} U$ for a regular open set $U \subset X \times Y$ and $R = \bigcup_\alpha(W_\alpha \setminus \operatorname{int}(W_\alpha))$, we obtain the required result. $\square$

Lemma 2.2. *If X and Y are non-empty topological spaces, then the following facts hold true*:

(a) *If U is a regular open subset of X and V is a regular open subset of Y, then $U \times V$ and $(U \times Y) \cup (X \times V)$ are regular open subsets of $X \times Y$.*
(b) *If $B \in \mathcal{M}(Y)$, then $X \times B \in \mathcal{M}(X \times Y)$. B is nowhere dense in Y if and only if $X \times B$ is nowhere dense in $X \times Y$.*
(c) *If $X \times Y$ is a Baire space, then also X and Y are Baire spaces.*
(d) *If X is Baire, then for any regular open sets $U, V \subseteq X$, we have $U \subseteq V$ if and only if $U \subseteq V$ a.e. $(\mathcal{M}(X))$.*
(e) *If $A_1 \subseteq A_2$ a.e. $(\mathcal{M}(X))$ and $B_1 \subseteq B_2$ a.e. $(\mathcal{M}(Y))$, then $A_1 \times A_2 \subseteq B_1 \times B_2$ a.e. $(\mathcal{M}(X \times Y))$. Similarly for equalities.*
(f) *If $X \times Y$ is a Baire space, $C \in \mathfrak{B}_c(X) \setminus \mathcal{M}(X)$ and $D \in \mathfrak{B}_c(Y) \setminus \mathcal{M}(Y)$, then $C \times D \notin \mathcal{M}(X \times Y)$.*
(g) *If $X \times Y$ is a Baire space, $C \times D \subseteq A \times B$ a.e. $(\mathcal{M}(X \times Y))$, where A, B, C, D have the Baire property in their respective spaces and C and D are non-meager, then $C \subseteq A$ a.e. $(\mathcal{M}(X))$ and $D \subseteq B$ a.e. $(\mathcal{M}(Y))$.*
(h) *If E is a regular open subset of $X \times Y$, then*

$$E = \bigcup \{A \times B\colon\ A \text{ is regular open in } X,\ B \text{ is regular open in } Y \text{ and } A \times B \subseteq E\}.$$

Proof. This is routine, so we omit most of the proofs. For parts (a) and (h), it is useful to note that because the closure operation satisfies the identities $\mathrm{cl}(A \times B) = \mathrm{cl}\, A \times \mathrm{cl}\, B$ and $\mathrm{cl}((A \times Y) \cup (X \times B)) = (\mathrm{cl}\, A \times Y) \cup (X \times \mathrm{cl}\, B)$ and the same identities hold with closure replaced by interior, these identities also hold for the composition $\mathrm{int}(\mathrm{cl}(\cdot))$. The other properties are easily established in the given order. □

Given arbitrary topological spaces X and Y, denote by $(\mathfrak{B}_c(X) \otimes \mathfrak{B}_c(Y)) \oplus \mathcal{M}(X \times Y)$ the system of all subsets H of $X \times Y$ such that there exist sets $P \in \mathfrak{B}_c(X) \otimes \mathfrak{B}_c(Y)$ with $H \triangle P \in \mathcal{M}(X \times Y)$. The following result explains partially the relation between product sets with the property of Baire and the coordinate sets with the Baire property.

Proposition 2.3. *Let X and Y be topological spaces. Then we have*

$$\big(\mathfrak{B}_c(X) \otimes \mathfrak{B}_c(Y)\big) \oplus \mathcal{M}(X \times Y) \subseteq \mathfrak{B}_c(X \times Y).$$

Moreover, if X or Y has a countable basis, then

$$\big(\mathfrak{B}_c(X) \otimes \mathfrak{B}_c(Y)\big) \oplus \mathcal{M}(X \times Y) = \mathfrak{B}_c(X \times Y).$$

Proof. Since $\mathfrak{B}_c(X \times Y)$ is a σ-algebra, the inclusion follows immediately from the definition of the Baire property.

Assume now that Y has a countable basis $\langle E_n \rangle_{n\in\mathbb{N}}$ of open sets. To check that $\mathfrak{B}_c(X \times Y) \subseteq (\mathfrak{B}_c(X) \otimes \mathfrak{B}_c(Y)) \oplus \mathcal{M}(X \times Y)$, it is enough to observe that if $U \subseteq X \times Y$ is open then $U = \bigcup_{n\in\mathbb{N}} (V_n \times E_n)$ where $V_n = \bigcup \{V\colon\ V \text{ is open in } X \text{ and } V \times E_n \subseteq U\}$ for $n \in \mathbb{N}$. □

We recall yet a definition introduced by Fremlin, Natkaniec and Recław in [10]. A pair (X, Y) of topological spaces is a *Kuratowski–Ulam pair* (briefly K–U pair) or it has the *Kuratowski–Ulam property*, if the Kuratowski–Ulam Theorem holds in $X \times Y$:

$$\forall E \subseteq X \times Y\big[E \in \mathcal{M}(X \times Y) \Longrightarrow \big\{x \in X\colon\ E_x \notin \mathcal{M}(Y)\big\} \in \mathcal{M}(X)\big].$$

Kuratowski and Ulam proved that if $\pi(Y) < \operatorname{add}(\mathcal{M}(X))$, then the pair (X, Y) is a K–U pair (see [22, Theorem 15.1]). In particular, if Y has a countable π-basis, then for each topological space X the pair (X, Y) is a K–U pair.

Recall the Banach Category Theorem (cf. [13, Theorem I.10.III.1]): in any topological space X, if A is a set which is covered by open sets U such that every $U \cap A$ is meager, then A is meager.

The properties of the density topology associated with a density for $(X, \Sigma, \mathcal{N})$ when $\mathcal{N} \subset \Sigma$ is a σ-ideal and every subset of X has a minimal Σ-cover modulo $\mathcal{N}$ are studied in some detail in [14]. The minimal cover property when $\Sigma = \mathfrak{B}_c(X)$ and $\mathcal{N} = \mathcal{M}(X)$ is a classical result of Szpilrajn-Marczewski (see Szpilrajn-Marczewski [27], Kuratowski [13, Corollary I.11.IV] or [14, Exercise 6.E.30, p. 221]). It is shown in [14, Proposition 6.37] (see also the Remark on p. 213) that the strong density topology (see Definition 2.5) is indeed a topology. For the convenience of the reader, we give the proof of this fact that is used repeatedly in this paper.

Proposition 2.4. *Let X be a Baire space and let $\delta \in \vartheta(\mathcal{M}(X))$ be arbitrary. Then for each collection $\mathcal{C} \subseteq \mathfrak{B}_c(X)$ such that $C \subseteq \delta(C)$ for each $C \in \mathcal{C}$, we have*

$$\bigcup \mathcal{C} \in \mathfrak{B}_c(X) \quad \text{and} \quad \bigcup \mathcal{C} \subseteq \delta\Big(\bigcup \mathcal{C}\Big).$$

Proof. Let U be the regular open set in $\bigvee\{C^\bullet\colon\ C \in \mathcal{C}\}$, where $C^\bullet$ denotes the equivalence class of C in $\mathfrak{B}_c(X)$ and $\bigvee$ is the sup operation in the algebra $\mathfrak{B}_c(X)/\mathcal{M}(X)$. For any $C \in \mathcal{C}$, we have $C^\bullet \leqslant U^\bullet$ and hence $C \subseteq U$ a.e. $(\mathcal{M}(X)$. This gives $C \subseteq \delta(C) \subseteq \delta(U)$ and hence

$$\bigcup \mathcal{C} \subseteq \delta(U).$$

There remains to check that $\delta(U) \setminus \bigcup \mathcal{C}$ is meager, or equivalently, that $U \setminus \bigcup \mathcal{C}$ is meager. Note that if U_C denotes the regular open set equivalent to C, then $\bigcup\{U_C\colon\ C \in \mathcal{C}\}$ is a dense open subset of U. Also, $U_C \cap (U \setminus \bigcup \mathcal{C}) \subseteq U_C \cap (U \setminus C) =_{\mathcal{M}} U_C \cap (U \setminus U_C) = \emptyset$. Hence $U \setminus \bigcup \mathcal{C}$ has a meager trace on each U_C and thus, by the Banach Category Theorem, it has a meager trace on $\bigcup\{U_C\colon\ C \in \mathcal{C}\}$ and hence is meager. □

Next we define the density and lifting topologies associated with a density and a lifting, respectively.

Definition 2.5. Let X be a Baire space, and let $\delta \in \vartheta(\mathcal{M}(X))$ be arbitrary. If

$$\tau_\delta := \big\{A \in \mathfrak{B}_c(X)\colon\ A \subseteq \delta(A)\big\},$$

then, due to Proposition 2.4, τ_δ is a topology on X, called the *strong* (*category*) *density topology* associated with δ. The family $\{\delta(A)\colon\ A \in \mathfrak{B}_c(X)\}$ itself forms a topological basis of another topology, called the *weak* (*category*) *density topology* t_δ associated with δ. Clearly, we have $t_\delta \subseteq \tau_\delta \subseteq \mathfrak{B}_c(X)$.

If δ is a lifting in $\Lambda(\mathcal{M}(X))$ then we call t_δ the *weak lifting topology* and τ_δ the *strong lifting topology*.

3. Products of canonical densities

We are going to present some basic properties of the product of the canonical densities.

Proposition 3.1. *If $X \times Y$ is a Baire space, then the following conditions hold true*:

(i) $\varphi_{X\times Y} \in \varphi_X \otimes \varphi_Y$ *and* $\varphi_{X\times Y}$ *is separately Baire additive*;
(ii) *for every* $E \in \mathfrak{B}_c(X \times Y)$

$$\varphi_{X\times Y}(E) = \bigcup\bigl\{\varphi_X(A) \times \varphi_Y(B)\colon\ A \times B \subseteq_{\mathcal{M}} E \ \&\ A \in \mathfrak{B}_c(X),\ B \in \mathfrak{B}_c(Y)\bigr\};$$

(iii) $t_{\varphi_X} \times t_{\varphi_Y} = t_{\varphi_{X\times Y}}$;
(iv) $\tau_{\varphi_X} \times \tau_{\varphi_Y} \subseteq \tau_{\varphi_{X\times Y}}$;
(v) $[\varphi_{X\times Y}(E)]_x \subseteq \varphi_Y([\varphi_{X\times Y}(E)]_x)$ *for every* $E \in \mathfrak{B}_c(X \times Y)$ *and* $x \in X$;
(vi) $[\varphi_{X\times Y}(E)]^y \subseteq \varphi_X([\varphi_{X\times Y}(E)]^y)$ *for every* $E \in \mathfrak{B}_c(X \times Y)$ *and* $y \in Y$.

If moreover $w(Y) < \mathrm{add}(\mathcal{M}(X))$, *then*

(a) *for each* $E \in \mathfrak{B}_c(X \times Y)$ *there exists* $M_E \in \mathcal{M}(X)$ *such that*

$$\bigl[\varphi_{X\times Y}(E)\bigr]_x = \varphi_Y\bigl([\varphi_{X\times Y}(E)]_x\bigr) \quad \textit{for every } x \notin M_E.$$

If also $w(X) < \mathrm{add}(\mathcal{M}(Y))$, *then we obtain moreover*

(b) *for each* $E \in \mathfrak{B}_c(X \times Y)$ *there exists* $N_E \in \mathcal{M}(Y)$ *such that*

$$\bigl[\varphi_{X\times Y}(E)\bigr]^y = \varphi_X\bigl(\bigl[\varphi_{X\times Y}(E)\bigr]^y\bigr) \quad \textit{for every } y \notin N_E.$$

Proof. To prove (i) we have to notice only that the product of two regular open sets is regular open and, that $\varphi_X(A) \times \varphi_Y(B) = A \times B$ a.e. $(\mathcal{M}(X \times Y))$. These properties follow from Lemma 2.2(a,e) for equalities. The separate additivity of $\varphi_{X\times Y}$ follows from the second part of Lemma 2.2(a).

