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Splitting of Conditional Expectations and Liftings in Product Spaces II

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Abstract

Let $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ be two probability spaces, R be their skew product on the product σ-algebra $\mathfrak{A} \otimes \mathfrak{B}$ and $\{(\mathfrak{A}_y, S_y) \colon y \in Y\}$ be a Q-disintegration of R. Then, let $\mathfrak{A} \circledast \mathfrak{B}$ be the σ-algebra generated $\mathfrak{A} \otimes \mathfrak{B}$ and by the family $\mathcal{M} := \{E \subset X \times Y \colon \exists\, N \in \mathfrak{B}_0\ \forall\, y \notin N\ \widehat{S_y}(E^y) = 0\}$ and $R_\circledast$ be the extension of R such that $\mathcal{M}$ becomes the family of R_*-zero sets ($\widehat{S_y}$ is the completion of S_y and $\mathfrak{B}_0 = \{B \in \mathfrak{B} : Q(B) = 0\}$). We prove that there exists a lifting π on $\mathcal{L}^\infty(R_\circledast)$ and liftings σ_y on $\mathcal{L}^\infty(\widehat{S_y})$, $y \in Y$, such that sections of π determined by Y are lifting invariant (in particular, the sections are measurable), i.e., $[\pi(f)]^y = \sigma_y\Big([\pi(f)]^y\Big)$ for every $y \in Y$ and every $f \in \mathcal{L}^\infty(R_\circledast)$. In general, if π is an arbitrary lifting on the product, then some sections of $\pi(f)$ may be even nonmeasurable. The main novelty of my paper lies in expanding the domain of the measure in the product to $\mathfrak{A} \circledast \mathfrak{B}$ and constructing on such a much larger abstract space the suitable lifting. Such expansions used to be made only in case of topological spaces, where product of marginal Borel sets was replaced by the Borel subsets of the product space. However, several topological technics are then applied, not approachable in the abstract case. The main theorem is a generalization of earlier lifting results, where either separability of $\mathfrak{A}$ in the Frechet–Nikodým pseudometric was assumed or $R \ll P \times Q$. In case of a separable P and in the case when $R \ll P \times Q$, a characterization of stochastic processes possessing an equivalent measurable version is presented. The theorem is a strong generalization of earlier results (see the introduction) where it was proved only that the lifting modification of a measurable stochastic process (via the lifting constructed there) is again measurable.

Keywords: disintegration; regular conditional probability; lifting

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1. Introduction

Let $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ be probability spaces and π_X, π_Y be the projections of $X \times Y$ onto X and Y, respectively. The algebra on $X \times Y$ generated by $\pi_X^{-1}(\mathfrak{A})$ and $\pi_Y^{-1}(\mathfrak{B})$ is denoted by $\mathfrak{A} \times \mathfrak{B}$ and the σ-algebra generated by $\mathfrak{A} \times \mathfrak{B}$ is denoted by $\mathfrak{A} \otimes \mathfrak{B}$. A measure R on $(X \times Y, \mathfrak{A} \otimes \mathfrak{B})$ such that $\pi_X(R) = P$ and $\pi_Y(R) = Q$ is called a skew product of P and Q. We will use the notation $R \in P \circledast Q$. $\widehat{P}$ is the completion of P and $\mathfrak{A}_0 = \{A \in \mathfrak{A} : P(A) = 0\}$. If $E \subset X \times Y$, then $E^y := \{x \in X : (x, y) \in E\}$. If $f : X \times Y \to \mathbb{R}$, then $f^y(x) := f(x, y)$.

Assume that $\{(\mathfrak{A}, S_y) : y \in Y\}$ is a regular conditional expectation on $\mathfrak{A}$ with respect to Q (see Definition 1). The following problem was investigated in [1–3]: When does a lifting π for $\widehat{R}$ exist and liftings σ_y for $\widehat{S_y}$ such that $[\pi(E)]^y = \sigma_y([\pi(E)]^y)$ for every $E \in \mathfrak{A} \otimes \mathfrak{B}$

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and $y \in Y$. In other words, when the range of a set $E \in \mathfrak{A} \otimes \mathfrak{B}$ via lifting π has all Y-sections $[\pi(E)]^y = \{x \in X : (x, y) \in \pi(E)\}$ lifting invariant with respect to a given a priori liftings σ_y? The problem was solved in [2] in case of the direct product $P \times Q$ of P and Q and in [1] in case of R absolutely continuous with respect to $P \times Q$.

