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Various products of category densities and liftings

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ABSTRACT

We extend earlier work [M.R. Burke, N.D. Macheras, K. Musiał, W. Strauss, Category product densities and liftings, Topology Appl. 153 (2006) 1164–1191] of the authors on the existence of category liftings in the product of two topological spaces X and Y such that $X \times Y$ is a Baire space. For given densities ρ, σ on X and Y, respectively, we introduce two 'Fubini type' products $\rho \odot \sigma$ and $\rho \boxdot \sigma$ on $X \times Y$. We present a necessary and sufficient condition for $\rho \odot \sigma$ to be a density. Provided (X, Y) and (Y, X) have the Kuratowski–Ulam property, we prove for given category liftings ρ, σ on the factors the existence of a category lifting π on the product, dominating the density $\rho \boxdot \sigma$ and such that

$$\pi(A \times B) = \rho(A) \times \sigma(B) \quad \text{for Baire subsets } A \text{ of } X \text{ and } B \text{ of } Y, \quad \text{and}$$

$$\rho\big([\pi(E)]^y\big) = [\pi(E)]^y \quad \text{for all } y \in Y \text{ and Baire subsets } E \text{ of } X \times Y.$$

We show that further properties of consistency with the product structure cannot be expected.
We prove also that contrary to measure theoretical liftings, in case of Baire spaces there might exist countably additive liftings. This answers, assuming the existence of a compact cardinal, a question from [M.R. Burke, N.D. Macheras, K. Musiał, W. Strauss, Category product densities and liftings, Topology Appl. 153 (2006) 1164–1191]. The example we present is a version of an example of D.H. Fremlin of a space whose category algebra has a countably additive lifting.

0. Introduction

In [11] densities and liftings defined in products of two probability spaces and possessing section properties analogous to that described by the Fubini theorem in case of measures and measurable sets were considered. These properties have been then applied to prove the permanence of the measurability of stochastic processes under the modification by liftings [11]. In [2] the product situation for the σ-algebra $\mathfrak{B}_c(X)$ of all sets having the Baire property, selecting a representative element from each equivalence class of $\mathfrak{B}_c(X)$ modulo sets of the first category (S. Graf [8], D. Maharam [10] and J.C. Oxtoby [14])

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was investigated. Following J.C. Oxtoby's [14, p. 74] remark that "the suggestion to look for a category analogue has very often proved to be a useful guide", we have attempted to check if this can be interesting in case of our investigations.

It has been proven in [2] that given topological spaces X and Y such that the product space $X \times Y$ is Baire and given (strong) liftings ρ on $(X, \mathfrak{B}_c(X), \mathcal{M}(X))$ and σ on $(Y, \mathfrak{B}_c(Y), \mathcal{M}(Y))$ always there exists a (strong) lifting π_1 on $(X \times Y, \mathfrak{B}_c(X \times Y), \mathcal{M}(X \times Y))$ satisfying the product condition $\pi_1(A \times B) = \rho(A) \times \sigma(B)$ for all $A \in \mathfrak{B}_c(X)$, $B \in \mathfrak{B}_c(Y)$ and such that for each $E \in \mathfrak{B}_c(X \times Y)$ and each $x \in X$ the section property $[\pi_1(E)]_x = \sigma([\pi_1(E)]_x)$ holds true (see [2, Theorem 5.1]).

The second problem investigated also in [2] concerns the existence of a density θ_I on a Baire product $\prod_{i\in I} X_i$ of topological spaces such that if $\emptyset \neq J \subseteq I$ and $A \in \mathfrak{B}_c(X_J)$, then there is a $B \in \mathfrak{B}_c(X_J)$ such that $\theta_I(A \times X_{J^c}) = B \times X_{J^c}$. This is an obvious generalization of the two factor case. We say that such a density respects coordinates. The terminology is taken from measure products case, where it has been proposed by Fremlin [5].

The best-known result in case of finite measure products is due to Burke [1], who proved the existence of liftings respecting coordinates (no coordinate liftings are fixed in advance). In case of infinite product, Fremlin [5] proved the existence of liftings respecting coordinates if all the coordinate measure spaces are Maharam homogenous.

In this paper we continue the investigation of [2] by introducing Fubini type products for densities and liftings and studying their consistency with the product structure. Given two topological spaces X, Y such that $X \times Y$ is Baire and arbitrary densities υ and τ on $\mathfrak{B}_c(X)$ and $\mathfrak{B}_c(Y)$, we consider the density ξ on $\mathfrak{B}_c(X \times Y)$ of Proposition 4.1 from [2], satisfying the formula

$$\xi(E) = \bigcup\bigl\{\upsilon(A) \times \tau(B):\ A \in \mathfrak{B}_c(X),\ B \in \mathfrak{B}_c(Y),\ A \times B \subseteq_{\mathcal{M}} E\bigr\}.$$

We call ξ the $\boxtimes$-product of υ and τ, and denote it by $\upsilon \boxtimes \tau$.

This formula defining the $\boxtimes$-product density from its marginals (see Definition 2.2 and [2, Propositions 3.1 and 4.1]) makes clear the crucial difference between the measure and the category cases. A non-meager set with the Baire property in the product contains, up to a meager set, a rectangle with non-meager sides with the Baire property, while a famous result of P. Erdös and J.C. Oxtoby [3] exhibits an example of a set of positive measure in the product σ-algebra of quite arbitrary non-atomic *positive* measure spaces, containing, up to a set of measure zero, no rectangle of positive measure. That fact makes it clear that in the category case we should apply completely different methods than in case of measure product liftings. The latter is done, as a rule, by transfinite induction, relying crucially on the martingale theorem, not available in the category case.

We prove that under the mild condition that the pairs (X, Y) and (Y, X) satisfy the Kuratowski–Ulam property (see Section 1 for the definition), the map $\upsilon \odot \tau$ (see Definition 4.1) is a subdensity and the map $\upsilon \boxdot \tau$ (see Definition 4.7) is a density, called the $\boxdot$-product of its marginals. Both maps $\upsilon \odot \tau$ and $\upsilon \boxdot \tau$ have nice properties consistent with the product structure of

$$\bigl(X \times Y, \mathfrak{B}_c(X \times Y), \mathcal{M}(X \times Y)\bigr)$$

(see Proposition 4.4 and Theorem 4.9). If moreover the cardinality condition $w(Y) < \operatorname{add}(\mathcal{M}(X))$ holds true (such a condition holds e.g. true if X, Y are Polish spaces) and τ is the canonical density on Y, then $\upsilon \odot \tau = \upsilon \boxdot \tau$ (see Theorem 4.6). If X, Y are Polish spaces with the corresponding canonical densities φ_X, φ_Y we get additional properties:

$$\varphi_{X\times Y} = \varphi_X \boxtimes \varphi_Y \leqslant \varphi_X \odot \varphi_Y = \varphi_X \boxdot \varphi_Y \in \vartheta\bigl(\mathcal{M}(X \times Y)\bigr) \cap \varphi_X \otimes \varphi_Y,$$
$$\bigl[(\varphi_X \boxdot \varphi_Y)(E)\bigr]^y = \varphi_X\bigl(\bigl[(\varphi_X \boxdot \varphi_Y)(E)\bigr]^y\bigr)$$

for all $y \in Y$ and $E \in \mathfrak{B}_c(X \times Y)$, and $[(\varphi_X \boxdot \varphi_Y)(E)]_x \in \mathfrak{B}_c(Y)$ for all $x \in X$ and all $E \in \mathfrak{B}_c(X \times Y)$ (see Theorem 4.11).

The situation here is much better than in the measure-theoretic case, since in that case such results hold true only under a measurability condition (see [12, Definition 4.2]) that is automatically satisfied in the category case. We prove in [12, Remark 5.1], that this measurability condition sometimes fails in the measure-theoretic case.

Based on the properties of $\rho \boxdot \sigma$, for a lifting ρ on $(X, \mathfrak{B}_c(X), \mathcal{M}(X))$ and a lifting σ on $(Y, \mathfrak{B}_c(Y), \mathcal{M}(Y))$, we find that the maximal elements of the system of all densities possessing the properties of $\rho \boxdot \sigma$ and dominating $\rho \boxdot \sigma$ are liftings on $X \times Y$ being consistent with the product structure of $(X \times Y, \mathfrak{B}_c(X \times Y), \mathcal{M}(X \times Y)$. More precisely, we prove that given arbitrary topological spaces X and Y such that the product space $X \times Y$ is Baire and such that the pair (X, Y) satisfies the Kuratowski–Ulam property, and given (strong) liftings ρ on $(X, \mathfrak{B}_c(X), \mathcal{M}(X))$ and σ on $(Y, \mathfrak{B}_c(Y), \mathcal{M}(Y))$ there always exists a (strong) lifting π_2 on $(X \times Y, \mathfrak{B}_c(X \times Y), \mathcal{M}(X \times Y))$ dominating the density $\rho \boxdot \sigma$, satisfying the product condition $\pi_2(A \times B) = \rho(A) \times \sigma(B)$ for all $A \in \mathfrak{B}_c(X)$, $B \in \mathfrak{B}_c(Y)$ and such that for each $E \in \mathfrak{B}_c(X \times Y)$ and each $y \in Y$ the section property $[\pi_2(E)]^y = \rho([\pi_2(E)]^y)$ holds true (see Theorem 4.12). One should notice that it is impossible (besides some trivial cases) to have also the section property $[\pi_2(E)]_x = \sigma([\pi_2(E)]_x)$ for each $E \in \mathfrak{B}_c(X \times Y)$ and each $x \in X$, even if $X = Y$ (see Theorem 5.1). It is also impossible (besides some trivial cases) for the subdensity $\rho \odot \sigma$ to be a lifting (see Theorem 5.5).

In Section 6 we extend the result to finite products of topological spaces. For finite products of Polish spaces we prove that the $\odot$-product of the corresponding canonical densities coincides with the corresponding $\boxdot$-product, which respects coordinates and possesses nice section properties (see Theorem 6.8).

In the category products of more than two factors the existence of liftings respecting coordinates remains open.

In Section 7 we examine conditions under which there exist countably additive liftings or countably multiplicative densities and consequences for the existence of product densities with invariant sections. It is well known that in case of measure spaces countably additive liftings exist only in the case of purely atomic measures. (Cf. the comment after Proposition 7.8.)

1. Preliminaries

Throughout we assume that all topological spaces under consideration are non-empty. Let X be a topological space. The weight of X is denoted by $w(X)$. A family $\mathcal{U}$ of non-empty open sets in a topological space will be called a *pseudo-basis* (π-basis for short), if every non-empty open set in X contains an element $U \in \mathcal{U}$. The minimal cardinality of a π-basis will be denoted by $\pi(X)$. For each subset A of X we denote by $\operatorname{cl} A$ (or by $\overline{A}$) and by $\operatorname{int}(A)$ the topological closure and interior of A, respectively. A set $A \subseteq X$ is *nowhere dense* if $\operatorname{int}(\operatorname{cl} A) = \emptyset$. A set $M \subseteq X$ is *meager* or of the *first category* if it is expressible as a union of a sequence of nowhere dense sets. A set $A \subseteq X$ is of the *second category* if it is not meager. We recall the standard observation (see, e.g., [13]) that when Y is a dense subspace of X, for subsets A of Y we have that A is nowhere dense in Y if and only if A is nowhere dense in X, and A is meager in Y if and only if A is meager in X.

An open set $A \subseteq X$ is said to be *regular open* in X if it coincides with the interior of its closure. A set $A \subseteq X$ has the *Baire property* if it can be represented in the form $A = G \triangle N$, where G is open and N is meager. A topological space X is called a *Baire space* if every non-empty open set in X is non-meager. $\mathcal{M}(X)$ denotes the collection of all meager subsets of the topological space X and $\mathfrak{B}_c(X)$ denotes the σ-algebra of sets possessing the Baire property. $\operatorname{add}(\mathcal{M}(X)) := \min\{\operatorname{card}\mathfrak{J}\colon \mathfrak{J} \subset \mathcal{M}(X)$ and $\bigcup \mathfrak{J} \notin \mathcal{M}(X)\}$. For $A, B \in \mathfrak{B}_c(X)$ we write $A \subseteq B$ a.e. $(\mathcal{M}(X))$ or $A \subseteq_{\mathcal{M}} B$ if $A \setminus B \in \mathcal{M}(X)$ and similarly for equality in place of the inclusion.

It is crucial for this paper that we apply weaker functionals than densities. We define them now. Given a map $\upsilon : \mathfrak{B}_c(X) \to \mathcal{P}(X)$ we consider for every $A, B \in \mathfrak{B}_c(X)$ the following properties

(L1) $\upsilon(A) \in \mathfrak{B}_c(X)$ and $\upsilon(A) = A$ a.e. $(\mathcal{M}(X))$;
(L2) $A =_{\mathcal{M}} B$ implies $\upsilon(A) = \upsilon(B)$;
(N) $\upsilon(\emptyset) = \emptyset$ and $\upsilon(X) = X$;
(O) $A \subseteq B$ implies $\upsilon(A) \subseteq \upsilon(B)$;
(F) $\upsilon(A) \cap \upsilon(B) \subseteq \upsilon(A \cap B)$.
(ϑ) $\upsilon(A \cap B) = \upsilon(A) \cap \upsilon(B)$.
(U) $\upsilon(A^c) = [\upsilon(A)]^c$.

We call a $\upsilon \in \mathcal{P}(X)^{\mathfrak{B}_c(X)}$ satisfying (L1), (L2), and (N) a *primitive lifting* for $\mathcal{M}(X)$ and we denote by $P(\mathcal{M}(X))$ the class of all primitive liftings. A primitive lifting for $\mathcal{M}(X)$ will be called a *monotone lifting* for $\mathcal{M}(X)$ if it satisfies in addition the axiom (O). If a primitive lifting satisfies in addition (F), we call it a *subdensity* for $\mathcal{M}(X)$ and denote by $F(\mathcal{M}(X))$ the class of all subdensities. Any subdensity υ has the property $\upsilon(A) \cap \upsilon(B) = \emptyset$ if $A \cap B \in \mathcal{M}(X)$ for $A, B \in \mathfrak{B}_c(X)$. A monotone subdensity is called a *lower density*. The collection of all lower densities is denoted by $\vartheta(\mathcal{M}(X))$. A lower density satisfying (U) is a *lifting*. The family of all liftings is denoted by $\Lambda(\mathcal{M}(X))$.

Lower densities and liftings on $(X, \mathfrak{B}_c(X), \mathcal{M}(X))$ are defined exactly in the same way as densities and liftings for measure spaces (cf. [15, Chapter 28]). We call them *category lower densities* and *category liftings*, while we call the densities and liftings for measure spaces *measure-theoretic densities* and *measure-theoretic liftings*. If no confusion arises we say "density" instead of "category lower density" and "measure-theoretic lower density" and "lifting" instead of "category lifting" and "measure-theoretic lifting".

For maps δ, υ from $\mathfrak{B}_c(X)$ into $\mathcal{P}(X)$ we write $\delta \leqslant \upsilon$, if $\delta(A) \subseteq \upsilon(A)$ for all $A \in \mathfrak{B}_c(X)$.

For each $E \in \mathfrak{B}_c(X)$ we denote by $\varphi_X(E)$ the regular open set equivalent to E. $\varphi_X : \mathfrak{B}_c(X) \to \mathfrak{B}_c(X)$ defined in that way is a strong density (see [8, Section 9], [10, Section 4] or [14, p. 88]). φ_X will be called the *canonical density* on $(X, \mathfrak{B}_c(X), \mathcal{M}(X))$.

A set $A \in \mathfrak{B}_c(X) \setminus \mathcal{M}(X)$ is an $\mathcal{M}(X)$-*atom* of $\mathfrak{B}_c(X)$ if A cannot be decomposed into two disjoint elements of $\mathfrak{B}_c(X) \setminus \mathcal{M}(X)$. Notice that φ_X is a lifting precisely when every regular open set in X is clopen, i.e., precisely when X is extremally disconnected.