(ii) follows from (i) and from Lemma 2.2(h). Indeed, both sides of the formula in (ii) depend only on the class of E in the category algebra, so we may assume that E is a regular open set. Similarly, we may restrict A and B to vary over regular open sets in their respective spaces. But then by Lemma 2.2(a,d), we may write $A \times B \subseteq E$ instead of $A \times B \subseteq_{\mathcal{M}} E$. The formula now reduces to Lemma 2.2(h).

To prove (iii) let us notice that the inclusion $t_{\varphi_X} \times t_{\varphi_Y} \subseteq t_{\varphi_{X\times Y}}$ follows immediately from (i). The converse inclusion is a consequence of (ii). (iv) follows from (i). (v) and (vi) follow from the fact that the canonical densities are strong.

Moreover, if $w(Y) < \mathrm{add}(\mathcal{M}(X))$ (in this case, instead of assuming $X \times Y$ is Baire, we can make a simpler assumption that X and Y are Baire, since the fact that $X \times Y$ is Baire then follows from [21, Theorem 2]), then (a) follows from Lemma 2.1. One obtains (b) in a similar way. This completes the proof of the whole proposition. □

Corollary 3.2. *If $X \times Y$ is a Baire space, then there exists a density $\bar{\varphi}_{X\times Y} \in \vartheta_s(\mathcal{M}(X \times Y))$ with the following properties*:

(i) $\bar{\varphi}_{X\times Y} \in \varphi_X \otimes \varphi_Y$ *and* $\varphi_{X\times Y}(E) \subseteq \bar{\varphi}_{X\times Y}(E)$ *for all* $E \in \mathfrak{B}_c(X \times Y)$;
(ii) $[\bar{\varphi}_{X\times Y}(E)]_x = \varphi_Y([\bar{\varphi}_{X\times Y}(E)]_x)$ *for every* $x \in X$ *and* $E \in \mathfrak{B}_c(X \times Y)$;

If moreover $w(Y) < \mathrm{add}(\mathcal{M}(X))$, *then*

(iii) $[\bar{\varphi}_{X\times Y}(E)]^y \in \mathfrak{B}_c(X)$ *for every* $y \in Y$ *and* $E \in \mathfrak{B}_c(X \times Y)$.

Proof. The canonical densities φ_Y and $\varphi_{X\times Y}$ satisfy the condition (v) of Proposition 3.1. Let $E \in \mathfrak{B}_c(X \times Y)$ be an arbitrary set. We define $\bar{\varphi}_{X\times Y}(E)$ by setting for each $x \in X$

$$\big[\bar{\varphi}_{X\times Y}(E)\big]_x = \varphi_Y\big(\big[\varphi_{X\times Y}(E)\big]_x\big) \quad \text{for all } x \in X.$$

It can be easily seen, that $\bar{\varphi}_{X\times Y}$ satisfies the condition (i) and (ii). It is also obvious that $\bar{\varphi}_{X\times Y}$ is strong.

To prove (iii), let us fix a set $E \in \mathfrak{B}_c(X \times Y)$ and a $y \in Y$. If $w(Y) < \mathrm{add}(\mathcal{M}(X))$, then according to Lemma 2.1 there exists a set $M_E \in \mathcal{M}(X)$ such that

$$\big[\varphi_{X\times Y}(E)\big]_x = \varphi_Y\big(\big[\varphi_{X\times Y}(E)\big]_x\big) \quad \text{for each } x \notin M_E.$$

Consequently, we get for every $y \in Y$

$$\begin{aligned}
\big[\bar{\varphi}_{X\times Y}(E)\big]^y \cap M_E^c &= \big\{x \in X\colon\ (x, y) \in \bar{\varphi}_{X\times Y}(E)\big\} \cap M_E^c \\
&= \big\{x \in X\colon\ y \in \big[\bar{\varphi}_{X\times Y}(E)\big]_x\big\} \cap M_E^c \\
&= \big\{x \in X\colon\ y \in \varphi_Y\big(\big[\varphi_{X\times Y}(E)\big]_x\big)\big\} \cap M_E^c \\
&= \big\{x \in X\colon\ y \in \big[\varphi_{X\times Y}(E)\big]_x\big\} \cap M_E^c \\
&= \big[\varphi_{X\times Y}(E)\big]^y \cap M_E^c.
\end{aligned}$$

Since $[\varphi_{X\times Y}(E)]^y \cap M_E^c \in \mathfrak{B}_c(X)$, we get $[\bar{\varphi}_{X\times Y}(E)]^y \cap M_E^c \in \mathfrak{B}_c(X)$, hence $[\bar{\varphi}_{X\times Y}(E)]^y \in \mathfrak{B}_c(X)$. □

The following result follows immediately from Proposition 3.1 by induction.

Corollary 3.3. *Let X be a topological space such that for each $n \in \mathbb{N}$ the product space X^n is Baire. Then the canonical density $\varphi_X \in \vartheta_s(\mathcal{M}(X))$ is consistent.*

Corollary 3.4. *Let $(X, \mathfrak{B}_c(X), \mu)$ be a hyperstonian space. Then the canonical density $\varphi_X \in \vartheta_s(\mathcal{M}(X))$ is a consistent (strong) lifting.*

Proof. Since X is extremally disconnected, φ_X is a lifting. It follows from Corollary 3.3 that for each $n \in \mathbb{N}$ there exists a density $\varphi^n \in \vartheta_s(\mathcal{M}(X^n))$ such that

$$\varphi^n(A_1 \times \cdots \times A_n) = \varphi_X(A_1) \times \cdots \times \varphi_X(A_n) \tag{1}$$

for all $A_1, \ldots, A_n \in \mathfrak{B}_c(X)$. It follows from [11, Corollary 9.4] (see also [23]) that there exists a lifting $\rho^n \in \Lambda(\mathcal{M}(X^n))$ such that $\varphi^n(E) \subseteq \rho^n(E)$ for each $E \in \mathfrak{B}_c(X^n)$, hence ρ^n is strong. It follows from (1) and from the lifting properties of φ_X and ρ^n that φ_X is consistent. □

Corollary 3.5. *Let $\{X_i\colon i \in I\}$ be non-empty collection of non-empty topological spaces such that X_I is a Baire space. Then, the canonical density on X_I respects coordinates and is separately Baire additive.*

If $w(X_I) < \mathrm{add}(\mathcal{M}(X_I))$, then for each proper non-empty $J \subset I$ and each $E \in \mathfrak{B}_c(X_I)$ there is $M_{E,J^c} \in \mathcal{M}(X_{J^c})$ such that

$$\bigl[\varphi_{X_I}(E)\bigr]_{x_{J^c}} = \varphi_{X_{J^c}}\bigl(\bigl[\varphi_{X_I}(E)\bigr]_{x_{J^c}}\bigr) \quad \textit{for every } x_{J^c} \notin M_{E,J^c}.$$

Proof. If $I = K \cup L$ is a proper partition of I, the according to Proposition 3.1 we have $\varphi_{X_I} \in \varphi_{X_K} \otimes \varphi_{X_L}$ what means exactly that φ_{X_I} respects coordinates. Separate additivity of φ_{X_I} is a consequence of Lemma 2.2(a). The section property comes from Proposition 3.1. □

Remark 3.6. (a) It should be noted here that in general the τ*-additive product* (see e.g. [9] for the definition) of two category probability spaces is not a category probability space. In fact, let be given two category probability spaces $(X, \mathfrak{B}_c(X), \mu)$ and $(Y, \mathfrak{B}_c(Y), \nu)$. Assume if possible that their τ-additive product is a category probability space. Then we get

$$\mathcal{M}(X \times Y) = \bigl(\widehat{\mathfrak{B}}_\tau(X \times Y)\bigr)_0, \tag{2}$$

where $\widehat{\mathfrak{B}}_\tau(X \times Y)$ is the completion the σ-algebra $\mathfrak{B}(X \times Y)$ of Borel subsets of $X \times Y$ with respect to the τ-additive product $\mu \otimes_\tau \nu$ of μ and ν. But since the Fubini Theorem holds true for τ-additive products of probability measures (see Ressel [23]), it follows from (2) that (X, Y) is a K–U pair, what is not in general true according to [10, Example 2].

(b) The Radon product of two non-atomic hyperstonian spaces is not a category probability space. In fact, assume if possible that for given hyperstonian spaces $(X, \mathfrak{B}_c(X), \mu)$ and $(Y, \mathfrak{B}_c(Y), \nu)$ their Radon product is a category probability space. It then follows that

$$\mathcal{M}(X \times Y) = \bigl(\widehat{\mathfrak{B}}_R(X \times Y)\bigr)_0, \tag{3}$$

where by $\widehat{\mathfrak{B}}_R(X \times Y)$ is denoted the completion the σ-algebra $\mathfrak{B}(X \times Y)$ with respect to the Radon product $\mu \otimes_R \nu$ of μ and ν.

A well-known result of Erdős and Oxtoby [5] says that there exists $E \in \mathfrak{B}(X) \widehat{\otimes} \mathfrak{B}(Y)$ of positive measure such that for no $A \in \widehat{\mathfrak{B}}(X) \setminus (\widehat{\mathfrak{B}}(X))_0$ and no $B \in \widehat{\mathfrak{B}}(Y) \setminus (\widehat{\mathfrak{B}}(Y))_0$ the inclusion $A \times B \subseteq E$ a.e. $(\mu \widehat{\otimes} \nu)$ holds true. But since $E \in \mathfrak{B}(X) \widehat{\otimes} \mathfrak{B}(Y)$, we get $E \in \mathfrak{B}_c(X \times Y) \setminus \mathcal{M}(X \times Y)$. Hence there exist a non-empty set $G \in \mathcal{T} \times \mathcal{S}$ such that

$G \triangle E \in \mathcal{M}(X \times Y)$. Consequently, there exist non-empty sets $V \in \mathcal{T}$ and $W \in \mathcal{S}$ such that $V \times W \subseteq G$, hence $V \times W \subseteq E$—a.e. $(\mathcal{M}(X \times Y))$. So applying condition (3) we get

$$V \times W \subseteq E \quad \text{a.e. } (\mu \widehat{\otimes}_R \nu).$$

But since $E \in \mathfrak{B}(X) \widehat{\otimes} \mathfrak{B}(Y)$ this is the same as

$$V \times W \subseteq E \quad \text{a.e. } (\mu \widehat{\otimes} \nu),$$

what is impossible.

Notice that the above proof shows that even $\mathcal{M}(X \times Y) \subseteq (\widehat{\mathfrak{B}}_R(X \times Y))_0$ is false.