In this work, I investigate the case when regular conditional probability is replaced by the disintegration in the sense of [4] (see Definition 1). In the classical approach, disintegration often coincides with regular conditional probability, but the disintegration described in Definition 1 is more general. Each regular conditional probability is a desintegration but not conversely. Instead of the product σ-algebra, I consider the new σ-algebra $\mathfrak{A} \circledast \mathfrak{B}$ which is the σ-algebra generated by $\mathfrak{A} \otimes \mathfrak{B}$ and the family $\mathcal{M} := \{E \subset X \times Y\colon \exists\ N \in \mathfrak{B}_0\ \forall\ y \notin N\ \widehat{S_y}(E^y) = 0\}$. Then, I define an extension $R_\circledast$ of R on the whole $\mathfrak{A} \circledast \mathfrak{B}$ by the formula $R_\circledast(W) = R(V)$, whenever $V \in \mathfrak{A} \otimes \mathfrak{B}$ and $W \triangle V \in \mathcal{N}$. The measure $R_\circledast$ is complete. Then, I describe the procedure of building a lifting $\pi : \mathfrak{A} \circledast \mathfrak{B} \to \mathfrak{A} \circledast \mathfrak{B}$ such that $[\pi(E)]^y = \sigma_y\big([\pi(E)]^y\big)$ for every $E \in \mathfrak{A} \circledast \mathfrak{B}$ and $y \in Y$. The problem was examined in [5], but the proof was mistaken. It is the aim of this work to present the correct version of the lifting results presented in [5]. For the sake of completeness, I present, here, complete proofs.

In the proof of the main result, very important role plays splitting of the conditional expectation in the product space into a family of conditional expectations taken with respect to one of the coordinate spaces. More precisely, if $\mathfrak{C} \subset \mathfrak{A}$ and f is a bounded function measurable with respect to $\mathfrak{A} \otimes \mathfrak{B}$, then each version $\mathbb{E}_{\mathfrak{C} \otimes \mathfrak{B}}(f)$ has the following property: $[\mathbb{E}_{\mathfrak{C} \otimes \mathfrak{B}}(f)]^y$ is a version of $\mathbb{E}^y_{\mathfrak{C}}(f^y)$ (=the conditional expectation of f^y with respect to $\mathfrak{C}$ and S_y) for Q-a.e. $y \in Y$. The exceptional set depends on f. Theorem 1 gives the real possibility of calculating fiberwise conditional probabilities. It is, perhaps, the proper place to say that in case of σ-algebras with non-separable measured algebras, two versions of a regular conditional probability may be really different. If $(S_y)_{y \in Y}$, $(T_y)_{y \in Y}$ are two rcp on $(X, \mathfrak{A}, P)$, then clearly $P(\{y \in Y : S_y(A) \neq T_y(A)\}) = 0$ for each $A \in \mathfrak{A}$. Yet it may be impossible to find a set $N \in \mathfrak{B}_0$ such that $S_y(A) = T_y(A)$ for every $A \in \mathfrak{A}$ and every $y \notin N$. That can be interpreted as non-existence of the canonical conditional probability.

I assume that the disintegration is given in advance. Pachl [4] proved that a disintegration always exists in case of a compact (in the sense Marczewski [6]) measure P and arbitrary Q. That is a huge class of probabilities including Radon measures, probabilities on standard spaces and arbitrary products of such measures. In the literature, quite often the measure R on the product space is defined not on the product σ-algebra but on Borel subsets of the product of topological spaces, which is usually much larger. That is beyond the topic of the current paper and will be the subject of a separate investigation. When one is interested in the results of this paper only in case of regular conditional probabilities, then there are several papers guaranteeing their existence and describing several properties (cf. [7–11]).

In order to prove existence of a very special lifting on the σ-algebra $\mathfrak{A} \circledast \mathfrak{B}$, I use in its construction the whole family of liftings $\{\sigma_y : y \in Y\}$, each σ_y defined on the completion of $\mathfrak{A}_y$ with respect to S_y. Maharam [12] proved that liftings exist on an arbitrary complete σ-finite measure space. Therefore, the main result (Theorem 2) is formulated for complete measures. In case of non-complete measure spaces, sometimes a lifting exists, sometimes not and sometimes its existence depends on additional axioms. The lifting is a tool that permits it to overcome the "almost sure" nature of sets and functions. One of the main advantage of lifting is the following fact: if ρ is a lifting on $(X, \mathfrak{A}, P)$ and $\{A_t : t \in T\} \subset \mathfrak{A}$ is an arbitrary collection of sets satisfying the inclusion $A_t \subset \rho(A_t)$, then $\bigcup_{t \in T} A_t$ is measurable with respect to the completion of P and $\bigcup_{t \in T} A_t \subset \rho(\bigcup_{t \in T} A_t)$. The statement remains valid if the lifting is replaced by a lower density (see the beginning of