A map $\delta : \mathfrak{B}_c(X) \to \mathcal{P}(X)$ is called *strong* if for every non-empty open set $G \subseteq X$ we have $G \subseteq \delta(G)$.

The collection of all strong densities and of all strong liftings on $(X, \mathfrak{B}_c(X), \mathcal{M}(X))$ will be denoted by $\vartheta_s(\mathcal{M}(X))$ and by $\Lambda_s(\mathcal{M}(X))$, respectively.

Each topological space X admitting a strong density is a Baire space. In fact, assume that X is a topological space admitting a strong density φ. Then for each non-empty open set G we have $G \subseteq \varphi(G)$, from which it follows that $\varphi(G) \neq \emptyset$ and hence G is not meager.

If a Baire space X is a topological group, a map δ from $\mathfrak{B}_c(X)$ into $\mathcal{P}(X)$ is called *left invariant* for X, if $\delta(xE) = x\delta(E)$ for every $E \in \mathfrak{B}_c(X)$ and $x \in X$.

If I is a non-empty set and $\langle X_i \rangle_{i \in I}$ is a family of arbitrary topological spaces then, for each $\emptyset \neq J \subseteq I$ we denote by X_J the product topological space $\prod_{i \in J} X_i$. If $J = \emptyset$, then for simplicity of notation we identify $X_J \times Y$ with Y.

We say that a $\varphi \in \vartheta(\mathcal{M}(X_I))$ is *separately Baire additive* if for any non-empty sets $J, K \subseteq I$ with $J \cap K = \emptyset$ we have

$$\varphi(E \cup F) = \varphi(E) \cup \varphi(F) \quad \text{for all } E \in \mathfrak{B}_c(X_J) \times X_{J^c} \text{ and } F \in \mathfrak{B}_c(X_K) \times X_{K^c}.$$

For measure-theoretic densities this notion is due to Fremlin [5], where it is called the $(*)$ property.

For maps $\delta : \mathfrak{B}_c(X) \to \mathcal{P}(X)$, $\upsilon : \mathfrak{B}_c(Y) \to \mathcal{P}(Y)$ and $\xi : \mathfrak{B}_c(X \times Y) \to \mathcal{P}(X \times Y)$ we say that ξ is a *product* of δ and υ, and we write it as $\xi \in \delta \otimes \upsilon$ if

$$\xi(A \times B) = \delta(A) \times \upsilon(B) \quad \text{for all } A \in \mathfrak{B}_c(X) \text{ and } B \in \mathfrak{B}_c(Y).$$

We use similar notation for a map $\xi : \mathfrak{B}_c(X_I) \to \mathcal{P}(X_I)$ with maps $\delta_i : \mathfrak{B}_c(X_i) \to \mathcal{P}(X_i)$, writing $\xi \in \otimes_{i \in I} \delta_i$ if $\xi(\prod_{i \in I} A_i) = \prod_{i \in I} \delta_i(A_i)$ for each product set $\prod_{i \in I} A_i$ where $A_i \in \mathfrak{B}_c(X_i)$ and $A_i = X_i$ for all but finite collection of $i \in I$. If $I := [n] := \{1, \dots, n\}$ then we write $\xi \in \delta_1 \otimes \dots \otimes \delta_n$.

We say that $\varphi_I \in F(\mathcal{M}(X_I))$ *respects coordinates* if for each proper $\emptyset \neq J \subseteq I$ the inclusion $\varphi_I(\mathfrak{B}_c(X_J) \times X_{J^c}) \subseteq \mathfrak{B}_c(X_J) \times X_{J^c}$ holds true.

It can be easily seen that if φ_I respects coordinates then, for each $\emptyset \neq J \subseteq I$ there is a uniquely determined subdensity $\varphi_J \in F(\mathcal{M}(X_J))$ given by $\varphi_J(A) \times X_{J^c} = \varphi_I(A \times X_{J^c})$, for all $A \in \mathfrak{B}_c(X_J)$. And conversely, if for each $\emptyset \neq J \subseteq I$ there is a subdensity φ_J on $\mathfrak{B}_c(X_J)$ such that $\varphi_I(A \times X_{J^c}) = \varphi_J(A) \times X_{J^c}$, whenever $A \in \mathfrak{B}_c(X_J)$, then φ_I respects coordinates. From this point of view one could speak about a completely product subdensity instead of a subdensity respecting coordinates.

We recall a definition introduced by D.H. Fremlin, T. Natkaniec and I. Recław in [6]. A pair (X, Y) of topological spaces is a *Kuratowski–Ulam pair* (briefly K-U pair) or it has the *Kuratowski–Ulam property*, if the Kuratowski–Ulam theorem holds in $X \times Y$:

$$\forall E \subseteq X \times Y \quad \bigl[E \in \mathcal{M}(X \times Y) \;\Rightarrow\; \bigl\{x \in X:\ E_x \notin \mathcal{M}(Y)\bigr\} \in \mathcal{M}(X)\bigr].$$

Kuratowski and Ulam proved that if $\pi(Y) < \mathrm{add}(\mathcal{M}(X))$, then the pair (X, Y) is a K-U pair (see [14, Theorem 15.1]). In particular, if Y has a countable π-basis, then for each topological space X the pair (X, Y) is a K-U pair.

Throughout this paper we assume that X and Y are topological spaces such that $X \times Y$ is a Baire space.

2. The box-cross product

Before the next result we need a proposition as a preparation. The notion of the upper hull appears in the paper [7] of J. Gapaillard, for the measure-theoretic case. For a map $\varpi : \mathfrak{B}_c(X) \to \mathcal{P}(X)$ we define the *upper hull of* ϖ by means of

$$\bigl(\varpi^m\bigr)(A) := \bigcup_{A \supseteq B \in \mathfrak{B}_c(X)} \varpi(B).$$

For given $\xi \in F(\mathcal{M}(X))$ we denote by $\Lambda_\xi(\mathcal{M}(X)) := \{\rho \in \Lambda(\mathcal{M}(X)):\ \xi \leqslant \rho\}$ the set of all liftings generated by ξ.

Proposition 2.1. *For given map $\varpi : \mathfrak{B}_c(X) \to \mathcal{P}(X)$ the map $\varpi^m : \mathfrak{B}_c(X) \to \mathcal{P}(X)$ has the following properties.*

(i) *ϖ^m satisfies condition* (O), *$\varpi \leqslant \varpi^m$, and for any map $\xi : \mathfrak{B}_c(X) \to \mathcal{P}(X)$ satisfying condition* (O) *and $\varpi \leqslant \xi$ follows $\varpi^m \leqslant \xi$;*
(ii) *$\varpi^m \in \vartheta(\mathcal{M}(X))$ for all $\varpi \in F(\mathcal{M}(X))$;*
(iii) *if $\varpi \in F(\mathcal{M}(X))$ satisfies also* (U), *then $\varpi^m = \varpi \in \Lambda(\mathcal{M}(X))$;*
(iv) *if $\varpi \in F(\mathcal{M}(X))$, then $\Lambda_\varpi(\mathcal{M}(X)) = \Lambda_{\varpi^m}(\mathcal{M}(X))$.*

Proof. (i) is obvious and ad (ii) note that in the same way as in [7] the map ϖ^m is a monotone lifting, hence $\varpi^m(A \cap B) \subseteq \varpi^m(A) \cap \varpi^m(B)$ for all $A, B \in \mathfrak{B}_c(X)$, and it is sufficient to show that $\varpi^m \in F(\mathcal{M}(X))$.

Indeed, let $A, B, C, D \in \mathfrak{B}_c(X)$ with $A \supseteq C$ and $B \supseteq D$ be given. It follows $A \cap B \supseteq C \cap D$, hence $\varpi^m(A \cap B) \supseteq \varpi(C \cap D) \supseteq \varpi(C) \cap \varpi(D)$ and for fixed D we have $\varpi^m(A \cap B) \supseteq (\bigcup_{A \supseteq C \in \mathfrak{B}_c(X)} \varpi(C)) \cap \varpi(D)$ hence $\varpi^m(A \cap B) \supseteq \varpi^m(A) \cap \varpi(D)$ for all D with $B \supseteq D \in \mathfrak{B}_c(X)$, consequently $\varpi^m(A \cap B) \supseteq \varpi^m(A) \cap \bigcup_{B \supseteq D \in \mathfrak{B}_c(X)} \varpi(D)$, i.e. $\varpi^m(A \cap B) \supseteq \varpi^m(A) \cap \varpi^m(B)$.

Ad (iii): We first check that $\varpi^m \in \Lambda(\mathcal{M}(X))$. By (ii) we have only to show that ϖ^m satisfies condition (U) and that $\varpi^m = \varpi$. For all $A \in \mathfrak{B}_c(X)$ we get by (i) that $\varpi(A) \subseteq \varpi^m(A)$, $\varpi(A^c) \subseteq \varpi^m(A^c)$ and by (U) for ϖ

$$X = \varpi(A) \cup \varpi\bigl(A^c\bigr) \subseteq \varpi^m(A) \cup \varpi^m\bigl(A^c\bigr) \subseteq X.$$

Together with the consequence of (ii) that $\varpi^m(A) \cap \varpi^m(A^c) = \emptyset$, these imply that $\varpi(A) = \varpi^m(A)$ and $[\varpi^m(A)]^c = \varpi^m(A^c)$.

Item (iv) follows from the minimality condition satisfied by ϖ^m according to (i). $\square$

Definition 2.2. For arbitrary maps $\upsilon : \mathfrak{B}_c(X) \to \mathcal{P}(X)$ and $\tau : \mathfrak{B}_c(Y) \to \mathcal{P}(Y)$ define the map $\upsilon \boxtimes \tau : \mathfrak{B}_c(X \times Y) \to \mathcal{P}(X \times Y)$ by means of

$$(\upsilon \boxtimes \tau)(E) := \bigcup \bigl\{\upsilon(A) \times \tau(B):\ A \in \mathfrak{B}_c(X),\ B \in \mathfrak{B}_c(Y),\ A \times B \subseteq_{\mathcal{M}} E\bigr\}$$

for every $E \in \mathfrak{B}_c(X \times Y)$.

The next result improves Proposition 4.1 from [2].

Proposition 2.3. *For arbitrary maps* $\upsilon : \mathfrak{B}_c(X) \to \mathcal{P}(X)$ *and* $\tau : \mathfrak{B}_c(Y) \to \mathcal{P}(Y)$ *we get*

(i) $\upsilon \boxtimes \tau$ *always satisfies condition* (O);
(ii) $\upsilon \boxtimes \tau = \upsilon^m \boxtimes \tau^m$;
(iii) *suppose that* X, Y *are topological groups and* $X \times Y$ *is Baire. If* υ *is left invariant for* X *and* τ *is left invariant for* Y, *then* $\upsilon \boxtimes \tau$ *is left invariant for* $X \times Y$;
(iv) *if* υ *and* τ *are monotone liftings, then* $\upsilon \boxtimes \tau \in \upsilon \otimes \tau$ *and* $\upsilon \boxtimes \tau$ *is separately additive*;
(v) *if* υ *and* τ *satisfy condition* (F), *so* $\upsilon \boxtimes \tau$ *does*;
(vi) *if* υ *and* τ *are strong, then* $\upsilon \boxtimes \tau$ *is also strong*;
(vii) *if* $\upsilon \in F(\mathcal{M}(X))$ *and* $\tau \in F(\mathcal{M}(Y))$, *then*:

$$\upsilon \boxtimes \tau \in \vartheta\big(\mathcal{M}(X \times Y)\big) \cap \big(\upsilon^m \otimes \tau^m\big);$$
$$\big[(\upsilon \boxtimes \tau)(E)\big]_x \in \mathfrak{B}_c(Y) \quad \textit{and} \quad \big[(\upsilon \boxtimes \tau)(E)\big]_x \subseteq \tau^m\big(\big[(\upsilon \boxtimes \tau)(E)\big]_x\big) \quad \textit{for every } E \in \mathfrak{B}_c(X \times Y) \textit{ and } x \in X;$$
$$\big[(\upsilon \boxtimes \tau)(E)\big]^y \in \mathfrak{B}_c(X) \quad \textit{and} \quad \big[(\upsilon \boxtimes \tau)(E)\big]^y \subseteq \upsilon^m\big(\big[(\upsilon \boxtimes \tau)(E)\big]^y\big) \quad \textit{for every } E \in \mathfrak{B}_c(X \times Y) \textit{ and } y \in Y.$$

Proof. Condition (i) is obvious by definition of $\upsilon \boxtimes \tau$.

Ad (ii): For $E \in \mathfrak{B}_c(X \times Y)$ we get with sets $A, C \in \mathfrak{B}_c(X)$ and $B, D \in \mathfrak{B}_c(Y)$ that

$$\begin{aligned}\big(\upsilon^m \boxtimes \tau^m\big)(E) &= \bigcup_{A \times B \subseteq_{\mathcal{M}} E} \Big(\bigcup_{C \subseteq A} \upsilon(C) \times \bigcup_{D \subseteq B} \tau(D)\Big)\\ &= \bigcup_{A \times B \subseteq_{\mathcal{M}} E} \bigcup_{C \subseteq A} \bigcup_{D \subseteq B} \big(\upsilon(C) \times \tau(D)\big) = (\upsilon \boxtimes \tau)(E).\end{aligned}$$

Ad (iii): For $(x, y) \in X \times Y$ and $E \in \mathfrak{B}_c(X \times Y)$ we get

$$\begin{aligned}(x, y)(\upsilon \boxtimes \tau)(E) &= (x, y) \bigcup \big\{\upsilon(A) \times \tau(B):\ A \in \mathfrak{B}_c(X),\ B \in \mathfrak{B}_c(Y),\ A \times B \subseteq_{\mathcal{M}} E\big\}\\ &= \bigcup \big\{x\upsilon(A) \times y\tau(B):\ A \in \mathfrak{B}_c(X),\ B \in \mathfrak{B}_c(Y),\ A \times B \subseteq_{\mathcal{M}} E\big\}\\ &= \bigcup \big\{\upsilon(xA) \times \tau(yB):\ A \in \mathfrak{B}_c(X),\ B \in \mathfrak{B}_c(Y),\ A \times B \subseteq_{\mathcal{M}} E\big\}\\ &= \bigcup \big\{\upsilon(xA) \times \tau(yB):\ xA \in \mathfrak{B}_c(X),\ yB \in \mathfrak{B}_c(Y),\ xA \times yB \subseteq_{\mathcal{M}} (x, y)E\big\}\\ &= (\upsilon \boxtimes \tau)\big((x, y)E\big).\end{aligned}$$

Ad (iv): Apply [2, Lemma 2.2(g)] to find $\upsilon \boxtimes \tau \in \upsilon \otimes \tau$. Looking at the proof given in [2, Proposition 4.1(vii)] for densities instead of monotone υ, τ, we see that monotonicity suffices to ensure separate additivity for $\upsilon \boxtimes \tau$.

Ad (v): If υ and τ satisfy condition (F) we find with $A, C \in \mathfrak{B}_c(X)$ and $B, D \in \mathfrak{B}_c(Y)$ and $E, F \in \mathfrak{B}_c(X \times Y)$ that

$$\begin{aligned}(\upsilon \boxtimes \tau)(E) \cap (\upsilon \boxtimes \tau)(F) &= \Big(\bigcup_{A \times B \subseteq_{\mathcal{M}} E} \big[\upsilon(A) \times \tau(B)\big]\Big) \cap \Big(\bigcup_{C \times D \subseteq_{\mathcal{M}} F} \big[\upsilon(C) \times \tau(D)\big]\Big)\\ &\subseteq \bigcup_{(A \cap C) \times (B \cap D) \subseteq_{\mathcal{M}} E \cap F} \big[\upsilon(A) \cap \upsilon(C)\big] \times \big[\tau(B) \cap \tau(D)\big]\\ &\subseteq \bigcup_{(A \cap C) \times (B \cap D) \subseteq_{\mathcal{M}} E \cap F} \big[\upsilon(A \cap C) \times \tau(B \cap D)\big]\\ &\subseteq (\upsilon \boxtimes \tau)(E \cap F).\end{aligned}$$

Ad (vi): It follows in the same way as in [2, Proposition 4.1].