One can in fact see that the validity of the above inclusion yields the K–U property of (X, Y) and (Y, X). Indeed, if $E \in \mathcal{M}(X \times Y)$, then the Fubini theorem yields $\{x \in X\colon E_x \notin \mathcal{M}(Y)\} = \{x \in X\colon E_x \notin \mathfrak{B}_c(Y)_0\} \in \mathfrak{B}_c(X)_0 = \mathcal{M}(X)$. Similarly for (Y, X).

(c) Assume that $(X, \mathfrak{B}_c(X), \mu)$ and $(Y, \mathfrak{B}_c(Y), \nu)$ are hyperstonian. If (X, Y) is a K–U pair, then

$$\big(\widehat{\mathfrak{B}}_R(X \times Y)\big)_0 \subseteq \mathcal{M}(X \times Y). \tag{4}$$

In fact, let us fix a set $E \in (\widehat{\mathfrak{B}}_R(X \times Y))_0$. Then there exists a set $F \in (\mathfrak{B}(X \times Y))_0$ such that $E \subseteq F$, hence $F \in \mathfrak{B}_c(X \times Y)$. Since $X \times Y$ is a K–U pair there exists a set $N_F \in \mathcal{M}(X)$ such that $F_x \in \mathcal{M}(Y) = \mathfrak{B}_c(Y)_0$ for each $x \notin N_F$, hence $F \in \mathcal{M}(X \times Y)$ and so $E \in \mathcal{M}(X \times Y)$.

(d) Corollary 3.4 shows that in case of hyperstonian probability spaces the category strong liftings have a better behavior than the measure theoretic ones under the product formation, since a category strong product lifting always exists and has nice properties, while the existence of a measure theoretic strong product lifting remains an open problem (see [9, 453Z, Problem (a)]).

4. Products of two arbitrary densities

Proposition 4.1. *Assume that $X \times Y$ is a Baire space. Given densities $\delta \in \vartheta(\mathcal{M}(X))$ and $\upsilon \in \vartheta(\mathcal{M}(Y))$, we set*

$$\xi(E) := \bigcup \big\{\delta(A) \times \upsilon(B)\colon A \times B \subseteq E \text{ a.e. } \big(\mathcal{M}(X \times Y)\big)\big\}$$

for every $E \in \mathfrak{B}_c(X \times Y)$. Then $\xi \in \vartheta(\mathcal{M}(X \times Y))$ and satisfies the following conditions:

(i) $\xi \in \delta \otimes \upsilon$;
(ii) $t_\xi = t_\delta \times t_\upsilon$;
(iii) $\tau_\xi \supseteq \tau_\delta \times \tau_\upsilon$;
(iv) $[\xi(E)]_x \in \mathfrak{B}_c(Y)$ *and* $[\xi(E)]_x \subseteq \upsilon([\xi(E)]_x)$ *for every* $E \in \mathfrak{B}_c(X \times Y)$ *and* $x \in X$;
(v) $[\xi(E)]^y \in \mathfrak{B}_c(X)$ *and* $[\xi(E)]^y \subseteq \delta([\xi(E)]^y)$ *for every* $E \in \mathfrak{B}_c(X \times Y)$ *and* $y \in Y$;
(vi) *if δ and υ are strong, then ξ is also strong*;
(vii) *ξ is separately Baire additive.*

Proof. $\xi(\emptyset) = \emptyset$ by Lemma 2.2(f), $\xi(X \times Y) = X \times Y$ and, $E = F$ a.e. $(\mathcal{M}(X \times Y))$ implies $\xi(E) = \xi(F)$. It is also easy to check that ξ preserves intersections. We have to check yet if $\xi(E) \in \mathfrak{B}_c(X \times Y)$ for all $E \in \mathfrak{B}_c(X \times Y)$ and

$$\xi(E) =_{\mathcal{M}} E \quad \text{for all } E \in \mathfrak{B}_c(X \times Y). \tag{5}$$

Clearly it suffices to prove (5). To check if property (5) holds true, we need first to prove that ξ satisfies condition (i). To this aim, notice first that if $E = A \times B$ a.e. $(\mathcal{M}(X \times Y))$, then directly from the definition of ξ follows the inclusion $\delta(A) \times \upsilon(B) \subseteq \xi(A \times B)$. The converse inclusion follows from Lemma 2.2(g). Indeed, if $C \times D \subseteq A \times B$ a.e. $(\mathcal{M}(X \times Y))$ and all the sets have the Baire property, then it follows from Lemma 2.2(g) that $\delta(C) \times \upsilon(D) \subseteq \delta(A) \times \upsilon(B)$, what immediately yields $\xi(A \times B) \subseteq \delta(A) \times \upsilon(B)$. This proves (i).

To check if property (5) always holds true, let E be a regular open subset of $X \times Y$. That $\xi(E) = E$ a.e. $(\mathcal{M}(X \times Y))$ can be seen as follows. If $U \times V$ is a basic open set disjoint from E, then, using condition (i) and the fact that ξ preserves intersections, we get

$$\begin{aligned}\xi(E) \cap (U \times V) &=_{\mathcal{M}} \xi(E) \cap \bigl(\delta(U) \times \upsilon(V)\bigr) \\ &= \xi(E) \cap \xi(U \times V) = \xi\bigl(E \cap (U \times V)\bigr) = \emptyset.\end{aligned}$$

By the Banach Category Theorem, we get $\xi(E) \subseteq \operatorname{cl} E$ a.e. $(\mathcal{M}(X \times Y))$ and hence $\xi(E) \subseteq E$ a.e. $(\mathcal{M}(X \times Y))$.

Similarly, for each basic open set $U \times V \subseteq E$, applying condition (i), we see that

$$(U \times V) \cap \bigl(E \setminus \xi(E)\bigr) \subseteq (U \times V) \setminus \bigl(\delta(U) \times \upsilon(V)\bigr)$$

is meager by Lemma 2.2(e) and hence, by the Banach Category Theorem, $E \setminus \xi(E)$ is meager. Consequently, $\xi(E) = E$ a.e. $(\mathcal{M}(X \times Y))$ and $E \in \mathfrak{B}_c(X \times Y)$.

Inclusion $t_\delta \times t_\upsilon \subseteq t_\xi$ follows from condition (i), while the inverse inclusion follows from the definition of ξ, hence condition (ii) holds true. Condition (iii) follows from (i).

To prove condition (iv), let us fix a set $E \in \mathfrak{B}_c(X \times Y)$ and $x \in X$. Then, let

$$\mathfrak{B}_x := \bigl\{B \in \mathfrak{B}_c(Y)\colon \exists A \in \mathfrak{B}_c(X)\ A \times B \subseteq E \text{ a.e. } \bigl(\mathcal{M}(X \times Y)\bigr) \ \&\ x \in \delta(A)\bigr\}.$$

Now we have

$$\begin{aligned}\bigl[\xi(E)\bigr]_x &= \bigcup\bigl\{\bigl[\delta(A) \times \upsilon(B)\bigr]_x\colon\ A \times B \subseteq E \text{ a.e. } \bigl(\mathcal{M}(X \times Y)\bigr)\bigr\} \\ &= \bigcup_{B \in \mathfrak{B}_x} \upsilon(B) \subseteq \upsilon\biggl(\bigcup_{B \in \mathfrak{B}_x} \upsilon(B)\biggr) = \upsilon\bigl(\bigl[\xi(E)\bigr]_x\bigr),\end{aligned}$$

where the relation $\bigcup_{B \in \mathfrak{B}_x} \upsilon(B) \in \mathfrak{B}_c(Y)$ and the inclusion follow from Proposition 2.4.

If $\mathfrak{B}_x = \emptyset$, then $[\xi(E)]_x = \emptyset$. In both cases condition (iv) holds true. Consequently, for each $E \in \mathfrak{B}_c(X \times Y)$ all sections $[\xi(E)]_x$ of the set $\xi(E)$ are in $\mathfrak{B}_c(Y)$.

To prove condition (vi), fix an open subset G of $X \times Y$. There exists a family $\langle G_i \times U_i\rangle_{i \in I}$ of open rectangles in $\mathfrak{B}_c(X \times Y)$ such that $G = \bigcup_{i \in I} G_i \times U_i$. Since δ and υ are strong densities, we get

$$G \subseteq \bigcup_{i \in I} \delta(G_i) \times \upsilon(U_i) \subseteq \xi(G),$$

hence ξ is strong.

To show (vii), let $E \in \mathfrak{B}_c(X)$ and $F \in \mathfrak{B}_c(Y)$. Notice then that if $A \in \mathfrak{B}_c(X)$ and $B \in \mathfrak{B}_c(Y)$ satisfy $A \times B \subseteq (E \times Y) \cup (X \times F)$ a.e. ($\mathcal{M}(X \times Y)$), then $(A \setminus E) \times (B \setminus F) \subseteq (A \times B) \setminus ((E \times Y) \cup (X \times F)) \in \mathcal{M}(X \times Y)$. Thus, because the sets $A \setminus E$ and $B \setminus F$ have the property of Baire, we get either $A \subseteq E$ a.e. ($\mathcal{M}(X)$) or $B \subseteq F$ a.e. ($\mathcal{M}(Y)$), by Lemma 2.2(f). Hence,

$$\begin{aligned} &\xi\big((E \times Y) \cup (X \times F)\big) \\ &\quad = \bigcup \big\{\delta(A) \times \upsilon(B) \colon A \times B \subseteq_{\mathcal{M}} (E \times Y) \cup (X \times F)\big\} \\ &\quad \subseteq \big(\delta(E) \times Y\big) \cup \big(X \times \upsilon(F)\big) \subseteq \xi(E \times Y) \cup \xi(X \times F), \end{aligned}$$

and so condition (vii) holds true. □

Proposition 4.2. *Assume that $X \times Y$ is a Baire space and that we are given densities $\delta \in \vartheta(\mathcal{M}(X))$, $\upsilon \in \vartheta(\mathcal{M}(Y))$ and $\zeta \in \vartheta(\mathcal{M}(X \times Y))$ such that for each $E \in \mathfrak{B}_c(X \times Y)$ and each $x \in X$*

$$\big[\zeta(E)\big]_x \in \mathfrak{B}_c(Y) \quad \textit{and} \quad \big[\zeta(E)\big]_x \subseteq \upsilon\big(\big[\zeta(E)\big]_x\big). \tag{6}$$

If $\zeta_1 : \mathfrak{B}_c(X \times Y) \to \mathcal{P}(X \times Y)$ is defined by $[\zeta_1(E)]_x = \upsilon([\zeta(E)]_x)$, then

(a) *$\zeta_1 \in \vartheta(\mathcal{M}(X \times Y))$ and $\zeta(E) \subseteq \zeta_1(E)$ for every $E \in \mathfrak{B}_c(X \times Y)$;*
(b) *If $\zeta \in \delta \otimes \upsilon$, then $\zeta_1 \in \delta \otimes \upsilon$;*
(c) *If $\zeta \in \delta \otimes \upsilon$ and ζ is separately Baire additive, then also ζ_1 is separately Baire additive*;
(d) *If ζ is strong, then also ζ_1 is strong.*

Proof. Due to (6), we have $\zeta(E) \subseteq \zeta_1(E)$ and consequently $\zeta_1(E) \in \mathfrak{B}_c(X \times Y)$ and $\zeta_1(E) \triangle E \in \mathcal{M}(X \times Y)$. Other density properties are immediate. To show condition (b), let $A \in \mathfrak{B}_c(X)$, $B \in \mathfrak{B}_c(Y)$ and $x \in X$ be arbitrary. Then,

$$\begin{aligned} \big[\zeta_1(A \times B)\big]_x &= \upsilon\big(\big[\zeta(A \times B)\big]_x\big) = \upsilon\big(\big[\delta(A) \times \upsilon(B)\big]_x\big) \\ &= \begin{cases} \upsilon(B) & \text{if } x \in \delta(A), \\ \emptyset & \text{if } x \notin \delta(A). \end{cases} \end{aligned}$$

To show (c) take sets $A \times Y \in \mathfrak{B}_c(X) \times Y$ and $X \times B \in X \times \mathfrak{B}_c(Y)$.