Section 3). In the language of bounded measurable functions, it looks like the following: If $\{f_t : t \in T\}$ is essentially bounded collection of measurable functions on a $(X, \mathfrak{A}, P)$ satisfying the pointwise inequality $f_t \leq \rho(f_t)$, then (pointwise) $\sup_{t\in T} f_t$ is a $\widehat{P}$-measurable function and $\sup_{t\in T} f_t \leq \rho(\sup_{t\in T} f_t)$, pointwise. The advanced presentation of the liftings and several applications can be found in [13,14]. A basic approach can be found in [15].

Finally, I apply the lifting to the process of regularization of stochastic processes: When a stochastic process possesses an equivalent measurable version (sometimes also separable). There are several papers devoted to the problem (cf. [16–20]).

In particular, Talagrand proved in [19] that if $\{\xi_y : y \in Y\}$ is a stochastic process on $(X, \mathfrak{A}, P)$ and Y is endowed with a separable pseudometric, then the process possesses a separable modification (produced with the help of so called consistent lifting). Instead of the consistent liftings, I apply admissibly generated liftings (each consistent lifting is also admissibly generated), which always exist on complete probability spaces (see Definition 2). In [20], Talagrand presents an example of a process and a lifting that produces a non-measurable version of the measurable process. Applying the main lifting result of this paper, I prove that in case when either $\mathfrak{A}$ is separable in the Fréchet–Nikodým pseudo-metric with respect to P or $R \ll P \times Q$, then the process has an equivalent measurable version if and only if it is measurable with respect to $\mathfrak{A} \circledast \mathfrak{B}$. That is an essential strengthening of [3] Theorem 5.1 and [1] Theorem 6.1, where it was proved only that the constructed there lifting of a measurable process preserves measurability.

2. Preliminaries

Throughout this paper, $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ are fixed probability spaces. The notation is as presented in the introduction. If $(Z, \mathfrak{Z}, T)$ is a measure space, then $(Z, \widehat{\mathfrak{Z}}, \widehat{T})$ denotes its completion and $\mathfrak{Z}_0 := \{A \in \mathfrak{Z} : T(A) = 0\}$. T^* and T_* are respectively the outer and inner measures induced by T. $\mathcal{L}^\infty(T)$ is the collection of bounded T-measurable real-valued functions.

Definition 1. *Let $R \in P \circledast Q$ be a probability measure. Assume that for every $y \in Y$ there exists a σ-algebra $\mathfrak{A}_y \subset \mathfrak{A}$ and a probability S_y on $\mathfrak{A}_y$ with the following properties:*

(Dis1) *for each $A \in \mathfrak{A}$ there exists $N \in \mathfrak{B}_0$ such that $A \in \mathfrak{A}_y$ for all $y \notin N$ and the function $Y \setminus N \ni y \longrightarrow S_y(A)$ is $\mathfrak{B}$-measurable on $Y \setminus N$;*

(Dis2) *If $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, then*

$$\int_B S_y(A)\, dQ(y) = R(A \times B)\,.$$

The family $\mathbb{S} := \{(\mathfrak{A}_y, S_y) : y \in Y\}$ is called a Q-disintegration of R (or a disintegration of R with respect to Q). If $\mathfrak{A}_y = \mathfrak{A}$ for all $y \in Y$, then $\{(\mathfrak{A}, S_y) : y \in Y\}$ is called a regular conditional probability on $\mathfrak{A}$ with respect to Q.

We say that P is approximated by a family $\mathfrak{D} \subset \mathfrak{A}$ (or P is inner regular with respect to $\mathfrak{D}$) if for each $A \in \mathfrak{A}$ and $\varepsilon > 0$ there exists $D \in \mathfrak{D}$ such that $D \subset A$ and $P(A \setminus D) < \varepsilon$. $\mathfrak{D}$ is compact, if $\bigcap_n D_n = \emptyset$ for each decreasing sequence $\{D_n \in \mathfrak{D} : n \in \mathbb{N}\}$ yields $\bigcap_{n=1}^m D_n = \emptyset$ for a certain $m \in \mathbb{N}$. P is compact if it is approximated by a compact family (see [6]).