Ad (vii): For $\upsilon \in F(\mathcal{M}(X))$ and $\tau \in F(\mathcal{M}(Y))$ we get $\upsilon^m \in \vartheta(\mathcal{M}(X))$ and $\tau^m \in \vartheta(\mathcal{M}(Y))$ by Proposition 2.1(ii), hence $\upsilon^m \boxtimes \tau^m \in \vartheta(\mathcal{M}(X \times Y))$ and we get (vii) for $\upsilon^m \boxtimes \tau^m$ instead of $\upsilon \boxtimes \tau$, by [2, Proposition 4.1]. But both are the same by (ii). This completes the proof. □

3. Lifting of sections

Definition 3.1. For given $\tau \in \mathcal{P}(Y)^{\mathfrak{B}_c(Y)}$ we define the set

$$\tau_\bullet(E) := \big\{(x, y) \in X \times Y:\ E_x \in \mathfrak{B}_c(Y) \wedge y \in \tau(E_x)\big\}$$

for all $E \in \mathfrak{B}_c(X \times Y)$. Similarly for given $\upsilon \in \mathcal{P}(X)^{\mathfrak{B}_c(X)}$ we define the set

$$\upsilon^\bullet(E) := \big\{(x, y) \in X \times Y:\ E^y \in \mathfrak{B}_c(X) \wedge x \in \upsilon\big(E^y\big)\big\}$$

for all $E \in \mathfrak{B}_c(X \times Y)$.

Lemma 3.2. *If $E \in \mathfrak{B}_c(X \times Y)$ and (X, Y) is a K-U pair, then $\{x\colon\ E_x \notin \mathfrak{B}_c(Y)\} \in \mathcal{M}(X)$.*

Proof. Let U be a regular open set such that $E \triangle U \in \mathcal{M}(X \times Y)$. According to the K-U assumption, we have $\{x\colon\ [E \triangle U]_x \notin \mathcal{M}(Y)\} \in \mathcal{M}(X)$. As U is open we have $\{x\colon\ E_x \notin \mathfrak{B}_c(Y)\} \subseteq \{x\colon\ [E \triangle U]_x \notin \mathcal{M}(Y)\}$. □

Proposition 3.3. *For given $\tau \in P(\mathcal{M}(Y))$ we get for $\tau_\bullet$ the following properties.*

(i) *For $A \in \mathfrak{B}_c(X)$ and $B \in \mathfrak{B}_c(Y)$ we get $\tau_\bullet(A \times B) = A \times \tau(B)$, in particular $\tau_\bullet$ does not satisfy* (L2);
(ii) *$\tau_\bullet$ satisfies condition* (N): *$\tau_\bullet(\emptyset) = \emptyset$ and $\tau_\bullet(X \times Y) = X \times Y$;*
(iii) *if τ satisfies (ϑ), then $\tau_\bullet(E) \cap \tau_\bullet(F) = \tau_\bullet(E \cap F)$ for all $E, F \in \mathfrak{B}_c(X \times Y)$ such that $E_x, F_x \in \mathfrak{B}_c(Y)$ for all $x \in X$;*
(iv) *if τ satisfies condition* (F), *then $\tau_\bullet$ satisfies* (F) *too;*
(v) *if $\tau \in \Lambda(\mathcal{M}(Y))$, then $\tau_\bullet(E^c) = [\tau_\bullet(E)]^c$ for all sets $E \subseteq X \times Y$ such that $E_x \in \mathcal{B}_c(Y)$ for all $x \in X$.*

If (X, Y) has the Kuratowski–Ulam property, then the following properties are also satisfied:

(vi) *if $\tau \in \vartheta(\mathcal{M}(Y))$, then $\tau_\bullet$ satisfies* (L1);
(vii) *if $E, F \in \mathfrak{B}_c(X \times Y)$, then $E = F$ a.e. $(\mathcal{M}(X \times Y))$ implies for all $y \in Y$ the equality $[\tau_\bullet(E)]^y = [\tau_\bullet(F)]^y$ a.e. $(\mathcal{M}(X))$.*

If also (Y, X) has the Kuratowski–Ulam property we get in addition:

(viii) *if $\tau \in \vartheta(\mathcal{M}(Y))$, then for each $E \in \mathfrak{B}_c(X \times Y)$ there exists a set $M_E \in \mathcal{M}(Y)$ such that $[\tau_\bullet(E)]^y \in \mathfrak{B}_c(X)$ and $[\tau_\bullet(E)]^y =_{\mathcal{M}} E^y$, for all $y \notin M_E$.*

Proof. The assertions (i)–(iii) can be easily proven.

Ad (iv): If $E, F \in \mathfrak{B}_c(X \times Y)$ and $(x, y) \in X \times Y$, then

$$\begin{aligned}(x, y) \in \tau_\bullet(E) \cap \tau_\bullet(F) \quad &\Rightarrow \quad E_x, F_x \in \mathfrak{B}_c(Y) \ \wedge\ y \in \tau(E_x) \cap \tau(F_x)\\ &\Rightarrow \quad (E \cap F)_x \in \mathfrak{B}_c(Y) \ \wedge\ y \in \tau\big([E \cap F]_x\big) = \big[\tau_\bullet(E \cap F)\big]_x,\end{aligned}$$

hence $(x, y) \in \tau_\bullet(E \cap F)$.

Ad (v): We have

$$\begin{aligned}\tau_\bullet\big(E^c\big) &= \big\{(x, y) \in X \times Y\colon\ y \in \tau\big(\big[E^c\big]_x\big)\big\} = \big\{(x, y) \in X \times Y\colon\ y \notin \tau(E_x)\big\}\\ &= \big\{(x, y) \in X \times Y\colon\ y \in \tau(E_x)\big\}^c = \big[\tau_\bullet(E)\big]^c.\end{aligned}$$

Ad (vi): Choose a $\upsilon \in \vartheta(\mathcal{M}(X))$. Let $E \in \mathfrak{B}_c(X \times Y)$ be arbitrary. According to Proposition 2.3(vii) we get $\xi := \upsilon \boxtimes \tau \in \vartheta(\mathcal{M}(X \times Y))$ and $[\xi(E)]_x \subseteq \tau([\xi(E)]_x)$ for every $x \in X$. Due to the K-U property of (X, Y) there is $N_E \in \mathcal{M}(X)$ such that $[\xi(E)]_x =_{\mathcal{M}} E_x$, for every $x \notin N_E$. Define ξ_1 by $[\xi_1(E)]_x = \tau([\xi(E)]_x)$. By [2, Proposition 4.2(a)], $\xi_1 \in \vartheta(\mathcal{M}(X \times Y))$ and for $x \in N_E^c$ follows $[\xi_1(E)]_x = \tau([\xi(E)]_x) = \tau(E_x) = [\tau_\bullet(E)]_x$. Thus, $\tau_\bullet(E) \triangle \xi_1(E) \subseteq N_E \times Y$ and so $\tau_\bullet(E) \in \mathfrak{B}_c X \times Y$. It follows that $\tau_\bullet(E) \overset{\mathcal{M}}{=} E$.

Ad (vii): Since (X, Y) satisfies the Kuratowski–Ulam property there exists a set $N_E \in \mathcal{M}(X)$ such that $E_x = F_x$ a.e. $(\mathcal{M}(Y))$ for all $x \notin N_E$, what yields $[\tau_\bullet(E)]^y \setminus N_E = [\tau_\bullet(F)]^y \setminus N_E$.

Condition (viii) is immediate from (vi) and Lemma 3.2 in case of (Y, X) possessing the K-U property. □

Proposition 3.4. *If (X, Y) has the Kuratowski–Ulam property then for $\tau \in \vartheta(\mathcal{M}(Y))$ there exists $\varphi \in \vartheta(\mathcal{M}(X \times Y))$ such that for every $E \in \mathfrak{B}_c(X \times Y)$ and every $x \in X$ we get $[\varphi(E)]_x \in \mathcal{B}_c(Y)$ and $\tau([\varphi(E)]_x) = [\varphi(E)]_x$.*

Proof. Define $\varphi\colon \mathfrak{B}_c(X \times Y) \to \mathcal{P}(X \times Y)$ by $\varphi(E) := \tau_\bullet(\varphi_{X \times Y}(E))$ for each $E \in \mathfrak{B}_c(X \times Y)$. It follows from Proposition 3.3(vi) that $\varphi(E) = \varphi_{X \times Y}(E) = E$ a.e. $(\mathcal{M}(X \times Y))$, for all $E \in \mathfrak{B}_c(X \times Y)$, i.e. φ satisfies (L1).

For $E, F \in \mathfrak{B}_c(X \times Y)$ with $E = F$ a.e. $(\mathcal{M}(X \times Y))$ we get $\varphi_{X \times Y}(E) = \varphi_{X \times Y}(F)$, hence $\varphi(E) = \varphi(F)$, i.e. (L2) for φ.

By Proposition 3.3(ii) φ satisfies condition (N), hence $\varphi \in P(\mathcal{M}(X \times Y))$. Since τ and $\varphi_{X \times Y}$ satisfy condition (ϑ) and moreover $\varphi_{X \times Y}(E)$ is open, we have $[\varphi_{X \times Y}(E)]_x \in \mathfrak{B}_c(X)$ for all $x \in X$, it follows by Proposition 3.3(iii), that φ also satisfies (ϑ). □

Definition 3.5. Once the basic topological spaces X and Y are fixed, we say that $\tau \in P(\mathcal{M}(Y))$ *generates X-measurable sections*, if $[\tau_\bullet(E)]^y \in \mathfrak{B}_c(X)$ for all $E \in \mathfrak{B}_c(X \times Y)$ and all $y \in Y$.

Proposition 3.6. *Let (X, Y) be a K-U pair, and let $\tau \in \vartheta(\mathcal{M}(Y))$ be arbitrary. Then τ generates X-measurable sections if and only if the $\varphi \in \vartheta(\mathcal{M}(X \times Y))$ from Proposition 3.4 can be taken such that $[\varphi(E)]^y \in \mathfrak{B}_c(X)$ for every $y \in Y$.*

Proof. Let be given $\varphi \in \vartheta(\mathcal{M}(X \times Y))$ such that $\tau([\varphi(E)]_x) = [\varphi(E)]_x$ for all $E \in \mathfrak{B}_c(X \times Y)$ and all $x \in X$. Note that for every $E \in \mathfrak{B}_c(X \times Y)$ there exists a set $P_E \in \mathcal{M}(X)$ such that $E_x \in \mathfrak{B}_c(Y)$ and $E_x = [\varphi(E)]_x$ a.e. $(\mathcal{M}(Y))$ for all $x \notin P_E$. Now for all $y \in Y$ we obtain

$$\begin{aligned}\left[\tau_\bullet(E)\right]^y \cap P_E^c &= \left\{x \in P_E^c\colon\ y \in \tau(E_x)\right\} = \left\{x \in P_E^c\colon\ y \in \tau\left(\left[\varphi(E)\right]_x\right)\right\} \\ &= \left\{x \in P_E^c\colon\ y \in \left[\varphi(E)\right]_x\right\} = \left[\varphi(E)\right]^y \cap P_E^c,\end{aligned}$$

hence $[\tau_\bullet(E)]^y \triangle [\varphi(E)]^y \in \mathcal{M}(X)$. So it follows that $[\varphi(E)]^y \in \mathfrak{B}_c(X)$ if and only if $[\tau_\bullet(E)]^y \in \mathfrak{B}_c(X)$ for every $y \in Y$. □

Corollary 3.7. *If $w(Y) < \operatorname{add}(\mathcal{M}(X))$, then the canonical density φ_Y generates X-measurable sections and $(\varphi_Y)_\bullet(E) \in \mathfrak{B}_c(X \times Y)$ for every $E \in \mathfrak{B}_c(X \times Y)$. Moreover, if (Y, X) has the Kuratowski–Ulam property, then $E^y =_{\mathcal{M}} [(\varphi_Y)_\bullet(E)]^y$, for $\mathcal{M}(Y)$-almost every $y \in Y$.*

Proof. The result follows immediately from Proposition 3.6 and from [2, Proposition 3.1]. The equality $E^y =_{\mathcal{M}} [(\varphi_Y)_\bullet(E)]^y$ is a direct consequence of the measurability of the set $(\varphi_Y)_\bullet(E)$ and of the Kuratowski–Ulam property of (Y, X). □

4. Fubini type products

Definition 4.1. For given $\upsilon \in P(\mathcal{M}(X))$ and $\tau \in P(\mathcal{M}(Y))$ we define a mapping $\upsilon \odot \tau : \mathfrak{B}_c(X \times Y) \to \mathcal{P}(X \times Y)$ by the formula

$$\upsilon \odot \tau(E) := \left\{(x, y) \in X \times Y\colon\ \left[\tau_\bullet(E)\right]^y \in \mathfrak{B}_c(X) \text{ and } x \in \upsilon\left(\left[\tau_\bullet(E)\right]^y\right)\right\}$$

for all $E \in \mathfrak{B}_c(X \times Y)$.

In a similar way we define a mapping $\tau \odot_t \upsilon : \mathfrak{B}_c(X \times Y) \to \mathcal{P}(X \times Y)$ by

$$\tau \odot_t \upsilon(E) := \left\{(x, y) \in X \times Y\colon\ \left[\upsilon^\bullet(E)\right]_x \in \mathfrak{B}_c(Y) \text{ and } y \in \tau\left(\left[\upsilon^\bullet(E)\right]_x\right)\right\}$$

for all $E \in \mathfrak{B}_c(X \times Y)$.

Lemma 4.2. *For given $\upsilon \in P(\mathcal{M}(X))$ and $\tau \in P(\mathcal{M}(Y))$ we have*

(i) *$\upsilon([\tau_\bullet(E)]^y) = [(\upsilon \odot \tau)(E)]^y$ for every $y \in Y$ with $[\tau_\bullet(E)]^y \in \mathfrak{B}_c(X)$;*
(ii) *$[(\upsilon \odot \tau)(E)]^y = \upsilon([(\upsilon \odot \tau)(E)]^y)$ for all $E \in \mathfrak{B}_c(X \times Y)$ and all $y \in Y$;*
(iii) *$\upsilon \odot \tau \in \upsilon \otimes \tau$.*

If (X, Y) satisfies the Kuratowski–Ulam property, then

(iv) *$\upsilon \odot \tau$ satisfies (L2);*
(v) *if $\tau \in \vartheta(\mathcal{M}(Y))$ then $\upsilon^\bullet(\tau_\bullet(E)) = (\upsilon \odot \tau)(E)$ for every $E \in \mathfrak{B}_c(X \times Y)$.*

Proof. Condition (i) is easily seen. Taking into account the fact that $[\tau_\bullet(E)]^y \notin \mathfrak{B}_c(X)$ yields $[(\upsilon \odot \tau)(E)]^y = \emptyset$, we obtain condition (ii).

By Proposition 3.3(i) we have $\upsilon^\bullet(\tau_\bullet(A \times B)) = \upsilon^\bullet(A \times \tau(B)) = \upsilon(A) \times \tau(B)$, and this implies the product property (iii).

Condition (iv) is a consequence of Proposition 3.3(vii).