We have then $\zeta(A \times Y \cup X \times B) = \zeta(A \times Y) \cup \zeta(X \times B)$. Since $\zeta \in \delta \otimes \upsilon$, we have $\zeta(A \times Y) = \delta(A) \times Y$ what yields $[\zeta(A \times Y)]_x$ equals $\emptyset$ or Y. Hence

$$\begin{aligned} &\big[\zeta_1(A \times Y \cup X \times B)\big]_x \\ &\quad = \upsilon\big(\big[\zeta(A \times Y \cup X \times B)\big]_x\big) = \upsilon\big(\big[\zeta(A \times Y) \cup \zeta(X \times B)\big]_x\big) \\ &\quad = \upsilon\big([\zeta(A \times Y)]_x \cup \big[\zeta(X \times B)\big]_x\big) \\ &\quad = \upsilon\big(\big[\zeta(A \times Y)\big]_x\big) \cup \upsilon\big(\big[\zeta(X \times B)\big]_x\big) \\ &\quad = \big[\zeta_1(A \times Y)\big]_x \cup \big[\zeta_1(X \times B)\big]_x \end{aligned}$$

and so ζ_1 is separately Baire additive. (d) follows from (a). This completes the whole proof. □

Remark 4.3. If υ is a Borel density, i.e. $\upsilon(B)$ is Borel if $B \in \mathfrak{B}_c(Y)$, then all X-sections of ξ_1 are Borel sets (ξ is defined as in Proposition 4.1). In spite of this ξ may be not Borel, at least when CH is assumed (see Example 1.7 of [2] with $\mathcal{N} =$ the ideal of meager sets).

Lemma 4.4. *Assume that $X \times Y$ is a Baire space. Then, given arbitrary densities $\delta \in \vartheta(\mathcal{M}(X))$ and $\upsilon \in \vartheta(\mathcal{M}(Y))$ there exists $\psi_1 \in \vartheta(\mathcal{M}(X \times Y))$ satisfying for each $x \in X$ and $E \in \mathfrak{B}_c(X \times Y)$ the following conditions*:

(j) $\xi_1(E) \subseteq \psi_1(E)$;
(jj) $[\psi_1(E)]_x \cup [\psi_1(E^c)]_x = Y$ *a.e.* $(\mathcal{M}(Y))$;
(jjj) $[\psi_1(E)]_x = \upsilon([\psi_1(E)]_x)$;
(jv) $\forall\, C \in \mathfrak{B}_c(X)\ [\psi_1(C \times Y)]_x \in \{\emptyset, Y\}\ \&\ [\psi_1(C \times Y)]_x \cup [\psi_1(C^c \times Y)]_x = Y$;
(v) *if δ and υ are strong, then ψ_1 is also strong.*

If moreover $w(Y) < \mathrm{add}(\mathcal{M}(X))$, $\delta = \varphi_X$ and $\upsilon = \varphi_Y$, then there exists a density $\bar{\psi}_1 \in \vartheta_s(\mathcal{M}(X \times Y))$ satisfying the properties (j)–(v) *with $\bar{\psi}_1$ and $\bar{\varphi}_{X\times Y}$ instead of ψ_1 and ξ_1, respectively, and the additional property*

(vj) $[\bar{\psi}_1(E)]^y \in \mathfrak{B}_c(X)$ *for all $E \in \mathfrak{B}_c(X \times Y)$ and $y \in Y$.*

Proof. Let

$$\begin{aligned}\Phi := \Big\{ \varphi \in \vartheta\big(\mathcal{M}(X \times Y)\big) \colon\ & \forall E \in \mathfrak{B}_c(X \times Y)\ \xi_1(E) \subseteq \varphi(E)\\ & \&\ \forall x \in X\ \forall E \in \mathfrak{B}_c(X \times Y)\ [\varphi(E)]_x \subseteq \upsilon\big(\big[\varphi(E)\big]_x\big)\\ & \&\ \forall C \in \mathfrak{B}_c(X)\ \forall x \in X\big[\varphi(C \times Y)\big]_x \in \{\emptyset, Y\}\Big\}.\end{aligned}$$

Notice first that Proposition 4.1 yields $\xi_1 \in \Phi$ and so $\Phi \neq \emptyset$. We consider Φ with inclusion as the partial order: $\varphi \leqslant \tilde{\varphi}$ if $\varphi(E) \subseteq \tilde{\varphi}(E)$ for each $E \in \mathfrak{B}_c(X \times Y)$.

Claim 1. *There exists a maximal element in Φ.*

Proof. The only fact we have to prove is showing that each chain $\{\varphi_\alpha\}_{\alpha\in A} \subseteq \Phi$ has a dominating element in Φ. The obvious candidate is φ given for each $E \in \mathfrak{B}_c(X \times Y)$ by

$$\varphi(E) = \bigcup_{\alpha\in A} \varphi_\alpha(E)\,.$$

Let us prove first the Baire property of $\varphi(E)$. To do it notice first that

$$\varphi\big(E^c\big) = \bigcup_{\alpha\in A} \varphi_\alpha\big(E^c\big).$$

and suppose, there exists $(x, y) \in \varphi(E) \cap \varphi(E^c)$. In such a case there exist $\alpha \in A$ and $\bar{\alpha} \in A$ such that $(x, y) \in \varphi_\alpha(E)$ and $(x, y) \in \varphi_{\bar{\alpha}}(E^c)$. Since A is linearly ordered, we have $\alpha \leqslant \bar{\alpha}$ or conversely. Assume that $\alpha \leqslant \bar{\alpha}$, then $(x, y) \in \varphi_{\bar{\alpha}}(E) \cap \varphi_{\bar{\alpha}}(E^c)$, contradicting the disjointness of these two sets. Thus,

$$\varphi(E) \cap \varphi\big(E^c\big) = \emptyset.$$

Hence, $\varphi(E) \subseteq [\varphi(E^c)]^c$ and so if an $\alpha \in A$ is fixed, then

$$\varphi_\alpha(E) \subseteq \varphi(E) \subseteq \left[\varphi\left(E^c\right)\right]^c \subseteq \left[\varphi_\alpha\left(E^c\right)\right]^c$$

for each $E \in \mathfrak{B}_c(X \times Y)$. Since $\mathcal{M}(X \times Y)$ is complete and $\varphi_\alpha \in \vartheta(\mathcal{M}(X \times Y))$, this proves the Baire property of $\varphi(E)$. Consider now the section properties of $\varphi(E)$. For fixed $x \in X$

$$\left[\varphi(E)\right]_x = \bigcup_{\alpha \in A}\left[\varphi_\alpha(E)\right]_x \subseteq \bigcup_{\alpha \in A} \upsilon\left(\left[\varphi_\alpha(E)\right]_x\right)$$

and so—in virtue of Proposition 2.4—the set $[\varphi(E)]_x$ is in $\mathfrak{B}_c(Y)$. It is clear that the inclusion $[\varphi(E)]_x \subseteq \upsilon([\varphi(E)]_x)$ is satisfied also. □

Now, we take as ψ_1 an arbitrary maximal element of Φ. To prove all its properties we can follow the proof of Lemma 2.8 from [20]. But for the sake of completeness we present here the important steps.

Claim 2. *For every $E \in \mathfrak{B}_c(X \times Y)$ and every $x \in X$*

$$\left[\psi_1(E)\right]_x = \upsilon\left(\left[\psi_1(E)\right]_x\right).$$

Proof. Set for each $x \in X$ and $E \in \mathfrak{B}_c(X \times Y)$

$$\left[\tilde{\psi}(E)\right]_x = \upsilon\left(\left[\psi_1(E)\right]_x\right).$$

Clearly $\psi_1(F) \subseteq \tilde{\psi}(F)$ for each F. Moreover the equality $\psi_1(E) \cap \psi_1(E^c) = \emptyset$ yields for each x the relation $\upsilon([\psi_1(E)]_x) \cap \upsilon([\psi_1(E^c)]_x) = \emptyset$. As a consequence, we get $\tilde{\psi}(E^c) \subseteq (\tilde{\psi}(E))^c$. Hence

$$\psi_1\left(E^c\right) \subseteq \tilde{\psi}\left(E^c\right) \subseteq \left[\tilde{\psi}(E)\right]^c \subseteq \left[\psi_1(E)\right]^c$$

and so $\tilde{\psi}(E) \in \mathfrak{B}_c(X \times Y)$. It follows that $\tilde{\psi} \in \Phi$ and consequently $\psi_1 = \tilde{\psi}$ and ψ_1 satisfies (iii). □

Claim 3. *For each $x \in X$ and $C \in \mathfrak{B}_c(X)$*

$$\left[\psi_1(C \times Y)\right]_x \cup \left[\psi_1(C^c \times Y)\right]_x = Y.$$

Proof. According to the definition of Φ we have the relation $[\psi_1(C \times Y)]_x \in \{\emptyset, Y\}$ for each x and $C \in \mathfrak{B}_c(X)$. Suppose that for some x_0 and $C_0 \in \mathfrak{B}_c(X)$ the equality $[\psi_1(C_0 \times Y)]_{x_0} \cup [\psi_1(C_0^c \times Y)]_{x_0} = \emptyset$ holds true. Then define $\hat{\psi} \in \vartheta(\mathcal{M}(X \times Y))$ by the equality

$$\left[\hat{\psi}(E)\right]_x = \begin{cases} [\psi_1(E)]_x & \text{if } x \neq x_0, \\ [\psi_1(E \cup (C_0 \times Y))]_{x_0} & \text{if } x = x_0. \end{cases}$$

It is clear that $\psi_1(E) \subseteq \hat{\psi}(E)$ for each $E \in \mathfrak{B}_c(X \times Y)$ and $[\hat{\psi}(C \times Y)]_x \in \{\emptyset, Y\}$ for each $x \in X$ and $C \in \mathfrak{B}_c(X)$. Consequently $\hat{\psi} \in \Phi$. Since $[\hat{\psi}(C_0^c \times Y)]_{x_0} = Y \neq [\psi_1(C_0^c \times Y)]_{x_0} = \emptyset$, it follows that $\hat{\psi} \neq \psi_1$ what contradicts the maximality of ψ_1. This completes the proof of the claim and shows that ψ_1 satisfies (jv). □

Claim 4. *For each $x \in X$ and $E \in \mathfrak{B}_c(X \times Y)$*

$$\big[\psi_1(E)\big]_x \cup \big[\psi_1\big(E^c\big)\big]_x = Y \quad a.e. \ \big(\mathcal{M}(Y)\big).$$

Proof. If not, then there exist $H \in \mathfrak{B}_c(X \times Y)$ and $x_0 \in X$ such that $([\psi_1(H)]_{x_0} \cup [\psi_1(H^c)]_{x_0})^c \notin \mathcal{M}(Y)$. Let

$$W := \upsilon\big[\big(\big[\psi_1(H)\big]_{x_0} \cup \big[\psi_1\big(H^c\big)\big]_{x_0}\big)^c\big]$$

and let

$$\big[\hat{\psi}(E)\big]_x = \begin{cases} [\psi_1(E)]_x & \text{if } x \neq x_0, \\ [\psi_1(E)]_{x_0} \cup (W \cap [\psi_1(H \cup E)]_{x_0}) & \text{if } x = x_0. \end{cases}$$

It is clear, that $\psi_1(E) \subseteq \hat{\psi}(E)$ for each $E \in \mathfrak{B}_c(X \times Y)$. In particular $\hat{\psi}(X \times Y) = X \times Y$. It can be easily proved that $\hat{\psi} \in \vartheta(\mathcal{M}(X \times Y))$.