Disintegration not always exists. The following result of Pachl describes the relation between compactness and existence of disintegration. It is also a partial justification of investigating of behaviour of liftings in product spaces. It proves that the class of spaces admitting disintegrations is very large, in particular it contains all compact measures.

Proposition 1 ([4] Theorem 2.2, Theorem 3.5)**.** *Let $(X, \mathfrak{A}, P)$ be a probability space. P is compact if and only if for every complete probability space $(Y, \mathfrak{B}, Q)$ and every $R \in P \circledast Q$ there exists a Q-disintegration of R. If P is compact, then the disintegration $\mathbb{S} = \{(\mathfrak{A}_y, S_y) : y \in Y\}$ can be chosen in such a way that P and all measures S_y are approximated also by the same compact class $\mathcal{K} \subset \bigcap_{y \in Y} \mathfrak{A}_y$.*

Let

$$\mathcal{N} := \{E \subset X \times Y \colon \forall\, y \in Y\ E^y \in \mathfrak{A}_y\ \exists\, N \in \mathfrak{B}_0\ \forall\, y \notin N\ S_y(E^y) = 0\}$$
$$\widehat{\mathcal{N}} := \{E \subset X \times Y \colon \forall\, y \in Y\ E^y \in \widehat{\mathfrak{A}_y}\ \exists\, N \in \mathfrak{B}_0\ \forall\, y \notin N\ \widehat{S}_y(E^y) = 0\}$$

Elements of $\mathcal{N}$ and $\widehat{\mathcal{N}}$ form σ-ideals in $\mathcal{P}(X \times Y)$. If $\mathfrak{C} \subset \mathfrak{A}$ is a σ-algebra, then $\mathfrak{C} \divideontimes_{\mathbb{S}} \mathfrak{B} := \{W \triangle N \colon W \in \mathfrak{C} \otimes \mathfrak{B}\ \ \&\ N \in \mathcal{N}\}$ and $\mathfrak{C} \divideontimes_{\widehat{\mathbb{S}}} \mathfrak{B} := \{W \triangle N \colon W \in \mathfrak{C} \otimes \mathfrak{B}\ \ \&\ N \in \widehat{\mathcal{N}}\}$ are in general larger than $\mathfrak{A} \otimes \mathfrak{B}$. S_y-measurability ($\widehat{S}_y$-measurability) of E^y for every element of $E \in \mathfrak{C} \divideontimes_{\mathbb{S}} \mathfrak{B}$ ($E \in \mathfrak{C} \divideontimes_{\widehat{\mathbb{S}}} \mathfrak{B}$) is the property needed for the main result of the paper.

Because of the condition (Dis1), one cannot define in the same way the algebra $\mathfrak{A} \divideontimes_{\mathbb{S}} \mathfrak{B}$ or $\mathfrak{A} \divideontimes_{\widehat{\mathbb{S}}} \mathfrak{B}$. In order to omit the difficulty, we have to assume something more about $\mathfrak{C}$ and to define a larger algebra. Our general assumption, being a consequence of Proposition 1, is the following:

[C1] $\mathfrak{C} \subset \bigcap_{y \in Y} \mathfrak{A}_y$;

[C2] $\mathfrak{C}$ is P-dense in $\mathfrak{A}$ and S_y-dense in each $\mathfrak{A}_y$.

Then, let

$$\mathcal{M} := \{E \subset X \times Y \colon \exists\, N \in \mathfrak{B}_0\ \forall\, y \notin N\ \widehat{S}_y(E^y) = 0\}.$$

If $\mathfrak{A}_y = \mathfrak{A}$ for every $y \in Y$, then $\mathcal{M}$ is called a family of R-left nil sets (compare with [21] 3.2.2, Satz 1, Satz 2). One can also compare with [22], where a similar family of sets (called nil sets) in case of the direct product of measures are defined in a different way. We will keep that terminology also in this more general situation.

The measure R can be uniquely extended to the measure $R_{\divideontimes}$ on the σ-algebra

$$\mathfrak{A} \divideontimes \mathfrak{B} := \{V = W \triangle N \colon W \in \mathfrak{A} \otimes \mathfrak{B}\ \&\ N \in \mathcal{M}\}.$$

It is defined by the formula $R_{\divideontimes}(V) = \int_Y \widehat{S}_y(V^y)\, d\widehat{Q}(y)$ or, equivalently, by setting $R_{\divideontimes}(V) = R(W)$. $R_{\divideontimes}$ is complete and it is an extension of $\widehat{R}$.