To prove condition (v), let us fix an arbitrary $E \in \mathfrak{B}_c(X \times Y)$. Since (X, Y) has the K-U property and $\tau \in \vartheta(\mathcal{M}(Y))$, it follows by Proposition 3.3(vi), that $\tau_\bullet(E) \in \mathfrak{B}_c(X \times Y)$, hence we may define $\upsilon^\bullet(\tau_\bullet(E))$. The equality $\upsilon^\bullet(\tau_\bullet(E)) = (\upsilon \odot \tau)(E)$ follows easily from the definition of $\upsilon \odot \tau$. □

Remark 4.3. Let (X, Y) be a K-U pair. Since $\upsilon \odot \tau$ satisfies (L2) by Lemma 4.2, and since for every $E \in \mathfrak{B}_c(X \times Y)$ there exists a (regular) open subset G of $X \times Y$ with $E = G$ a.e. $(\mathcal{M}(X \times Y))$, we may restrict ourselves to work with $(\upsilon \odot \tau)(G)$ for G open subset of $X \times Y$ only, what we will do below without any further comment. Note that for every (regular) open subset G of $X \times Y$ we have G_x open in Y, hence $G_x \in \mathfrak{B}_c(Y)$, for all $x \in X$ and this simplifies the definition of $(\upsilon \odot \tau)(G)$.

Proposition 4.4. *Assume that (X, Y) and (Y, X) have the Kuratowski–Ulam property. Then for $\upsilon \in \vartheta(\mathcal{M}(X))$ and $\tau \in \vartheta(\mathcal{M}(Y))$ we get the following properties.*

(i) *$\upsilon \odot \tau \in F(\mathcal{M}(X \times Y)) \cap (\upsilon \otimes \tau)$;*
(ii) *τ generates X-measurable sections if and only if $\upsilon \odot \tau \in \vartheta(\mathcal{M}(X \times Y))$ and in this case $\upsilon \boxtimes \tau \leqslant \upsilon \odot \tau$;*
(iii) *if $\upsilon \in \Lambda(\mathcal{M}(X))$, then for every $E \in \mathfrak{B}_c(X \times Y)$ there is a set $K_E \in \mathcal{M}(Y)$ such that for all $y \notin K_E$*

$$\left[(\upsilon \odot \tau)\left(E^c\right)\right]^y = \left(\left[(\upsilon \odot \tau)(E)\right]^c\right)^y \quad \text{and} \quad \left[(\upsilon \odot \tau)(E)\right]^y \cup \left[(\upsilon \odot \tau)\left(E^c\right)\right]^y = X.$$

If moreover also $\tau \in \Lambda(\mathcal{M}(Y))$, then K_E can be chosen in such a way that

$$\big[(\upsilon \odot \tau)(E)\big]^y \cup \big[(\upsilon \odot \tau)\big(E^c\big)\big]^y = \emptyset \quad \textit{for all } y \in K_E.$$

Proof. The product property in (i) follows from Lemma 4.2(iii). (L2) for $\upsilon \odot \tau$ follows from Lemma 4.2(iv) and (N) for $\upsilon \odot \tau$ is clear by Proposition 3.3(ii).

Applying K-U of (X, Y) and (Y, X) we have by Proposition 3.3(vi) $\tau_\bullet(E) \in \mathfrak{B}_c(X \times Y)$ and $\upsilon^\bullet(\tau_\bullet(E)) = \tau_\bullet(E)$ a.e. $(\mathcal{M}(X \times Y))$. Again $\tau_\bullet(E) = E$ a.e. $(\mathcal{M}(X \times Y))$ by Proposition 3.3(vi). Thus, $(\upsilon \odot \tau)(E) = E$ a.e. $(\mathcal{M}(X \times Y))$, since $\upsilon^\bullet(\tau_\bullet(E)) = (\upsilon \odot \tau)(E)$. Consequently, $\upsilon \odot \tau \in P(\mathcal{M}(X \times Y))$.

To prove the condition (F) let us fix arbitrary sets $E, F \in \mathfrak{B}_c(X \times Y)$. According to Remark 4.3 we may assume that E, F are open subsets of $X \times Y$. It then follows that the sections E_x and F_x have the Baire property for all $x \in X$, hence $\tau_\bullet(E \cap F) = \tau_\bullet(E) \cap \tau_\bullet(F)$ by Proposition 3.3(iii). If $[(\upsilon \odot \tau)(E)]^y$ and $[(\upsilon \odot \tau)(F)]^y$ are non-empty then the sets $[\tau_\bullet(E)]^y$ and $[\tau_\bullet(F)]^y$ have the Baire property. Whenever these sections do have the property of Baire, so does their intersection $[\tau_\bullet(E)]^y \cap [\tau_\bullet(F)]^y = [\tau_\bullet(E \cap F)]^y$ and we get

$$\begin{aligned}
\big[(\upsilon \odot \tau)(E)\big]^y \cap \big[(\upsilon \odot \tau)(F)\big]^y &= \upsilon\big(\big[\tau_\bullet(E)\big]^y\big) \cap \upsilon\big(\big[\tau_\bullet(F)\big]^y\big) \\
&= \upsilon\big(\big[\tau_\bullet(E)\big]^y \cap \big[\tau_\bullet(F)\big]^y\big) \\
&= \upsilon\big(\big[\tau_\bullet(E \cap F)\big]^y\big) \\
&= \big[(\upsilon \odot \tau)(E \cap F)\big]^y.
\end{aligned}$$

The above proves also the forward implication in (ii) and for (i) there remains only to note that if one of the sets $[(\upsilon \odot \tau)(E)]^y$, $[(\upsilon \odot \tau)(F)]^y$ is empty then we trivially have the inclusion

$$\big[(\upsilon \odot \tau)(E)\big]^y \cap \big[(\upsilon \odot \tau)(F)\big]^y \subseteq \big[(\upsilon \odot \tau)(E \cap F)\big]^y.$$

To show the second part of (ii), let $E \in \mathfrak{B}_c(X \times Y)$, $A \in \mathfrak{B}_c(X)$, and $B \in \mathfrak{B}_c(Y)$ with $A \times B \subseteq_{\mathcal{M}} E$. It follows that $\upsilon(A) \times \tau(B) = (\upsilon \odot \tau)(A \times B) \subseteq (\upsilon \odot \tau)(E)$, the equality by Lemma 4.2(iii) and the inclusion since $\upsilon \odot \tau$ is monotone. By definition of $\upsilon \boxtimes \tau$ this implies $\upsilon \boxtimes \tau \leqslant \upsilon \odot \tau$.

For the converse implication in (ii), we prove the contrapositive. If τ does not generate X-measurable sections, then we can find $X_1 \in \mathfrak{B}_c(X)$ with $X_1^c \notin \mathcal{M}_c(X)$, $\bar{y} \in Y$ and $X_1 \times Y \supset E \in \mathfrak{B}_c(X \times Y)$ such that $[\tau_\bullet(E)]^{\bar{y}} \notin \mathfrak{B}_c(X)$. Setting $E_1 = E \cup (X_1^c \times Y)$ and $E_2 = X_1^c \times Y$ we have $(\upsilon \odot \tau)(E_1 \cap E_2)^{\bar{y}} = \upsilon(X_1^c) \neq \emptyset$ while

$$(\upsilon \odot \tau)(E_1)^{\bar{y}} \cap (\upsilon \odot \tau)(E_2)^{\bar{y}} = \emptyset.$$

Thus, $\upsilon \odot \tau$ is not a density. This completes the proof of (ii).

Ad (iii): Let $E \in \mathfrak{B}_c(X \times Y)$ be arbitrary. Then, according to (i) we have $(\upsilon \odot \tau)(E) \cup (\upsilon \odot \tau)(E^c) =_{\mathcal{M}} X \times Y$. Since (Y, X) has the K-U property, there is $K_E \in \mathcal{M}(Y)$ such that

$$\big[(\upsilon \odot \tau)(E)\big]^y \cup \big[(\upsilon \odot \tau)\big(E^c\big)\big]^y =_{\mathcal{M}} X \quad \text{for every } y \notin K_E.$$

But $\upsilon \in \Lambda(\mathcal{M}(X))$ and so taking into account Lemma 4.2(ii) we obtain

$$\big[(\upsilon \odot \tau)(E)\big]^y \cup \big[(\upsilon \odot \tau)\big(E^c\big)\big]^y = X \quad \text{for every } y \notin K_E,$$

hence

$$\big[(\upsilon \odot \tau)\big(E^c\big)\big]^y = \big(\big[(\upsilon \odot \tau)(E)\big]^c\big)^y \quad \text{for all } y \notin K_E.$$

Assume now also that $\tau \in \Lambda(\mathcal{M}(Y))$ and that E is regular open, we get then from Proposition 3.3(v) that $[\tau_\bullet(E)]^y \cup [\tau_\bullet(E^c)]^y = X$, for every $y \in Y$. Consequently, $[\tau_\bullet(E)]^y \in \mathfrak{B}_c(X)$ if and only if $[\tau_\bullet(E^c)]^y \in \mathfrak{B}_c(X)$. It follows from Lemma 3.2 applied in case of (Y, X) possessing the K-U property that $K_E := \{y \in Y\colon\ [\tau_\bullet(E)]^y \notin \mathfrak{B}_c(Y)\} \in \mathcal{M}(X)$. Now it is obvious that

$$\big[(\upsilon \odot \tau)(E)\big]^y \cup \big[(\upsilon \odot \tau)\big(E^c\big)\big]^y = X \quad \text{for every } y \notin K_E,$$

hence

$$\big[(\upsilon \odot \tau)\big(E^c\big)\big]^y = \big(\big[(\upsilon \odot \tau)(E)\big]^c\big)^y \quad \text{for every } y \notin K_E.$$

Clearly

$$\big[(\upsilon \odot \tau)(E)\big]^y \cup \big[(\upsilon \odot \tau)\big(E^c\big)\big]^y = \emptyset \quad \text{for every } y \in K_E. \qquad \square$$

Proposition 4.5. *Suppose that X, Y are topological groups such that (X, Y) is a K-U pair. If $\upsilon \in \vartheta(\mathcal{M}(X))$ is left invariant for X and $\tau \in \vartheta(\mathcal{M}(Y))$ is left invariant for Y, then $\upsilon \odot \tau$ is left invariant for $X \times Y$.*

Proof. Let $(x_0, y_0) \in X \times Y$ and E regular open (this will suffice as explained in Remark 4.3) in $X \times Y$. First we have $[(x_0, y_0)E]_x = \{y_0\eta\colon\ \eta \in E_{x_0^{-1}x}\}$ for all $x \in X$. Hence

$$\begin{aligned}\tau_\bullet\big((x_0, y_0)E\big) &= \big\{(x, y) \in X \times Y\colon\ y \in \tau\big(\big[(x_0, y_0)E\big]_x\big)\big\}\\ &= \big\{(x, y) \in X \times Y\colon\ y_0^{-1}y \in \tau(E_{x_0^{-1}x})\big\}\\ &= \big\{(x_0, y_0)(\xi, \eta)\colon\ \eta \in \tau(E_\xi)\big\} = (x_0, y_0)\tau_\bullet(E),\end{aligned}$$

i.e. we have shown $\tau_\bullet((x_0, y_0)E) = (x_0, y_0)\tau_\bullet(E)$.

For $(x, y) \in X \times Y$ put $\xi := x_0^{-1}x$ and $\eta := y_0^{-1}y$ and get first $[\tau_\bullet((x_0, y_0)E)]^y = x_0[\tau_\bullet(E)]^\eta$, therefore both or neither of $[\tau_\bullet((x_0, y_0)E)]^y$, $[\tau_\bullet(E)]^\eta$ have the Baire property and when they both have the property we have $\upsilon([\tau_\bullet((x_0, y_0)E)]^y) = x_0\upsilon([\tau_\bullet(E)]^\eta)$ and hence $x \in \upsilon([\tau_\bullet((x_0, y_0)E)]^y) \Leftrightarrow \xi \in \upsilon([\tau_\bullet(E)]^\eta)$. This implies

$$(\upsilon \odot \tau)\big((x_0, y_0)E\big) = \big\{(x_0, y_0)(\xi, \eta)\colon\ (\xi, \eta) \in (\upsilon \odot \tau)(E)\big\} = (x_0, y_0)\big((\upsilon \odot \tau)(E)\big). \qquad \square$$

Theorem 4.6. *If $w(Y) < \operatorname{add}(\mathcal{M}(X))$ and (Y, X) is a K-U pair, then $\upsilon \odot \varphi_Y \in \vartheta(\mathcal{M}(X \times Y))$ for arbitrary $\upsilon \in \vartheta(\mathcal{M}(X))$. In particular $\varphi_X \odot \varphi_Y \in \vartheta(\mathcal{M}(X \times Y))$ and $\varphi_{X\times Y} = \varphi_X \boxtimes \varphi_Y \leqslant \varphi_X \odot \varphi_Y$.*

Proof. It is immediate from Corollary 3.7 in connection with Proposition 4.4(ii) that $\upsilon \odot \varphi_Y \in \vartheta(\mathcal{M}(X \times Y))$ and $\varphi_X \boxtimes \varphi_Y \leqslant \varphi_X \odot \varphi_Y$. The equality $\varphi_{X\times Y} = \varphi_X \boxtimes \varphi_Y$ follows from [2, Proposition 3.1]. $\square$

Definition 4.7. For $\upsilon \in \mathcal{P}(X)^{\mathfrak{B}_c(X)}$ and $\tau \in \mathcal{P}(Y)^{\mathfrak{B}_c(Y)}$ we define the $\boxdot$*-product* $\upsilon \boxdot \tau : \mathfrak{B}_c(X \times Y) \to \mathcal{P}(X \times Y)$ by $\upsilon \boxdot \tau := (\upsilon \odot \tau)^m$.

Lemma 4.8. *For $\varphi \in F(\mathcal{M}(X \times Y))$ and $\upsilon \in O(\mathcal{M}(X))$ the following conditions hold true:*

(i) *From $\upsilon([\varphi(E)]^y) \supseteq [\varphi(E)]^y$ for all $E \in \mathfrak{B}_c(X \times Y)$ and all $y \in Y$ follows $\upsilon([\varphi^m(E)]^y) \supseteq [\varphi^m(E)]^y$ for all $E \in \mathfrak{B}_c(X \times Y)$ and all $y \in Y$.*

If (Y, X) is a K-U pair, then

(ii) *if for each $E \in \mathfrak{B}_c(X \times Y)$ exists a set $N_E \in \mathcal{M}(Y)$ such that $\upsilon([\varphi(E)]^y) \subseteq [\varphi(E)]^y$ for every $y \notin N_E$ then there exists a set $\widehat{N}_E \in \mathcal{M}(Y)$ such that $N_E \subseteq \widehat{N}_E$ and $\upsilon([\varphi^m(E)]^y) \subseteq [\varphi^m(E)]^y$ for every $y \notin \widehat{N}_E$*

and

(iii) *if for each $E \in \mathfrak{B}_c(X \times Y)$ exists a set $M_E \in \mathcal{M}(Y)$ such that $\upsilon([\varphi(E)]^y) = [\varphi(E)]^y$ for every $y \notin M_E$ then there exists a set $\widehat{M}_E \in \mathcal{M}(Y)$ such that $M_E \subseteq \widehat{M}_E$ and $\upsilon([\varphi^m(E)]^y) = [\varphi^m(E)]^y$ for every $y \notin \widehat{M}_E$.*

Proof. Ad (i): For all $E, F \in \mathfrak{B}_c(X \times Y)$ with $F \subseteq E$ we get $[\varphi^m(E)]^y \supseteq [\varphi(F)]^y$, hence $\upsilon([\varphi^m(E)]^y) \supseteq \upsilon([\varphi(F)]^y) \supseteq [\varphi(F)]^y$ for every $y \in Y$. This implies

$$\upsilon\big(\big[\varphi^m(E)\big]^y\big) \supseteq \big[\varphi^m(E)\big]^y \quad \text{for all } y \in Y.$$

Ad (ii): For $E \in \mathfrak{B}_c(X \times Y)$ we have $\varphi^m(E) = \varphi(E)$ a.e. $(\mathcal{M}(X \times Y))$. Since (Y, X) is a K-U pair there exists a set $\widetilde{N}_E \in \mathcal{M}(Y)$ with $[\varphi^m(E)]^y = [\varphi(E)]^y$ a.e. $(\mathcal{M}(Y))$ for every $y \notin \widetilde{N}_E$. Put $\widehat{N}_E := N_E \cup \widetilde{N}_E$. This implies $\upsilon([\varphi^m(E)]^y) = \upsilon([\varphi(E)]^y) \subseteq [\varphi(E)]^y \subseteq [\varphi^m(E)]^y$ for every $y \notin \widehat{N}_E$.