Since $[\hat{\psi}(H^c)]_{x_0} = [\psi_1(H^c)]_{x_0} \cup W \neq [\psi_1(H^c)]_{x_0}$, we see that $\hat{\psi}$ and ψ_1 are different densities.

In order to get a contradiction with our hypothesis it is enough to show that $[\hat{\psi}(E)]_{x_0} \subseteq \upsilon([\hat{\psi}(E)]_{x_0})$ and $[\hat{\psi}(C \times Y)]_{x_0} \in \{\emptyset, Y\}$ for every $E \in \mathfrak{B}_c(X \times Y)$ and every $C \in \mathfrak{B}_c(X)$, but this is immediate. If $E \in \mathfrak{B}_c(X \times Y)$, then

$$\begin{aligned} \upsilon\big(\big[\hat{\psi}(E)\big]_{x_0}\big) &\supseteq \upsilon\big(\big[\psi_1(E)\big]_{x_0}\big) \cup \upsilon\big(W \cap \big[\psi_1(H \cup E)\big]_{x_0}\big) \\ &= \big[\psi_1(E)\big]_{x_0} \cup \big[\upsilon(W) \cap \upsilon\big(\big[\psi_1(H \cup E)\big]_{x_0}\big)\big] \\ &= \big[\psi_1(E)\big]_{x_0} \cup \big(W \cap \big[\psi_1(H \cup E)\big]_{x_0}\big) = \big[\hat{\psi}(E)\big]_{x_0}. \end{aligned}$$

If $C \in \mathfrak{B}_c(X)$, then

$$\big[\hat{\psi}(C \times Y)\big]_{x_0} = \big[\psi_1(C \times Y)\big]_{x_0} \cup \big(W \cap \big[\psi_1\big(H \cup (C \times Y)\big)\big]_{x_0}\big)$$

and

$$\big[\hat{\psi}\big(C^c \times Y\big)\big]_{x_0} = \big[\psi_1\big(C^c \times Y\big)\big]_{x_0} \cup \big(W \cap \big[\psi_1\big(H \cup \big(C^c \times Y\big)\big)\big]_{x_0}\big).$$

If $[\psi_1(C \times Y)]_{x_0} = Y$, then $[\hat{\psi}(C \times Y)]_{x_0} = Y$ either. If $[\psi_1(C \times Y)]_{x_0} = \emptyset$, then, according to Claim 3, $[\psi_1(C^c \times Y)]_{x_0} = Y$ and so $[\hat{\psi}(C^c \times Y)]_{x_0} = Y$. Consequently, $[\hat{\psi}(C \times Y)]_{x_0} = \emptyset$. This completes the proof of the claim and shows that ψ_1 satisfies (ii). □

Since for each $E \in \mathfrak{B}_c(X \times Y)$ we have $\xi_1(E) \subseteq \psi_1(E)$ and since according to Proposition 4.1 the density ξ_1 is strong, provided δ and υ are strong, it follows that ψ_1 satisfies (v).

If $w(Y) < \mathrm{add}(\mathcal{M}(X))$ and $\upsilon = \varphi_Y$, then we can consider the set $\overline{\Phi}$ to be the same with Φ but with $\bar{\varphi}_{X \times Y}$ instead of ξ_1. Notice that $\overline{\Phi} \neq \emptyset$, since according to Corollary 3.2 we have $\bar{\varphi}_{X \times Y} \in \overline{\Phi}$. It follows in the same way as above that there exists a density $\bar{\psi}_1 \in \vartheta_s((\mathcal{M}(X \times Y))$ satisfying conditions (j)–(v) with $\bar{\psi}_1$ and $\bar{\varphi}_{X \times Y}$ instead of ψ_1 and ξ_1, respectively.

In order to prove the Baire property of the Y-sections of $\bar{\psi}_1$ notice, that since $\bar{\varphi}_{X \times Y}$ and $\bar{\psi}_1$ are densities in the same space, the equality $\bar{\varphi}_{X \times Y}(E) = \bar{\psi}_1(E)$ a.e. $(\mathcal{M}(X \times Y))$ holds true. It follows then from the Kuratowski–Ulam Theorem that there is $M_E \in \mathcal{M}(X)$ such that for all $x \notin M_E$

$$\big[\bar{\varphi}_{X\times Y}(E)\big]_x \bigtriangleup \big[\bar{\psi}_1(E)\big]_x \in \mathcal{M}(Y) \quad \text{and}$$
$$\big[\bar{\varphi}_{X\times Y}(E)\big]_x \cup \big[\bar{\varphi}_{X\times Y}\big(E^c\big)\big]_x = Y \quad \text{a.e. } (\mathcal{M}(Y)).$$

If $x \notin M_E$, then

$$\big[\bar{\psi}_1(E)\big]_x = \varphi_Y\big(\big[\bar{\psi}_1(E)\big]_x\big) = \varphi_Y\big(\big[\bar{\varphi}_{X\times Y}(E)\big]_x\big) = \big[\bar{\varphi}_{X\times Y}(E)\big]_x.$$

Hence,

$$\bar{\varphi}_{X\times Y}(E) \setminus (M_E \times Y) = \bar{\psi}_1(E) \setminus (M_E \times Y).$$

Since all sections $[\bar{\varphi}_{X\times Y}(E)]^y$ have the Baire property, the same holds true for the sections $[\bar{\psi}_1(E)]^y$. This completes the proof of the whole lemma. □

We are going to formulate now two suggesting themselves questions.

Question 4.5. Let $X \times Y$ be a Baire space. Given densities $\delta \in \vartheta(\mathcal{M}(X))$ and $\upsilon \in \vartheta(\mathcal{M}(Y))$, does there exist a density $\zeta \in \vartheta(\mathcal{M}(X \times Y)) \cap (\delta \otimes \upsilon)$ satisfying the properties

(i) $[\zeta(E)]_x \in \mathfrak{B}_c(Y)$ and $[\zeta(E)]^y \in \mathfrak{B}_c(Y)$ for all $E \in \mathfrak{B}_c(X \times Y)$, $(x, y) \in X \times Y$;
(ii) $\forall E \in \mathfrak{B}_c(X \times Y)\ \exists N_E \in \mathcal{M}(X)\ [\zeta(E)]_x = \upsilon([\zeta(E)]_x)\ \forall x \notin N_E$?

Proposition 3.1 proves that if Y has a countable basis for its topology and if δ and υ are the canonical densities, then the answer is affirmative. It will follow from Theorem 6.8 that if ζ and υ are liftings, then in general the answer is negative.

It follows from [26], Corollary 6, that in case of Polish spaces X and Y such that both Boolean algebras $\mathfrak{B}_c(X)$ and $\mathfrak{B}_c(Y)$ are non-atomic there are no densities $\delta \in \vartheta(\mathcal{M}(X))$, $\upsilon \in \vartheta(\mathcal{M}(Y))$ and $\xi \in \vartheta(\mathcal{M}(X \times Y))$ satisfying the following conditions:

(i) $[\xi(E)]_x = \upsilon([\xi(E)]_x)$ for each $E \in \mathfrak{B}_c(X \times Y)$ and $x \in X$;
(ii) $[\xi(E)]^y = \delta([\xi(E)]^y)$ for each $E \in \mathfrak{B}_c(X \times Y)$ and $y \in Y$.

But the following question remains open.

Question 4.6. Let $X \times Y$ be a Baire space and let $\delta \in \vartheta(\mathcal{M}(X))$ and $\upsilon \in \vartheta(\mathcal{M}(Y))$ be arbitrary. Does there exist $\xi \in \vartheta(\mathcal{M}(X \times Y)) \cap (\delta \otimes \upsilon)$ such that for each $E \in \mathfrak{B}_c(X \times Y)$ there exist $N_E \in \mathcal{M}(X)$ and $M_E \in \mathcal{M}(Y)$ with $[\xi(E)]_x = \upsilon([\xi(E)]_x)$ for each $x \notin N_E$ and $[\xi(E)]^y = \delta([\xi(E)]^y)$ for each $y \notin M_E$?

Proposition 3.1 proves that if X and Y have countable bases for their topologies and if δ and υ are the canonical densities, then the answer is affirmative.

The following remarks show that the category densities behave better than the measure-theoretic ones under formation of products.

Remark 4.7. (a) In general a result analogous to Proposition 4.1 is false for measure-theoretic densities. More precisely, given non-atomic complete probability spaces (X, Σ, μ)

and (Θ, T, ν), and densities $\delta \in \vartheta(\mu)$ and $\upsilon \in \vartheta(\nu)$, the map $\xi : \Sigma \widehat{\otimes} T \to \mathcal{P}(X \times \Theta)$ defined by

$$\xi(E) := \bigcup \bigl\{\delta(A) \times \upsilon(B)\colon\ A \times B \subseteq E \text{ a.e. } (\mu \widehat{\otimes} \nu)\bigr\}$$

cannot be a density for $\mu \widehat{\otimes} \nu$.

In fact, suppose that $\xi \in \vartheta(\mu \widehat{\otimes} \nu)$. If $E \in \Sigma \widehat{\otimes} T$ then $E = \xi(E)$ a.e. $(\mu \widehat{\otimes} \nu)$, hence for each $E \in (\Sigma \widehat{\otimes} T)_+$ there exist $A \in \Sigma_+$ and $B \in T_+$ such that $A \times B \subseteq E$ a.e. $(\mu \widehat{\otimes} \nu)$, a contradiction to a well-known result of Erdős and Oxtoby [5] saying that there exists $E \in (\Sigma \widehat{\otimes} T)_+$ such that there are no $A \in \Sigma_+$ and $B \in T_+$ satisfying condition $A \times B \subseteq E$ a.e. $(\mu \widehat{\otimes} \nu)$.

(b) Given non-atomic complete probability spaces (X, Σ, μ) and (Θ, T, ν), and arbitrary densities $\delta \in \vartheta(\mu)$, $\upsilon \in \vartheta(\nu)$ and $\xi \in \vartheta(\mu \widehat{\otimes} \nu)$, condition (ii) from Proposition 4.1 cannot be true.

In fact, assume if possible that condition (ii) holds true. It then follows that for each $E \in (\Sigma \widehat{\otimes} T)_+$ there exists a family $\langle A_i \times B_i \rangle_{i \in I}$ of measurable rectangles of positive measure such that $\xi(E) = \bigcup_{i \in I} [\delta(A_i) \times \upsilon(B_i)]$, hence there exist $A_{i_0} \in \Sigma_+$ and $B_{i_0} \in T_+$ such that $A_{i_0} \times B_{i_0} \subseteq E$ a.e. $(\mu \widehat{\otimes} \nu)$, which again contradicts [5].