Clearly $\mathfrak{C} \divideontimes_{\widehat{\mathbb{S}}} \mathfrak{B} \subset \mathfrak{A} \divideontimes \mathfrak{B}$ and if $E \in \mathfrak{A} \divideontimes \mathfrak{B}$, then there exists $F \in \mathfrak{C} \otimes \mathfrak{B}$ with $R_{\divideontimes}(E \triangle F) = 0$.

The following fact about skew products is known in case of regular conditional probabilities. Its proof is standard.

Proposition 2. *Let $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ be probability spaces, $R \in P \circledast Q$ and $\{(\mathfrak{A}_y, S_y) : y \in Y\}$ be a Q-disintegration of R. If f is a bounded real valued $\mathfrak{A} \divideontimes \mathfrak{B}$-measurable function on $X \times Y$, then*

(a) the function f^y is $\widehat{\mathfrak{A}}_y$-measurable for $Q - a.e.\ y \in Y$;
(b) if $f = 0$ $R_{\divideontimes}$-a.e., then $f^y = 0$ $\widehat{S}_y$-a.e. for $Q - a.e.\ y \in Y$;
(c) the function $y \longmapsto \int_X f^y(x) d\widehat{S}_y(x)$ is $\widehat{\mathfrak{B}}$-measurable;
(d) the equality

$$\int_{X \times Y} f(x, y) dR_{\divideontimes}(x, y) = \int_Y \int_X f^y(x) d\widehat{S}_y(x) d\widehat{Q}(y) \tag{1}$$

holds true.

In particular, if $E \in \mathfrak{A} \divideontimes \mathfrak{B}$, then $R_{\divideontimes}(E) = \int_Y \widehat{S}_y(E^y)\, d\widehat{Q}(y)$.

If $(Z, \mathfrak{Z}, T)$ is a probability space, $f \in \mathcal{L}^\infty(T)$ and $\mathfrak{D} \subset \mathfrak{Z}$ is arbitrary, then $\mathbb{E}_{\mathfrak{D}}(f)$ denotes the conditional expectation of f with respect to $\mathfrak{D}$. If $f \in \mathcal{L}^\infty(S_y)$, then the conditional expectation of f will be denoted by $\mathbb{E}^y_{\mathfrak{D}}(f)$. The notations $\mathbb{E}_{\mathfrak{D}}$ and $\mathbb{E}^y_{\mathfrak{D}}$ are fixed till the end of the paper.

The following theorem is the key result in our examination of liftings in products.

Theorem 1 ([5] Theorem 2.1)**.** *Assume that $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ are probability spaces, $R \in P \circledast Q$ and $\{(\mathfrak{A}_y, S_y) : y \in Y\}$ is a Q-disintegration of R. Let $\mathfrak{C}$ be a sub-σ-algebra of $\bigcap_{y \in Y} \mathfrak{A}_y$ and $\mathfrak{M} := \mathfrak{C} \otimes \mathfrak{B}$. If $f \in \mathcal{L}^\infty(\widehat{R})$ is arbitrary, then each version $\mathbb{E}_{\mathfrak{M}}(f)$ of the conditional expectation of f with respect to $\mathfrak{M}$ is such that $[\mathbb{E}_{\mathfrak{M}}(f)]^y$ for Q-almost all $y \in Y$ is a version of the conditional expectation $\mathbb{E}^y_{\mathfrak{C}}(f^y)$ of f^y with respect to $\mathfrak{C}$ and S_y. Moreover, the function $(x, y) \longrightarrow \mathbb{E}^y_{\mathfrak{C}}(f^y)(x)$ is then measurable with respect to $\mathfrak{M}$ on a set $X \times (Y \setminus N_f)$, where $N_f \in \mathfrak{B}_0$.*

Remark 1. *The proof of [5] Theorem 2.1 remains valid also for $f \in \mathcal{L}^\infty(R_{\circledast})$, but I will apply the original version.*