Ad (iii): For all $E, F \in \mathfrak{B}_c(X \times Y)$ with $F \subseteq E$ and all $y \notin M_E$ we have $\upsilon([\varphi^m(F)]^y) \supseteq \upsilon([\varphi(F)]^y) = [\varphi(F)]^y$ and this implies $\upsilon([\varphi^m(E)]^y) \supseteq [\varphi^m(E)]^y$. Put $\widehat{M}_E := M_E \cup \widehat{N}_E$. It then follows by (ii) the inverse equation for all $y \notin \widehat{M}_E$. $\square$

Theorem 4.9. *Let (X, Y) and (Y, X) be K-U pairs and let $\upsilon \in \vartheta(\mathcal{M}(X))$ and $\tau \in \vartheta(\mathcal{M}(Y))$ be arbitrary densities. Then*

(i) *$\upsilon \odot \tau \leqslant \upsilon \boxdot \tau \in \vartheta(\mathcal{M}(X \times Y)) \cap \upsilon \otimes \tau$ and $\varphi \geqslant \upsilon \boxdot \tau$ for all $\varphi \in \vartheta(\mathcal{M}(X \times Y))$ with $\varphi \geqslant \upsilon \odot \tau$;*
(ii) *$\upsilon \boxtimes \tau \leqslant \upsilon \boxdot \tau$;*
(iii) *$[(\upsilon \boxdot \tau)(E)]^y \subseteq \upsilon([(\upsilon \boxdot \tau)(E)]^y)$ for all $y \in Y$ and all $E \in \mathfrak{B}_c(X \times Y)$;*
(iv) *if υ and τ are strong, then $\upsilon \boxdot \tau$ is strong;*
(v) *for each $E \in \mathfrak{B}_c(X \times Y)$ there exists $M_E \in \mathcal{M}(Y)$ such that*

$$\upsilon\big(\big[(\upsilon \boxdot \tau)(E)\big]^y\big) = \big[(\upsilon \boxdot \tau)(E)\big]^y \quad \textit{for every } y \notin M_E;$$

and

(vi) *if $\upsilon \in \Lambda(\mathcal{M}(X))$, then for every $E \in \mathfrak{B}_c(X \times Y)$ there is a set $K_E \in \mathcal{M}(Y)$ such that $[(\upsilon \boxdot \tau)(E^c)]^y = ([(\upsilon \boxdot \tau)(E)]^c)^y$ for all $y \notin K_E$.*

Proof. Ad (i): All properties listed in this item with the exception of the product property $\upsilon \boxdot \tau \in \upsilon \otimes \tau$ are obvious from the Propositions 4.4 and 2.1.

To prove the product property, let us fix arbitrary $A \in \Sigma$ and $B \in T$. If $F \subseteq A \times B$, then $\tau_\bullet(F) \subseteq A \times \tau(B) = \tau_\bullet(A \times B)$, by Proposition 3.3(i). It follows that if $y \in \tau(B)$ is arbitrary and $[\tau_\bullet(F)]^y \in \mathfrak{B}_c(X)$, then $[(\upsilon \odot \tau)(F)]^y = \upsilon([\tau_\bullet(F)]^y) \subseteq \upsilon([A \times \tau(B)]^y) = \upsilon(A)$. Otherwise $[(\upsilon \odot \tau)(F)]^y = \emptyset$. Hence, $\upsilon \odot \tau(F) \subseteq \upsilon(A) \times \tau(B) = \upsilon \odot \tau(A \times B)$. And so $(\upsilon \boxdot \tau)(A \times B) \subseteq (\upsilon \odot \tau)(A \times B)$.

Conversely, by Lemma 4.2(iii) and Proposition 2.1(i) we get $\upsilon(A) \times \tau(B) = (\upsilon \odot \tau)(A \times B) \subseteq (\upsilon \boxdot \tau)(A \times B)$. Hence $\upsilon \boxdot \tau(A \times B) = \upsilon(A) \times \tau(B)$.

Ad (ii): Let us fix an arbitrary $E \in \mathfrak{B}_c(X \times Y)$. Then applying $\upsilon \boxdot \tau \in \upsilon \otimes \tau$ from (i) we get

$$\begin{aligned}(\upsilon \boxtimes \tau)(E) &= \bigcup\{\upsilon(A) \times \tau(B):\ A \times B \in \mathfrak{B}_c(X \times Y),\ A \times B \subseteq_{\mathcal{M}} E\}\\ &= \bigcup\{(\upsilon \boxdot \tau)(A \times B):\ A \times B \in \mathfrak{B}_c(X \times Y),\ A \times B \subseteq_{\mathcal{M}} E\}\\ &\subseteq (\upsilon \boxdot \tau)(E).\end{aligned}$$

Condition (iii) follows from Lemmas 4.2(ii) and 4.8(i), whereas (v) follows from Lemmas 4.2(ii) and 4.8(iii).

Condition (iv) follows in the same way as condition (vi) of Proposition 4.1 from [2].

(vi) is a direct consequence of Proposition 4.4(iii) since $(\upsilon \boxdot \tau) \geqslant (\upsilon \odot \tau)$. □

Proposition 4.10. *Suppose that X, Y are topological groups such that (X, Y) is a K-U pair. If $\upsilon \in \vartheta(\mathcal{M}(X))$ is left invariant for X and $\tau \in \vartheta(\mathcal{M}(Y))$ is left invariant for Y, then $\upsilon \boxdot \tau$ is left invariant for $X \times Y$.*

Proof. For $E \in \mathfrak{B}_c(X \times Y)$ and $(x_0, y_0) \in X \times Y$ we have $(\upsilon \boxdot \tau)((x_0, y_0)E) = \bigcup\{(\upsilon \odot \tau)(F):\ F \subseteq (x_0, y_0)E\} = \bigcup\{(\upsilon \odot \tau)((x_0, y_0)G):\ G \subseteq E\} = \bigcup\{(x_0, y_0)(\upsilon \odot \tau)(G):\ G \subseteq E\} = (x_0, y_0)\bigcup\{(\upsilon \odot \tau)(G):\ G \subseteq E\} = (x_0, y_0)(\upsilon \boxdot \tau)(E)$, where we used Proposition 4.5. □

Theorem 4.11. *If X, Y are Polish spaces, then*

(i) $\varphi_{X \times Y} = \varphi_X \boxtimes \varphi_Y \leqslant \varphi_X \boxdot \varphi_Y = \varphi_X \odot \varphi_Y \in \vartheta(\mathcal{M}(X \times Y)) \cap \varphi_X \otimes \varphi_Y$;
(ii) $[(\varphi_X \boxdot \varphi_Y)(E)]^y = \varphi_X([(\varphi_X \boxdot \varphi_Y)(E)]^y)$ *for all $y \in Y$ and $E \in \mathfrak{B}_c(X \times Y)$*;
(iii) $[(\varphi_X \boxdot \varphi_Y)(E)]_x \in \mathfrak{B}_c(Y)$ *for all $x \in X$ and all $E \in \mathfrak{B}_c(X \times Y)$.*

Proof. (i) follows from Theorem 4.6 and Lemma 4.2(iii). Condition (ii) follows from (i) and from Lemma 4.2(ii).

Ad (iii): Let us fix an arbitrary $E \in \mathfrak{B}_c(X \times Y)$. We then have $(\varphi_X \boxdot \varphi_Y)(E) \triangle \varphi_{X \times Y}(E) \in \mathcal{M}(X \times Y)$. Since (Y, X) is a K-U pair, there exists a set $M_1 \in \mathcal{M}(Y)$ such that

$$\big[(\varphi_X \boxdot \varphi_Y)(E) \triangle \varphi_{X \times Y}(E)\big]^y \in \mathcal{M}(X) \quad \text{for all } y \notin M_1.$$

But according to [2, Proposition 3.1], there exists a set $M_2 \in \mathcal{M}(Y)$ such that

$$\big[\varphi_{X \times Y}(E)\big]^y = \varphi_X\big(\big[\varphi_{X \times Y}(E)\big]^y\big) \quad \text{for all } y \notin M_2.$$

For $M := M_1 \cup M_2$ we get from the above

$$\varphi_X\big(\big[(\varphi_X \boxdot \varphi_Y)(E)\big]^y\big) = \big[\varphi_{X \times Y}(E)\big]^y \quad \text{for all } y \notin M. \tag{1}$$

Then for arbitrary $x \in X$ we get

$$\begin{aligned}\big[(\varphi_X \boxdot \varphi_Y)(E)\big]_x \cap M^c &= \big\{y \in Y:\ x \in \big[(\varphi_X \boxdot \varphi_Y)(E)\big]^y\big\} \cap M^c\\ &\overset{\text{(ii)}}{=} \big\{y \in Y:\ x \in \varphi_X\big(\big[(\varphi_X \boxdot \varphi_Y)(E)\big]^y\big)\big\} \cap M^c\\ &\overset{(1)}{=} \big\{y \in Y:\ x \in \big[\varphi_{X \times Y}(E)\big]^y\big\} \cap M^c\\ &= \big[\varphi_{X \times Y}(E)\big]_x \cap M^c.\end{aligned}$$

Since $[\varphi_{X \times Y}(E)]_x \cap M^c \in \mathfrak{B}_c(Y)$, we get $[(\varphi_X \boxdot \varphi_Y)(E)]_x \cap M^c \in \mathfrak{B}_c(Y)$, hence $[(\varphi_X \boxdot \varphi_Y)(E)]_x \in \mathfrak{B}_c(Y)$ for arbitrary $x \in X$ by completion of $\mathfrak{B}_c(Y)$. Consequently, condition (iii) holds true. □

Property (ii) of the above theorem improves the corresponding property for the $\boxtimes$-product $\varphi_X \boxtimes \varphi_Y$.

Theorem 4.12. *Assume the K-U property of (X, Y) and (Y, X). If $\rho \in \Lambda(\mathcal{M}(X))$ and $\sigma \in \Lambda(\mathcal{M}(Y))$ then, there exists $\pi_2 \in \Lambda(\mathcal{M}(X \times Y))$ such that*:

(i) $\pi_2 \in \rho \otimes \sigma$ *and* $\pi_2 \geqslant \rho \boxdot \sigma$;
(ii) $[\pi_2(E)]^y = \rho([\pi_2(E)]^y)$ *for all $y \in Y$ and $E \in \mathfrak{B}_c(X \times Y)$*;

(iii) *if ρ and σ are strong, then π_2 is strong*;
(iv) *for each $E \in \mathfrak{B}_c(X \times Y)$ there exists $M_E \in \mathcal{M}(Y)$ such that*

$$\left[\pi_2(E)\right]^y = \left[\rho \boxdot \sigma(E)\right]^y \quad \textit{for every } y \notin M_E.$$

Proof. Let

$$\Phi := \left\{\varphi \in \vartheta\left(\mathcal{M}(X \times Y)\right)\colon\ \forall y \in Y\ \forall E \in \mathfrak{B}_c(X \times Y)\ \left[\varphi(E)\right]^y \subseteq \rho\left(\left[\varphi(E)\right]^y\right) \text{ and } \rho \boxdot \sigma(E) \subseteq \varphi(E)\right\}.$$

Notice first that $\Phi \neq \emptyset$ since by Theorem 4.9 we have $\rho \boxdot \sigma \in \Phi$.

We consider Φ with inclusion as the partial order: $\varphi \leqslant \widetilde{\varphi}$ if $\varphi(E) \subseteq \widetilde{\varphi}(E)$ for each $E \in \mathfrak{B}_c(X \times Y)$. One can easily see that there exists a maximal element in Φ, which we denote by π_2. We shall prove first that if $E \in \mathfrak{B}_c(X \times Y)$ and $y \in Y$ then $[\pi_2(E)]^y \cup [\pi_2(E^c)]^y =_{\mathcal{M}} X$. So suppose that there is $H \in \mathfrak{B}_c(X \times Y)$ and $y_0 \in Y$ such that $W := \rho[([\pi_2(H)]^{y_0} \cup [\pi_2(H^c)]^{y_0})^c] \neq \emptyset$. Then set for each $E \in \mathfrak{B}_c(X \times Y)$

$$\left[\widehat{\pi}(E)\right]^y := \begin{cases} [\pi_2(E)]^y & \text{if } y \neq y_0, \\ [\pi_2(E)]^{y_0} \cup (W \cap [\pi_2(H \cup E)]^{y_0}) & \text{if } y = y_0. \end{cases}$$

It is clear, that $\pi_2(E) \subseteq \widehat{\pi}(E)$ for each $E \in \mathfrak{B}_c(X \times Y)$ and $\widehat{\pi} \in \vartheta(\mathcal{M}(X \times Y))$. It follows directly from the definition that $[\widehat{\pi}(H^c)]^{y_0} = [\pi_2(H^c)]^{y_0} \cup W \neq [\pi_2(H^c)]^{y_0}$ and so π_2 and $\widehat{\pi}$ are different densities. In order to get a contradiction with our hypothesis it is enough to show that $[\widehat{\pi}(E)]^{y_0} \subseteq \rho([\widehat{\pi}(E)]^{y_0})$, but this is immediate. If $E \in \mathfrak{B}_c(X \times Y)$, then

$$\begin{aligned}
\rho\left(\left[\widehat{\pi}(E)\right]^{y_0}\right) &= \rho\left(\left[\pi_2(E)\right]^{y_0}\right) \cup \rho\left(W \cap \left[\pi_2(H \cup E)\right]^{y_0}\right) \\
&\supseteq \left[\pi_2(E)\right]^{y_0} \cup \left[\rho(W) \cap \rho\left(\left[\pi_2(H \cup E)\right]^{y_0}\right)\right] \\
&\supseteq \left[\pi_2(E)\right]^{y_0} \cup \left(W \cap \left[\pi_2(H \cup E)\right]^{y_0}\right) \\
&= \left[\widehat{\pi}(E)\right]^{y_0}.
\end{aligned}$$

To finish the proof of the first part let us notice that $\widetilde{\pi}$ defined by $[\widetilde{\pi}(E)]^y := \rho([\pi_2(E)]^y)$ also is an element of Φ and so the condition (ii) is satisfied.

According to Theorem 4.9(vi) for each $E \in \mathfrak{B}_c(X \times Y)$ there exists $M_E \in \mathcal{M}(Y)$ such that

$$\left[(\rho \boxdot \sigma)\left(E^c\right)\right]^y = \left(\left[(\rho \boxdot \sigma)(E)\right]^c\right)^y \quad \text{for every } y \notin M_E. \tag{2}$$

Now, if $y \notin M_E$, then by (i) we have $[\pi_2(E)]^y \supseteq [(\rho \boxdot \sigma)(E)]^y$. Eq. (2) yields now the required equality. □

5. Non-existence results

There is now a natural question: Can $\rho \odot \sigma$ be a lifting at least for some liftings ρ and σ? We are going to show that in general the answer to this question is to the negative.

In the sequel we denote by $\mathcal{P}(\mathbb{N})$ the space of all subsets of $\mathbb{N}$ endowed with the ordinary product metric topology.

It follows in the same way as in Theorem 6.8 and Corollary 6.9 in [2] that the following two results hold true. We note that the proof of Theorem 6.8 in [2] was not presented in an entirely clear way. It is given as an immediate consequence of Proposition 6.7 of [2] but the obvious proof requires the version of Proposition 6.7 where y is the point $\bar{y}$ given by Theorem 6.8(j). The reason we have this is that the proof of Proposition 6.7 works as long as $\mathcal{U} = \{A \in \mathfrak{B}_c(Y)\colon\ y \in \theta(A)\}$ is not countably complete. By the proof of Proposition 6.4, this is true in the context of Theorem 6.8 for any choice of y because of the small cellularity.