5. Existence of liftings in products of two spaces, with sections possessing the Baire property

As we have proven in previous sections when densities are under consideration, then there exist always product densities with nice measurability properties. In Proposition 3.1 and in Proposition 4.1 the existence of a product density with measurable sections satisfying an inclusion has been proven. There is now a question whether similar results in case of liftings can be achieved. We solve this problem in the next theorem.

Theorem 5.1. *Assume that $X \times Y$ is a Baire space. Then given arbitrary liftings $\rho \in \Lambda(\mathcal{M}(X))$ and $\sigma \in \Lambda(\mathcal{M}(Y))$, there exists a lifting $\pi_1 \in \Lambda(\mathcal{M}(X \times Y))$ such that*

(i) $\pi_1 \in \rho \otimes \sigma$;
(ii) $[\pi_1(E)]_x = \sigma([\pi_1(E)]_x)$ *for all* $E \in \mathfrak{B}_c(X \times Y)$ *and all* $x \in X$;
(iii) *if ρ and σ are strong, then π_1 is strong.*

Proof. Applying Lemma 4.4 with $\delta = \rho$ and $\upsilon = \sigma$, we obtain a density $\pi_1 \in \vartheta(\mathcal{M}(X \times Y))$ such that

$$\rho(A) \times \sigma(B) \subseteq \pi_1(A \times B) \quad \text{for all } A \in \mathfrak{B}_c(X) \text{ and } B \in \mathfrak{B}_c(Y), \tag{7}$$

$$\bigl[\pi_1(E)\bigr]_x \cup \bigl[\pi_1\bigl(E^c\bigr)\bigr]_x = Y \quad \text{a.e. } (\mathcal{M}(Y)) \quad \text{for all } x \in X \text{ and } E \in \mathfrak{B}_c(X \times Y), \tag{8}$$

and

$$\bigl[\pi_1(E)\bigr]_x = \sigma\bigl(\bigl[\pi_1(E)\bigr]_x\bigr) \quad \text{for all } x \in X \text{ and } E \in \mathfrak{B}_c(X \times Y). \tag{9}$$

Standard calculation proves that $\pi_1(A \times B) = \rho(A) \times \sigma(B)$, whenever $A \in \mathfrak{B}_c(X)$ and $B \in \mathfrak{B}_c(Y)$. Consequently, we get condition (i) of the theorem.

We are going to prove now that π_1 is a lifting. To do it notice that as a consequence of (8) we get for each x the equality $[\pi_1(E^c)]_x = ([\pi_1(E)]_x)^c$ a.e. $(\mathcal{M}(Y))$. Hence

$$\sigma\big(\big[\pi_1\big(E^c\big)\big]_x\big) = \sigma\big[\big(\big[\pi_1(E)\big]_x\big)^c\big]. \tag{10}$$

Taking into account (9), (10) and the lifting properties of σ we see that

$$\big[\pi_1\big(E^c\big)\big]_x = \big(\big[\pi_1(E)\big]^c\big)_x.$$

This implies $\pi_1(E^c) = [\pi_1(E)]^c$ and so $\pi_1 \in \Lambda(\mathcal{M}(X \times Y))$.

Condition (iii) follows from (i) in the same way as in the proof of condition (vi) in Proposition 4.1. □

Remark 5.2. A result analogous to that from Theorem 5.1 fails for measure-theoretic strong liftings (see [17, Section 3, Remark 5]). The best possible result for measure-theoretic strong liftings is the following theorem from [16]:

Given complete topological probability spaces $(X, \mathcal{T}, \Sigma, \mu)$ and $(\Theta, \mathcal{S}, T, \nu)$, such that the first one admits a strong lifting ρ for μ and the second one admits a strong admissibly generated lifting σ for ν (see [16] or [20] or [25] for the definition), there exists a strong lifting π for $\mu \widehat{\otimes} \nu$ satisfying the conclusions of Theorem 5.1. The corresponding best possible result for measure-theoretic liftings can be found in [20, Theorem 2.13].

This shows again the better behavior of the category strong liftings than the measure-theoretic ones under formation of products.

6. Countably additive liftings

It is a consequence of [26] that the lifting π_1 in Theorem 5.1 cannot have, in general, all Y-sections ρ-invariant. We are going to settle in this section whether it can have all Y-sections with the property of Baire. This is related to the following question which we deal with first.

Question 6.1. Let Y be a non-empty Baire Tychonoff space without isolated points. Is it possible that there is a lifting for $(Y, \mathfrak{B}_c(Y), \mathcal{M}(Y))$ which is a σ-homomorphism?

It follows from the results below that a counterexample would have to have the property that every meager set is nowhere dense. Moreover, the cellularity of every open set would have to be at least equal to the first measurable cardinal. (Recall that a cardinal κ is measurable if $P(\kappa)$ carries a diffuse κ-additive probability measure, or equivalently, there is a κ-additive free ultrafilter on κ.) From the latter property it follows that the negative answer to the question for all spaces is consistent relative to ZFC.

Let us say that $\theta : \mathfrak{B}_c(Y) \to \mathfrak{B}_c(Y)$ is a *selector* if it chooses a representative from each class modulo $\mathcal{M}$, i.e., $\theta(E) =_{\mathcal{M}} E$ and $\theta(E) = \theta(F)$ whenever $E =_{\mathcal{M}} F$.

Proposition 6.2. *Suppose Y is a regular Baire space in which some non-empty open set has a dense meager subset. Then for any selector $\theta : \mathfrak{B}_c(Y) \to \mathfrak{B}_c(Y)$, there is a decreasing sequence $\{A_n\}$ in $\mathfrak{B}_c(Y)$ such that $\theta(\bigcap_n A_n) \neq \bigcap_n \theta(A_n)$.*

Proof. (Cf. the proof of [7, Lemma 20].) Let F_n, $n \in \mathbb{N}$, be closed nowhere dense subsets of Y such that $\bigcup_n F_n$ covers $\theta(\emptyset)$ and covers a dense subset of some non-empty open set $U \subseteq Y$. Assume we have fixed for each first category set a sequence of closed nowhere dense sets covering it and we have also fixed, for each non-empty open set V, a non-empty open set W such that $\overline{W} \subseteq V$. Consider the following strategy for player I in the Banach–Mazur game described in Section 2. Player I's first move is $U_1 = U$. Suppose both players have made n moves $U_1 \supseteq V_1 \supseteq \cdots \supseteq U_n \supseteq V_n$. Write F_{mk}, $k \in \mathbb{N}$, for the closed nowhere dense sets covering $V_m \triangle \theta(V_m)$, $m \leqslant n$, which were fixed above. Corresponding to the non-empty open set

$$V_n \setminus \biggl(\bigcup_{m \leqslant n} F_m \cup \bigcup_{m \leqslant n,\, k \leqslant n} F_{mk}\biggr),$$

there was fixed above a non-empty open subset U_{n+1} whose closure is contained inside it. This is player I's next move.

Because Y is Baire, there is a play of the game which is not winning for player I. Fix such a play of the game and consider the set $K = \bigcap_n V_n$. We have $K \neq \emptyset$ by assumption. Since $K = \bigcap_n \overline{U}_n$, K is closed. Since $K \subseteq U$ and K is disjoint from $\bigcup_n F_n$, K is nowhere dense. For each $n \in \mathbb{N}$, since K is disjoint from $\bigcup_k F_{nk}$, we have $K \cap (V_n \triangle \theta(V_n)) = \emptyset$ and hence $K \subseteq \bigcap_n \theta(V_n)$. Together with $K \cap \theta(\emptyset) = \emptyset$, the last inclusion shows that $\bigcap_n \theta(V_n) \setminus \theta(\emptyset)$ is not empty and hence $\theta(\bigcap_n V_n) = \theta(K) = \theta(\emptyset) \neq \bigcap_n \theta(V_n)$. □

Remark 6.3. In any T_1 space without isolated points, a set which is discrete in the subspace topology is nowhere dense. Hence Proposition 6.2 covers all T_1 spaces without isolated sets which have a σ-discrete dense set. In particular it covers metric spaces without isolated points. (Each metric space has a dense set $\bigcup_n D_n$, where D_n is a maximal set of points whose pairwise distances are at least $1/n$.)

If $\mathbf{B}$ is a Boolean algebra, then a set $S \subseteq \mathbf{B} \setminus \{0\}$ is a *cellular family* if $x \wedge y = 0$ for all distinct $x, y \in S$. We define the *cellularity* of $\mathbf{B}$ to be $\sup\{\operatorname{card} S\colon\ S$ is a cellular family$\}$. $\mathbf{B}{\restriction}a$ denotes the induced Boolean algebra on $\{x \in \mathbf{B}\colon\ x \leqslant a\}$.

In the proposition below $(Y, \Sigma, \mathcal{N})$ is a measurable space with a σ-ideal $\mathcal{N}$ of subsets of Y that is generated by $\mathcal{N} \cap \Sigma$. To avoid trivialities we also assume $Y \notin \mathcal{N}$. Notice that then the quotient algebra $\Sigma/\mathcal{N}$ satisfies $0 \neq 1$.

Proposition 6.4. *Let $\mathbf{A} = \Sigma/\mathcal{N}$. If some non-zero $a \in \mathbf{A}$ satisfies that $\mathbf{A}{\restriction}a$ is complete and non-atomic, and the cellularity of $\mathbf{A}{\restriction}a$ does not carry any countably complete free ultrafilter, then no lifting for $(Y, \Sigma, \mathcal{N})$ is a σ-homomorphism.*

Proof. Let $a \in \mathbf{A}$ be as in the hypothesis. Let $\theta : \mathbf{A} \to \Sigma$ be a lifting. Fix any point $p \in \theta(a)$. Since $\mathbf{A}{\restriction}a$ is non-atomic, for any non-zero $b \leqslant a$, there are two disjoint non-zero members of $\mathbf{A}$ which are $\leqslant b$. Hence, there is a non-zero $b' \leqslant b$ such that $p \notin \theta(b')$. Thus, there is a cellular family S in $\mathbf{A}{\restriction}a$ such that $p \notin \theta(b)$ for each $b \in S$ and $\bigvee S = a$. Let

$$F = \Bigl\{S' \subseteq S\colon\ p \in \theta\Bigl(\bigvee S'\Bigr)\Bigr\}.$$

F is a free ultrafilter on S. By assumption, F is not countably complete. Hence, we may write $S = \bigcup_n S'_n$ so that $p \notin \theta(\bigvee S'_n)$ for each $n \in \mathbb{N}$. Let $a_n = \bigvee S'_n$. Then $p \in \theta(a) = \theta(\bigvee_n a_n)$ whereas $p \notin \bigcup_n \theta(a_n)$. $\square$

Remark 6.5. Let Y be a non-void Baire Hausdorff space without isolated points. The structure $(Y, \mathfrak{B}_c(Y), \mathcal{M}(Y))$ satisfies the assumption if some non-empty open set $U \subseteq Y$ has cellularity below the first measurable cardinal.

In the sequel we denote by $P(\mathbb{N})$ the space of all subsets of $\mathbb{N}$ endowed with the ordinary product metric topology.

Proposition 6.6. *Let Y be a Baire space and let $\mathcal{U}$ be a non-countably-complete ultrafilter on $\mathfrak{B}_c(Y)$ extending the filter of dense open sets. There is then a set $E \subseteq P(\mathbb{N}) \times Y$ which is a union of countably many open rectangles such that $\{x \in P(\mathbb{N})\colon\ E_x \in \mathcal{U}\}$ is a free ultrafilter.*

Proof. Fix a pairwise disjoint family $\{A_n\colon\ n \in \mathbb{N}\}$ of open sets in Y such that $A_n \notin \mathcal{U}$ for each $n \in \mathbb{N}$ and $\bigcup_n A_n$ is dense in Y.

$$E = \bigcup_{n \in \mathbb{N}} \{x \subseteq \mathbb{N}\colon\ n \in x\} \times A_n$$

is as desired. $\square$

Proposition 6.7. *Let Y be a Baire space such that $(P(\mathbb{N}), Y)$ is a Kuratowski–Ulam pair. For any lifting $\theta \in \Lambda(\mathcal{M}(Y))$ which is not a σ-homomorphism, there is a point $y \in Y$ and there is a set $E \subseteq P(\mathbb{N}) \times Y$ which is the union of countably many open rectangles and is such that for no representative S of the category class of E do we have that S^y has the property of Baire and $\theta(S_x) = S_x$ for a residual set of $x \in P(\mathbb{N})$.*

Proof. Note that if $\theta(\bigcup_n A_n) \neq \bigcup_n \theta(A_n)$ for some sets $A_n \in \mathfrak{B}_c(Y)$, then for any $p \in \theta(\bigcup_n A_n) \setminus \bigcup_n \theta(A_n)$, the collection $\mathcal{U} := \{A \in \mathfrak{B}_c(Y)\colon\ p \in \theta(A)\}$ is a non-countably-complete ultrafilter on $\mathfrak{B}_c(Y)$. If E and A_n's are taken from Proposition 6.6, then

$$\bigl\{x \in P(\mathbb{N})\colon\ E_x \in \mathcal{U}\bigr\} = \Bigl\{x \in P(\mathbb{N})\colon\ \bigcup_{n \in x} A_n \in \mathcal{U}\Bigr\}$$

is a free ultrafilter. Suppose S is in the category class of E and $\theta(S_x) = S_x$ for a residual set of $x \in P(\mathbb{N})$. The Kuratowski–Ulam Theorem ensures that $E_x = S_x$ a.e. $(\mathcal{M}(Y))$, for a residual set of $x \in P(\mathbb{N})$. Hence, $W := \{x \in P(\mathbb{N})\colon\ \theta(E_x) = \theta(S_x) = S_x\}$ is residual. If $x \in W$, then $x \in S^y$ means that $y \in S_x = \theta(E_x)$, i.e., $E_x \in \mathcal{U}$. By Proposition 6.6 $\{x \in P(\mathbb{N})\colon\ E_x \in \mathcal{U}\}$ is a free ultrafilter. Since $\{x \in P(\mathbb{N})\colon\ E_x \in \mathcal{U}\} \supseteq W$, it possesses also the property of Baire. On the other hand according to a well-known result of Sierpinski (cf. [1, Theorem 4.1.1]), the set $\{x \in P(\mathbb{N})\colon\ E_x \in \mathcal{U}\}$ does not have the property of Baire. $\square$

Applying Proposition 6.7 we obtain the main non-existence result of this paper.

Theorem 6.8. *Let Y be a separable metric space without isolated points. Then there exist no lifting $\sigma \in \Lambda(\mathcal{M}(Y))$ and density $\varphi \in \vartheta(\mathcal{M}(P(\mathbb{N}) \times Y))$ satisfying the following two conditions*:

(j) *there exists $\bar{y} \in Y$ such that for each $E \in \mathfrak{B}_c(P(\mathbb{N}) \times Y)$*

$$\bigl[\varphi(E)\bigr]^{\bar{y}} \in \mathfrak{B}_c\bigl(P(\mathbb{N})\bigr).$$

(jj) *for each $E \in \mathfrak{B}_c(P(\mathbb{N}) \times Y))$ there exists a set $N_E \in \mathcal{M}(P(\mathbb{N}))$ such that*

$$\bigl[\varphi(E)\bigr]_x = \sigma\bigl(\bigl[\varphi(E)\bigr]_x\bigr) \quad \textit{for each } x \notin N_E.$$

Corollary 6.9. *Let Y be a separable metric space without isolated points. If ρ, σ and π_1 are liftings satisfying Theorem* 5.1 (*with $X = P(\mathbb{N})$*), *then for each $y \in Y$ there exists $E \in \mathfrak{B}_c(P(\mathbb{N}) \times Y)$ such that $[\pi_1(E)]^y \notin \mathfrak{B}_c(P(\mathbb{N}))$.*

It follows from the above corollary, that Theorem 5.1 cannot be in general improved.

Corollary 6.10. *Let Y be a separable metric space without isolated points and let $\rho \in \Lambda(\mathcal{M}(P(\mathbb{N})))$, $\sigma \in \Lambda(\mathcal{M}(Y))$ and $\pi_2 \in \Lambda(\mathcal{M}(P(\mathbb{N}) \times Y))$ be such that*

(k) $\pi_2 \in \rho \otimes \sigma$;
(kk) $[\pi_2(E)]^y = \rho([\pi_2(E)]^y)$ *for all $E \in \mathfrak{B}_c(P(\mathbb{N}) \times Y)$ and all $y \in Y$.*

Then, there exists $E \in \mathfrak{B}_c(P(\mathbb{N}) \times Y)$ such that

$$\bigl\{x \in P(\mathbb{N}) \colon \bigl[\pi_2(E)\bigr]_x \neq \rho\bigl(\bigl[\pi_2(E)\bigr]_x\bigr)\bigr\} \notin \mathcal{M}\bigl(P(\mathbb{N})\bigr).$$

Proof. According to Theorem 5.1 liftings ρ, σ and π_2 satisfying (k) and (kk) exist. If we assume that for each $E \in \mathfrak{B}_c(P(\mathbb{N}) \times Y)$ we have $[\pi_2(E)]_x = \rho([\pi_2(E)]_x)$ for almost all $x \in P(\mathbb{N})$, then we get a contradiction with Theorem 6.8. □

In the context of the preceding results it is natural to ask the following two questions:

Question 6.11. Let $X \times Y$ be a Baire space. Assume also, if necessary, that (X, Y) and (Y, X) are K–U pairs. Do there exist (strong) liftings $\rho \in \Lambda(\mathcal{M}(X))$, $\sigma \in \Lambda(\mathcal{M}(Y))$ and $\pi \in \Lambda(\mathcal{M}(X \times Y)) \cap (\rho \otimes \sigma)$ such that for each $E \in \mathfrak{B}_c(X \times Y)$ there exist sets $N_E \in \mathcal{M}(X)$ and $M_E \in \mathcal{M}(Y)$ with the property that whenever $x \notin N_E$ and $y \notin M_E$ then

$$\bigl[\pi(E)\bigr]_x = \sigma\bigl(\bigl[\pi(E)\bigr]_x\bigr) \quad \text{and} \quad \bigl[\pi(E)\bigr]^y = \rho\bigl(\bigl[\pi(E)\bigr]^y\bigr)?$$

Question 6.12. Let $X \times Y$ be a Baire space. Assume also, if necessary, that (X, Y) and (Y, X) are K–U pairs. Do there exist (strong) liftings $\rho \in \Lambda(\mathcal{M}(X))$, $\sigma \in \Lambda(\mathcal{M}(Y))$ and $\pi \in \Lambda(\mathcal{M}(X \times Y)) \cap (\rho \otimes \sigma)$ such that for each $E \in \mathfrak{B}_c(X \times Y)$ and for each $(x, y) \in X \times Y$ we have $[\pi(E)]_x \in \mathfrak{B}_c(Y)$ and $[\pi(E)]^y \in \mathfrak{B}_c(X)$?

7. Densities in arbitrary products

In this section we are going to present a generalization of Proposition 4.1 to the case of arbitrary Baire products of topological spaces. We start with an easy generalization of Lemma 2.2 to the case of arbitrary products.

Lemma 7.1. *Let $\{X_i\colon i \in I\}$ be a non-empty collection of topological spaces.*

(e) *If $C_i \subseteq_{\mathcal{M}} A_i$ in X_i, $i \in I$, and $C_i = A_i = X_i$ for all $i \in I \setminus F$ where $F \subseteq I$ is at most countable, then $\prod_{i\in I} C_i \subseteq_{\mathcal{M}} \prod_{i\in I} A_i$.*

(f) *If $X_I = \prod_{i\in I} X_i$ is Baire, then the product $\prod_{i\in I} C_i$ of non-meager sets $C_i \subseteq X_i$ having the Baire property and satisfying $C_i = X_i$ for all $i \in I \setminus F$ for some finite $F \subseteq I$, is non-meager.*

(g) *If X_I is Baire, and $\prod_{i\in I} C_i \subseteq_{\mathcal{M}} \prod_{i\in I} A_i$, where $C_i, A_i \subseteq X_i$ are sets having the property of Baire, $C_i = A_i = X_i$ for all $i \in I \setminus F$ for some finite F and the sets C_i are not meager, then for each $i \in I$ we have $C_i \subseteq_{\mathcal{M}} A_i$.*

Proof. (e) $\prod_{i\in I} C_i \setminus \prod_{i\in I} A_i \subseteq \bigcup_{i\in F}(C_i \setminus A_i) \times \prod_{j\in I\setminus\{i\}} X_j$ is meager.

(f) For $i \in F$, let $U_i \subseteq X_i$ be an open set such that $C_i \triangle U_i$ is meager. Set $U_i = X_i$ for $i \in I \setminus F$ and define the basic open set $U = \prod_{i\in I} U_i$. By (f), the set

$$M = \bigcup_{i\in F}(U_i \setminus C_i) \times \prod_{j\in I\setminus\{i\}} X_j$$

is meager. Then, because the sets C_i are non-meager, the sets U_i are not empty. Since X_I is a Baire space, U is non-meager. Thus, $U \setminus M$ is non-meager and the conclusion follows from

$$U \setminus M \subseteq \prod_{i\in I} C_i.$$

(g) If $C_i \setminus A_i \notin \mathcal{M}$, then, by (f), $(C_i \setminus A_i) \times \prod_{j\in I\setminus\{i\}} C_j \notin \mathcal{M}$, contradicting the inclusion $\prod_{i\in I} C_i \subseteq_{\mathcal{M}} \prod_{i\in I} A_i$. □

Theorem 7.2. *Let $\{X_i\colon i \in I\}$ be a non-empty collection of topological spaces such that X_I is a Baire space. Moreover, let $\{\delta_i \in \vartheta(\mathcal{M}(X_i))\colon i \in I\}$ be a collection of densities. For each $\emptyset \neq J \subseteq I$ and $E \in \mathfrak{B}_c(X_J)$ put*

$$\xi_J(E) = \bigcup \left\{ \prod_{i\in K} \delta_i(A_i) \times X_{J\setminus K}\colon \prod_{i\in K} A_i \times X_{J\setminus K} \subseteq E \right.$$
$$\left. \text{a.e. } \big(\mathcal{M}(X_J)\big),\, K \in \mathrm{Fin}(J) \right\},$$

where Fin(J) *denotes the collection of all non-empty finite subsets of $J \subseteq I$. Then for each non-empty subset J of I ξ_J is a density in $\vartheta(\mathcal{M}(X_J))$ satisfying the following conditions*:

(i) *ξ_J respects coordinates*;

(ii) $\xi_J \in \xi_K \otimes \xi_{J\setminus K}$ *if* K *is a non-empty proper subset of* J;
(iii) $\xi_J \in \bigotimes_{i\in J} \delta_i$;
(iv) ξ_J *is separately Baire additive*;
(v) $[\xi_J(E)]_{x_K} \subseteq \xi_{J\setminus K}([\xi_J(E)]_{x_K})$ *for each* $E \in \mathfrak{B}_c(X_J)$, *if* K *is a non-empty proper subset of* J;
(vi) *if for each* $i \in I$ *the density* δ_i *is strong, then* ξ_J *is also strong.*

Proof. Let us fix a non-empty subset J of I. Exactly as in Proposition 4.1 one can prove that $\xi_J \in \vartheta(\mathcal{M}(X_J))$. To show condition (ii), we are going to prove first the following fact:

Claim. *Let* $J = K \cup L$ *be a proper decomposition of* J *and let* $\phi(\xi_K, \xi_L)$ *be the density from Proposition* 4.1, *when* δ *and* υ *are replaced by* ξ_K *and* ξ_L, *respectively. Then* $\xi_J = \phi(\xi_K, \xi_L)$.