Theorem 5.1. *Let X be a Baire separable metric space without isolated points. If $\rho \in \Lambda(\mathcal{M}(X))$ and $\varphi \in \vartheta(\mathcal{M}(X \times \mathcal{P}(\mathbb{N})))$ are such that for each $E \in \mathfrak{B}_c(X \times \mathcal{P}(\mathbb{N}))$ there exists a set $M_E \in \mathcal{M}(\mathcal{P}(\mathbb{N}))$ such that*

$$\left[\varphi(E)\right]^y = \rho\left(\left[\varphi(E)\right]^y\right) \quad \textit{for each } y \notin M_E,$$

then for each $x \in X$ there exists a set $E \in \mathfrak{B}_c(X \times \mathcal{P}(\mathbb{N}))$ such that

$$\left[\varphi(E)\right]_x \notin \mathfrak{B}_c\left(\mathcal{P}(\mathbb{N})\right).$$

Corollary 5.2. *Let X be a Baire separable metric space without isolated points. If ρ, σ and π_2 are liftings satisfying Theorem* 4.12 (*with $Y = \mathcal{P}(\mathbb{N})$*), *then for each $x \in X$ there exists $E \in \mathfrak{B}_c(X \times \mathcal{P}(\mathbb{N}))$ such that $[\pi_2(E)]_x \notin \mathfrak{B}_c(\mathcal{P}(\mathbb{N}))$.*

There exists also a set $E \in \mathfrak{B}_c(X \times \mathcal{P}(\mathbb{N}))$ such that

$$\left\{x \in X\colon\ \left[\pi_2(E)\right]_x \neq \sigma\left(\left[\pi_2(E)\right]_x\right)\right\} \notin \mathcal{M}(X).$$

Even if $\sigma \in \vartheta(\mathcal{M}(Y)) \setminus \Lambda(\mathcal{M}(Y))$ but $\rho \in \Lambda(\mathcal{M}(X))$, then there exists a set $E \in \mathfrak{B}_c(X \times \mathcal{P}(\mathbb{N}))$ such that $\{x \in X\colon\ [\rho \boxdot \sigma(E)]_x \neq \sigma([\rho \boxdot \sigma(E)]_x)\} \notin \mathcal{M}(X)$.

It follows from the above corollary, that Theorem 4.12 cannot be in general improved.

Proposition 5.3. *Let X be a separable metric space without isolated points. Then no $\sigma \in \Lambda(\mathcal{M}(X))$ generates $\mathcal{P}(\mathbb{N})$-measurable sections.*

Proof. Assume, if possible, that there exists $\sigma \in \Lambda(\mathcal{M}(X))$ generating $\mathcal{P}(\mathbb{N})$-measurable sections. It then follows by Proposition 3.4 that there exists a map

$$\psi \in \vartheta\big(\mathcal{M}\big(\mathcal{P}(\mathbb{N}) \times X\big)\big)$$

such that for each $E \in \mathfrak{B}_c(\mathcal{P}(\mathbb{N}) \times X)$ and all $y \in \mathcal{P}(\mathbb{N})$ we get $\sigma([\psi(E)]_y) = [\psi(E)]_y$. But then applying [2, Theorem 6.8], we infer that for each $x \in X$ there exists a set $E \in \mathfrak{B}_c(\mathcal{P}(\mathbb{N}) \times X)$ such that $[\psi(E)]^x \notin \mathfrak{B}_c(\mathcal{P}(\mathbb{N}))$, a contradiction to Proposition 3.6. □

Proposition 5.4. *Let (X, Y) and (Y, X) be K-U pairs, and let $\upsilon \in \Lambda(\mathcal{M}(X))$ and $\tau \in \Lambda(\mathcal{M}(Y))$ be arbitrary. Then the following conditions are equivalent.*

(i) *τ generates X-measurable sections*;
(ii) *$\upsilon \odot \tau \in \Lambda(\mathcal{M}(X \times Y))$.*

Proof. If τ generates X-measurable sections then for all $E \in \mathfrak{B}_c(X \times Y)$ and all $y \in Y$ we have $[\tau_\bullet(E)]^y \in \mathfrak{B}_c(X)$. It follows from Lemma 4.2(ii), that $\upsilon \odot \tau \in \vartheta(\mathcal{M}(X \times Y))$. According to Proposition 2.1(iii) it will suffice to show $(\upsilon \odot \tau)(E^c) = [(\upsilon \odot \tau)(E)]^c$ for all open subsets E of $X \times Y$ to get $\upsilon \odot \tau \in \Lambda(\mathcal{M}(X \times Y))$. We have

$$\begin{aligned}(x, y) \in (\upsilon \odot \tau)\big(E^c\big) \quad &\Leftrightarrow \quad x \in \upsilon\big(\big[\tau_\bullet\big(E^c\big)\big]^y\big) = \upsilon\big(\big[\tau_\bullet(E)^c\big]^y\big) = \upsilon\big(\big[\big[\tau_\bullet(E)\big]^y\big]\big)^c \\ &\Leftrightarrow \quad x \notin \upsilon\big(\big[\tau_\bullet(E)\big]^y\big) \quad \Leftrightarrow \quad (x, y) \in \big[(\upsilon \odot \tau)(E)\big]^c\end{aligned}$$

for all $(x, y) \in X \times Y$. Hence (i) implies (ii).

For the converse implication note that $(\upsilon \odot \tau)(E^c) = [(\upsilon \odot \tau)(E)]^c$ for all $E \in \mathfrak{B}_c(X \times Y)$ yields for each y either $[(\upsilon \odot \tau)(E)]^y \neq \emptyset$ or $[(\upsilon \odot \tau)(E^c)]^y \neq \emptyset$ and so $[\tau_\bullet(E)]^y \in \mathfrak{B}_c(X)$ for every y, i.e. τ generates X-measurable sections. □

The next result says that in many situations the Fubini type product as well as the box product of liftings is never a lifting.

Theorem 5.5. *Let X be a Baire separable metric space without isolated points. If $\tau \in \Lambda(\mathcal{M}(X))$ and $\upsilon \in \Lambda(\mathcal{M}(\mathcal{P}(\mathbb{N})))$. Then $\upsilon \odot \tau \in F(\mathcal{M}(\mathcal{P}(\mathbb{N}) \times X))$ but $\upsilon \odot \tau \notin \Lambda(\mathcal{M}(\mathcal{P}(\mathbb{N}) \times X))$.*

Proof. The existence of $\upsilon \odot \tau \in F(\mathcal{M}(\mathcal{P}(\mathbb{N}) \times X))$ follows from Proposition 4.4, and the rest from Proposition 5.3 in connection with Proposition 5.4. □

6. Densities in finite products

If $K, L \subseteq \mathbb{N}$ then write $K < L$ if $\sup\{k\colon k \in K\} < \inf\{l\colon l \in L\}$.

Throughout what follows, for an arbitrary $n \in \mathbb{N}$, $\langle X_i\rangle_{i\in[n]}$ is a finite sequence of topological spaces such that the product space $X_{[n]}$ is Baire and for each $k \in [n]$ with $1 < k \leqslant n$ the pair $(X_{[k-1]}, X_k)$ has the Kuratowski–Ulam property.

$\langle \upsilon_i\rangle_{i\in[n]}$ is a finite sequence such that $\upsilon_i \in \vartheta(\mathcal{M}(X_i))$ for $i \in [n]$.

Proposition 6.1. *If (X_1, X_2), (X_2, X_3), $(X_1, (X_2 \times X_3))$, $((X_1 \times X_2), X_3)$ are K-U pairs, then $\upsilon_1 \odot (\upsilon_2 \odot \upsilon_3) = (\upsilon_1 \odot \upsilon_2) \odot \upsilon_3$.*

Proof. Let us fix $E \in \mathfrak{B}_c(X_1 \times X_2 \times X_3)$. Then

$$\begin{aligned}(x_1, x_2, x_3) \in \big((\upsilon_1 \odot \upsilon_2) \odot \upsilon_3\big)(E) \quad &\Leftrightarrow \quad (x_1, x_2) \in (\upsilon_1 \odot \upsilon_2)\big(\big[(\upsilon_3)_\bullet(E)\big]^{x_3}\big) \\ &\Leftrightarrow \quad x_1 \in \upsilon_1\big(\big[(\upsilon_2)_\bullet\big(\big[(\upsilon_3)_\bullet(E)\big]^{x_3}\big)\big]^{x_2}\big) \\ &\Leftrightarrow \quad x_1 \in \upsilon_1\big\{\bar{x}_1 \in X_1\colon\ x_2 \in \upsilon_2\big(\big[\big[(\upsilon_3)_\bullet(E)\big]^{x_3}\big]_{\bar{x}_1}\big)\big\} \\ &\Leftrightarrow \quad x_1 \in \upsilon_1\big\{\bar{x}_1 \in X_1\colon\ y \in \upsilon\big\{\bar{x}_2 \in X_2\colon\ x_3 \in \upsilon_3\big([E]_{(\bar{x}_1, \bar{x}_2)}\big)\big\}\big\}\end{aligned}$$

and

$$\begin{aligned}(x_1,x_2,x_3)\in\big(\upsilon_1\odot(\upsilon_2\odot\upsilon_3)\big)(E) \quad&\Leftrightarrow\quad x_1\in\upsilon_1\big(\big[(\upsilon_2\odot\upsilon_3)_\bullet(E)\big]^{(x_2,x_3)}\big)\\ &\Leftrightarrow\quad x_1\in\upsilon_1\big\{\bar{x}_1\in X_1\colon\ (x_2,x_3)\in(\upsilon_2\odot\upsilon_3)\big([E]_{\bar{x}_1}\big)\big\}\\ &\Leftrightarrow\quad x_1\in\upsilon_1\big\{\bar{x}_1\in X_1\colon\ x_2\in\upsilon_2\big(\big[(\upsilon_3)_\bullet\big([E]_{\bar{x}_1}\big)\big]^{x_3}\big)\big\}\\ &\Leftrightarrow\quad x_1\in\upsilon_1\big\{\bar{x}_1\in X_1\colon\ x_2\in\upsilon_2\big(\big\{\bar{x}_2\in X_2\colon\ x_3\in\upsilon_3\big([E]_{(\bar{x}_1,\bar{x}_2)}\big)\big\}\big)\big\},\end{aligned}$$

where in the first argument, $(\upsilon_3)_\bullet$ is defined on subsets of $(X_1\times X_2)\times X_3$ while in the second argument, $(\upsilon_3)_\bullet$ is defined on subsets of $X_2\times X_3$. $\square$

Definition 6.2. If for each $k\leqslant n$ also the pair $(X_k, X_{[k-1]})$ is K-U, we define $\upsilon_1\odot\cdots\odot\upsilon_n$ recursively by $\upsilon_1\odot\cdots\odot\upsilon_{n+1}:=(\upsilon_1\odot\cdots\odot\upsilon_n)\odot\upsilon_{n+1}$ for all $n\in N$ and in case $n=2$ by Definition 4.1.

It follows by Proposition 4.4(i) and by induction on n, that $\upsilon_1\odot\cdots\odot\upsilon_n$ is a uniquely defined subdensity on $\mathfrak{B}_c(X_{[n]})$. We call $\odot_{i\in[n]}\upsilon_i$ the *$\odot$-product subdensity* of the densities υ_i.

Corollary 6.3. *The product $\upsilon_1\odot\cdots\odot\upsilon_n$ will remain unchanged if we put brackets in a different manner in this product, where accordingly assumptions over K-U are assumed.*

Theorem 6.4. *Assume that $n\in\mathbb{N}$ is quite arbitrary and that for all non-empty disjoint sets $K, L\subset[n]$ both pairs (X_K, X_L) and (X_L, X_K) have the Kuratowski–Ulam property. Then we have*

(i) *$\odot_{j\in[n]}\upsilon_j$ respects coordinates*;
(ii) *for each $J=[m]$, $1\leqslant m<n$, $E\in\mathfrak{B}_c(X_{[n]})$ and $x_{J^c}\in X_{J^c}$ we have*

$$\big[\odot_{j\in[n]}\upsilon_j(E)\big]_{x_{J^c}}=\odot_{j\in J}\upsilon_j\big(\big[\odot_{j\in[n]}\upsilon_j(E)\big]_{x_{J^c}}\big).$$

Proof. Ad (i): The proof of (i) follows by induction on $n\in\mathbb{N}\setminus\{1\}$.

Define $\eta_n:=\upsilon_1\odot\cdots\odot\upsilon_n$ recursively by $\eta_{n+1}:=\eta_n\odot\upsilon_{n+1}$ for $n\in\mathbb{N}$.

The case $n=2$ follows by Lemma 4.2(iii). For the inductive step from n to $n+1$, let $J:=\{i_1,\dots,i_k\}\subseteq[n+1]$ and $E=E_J\times X_{J^c}$, $E_J\in\mathfrak{B}_c(X_J)$. We distinguish two cases.

First case $n+1\in J^c$ implies $E=E_J\times X_{J^c\setminus\{n+1\}}\times X_{n+1}$, hence $\eta_{n+1}(E)=\eta_n(E_J\times X_{J^c\setminus\{n+1\}})\times X_{n+1}$ since $\eta_{n+1}\in\eta_n\otimes\upsilon_{n+1}$, implying $\eta_{n+1}(E)=E_J^*\times X_{J^c\setminus\{n+1\}}\times X_{n+1}$ by the inductive hypothesis.

Second case $n+1\in J$ implies $J^c\subseteq[n]$. Writing $X:=X_{[n]}$ we have

$$\begin{aligned}\eta_{n+1}(E)&=\big\{(x,x_{n+1})\in X\times X_{n+1}\colon\ x\in\eta_n\big(\big[(\upsilon_{n+1})_\bullet(E_J\times X_{J^c})\big]^{x_{n+1}}\big)\big\}\\ &=\big\{(x,x_{n+1})\in X\times X_{n+1}\colon\ x\in\eta_n\big(\big\{\bar{x}\in X\colon\ x_{n+1}\in\upsilon_{n+1}\big([E_J\times X_{J^c}]_{\bar{x}}\big)\big\}\big)\big\}\\ &=\big\{(x,x_{n+1})\in X\times X_{n+1}\colon\ x\in\eta_n\big(\big\{\widetilde{x}\in X_{[n]\setminus J^c}\colon\ x_{n+1}\in\upsilon_{n+1}\big([E_J]_{\widetilde{x}}\big)\big\}\times X_{J^c}\big)\big\}\\ &=\big\{(x,x_{n+1})\in X\times X_{n+1}\colon\ x\in X_{J^c}\times\eta_{J\setminus\{n+1\}}\big(\big\{\widetilde{x}\in X_{[n]\setminus J^c}\colon\ x_{n+1}\in\upsilon_{n+1}\big([E_J]_{\widetilde{x}}\big)\big\}\big)\big\}\\ &=X_{J^c}\times\big\{(\widehat{x},x_{n+1})\in X_{J\setminus\{n+1\}}\times X_{n+1}\colon\ \widehat{x}\in\eta_{J\setminus\{n+1\}}\big(\big[\upsilon_{n+1}\big([E_J]^{J\setminus\{n+1\}}\big)\big]^{x_{n+1}}\big)\big\}\\ &=X_{J^c}\times E_J^*,\end{aligned}$$

where $E_J^*\in\mathfrak{B}_c(X_J)$, because of K-U property of $(X_{J\setminus n+1}, X_{n+1})$ and Proposition 4.4(i). Hence η_{n+1} respects coordinates.

Condition (ii) follows from Corollary 6.3 and Proposition 4.4(i), if we notice that $\odot_{j\in[n]}\upsilon_j(E)=(\odot_{j\in J}\upsilon_j)\odot(\odot_{j\in J^c}\upsilon_j)(E)$. $\square$

Definition 6.5. Let $\langle X_i\rangle_{i\in I}$ be a non-empty family of topological spaces such that the product space X_I is Baire. For an arbitrary non-empty subset J of I and an arbitrary family $\langle\upsilon_i\rangle_{i\in J}$ of densities $\upsilon_i\in\vartheta(\mathcal{M}(X_i))$ define the map $\boxtimes_{i\in J}\upsilon_i\colon\ \mathfrak{B}_c(X_J)\to\mathfrak{B}_c(X_J)$ by means of

$$\boxtimes_{i\in J}\upsilon_i(E):=\bigcup\left\{\prod_{i\in K}\upsilon_i(A_i)\times X_{J\setminus K}\colon\ \prod_{i\in K}A_i\times X_{J\setminus K}\subseteq E\ \text{a.e.}\ \big(\mathcal{M}(X_J)\big),\ K\in Fin(J)\right\},$$

where $Fin(J)$ denotes the collection of all non-empty finite subsets of $J\subseteq I$. It follows from [2, Theorem 7.2], that $\boxtimes_{i\in J}\upsilon_i$ is a uniquely defined density on $\mathfrak{B}_c(X_J)$. We call $\boxtimes_{i\in J}\upsilon_i$ the *$\boxtimes$-product density* of the densities υ_i.

Definition 6.6. If for each $k\leqslant n$ also the pair $(X_k, X_{[k-1]})$ is K-U, we define $\upsilon_1\boxdot\cdots\boxdot\upsilon_n$ recursively by $\upsilon_1\boxdot\cdots\boxdot\upsilon_{n+1}:=(\upsilon_1\boxdot\cdots\boxdot\upsilon_n)\boxdot\upsilon_{n+1}$ for all $n\in N$ and in case $n=2$ by Definition 4.7.

It follows by Theorem 4.9 and by induction on n, that $\boxdot_{i\in[n]}\upsilon_i$ is a uniquely defined density on $\mathfrak{B}_c(X_{[n]})$. We call $\boxdot_{i\in[n]}\upsilon_i$ the *$\boxdot$-product density* of the densities υ_i.

Theorem 6.7. *Assume $n \in \mathbb{N}$ is arbitrary and that for all non-empty disjoint subsets K, L of $[n]$ both pairs (X_K, X_L) and (X_L, X_K) have the Kuratowski–Ulam property. Then*

(i) $\odot_{i\in[n]}\upsilon_j \leqslant \boxdot_{j\in[n]}\upsilon_j$;
(ii) $\boxtimes_{j\in[n]}\upsilon_j \leqslant \boxdot_{j\in[n]}\upsilon_j$;
(iii) *for each $J = [m]$, $1 \leqslant m < n$, $E \in \mathfrak{B}_c(X_{[n]})$ and $x_{J^c} \in X_{J^c}$ we have*

$$\big[\boxdot_{j\in[n]}\upsilon_j(E)\big]_{x_{J^c}} \subseteq \boxdot_{j\in J}\upsilon_j\big(\big[\boxdot_{j\in[n]}\upsilon_j(E)\big]_{x_{J^c}}\big);$$

(iv) *for each $J = [m]$, $1 \leqslant m < n$ and $E \in \mathfrak{B}_c(X_{[n]})$ there exists a set $M_E \in \mathcal{M}(X_{J^c})$ such that for all $x_{J^c} \notin M_E$ we have*

$$\big[\boxdot_{j\in[n]}\upsilon_j(E)\big]_{x_{J^c}} = \boxdot_{j\in J}\upsilon_j\big(\big[\boxdot_{j\in[n]}\upsilon_j(E)\big]_{x_{J^c}}\big);$$

(v) *if υ_j $(j \in [n])$ are strong, then $\boxdot_{j\in[n]}\upsilon_j$ is strong*;
(vi) $\boxdot_{j\in[n]}\upsilon_j \in \otimes_{j\in[n]}\upsilon_j$;
(vi) *for each non-empty proper subset M of $[n]$ and for each $E = E_M \times X_{M^c}$ with $E_M \in \mathfrak{B}_c(X_M)$ we have*

$$\boxdot_{i\in[n]}\upsilon_i(E_M \times X_{M^c}) \supseteq \boxdot_{j\in M}\upsilon_j(E_M) \times X_{M^c}.$$

If $\upsilon_j \in \Lambda(\mathcal{M}(X_j))$ for all $j \in [n]$, then there exists $\pi \in \Lambda(\mathcal{M}(X_{[n]})) \cap (\otimes_{j\in[n]}\upsilon_j)$ with $\pi \geqslant \boxdot_{j\in[n]}\upsilon_j$.

Proof. The existence of the density $\boxdot_{j\in K}\upsilon_j$ satisfying properties (i)–(vi) follows by Theorem 4.9 and by induction.

To show property (vii), let $E = E_M \times X_{M^c}$ with $E_M \in \mathfrak{B}_c(X_M)$. We get

$$\begin{aligned}\boxdot_{i\in[n]}\upsilon_i(E_M \times X_{M^c}) &= \bigcup\big\{\odot_{i\in[n]}\upsilon_i(F):\ F \subseteq E_M \times X_{M^c}\big\}\\ &\supseteq \bigcup\big\{\odot_{i\in[n]}\upsilon_i(F_M \times X_{M^c}):\ F_M \subseteq_{\mathcal{M}} E_M\big\}\\ &= \bigcup\big\{\odot_{j\in M}\upsilon_j(F_M) \times X_{M^c}:\ F_M \subseteq_{\mathcal{M}} E_M\big\}\\ &= \bigcup\big\{\odot_{j\in M}\upsilon_j(F_M):\ F_M \subseteq_{\mathcal{M}} E_M\big\} \times X_{M^c}\\ &= \boxdot_{j\in M}\upsilon_j(E_M) \times X_{M^c},\end{aligned}$$

hence property (vii) holds true. □

Theorem 6.8. *Let $\langle X_j\rangle_{j\in[n]}$ be a finite sequence of Polish spaces. Then*

(i) $\odot_{j\in[n]}\varphi_{X_j} \in \vartheta(\mathcal{M}(X_{[n]}))$;
(ii) $\varphi_{X_{[n]}} = \boxtimes_{j\in[n]}\varphi_{X_j} \leqslant \odot_{j\in[n]}\varphi_{X_j} = \boxdot_{j\in[n]}\varphi_{X_j}$;
(iii) $\boxdot_{j\in[n]}\varphi_{X_j}$ *respects coordinates*;
(iv) *for each $J = [m]$, $1 \leqslant m < n$, $E \in \mathfrak{B}_c(X_{[n]})$ and $x_{J^c} \in X_{J^c}$ we have*

$$\big[\boxdot_{j\in[n]}\upsilon_j(E)\big]_{x_{Jc}} = \boxdot_{j\in J}\upsilon_j\big(\big[\boxdot_{j\in[n]}\upsilon_j(E)\big]_{x_{Jc}}\big);$$

(v) *for each $J = [m]$, $1 \leqslant m < n$ we have*

$$\big[\boxdot_{j\in[n]}\upsilon_j(E)\big]_{x_J} \in \mathfrak{B}_c(X_{J^c}) \quad \textit{for all } E \in \mathfrak{B}_c(X_{[n]}) \textit{ and all } x_J \in X_J.$$

Proof. It follows by Theorems 4.6 and 6.4 that $\odot_{j\in[n]}\varphi_{X_j}$ is a density in $\vartheta(\mathcal{M}(X_{[n]}))$, hence condition (i) holds true.

Ad (ii): The first equality follows from [2, Proposition 3.1], the inequality follows by Theorem 4.6 and by induction, while the last equality follows by the definition of the $\boxdot$-product and by induction.

Condition (iii) is an immediate consequence of condition (ii) and of Theorem 6.4(i).

Condition (iv) follows from condition (ii) and from Theorem 6.4(ii).

Condition (v) consists of a slight modification of that of condition (iii) from Theorem 4.11. We have only to apply [2, Corollary 3.5] here, instead of [2, Proposition 3.1] there. □

7. Countably multiplicative densities and liftings

In this section, we address questions raised by the results in Section 6 of [2] and in [16].

We begin with some observations concerning [16]. The results of that paper are worded in the language of the structures which are called in [4] *measurable spaces with negligibles* which are triples $(X, \Sigma, \mathcal{I})$ where Σ is a σ-algebra of subsets of X and $\mathcal{I} \subseteq \Sigma$ is a σ-ideal. We also include in the definition the non-triviality condition $X \notin \mathcal{I}$. The notions of density and lifting are defined for measurable spaces with negligibles by replacing the sets having the property of Baire and the meager sets in our definitions by the members of the given σ-algebra and the given σ-ideal respectively. By analogy to category densities and liftings, we denote by $\vartheta(\mathcal{I})$ and $\Lambda(\mathcal{I})$ the collection of all densities and liftings on $(X, \Sigma, \mathcal{I})$, respectively.

Given densities δ, τ for measurable spaces with negligibles $(X, \Sigma, \mathcal{I})$, $(Y, T, \mathcal{J})$, respectively, we say that a set $E \subseteq X \times Y$ has (δ, τ)-*sub-invariant sections* if the vertical sections of E are in T, its horizontal sections are in Σ and for each $x \in X$, $y \in Y$ we have $E_x \supseteq \tau(E_x)$ and $E^y \supseteq \delta(E^y)$.

Given Boolean algebras A, B, we say that $\delta : A \to B$ is *continuous at zero*[3] if $\bigwedge_n \delta(a_n) = 0$ whenever $\{a_n\}$ is a decreasing sequence in A such that $\bigwedge_n a_n = 0$.

In [16, p. 475], the claim is made that if the quotient algebra $\Sigma/\mathcal{I}$ is ccc and non-atomic then a density δ for $(X, \Sigma, \mathcal{I})$ cannot be continuous at zero. As we verify in this section, the claim is in fact equivalent to the Souslin Hypothesis[4] and hence is independent of ZFC. This observation leads to some minor adjustments to the results of [16]. We describe the ones that are relevant to the present paper. The main result of [16] is the following theorem.

Theorem 7.1. *([16, Theorem 5]) Assume that $(X, \Sigma, \mathcal{I})$, $(Y, T, \mathcal{J})$ and $(X \times Y, \Xi, \mathcal{K})$ are measurable spaces with negligibles satisfying the following properties.*

- (*Rectangles with measurable sides are measurable*) $\Sigma \times T \subseteq \Xi$.
- (*Fubini property*) *For each* $K \in \mathcal{K}$,

$$\{x \in X:\ K_x \notin \mathcal{J}\} \in \mathcal{I} \quad \text{and} \quad \bigl\{y \in Y:\ K^y \notin \mathcal{I}\bigr\} \in \mathcal{J}.$$

Suppose also that we are given densities $\delta \in \vartheta(\mathcal{I})$, $\tau \in \vartheta(\mathcal{J})$ and $\varphi \in \vartheta(\mathcal{K})$ and that the sets $\varphi(E)$ have (δ, τ)-sub-invariant sections for $E \in \Xi$. Then at least one of the densities δ, τ is continuous at zero.

In [16] there was the additional assumption that the quotient algebras $\Sigma/\mathcal{I}$ and $T/\mathcal{J}$ are ccc, but this was not used in the proof.

In the setting of Baire topological spaces of interest to us, we get the following corollary. Recall that we assume throughout that $X \times Y$ is Baire.

Corollary 7.2. *Let X and Y be topological spaces such that both (X, Y) and (Y, X) are K-U pairs. Let*

$$\delta \in \vartheta\bigl(\mathcal{M}(X)\bigr), \qquad \tau \in \vartheta\bigl(\mathcal{M}(Y)\bigr), \qquad \varphi \in \vartheta\bigl(\mathcal{M}(X \times Y)\bigr).$$

If the sets $\varphi(E)$ have (δ, τ)-sub-invariant sections for $E \in \mathfrak{B}_c(X \times Y)$, then at least one of the densities δ, τ is continuous at zero.

Concerning the possibility of a density being continuous at zero, we note that the proof of Proposition 6.2 of [2], with only minor changes, gives the following statement.

Proposition 7.3. *Suppose Y is a regular Baire space in which some non-empty open set has a dense meager subset. If $\theta : \mathfrak{B}_c(Y) \to \mathfrak{B}_c(Y)$ satisfies* (L1) *and* (L2) *of Section* 1*, then θ is not continuous at zero.*

This proposition applies in particular to non-void Baire metric spaces without isolated points (cf. Remark 6.3 of [2]).

Corollary 7.4. *Let X and Y be non-void Tychonoff spaces without isolated points. If (X, Y) and (Y, X) are K-U pairs and each of X and Y has a dense meager subset, then there do not exist $\delta \in \vartheta(\mathcal{M}(X))$, $\tau \in \vartheta(\mathcal{M}(Y))$ and $\varphi \in \vartheta(\mathcal{M}(X \times Y))$ such that the sets $\varphi(E)$ have (δ, τ)-sub-invariant sections.*

Proof. Apply Proposition 7.3 and Corollary 7.2. $\square$

We now consider the question of when a measurable space with negligibles $(X, \Sigma, \mathcal{I})$ such that $\Sigma/\mathcal{I}$ is ccc can have a density which is continuous at zero. Relevant examples can be constructed using a standard topology on partial orders which we now recall. Let $(P, \leqslant)$ be a partial order. Equip P with the topology in which the basic open neighborhood of

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[3] In [16] the term $\mathcal{I}$-*continuous* is used for the property of a density $\delta : \Sigma \to \Sigma$ for a measurable space with negligibles $(X, \Sigma, \mathcal{I})$ that $\bigcap_{n=1}^{\infty} \delta(A_n) = \emptyset$ whenever $\{A_n\}$ is a decreasing sequence in Σ with $\bigcap_{n=1}^{\infty} A_n \in \mathcal{I}$. The fact that δ is a selector for the equivalence classes easily implies the equivalence of $\mathcal{I}$-continuity and continuity at zero in this context. If δ is moreover a lifting, then it is not hard to see that continuity at zero is equivalent to the countable multiplicativity requirement that $\delta(\bigcap_n A_n) = \bigcap_n \delta(A_n)$ whenever $\{A_n\}$ is a sequence in Σ and this is again equivalent to the countable additivity requirement, that is $\delta(\bigcup_n A_n) = \bigcup_n \delta(A_n)$ whenever $\{A_n\}$ is a sequence in Σ. An example of a density which is continuous at zero but not countably multiplicative can be constructed as follows. Start with a non-atomic measurable space with negligibles $(X, \Sigma, \mathcal{I})$ which has a countably multiplicative density δ. (See Example 7.12.) Fix any strictly (modulo $\mathcal{I}$) decreasing sequence $\{A_n\}$ in Σ with $\bigcap_n A_n \notin \mathcal{I}$. Add a new point p to get $\widetilde{X} = X \cup \{p\}$ and define $\widetilde{\Sigma} = \Sigma \cup \{E \cup \{p\}:\ E \in \Sigma\}$, $\widetilde{\mathcal{I}} = \{E \subseteq \widetilde{X}:\ E \cap X \in \mathcal{I}\}$. The density $\widetilde{\delta}$ for $(\widetilde{X}, \widetilde{\Sigma}, \widetilde{\mathcal{I}})$ defined by setting $\widetilde{\delta}(E) = \delta(E \cap X) \cup \{p\}$ when for some n we have $A_n \subseteq_{\mathcal{I}} E \cap X$, and $\widetilde{\delta}(E) = \delta(E \cap X)$ otherwise, is continuous at zero but not countably multiplicative.

[4] The Souslin Hypothesis states that every ccc dense linear ordering is separable, or equivalently no Souslin algebras exist.