Proof. Without any comments we are going to apply below Lemma 7.1(e), (g). We assume also, where necessary, that sets have the Baire property in the corresponding spaces.

$$\begin{aligned}
\xi_J(E) &= \bigcup\Biggl\{\prod_{i\in M}\delta_i(A_i)\times X_{J\setminus M}\colon \prod_{i\in M}A_i\times X_{J\setminus M}\subseteq_{\mathcal{M}} E,\ M\in \operatorname{Fin} J\Biggr\}\\
&= \bigcup\Biggl\{\prod_{i\in M\cap K}\delta_i(A_i)\times X_{K\setminus M}\times \prod_{i\in M\cap L}\delta_i(A_i)\times X_{L\setminus M}\colon\\
&\qquad \prod_{i\in M}A_i\times X_{J\setminus M}\subseteq_{\mathcal{M}} E,\ M\in \operatorname{Fin} J\Biggr\}\\
&\subseteq \bigcup\bigl\{\xi_K(A)\times\xi_L(B)\colon\ A\times B\subseteq_{\mathcal{M}} E\bigr\} = \phi(\xi_K,\xi_L)(E)\\
&= \bigcup\Biggl\{\Biggl(\bigcup\prod_{i\in P}\delta(A_i)\times X_{K\setminus P}\Biggr)\times\Biggl(\bigcup\prod_{j\in Q}\delta(B_j)\times X_{L\setminus Q}\Biggr)\colon\ A\times B\subseteq_{\mathcal{M}} E,\\
&\qquad \prod_{i\in P}\delta(A_i)\times X_{K\setminus P}\subseteq_{\mathcal{M}} A\ \&\ \prod_{j\in Q}\delta(B_j)\times X_{L\setminus Q}\subseteq_{\mathcal{M}} B,\\
&\qquad P\in\operatorname{Fin} K,\ Q\in \operatorname{Fin} L\Biggr\}\\
&\subseteq \bigcup\Biggl\{\bigcup\prod_{i\in P}\delta(A_i)\times\prod_{j\in Q}\delta(B_j)\times X_{J\setminus P\cup Q}\colon\\
&\qquad \prod_{i\in P}\delta(A_i)\times\prod_{j\in Q}\delta(B_j)\times X_{J\setminus P\cup Q}\subseteq_{\mathcal{M}} E, P\in\operatorname{Fin} K,\ Q\in\operatorname{Fin} L\Biggr\}\\
&= \xi_J(E). \qquad \square
\end{aligned}$$

Condition (ii) follows now from Proposition 4.1(i). Condition (i) is equivalent to (ii). Conditions (iv), (v) and (vi) follow exactly in the same way as in Proposition 4.1.

Condition (iii) follows directly from (ii).

This completes the proof of the theorem. $\square$

Notice that in general the density ξ_J above is not a lifting, even if all δ_i's are liftings. For example, if X and Y are infinite extremally disconnected compact spaces, then the canonical densities φ_X and φ_Y are liftings. However the formula for ξ_J in this case produces the canonical density $\varphi_{X\times Y}$ (see Proposition 3.1) which is not a lifting since $X \times Y$ is not extremally disconnected [4, Exercise 6.3.21].

We finish with the following open problem:

Question 7.3. Let $\{X_i\colon i \in I\}$ be an infinite collection of topological spaces such that X_I is Baire. Does there exist a lifting $\pi \in \Lambda_s(\mathcal{M}(X_I))$ respecting coordinates?

8. A lifting respecting coordinates in a weak sense

Besides liftings respecting coordinates one can consider also the following two other similar properties of a lifting $\theta \in \Lambda(\mathcal{M}(X \times Y))$:

(WRC$_1$) For every $A \in \mathfrak{B}_c(X)$ and $B \in \mathfrak{B}_c(Y)$, there are sets $C \subseteq X$ and $D \subseteq Y$ such that $\theta(A \times B) = C \times D \in \mathfrak{B}_c(X \times Y)$.

(WRC$_2$) For every $A \times B \in \mathfrak{B}_c(X \times Y)$, there are sets $C \subseteq X$ and $D \subseteq Y$ such that $\theta(A \times B) = C \times D \in \mathfrak{B}_c(X \times Y)$.

If we write (RC) for the property of respecting coordinates, then clearly (WRC$_1$) is a consequence of either (RC) or (WRC$_2$).

We give an example to show that a lifting for the category algebra of a Baire product $X \times Y$ can satisfy (WRC$_1$) without respecting coordinates. Other than the obvious implication mentioned above, the relationship of (WRC$_2$) to the other two properties is not clear to us. However, as we have already mentioned in the introduction, under some assumptions concerning coordinate spaces, the situation is simpler.

Proposition 8.1. *Let X and Y be arbitrary Baire spaces. If (X, Y) or (Y, X) satisfy the Kuratowski–Ulam property or if X and Y are weakly α-favorable, then all three properties coincide.*

Proof. In case of K–U the conclusion is easily seen, so we will present only the proof in case of weakly α-favorable spaces. It is enough to show that if $A \notin \mathfrak{B}_c(X)$, then $A \times Y \notin \mathfrak{B}_c(X \times Y)$. To do it notice first that $A \subset X$ is without the property of Baire if and only if there is a non-empty open set U such that both $A \cap U$ and $U \setminus A$ are everywhere second category in U (i.e. have second category intersection with every non-void open subset of U). Let us fix $A \notin \mathfrak{B}_c(X)$ and the corresponding U. In particular, $A \cap U$ and $U \setminus A$ are both dense in U.

We need in what follows the Oxtoby observation that when Z is a dense subspace of X and $C \subset Z$, then C is meager in Z if and only if C is meager in X. Taking $C = V \cap Z$ for non-empty open sets $V \subset X$ gives in particular that Z is a Baire space if and only if Z is everywhere second category in X.

We have that $A \cap U$ and $U \setminus A$ are thus Baire spaces. Because Y is weakly-α-favorable, $(A \cap U) \times Y$ and $(U \setminus A) \times Y$ are both Baire spaces. But these are dense in $U \times Y$, so they are everywhere second category in $U \times Y$. Thus, $A \times Y$ does not have the property of Baire because both it and its complement are everywhere second category in $U \times Y$. □

Example 8.2. Let X and Y be Baire normed spaces such that $X \times Y$ is not Baire (and hence is meager). (See [29] or [19].) Let $\widetilde{X}$ be the completion of X, so that $\widetilde{X}$ is a Banach space. Since $\widetilde{X}$ is weakly α-favorable, $\widetilde{X} \times Y$ is Baire (see [30]). Notice that $X \times Y$ is meager also as a subset of $\widetilde{X} \times Y$.

The space X is everywhere second category in itself and hence also in $\widetilde{X}$ in which it is dense. It does not have the property of Baire in any open set U of $\widetilde{X}$ because otherwise it would follow from Lemma 2.2(f) that $(U \cap X) \times Y$ is second category in $\widetilde{X} \times Y$.

Let θ be a translation invariant lifting for the category algebra of $\widetilde{X} \times Y$. For example, start with the canonical density $\varphi_{\widetilde{X} \times Y}$, which is translation-invariant since translations are homeomorphisms. If $\mathcal{U}$ is any ultrafilter on $\mathfrak{B}_c(\widetilde{X} \times Y)$ containing all neighborhoods of 0 and all residual sets, then setting $\theta(E) = \{u \in \widetilde{X} \times Y\colon\ E - u \in \mathcal{U}\}$ works. We have that θ is a strong lifting and respects coordinates. (Respecting coordinates follows easily from the fact that θ dominates the separately additive $\varphi_{\widetilde{X} \times Y}$ and is translation-invariant, since the former is the same as being invariant under translations of the form $u \mapsto u + (x, 0)$ and $u \mapsto u + (0, y)$). Let $\theta_{\widetilde{X}}$ and θ_Y be the marginal liftings induced by θ. We define a Boolean homomorphism τ on the category algebra of $\widetilde{X}$ into $P(\widetilde{X})$ as follows.

$$\tau(A) = \begin{cases} \theta_{\widetilde{X}}(A) \setminus X & \text{if } 0 \notin \theta_{\widetilde{X}}(A), \\ \theta_{\widetilde{X}}(A) \cup X & \text{if } 0 \in \theta_{\widetilde{X}}(A). \end{cases}$$

Then $\tau(A)$ fails to have the property of Baire whenever A is not meager or residual. In particular $\tau \notin \vartheta(\mathcal{M}(\widetilde{X}))$.

Letting $\mathcal{A}$ denote $\mathfrak{B}_c(\widetilde{X} \times Y)$, write $\mathcal{A} = \bigcup_{\xi<\kappa} \mathcal{A}_\xi$, where $\mathcal{A}_0$ is the algebra generated by the rectangles $A \times B$ ($A \in \mathfrak{B}_c(\widetilde{X})$, $B \in \mathfrak{B}_c(Y)$), for each ordinal $\xi < \kappa$, $\mathcal{A}_{\xi+1}$ is generated over $\mathcal{A}_\xi$ by adding a single element A_ξ, and $\mathcal{A}_\xi = \bigcup_{\eta<\xi} \mathcal{A}_\eta$ when ξ is a limit ordinal. Inductively define a lifting σ for $\mathcal{A}$ by taking $\sigma \restriction \mathcal{A}_0$ to be the unique Boolean homomorphism of the product algebra $\mathcal{A}_0$ into $\mathfrak{B}_c(\widetilde{X} \times Y)$ satisfying

$$\sigma(A \times B) = \tau(A) \times \theta_Y(B).$$

Then at a successor stage, let $\sigma \restriction \mathcal{A}_{\xi+1}$ be the unique extension of $\sigma \restriction \mathcal{A}_\xi$ satisfying

$$\sigma(A_\xi) = \theta(A_\xi) \cup \bigcup \bigl\{\sigma(A)\colon\ A \in \mathcal{A}_\xi,\ A \subseteq_{\mathcal{M}} A_\xi\bigr\} \\ \setminus \bigcup \bigl\{\sigma(A)\colon\ A \in \mathcal{A}_\xi,\ A \cap A_\xi \in \mathcal{M}(\widetilde{X} \times Y)\bigr\}.$$

That σ is a lifting follows by checking by induction on ξ that for each $A \in \mathcal{A}_\xi$, we have $\sigma(A) \mathbin{\triangle} \theta(A) \subseteq X \times Y$ and hence $\sigma(A) =_{\mathcal{M}} \theta(A)$. Then σ is as desired since it respects coordinates in the weaker sense. θ_Y is a marginal lifting of σ. The second marginal lifting does not exist.

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