$p \in P$ is the cone $U_p = \{q \in P\colon\ q \leqslant p\}$. A set is open in this topology precisely if it is downward closed. If $\mathcal{U}$ is a collection of open sets in P, then $\bigcap \mathcal{U}$ is open. It follows that if $\mathcal{U}$ is a collection of regular open sets in P, then $\bigcap \mathcal{U}$ is regular open. [Proof: Let $V = \bigcap \mathcal{U}$. For each $U \in \mathcal{U}$, we have $\operatorname{int}\operatorname{cl} V \subseteq \operatorname{int}\operatorname{cl} U = U$ and hence $V \subseteq \operatorname{int}\operatorname{cl} V \subseteq \bigcap \mathcal{U} = V$.]

If P equipped with this topology is a Baire space, then the canonical density on the category algebra of P preserves arbitrary intersections.

Let us say that a density which preserves arbitrary intersections is *completely multiplicative*.

The following example shows that in Proposition 6.4 of [2], "lifting" cannot be weakened to "density".

Example 7.5. Let $T = 2^{<\omega_1}$ (the set of all transfinite binary sequences of countable length), ordered by reverse inclusion. Then T, equipped with the partial order topology defined above, is a Baire space whose category algebra is non-atomic and has a completely multiplicative density.

Proof. T is Baire because if G_n, $n \in \mathbb{N}$, are dense open sets and U is a non-empty open set, then we can inductively choose $x_n \in U \cap G_n$ so that $x_n \subseteq x_{n+1}$. Then $p = \bigcup_n x_n$ satisfies $p \in U \cap \bigcap_n G_n$. The regular open algebra of T is non-atomic since every node in T has two immediate successors. $\square$

Example 7.6. Suppose there is a Souslin tree T, i.e., a tree of height ω_1 in which the chains and antichains are all countable. Then there is a non-atomic Baire ccc space whose category algebra has a completely multiplicative density.

We do not know whether the example can be made Tychonoff.

Proof. This follows by pruning T slightly and giving it the partial order topology corresponding to the reverse order on T. All of this is standard. To make this section self-contained, we recall the arguments. The levels of T are antichains and hence countable. It follows that uncountable subsets of T have elements of arbitrarily large height. The basic property of Souslin trees that we need is the following.

Fact 7.7. *Let $X \subseteq T$ be uncountable. Then there is a countable set $S \subseteq T$ such that for each $p \in S$, $U_p \cap X$ is countable and for each $p \in T \setminus \bigcup_{p\in S} U_p$, $U_p \cap X$ is uncountable.*

Proof. Let $A = \{p \in T\colon\ U_p \cap X$ is countable$\}$. Let S consist of the minimal nodes of A. Then S is an antichain and hence countable. S is as desired. $\square$

Fact 7.7 has the following consequences.

(a) There is a countable open set $A \subseteq T$ such that every open set of $T \setminus A$ is uncountable. In particular, since chains of T are countable, the regular open algebra of $T \setminus A$ is non-atomic.
Take $X = T$ in Fact 7.7. We have that $A = \bigcup_{p\in S} U_p$ is countable and hence for each $p \in T \setminus A$ we have that $U_p \setminus A$ is uncountable.

(b) If every non-empty open set of a Souslin tree T is uncountable then T is Baire.
The point is that every dense open set G includes all but countably many elements of T (and hence $U \cap \bigcap_n G_n$ includes all but countably many points of U for any non-empty open set U and dense open sets G_n). Indeed, let $X \subseteq T$ be uncountable. We will show that G cannot omit all of X. With S as in Fact 7.7, take any $q \in T \setminus \bigcup_{p\in S} U_p$ of height greater than the height of any element of S. (The difference is uncountable since it includes all but countably many points of X.) Then for any $r \geqslant q$, we have that $r \in T \setminus \bigcup_{p\in S} U_p$. (If $r \in U_p$ for some $p \in S$, then we must have either $p \leqslant q$ or $q \leqslant p$. The former is impossible because $q \notin U_p$. The latter is impossible because q was chosen to have height greater than then height of p.) Thus, $X \cap U_r$ is uncountable and in particular non-empty. This shows that $X \cap U_q$ is dense in U_q and hence has non-empty intersection with G.

Now given any Souslin tree T, we can replace T by $T \setminus A$ as in (a) to get a tree whose non-empty open sets are uncountable and whose regular open algebra is non-atomic. By (b), the resulting tree is Baire. This gives the desired example. $\square$

Proposition 7.8. *Let $(X, \Sigma, \mathcal{I})$ be a measurable space with negligibles. Assume that $\mathcal{A} = \Sigma/\mathcal{I}$ is ccc and that $\mathcal{A} \setminus \{0\}$ has a sequence of dense open sets* (*in the sense of the partial order topology defined above*) *$D_n \subseteq \mathcal{A} \setminus \{0\}$ such that $\bigcap_n D_n = \emptyset$. Then no selector $\theta : \mathcal{A} \to \Sigma$ is continuous at zero.*

Proof. Inductively choose maximal cellular families $A_n \subseteq D_n$ so that A_{n+1} refines A_n. There is a set Y with $X \setminus Y \in \mathcal{I}$ such that for each n, the sets $\theta(e) \cap Y$, for $e \in A_n$ are pairwise disjoint and for each $e \in A_{n+1}$, there is an $e' \in A_n$ such that $\theta(e) \cap Y \subseteq \theta(e') \cap Y$. Choose any point $x \in \bigcap_n (\bigcup \{\theta(e)\colon\ e \in A_n\}) \cap Y$. (There is such a point because Y and each of the unions is co-negligible.) For each n, there is unique $e_n \in A_n$ such that $x \in \theta(e_n)$. Then $\{e_n\}$ is decreasing, $\bigcap_n \theta(e_n)$ is non-empty but $\bigwedge_n e_n = 0$ since otherwise $\bigwedge_n e_n$ belongs to each D_n. $\square$

Notice that Proposition 7.8 generalizes a standard argument for the non-existence of a countably additive lifting for the measure algebra of a non-atomic probability space. (In this setting D_n consists of all non-zero elements of $\mathcal{A}$ of measure at most $1/n$.)

Recall that a Souslin algebra is a complete Boolean algebra B which is non-atomic, ccc and has the property that the intersection of any sequence of dense open sets of $B \setminus \{0\}$ contains a dense open set of $B \setminus \{0\}$. The existence of a Souslin algebra is equivalent to the existence of a Souslin tree and is independent of the axioms of ZFC. The next proposition records two simple well-known properties of Boolean algebras.

Proposition 7.9. *Let B be a complete Boolean algebra.*

(i) *If B is ccc and non-atomic and there is no $a \in B \setminus \{0\}$ such that $B \restriction a$ is Souslin, then $B \setminus \{0\}$ has a sequence of dense open sets $D_n \subseteq B \setminus \{0\}$ such that $\bigcap_n D_n = \emptyset$.*
(ii) *If $B \setminus \{0\}$ has a countable dense set, then there is no $a \in B \setminus \{0\}$ such that $B \restriction a$ is Souslin.*

Proof. (i) Each of the algebras $B \restriction a$ is ccc and non-atomic. Since they are not Souslin, it follows that below every element a of $B \setminus \{0\}$, there is a sequence of dense open sets $\{D_n\colon n \in \mathbb{N}\}$ of $(B \restriction a) \setminus \{0\}$ and a non-zero $b \leqslant a$ below which there are no elements of $\bigcap_n D_n$. We thus get a maximal cellular family F of elements $b \in B \setminus \{0\}$ for which there are dense-below-b open sets $D_n^b \subseteq \{c \leqslant b\colon\ c \neq 0\}$, $n \in \mathbb{N}$, such that $\bigcap_n D_n^b = \emptyset$. The sets $D_n = \bigcup_{b \in F} D_n^b$ are dense open in $B \setminus \{0\}$ and have empty intersection.

(ii) Each $B \restriction a$ has a countable dense set, so it suffices to show that B itself is not Souslin. B is ccc since $B \setminus \{0\}$ has a countable dense set. We may assume that B is also non-atomic since otherwise we are done by the definition of Souslin algebra. Let $\{a_n\colon n \in \mathbb{N}\}$ be a dense set in $B \setminus \{0\}$. Let D_n be the downward closure in $B \setminus \{0\}$ of the set of atoms of the algebra generated by $\{a_1, \ldots, a_n\}$. Each D_n is dense open and $\bigcap_{n=1}^\infty D_n = \emptyset$. Thus, B is not Souslin. $\square$

We now verify that Corollary 6 of [16] is correct. (Its proof in [16] contains an implicit error.) This result does not assume any separation properties for the spaces X and Y. If they are Tychonoff, then the result follows more readily from Corollary 7.4 using the fact that spaces with a countable π-base are separable.

Corollary 7.10. *Let X and Y be topological spaces each having a countable π-base and a non-atomic regular open algebra. Then there do not exist $\delta \in \vartheta(\mathcal{M}(X))$, $\tau \in \vartheta(\mathcal{M}(Y))$ and $\varphi \in \vartheta(\mathcal{M}(X \times Y))$ such that the sets $\varphi(E)$ have (δ, τ)-sub-invariant sections.*

Proof. The existence of the countable π-bases ensures that (X, Y) and (Y, X) are K-U pairs. (See for example the comments following Theorem 15.1 in [14].) The existence of the countable π-bases is also precisely the hypothesis of Proposition 7.9(ii) for the regular open algebras of X and Y. By Proposition 7.9 ((ii) and (i)), we can apply Proposition 7.8 to conclude that δ and τ are not continuous at zero. Now apply Corollary 7.2. $\square$

In [2, Question 6.1] asks whether the category algebra of a non-empty Baire space without isolated points can have a lifting which is countably additive. The question was answered affirmatively by D.H. Fremlin. With his permission we include the example here. Fremlin's example was not compact. For the following compact version the authors acknowledge a helpful discussion with W.A.R. Weiss. The example uses a compact cardinal. A regular uncountable cardinal κ is called *compact* if every κ-complete filter on any set S can be extended to a κ-complete ultrafilter on S. (U is κ-complete if $\bigcap_{\alpha<\lambda} A_\alpha \in U$ whenever $A_\alpha \in U$ for $\alpha < \lambda < \kappa$.) The text [9] has the basic facts about these, but we only need the definition. It was shown in [2] that the construction of an example requires a measurable cardinal. In terms of consistency strength, a compact cardinal is much more than a measurable. We do not know whether an example can be constructed from just a measurable cardinal.

Example 7.11. There is a compact Hausdorff space without isolated points whose category algebra has a countably additive lifting.

Proof. Fix a compact cardinal κ and define the space $X = \{0, 1\}^\kappa$ equipped with the order topology induced by the lexicographic order. The order is Dedekind complete with smallest and largest elements the constant sequences 0 and 1, respectively. Notice that for each point p we have that at least one of the sets

$$\{i < \kappa\colon\ p(i) = 0\}, \qquad \{i < \kappa\colon\ p(i) = 1\}$$

has cardinality κ. In the first case $(p, 1]$ has coinitiality κ (i.e., κ is the minimum cardinality of a set in $(p, 1]$ with no lower bound) and in the second $[0, p)$ has cofinality κ. Notice also that if $x < y$ are adjacent (i.e., there is no z such that $x < z < y$) then there is an $\alpha < \kappa$ such that

(i) $x(i) = y(i)$ for $i < \alpha$,
(ii) $x(\alpha) = 0$, $y(\alpha) = 1$,
(iii) $x(i) = 1$, $y(i) = 0$ for $\alpha < i < \kappa$.

Thus, $[0, x)$ has cofinality κ and $(y, 1]$ has coinitiality κ. To get the desired lifting, proceed as follows.

(a) For $p \in X$, define a κ-complete filter F_p extending the neighborhood filter at p.
If $\{i < \kappa\colon\ p(i) = 0\}$ has cardinality κ, let $I_p = (p, 1]$. Otherwise, let $I_p = [0, p)$. Then let F_p denote the filter generated by the neighborhoods of p and I_p.

(b) The filter of dense open sets is κ-complete.
Let G_α, $\alpha < \lambda < \kappa$, be dense open sets. For $x < y$ which are not adjacent we need to check that $\bigcap_{\alpha<\lambda} G_\alpha$ contains a non-empty open subinterval of (x, y). Recursively define $\langle x_\alpha\colon\ \alpha \leqslant \lambda\rangle$ and $\langle y_\alpha\colon\ \alpha \leqslant \lambda\rangle$ so that
- $x_0 = x$, $y_0 = y$,
- $\alpha < \beta \leqslant \lambda$ implies $x_\alpha < x_\beta < y_\beta < y_\alpha$,
- $(x_\alpha, y_\alpha) \subseteq G_\alpha$.

The induction continues at a limit stage α because if we let $x'_\alpha = \sup_{\beta<\alpha} x_\beta$ and $y'_\alpha = \inf_{\beta<\alpha} y_\beta$ then the cofinality of $[0, x'_\alpha)$ and the coinitiality of $(y'_\alpha, 1]$ are both $< \kappa$ and hence $x'_\alpha = y'_\alpha$ is impossible. Thus $x'_\alpha < y'_\alpha$ and the two points are not adjacent because then $[0, x'_\alpha)$ would have cofinality κ as noted above.

(c) For each point p, the filter generated by F_p and the dense open sets is κ-complete, so it extends to a κ-complete ultrafilter $\widetilde{F}_p$. Define a lifting from these ultrafilters in the usual way. The κ-completeness of the filters gives the κ-additivity of the lifting. □

We can extract from the foregoing proof a similar example for densities which can be obtained in ZFC.

Example 7.12. There is a non-empty compact Hausdorff space without isolated points whose category algebra has a countably multiplicative density.

Proof. This is similar to Example 7.11 except that we now require of κ only that it is a regular uncountable cardinal. The formula for the density θ is $\theta(E) = \{p \in X\colon\ E \in \widetilde{F}_p\}$, where this time $\widetilde{F}_p$ is the filter generated by F_p and the dense open sets rather than an ultrafilter extending it. The resulting density is κ-multiplicative. □

References

[1] M.R. Burke, Consistent liftings, unpublished notes of 1995/01/23.
[2] M.R. Burke, N.D. Macheras, K. Musiał, W. Strauss, Category product densities and liftings, Topology Appl. 153 (2006) 1164–1191.
[3] P. Erdös, J.C. Oxtoby, Partitions of the plane into sets having positive measure in every non-null product set, Trans. Amer. Math. Soc. 79 (1955) 91–102.
[4] D.H. Fremlin, Measure-additive coverings and measurable selectors, Dissertationes Math. 260 (1987).
[5] D.H. Fremlin, Measure Theory, vol. 3, 2002, Torres Fremlin (Ed.).
[6] D.H. Fremlin, T. Natkaniec, I. Recław, Universally Kuratowski–Ulam spaces, Fund. Math. 165 (2000) 239–247.
[7] J. Gapaillard, Relevements monotones, Arch. Math. 24 (1973) 169–178.
[8] S. Graf, A selection theorem for Boolean correspondences, J. Reine Angew. Math. 295 (1977) 169–186.
[9] T. Jech, Set Theory, Academic Press, New York, 1978.
[10] D. Maharam, Category, Boolean algebras and measures, in: Lecture Notes in Math., vol. 609, Springer-Verlag, 1977, pp. 124–135.
[11] K. Musiał, W. Strauss, N.D. Macheras, Product liftings and densities with lifting invariant and density invariant sections, Fund. Math. 166 (2000) 281–303.
[12] K. Musiał, W. Strauss, N.D. Macheras, Fubini type products for densities and liftings, J. Theoret. Probab. 20 (2007) 545–560.
[13] J.C. Oxtoby, Cartesian products of Baire spaces, Fund. Math. XLIX (1961) 157–166.
[14] J.C. Oxtoby, Measure and Category, Springer-Verlag, New York, Heidelberg, Berlin, 1980.
[15] W. Strauss, N.D. Macheras, K. Musiał, Liftings, in: Handbook of Measure Theory, Elsevier, 2002, pp. 1131–1184.
[16] W. Strauss, N.D. Macheras, K. Musiał, Non-existence of certain types of liftings and densities in product spaces with σ-ideals, Real Anal. Exchange 29 (1) (2003/2004) 473–479.