3. Liftings in Product Spaces with Disintegration

The notions of lower densities and liftings are used in a little bit more general context than in [13]. More precisely, if $(Z, \mathfrak{Z}, T)$ is a probability space and $\mathfrak{D}$ is a sub-σ-algebra of $\mathfrak{Z}$, then $\delta : \mathfrak{D} \to \mathfrak{Z}$ is a lower density if $\delta(\emptyset) = \emptyset$, $\delta(Z) = Z$, $\delta(A \cap B) = \delta(A) \cap \delta(B)$, $T(\delta(A) \triangle A) = 0$ and $\delta(A) = \delta(B)$ whenever $A \equiv B$. The collection of such lower densities is denoted by $\vartheta(T|_{\mathfrak{D}}; T|_{\mathfrak{Z}})$. If $\mathfrak{D} = \mathfrak{Z}$, then we write simply $\vartheta(T)$. Since all densities considered in this paper are lower densities, I will use the word "density" instead of "lower density". The collection of all liftings on $(Z, \mathfrak{Z}, T)$ is written as $\Lambda(T)$.

We begin with a few facts, known for ordinary densities.

Lemma 1. *Let $(Z, \mathfrak{Z}, T)$ be a probability space and $\delta \in \vartheta(\widehat{T})$. If $\{C_i : i \in I\} \subset \mathfrak{Z}$ is such that $C_i \subset \delta(C_i)$ for every $i \in I$, then $\bigcup_{i \in I} C_i \in \widehat{\mathfrak{Z}}$ and $\bigcup_{i \in I} C_i \subset \delta(\bigcup_{i \in I} C_i)$.*

Lemma 2 ([23])**.** *Let $(Z, \mathfrak{Z}, T)$ be a probability space and let $\mathfrak{C}$ and $\mathfrak{D}$ be sub-σ-algebras of $\mathfrak{Z}$ such that $\mathfrak{C} \subset \mathfrak{D}$ and $\mathfrak{D}_0 \subset \mathfrak{C}$. Moreover, let $\delta \in \vartheta(T|_{\mathfrak{C}}; T|_{\mathfrak{Z}})$ be arbitrary and let $M \in \mathfrak{D} \setminus \mathfrak{C}$. If $M_1 \supset M$ and $M_2 \supset M^c$ are $\mathfrak{C}$-envelopes of M and M^c, respectively, then the formula*

$$\widetilde{\delta}\Big[(G \cap M) \cup (H \cap M^c)\Big] :=$$
$$\Big[M \cap \delta\Big((G \cap M_1) \cup (H \cap M_1^c)\Big)\Big] \cup \Big[M^c \cap \delta\Big((H \cap M_2) \cup (G \cap M_2^c)\Big)\Big]$$

defines a density $\widetilde{\delta} \in \vartheta(T|_{\sigma(\mathfrak{C} \cup \{M\}}; T|_{\mathfrak{Z}}))$ that is an extension of δ.

Let $(\mathfrak{Z}_m)_m$ be an increasing sequence of σ-algebras such that $\mathfrak{Z} = \sigma(\bigcup_m \mathfrak{Z}_m)$. If $\{\delta_m \in \vartheta(T|_{\mathfrak{Z}_m}; T|_{\mathfrak{Z}})\}_m$ is a sequence of densities such that $\delta_{m+1}|_{\mathfrak{Z}_m} = \delta_m$, $m \in \mathbb{N}$, then the formula

$$\delta(A) = \bigcap_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \bigcap_{m > n} \delta_m\Big(\{\mathbb{E}_{\mathfrak{Z}_m}(\chi_A) > 1 - 2^{-k}\}\Big) \quad \text{for } A \in \mathfrak{Z}$$

defines a density $\delta \in \vartheta(T)$.

Lemma 3. *Let $(Z, \mathfrak{Z}, T)$ be a probability space and let $\mathfrak{C}$ be a sub-σ-algebra of $\mathfrak{Z}$, that is T-dense in $\mathfrak{Z}$. If $\rho : \mathfrak{C} \to \mathfrak{Z}$ is a density (lifting), then the formula $\widetilde{\rho}(E) = \rho(F)$, where $T(E \triangle F) = 0$, defines a density (lifting) $\widetilde{\rho} : \mathfrak{Z} \to \mathfrak{Z}$.*

Definition 2. *Let $(Z, \mathfrak{Z}, T)$ be a probability space. A density $\tau \in \vartheta(T)$ is called an* admissible density *(see [2]) if it can be constructed with the help of the transfinite induction in the way described below.*

Let $\mathfrak{Z} = \sigma\{M_\alpha : \alpha < \kappa\}$ be numbered by the ordinals less than κ, where κ is the first ordinal with this property, $M_0 = \emptyset$ and, for each $1 \leq \gamma < \kappa$ let $\mathfrak{Z}_\gamma = \sigma(\{M_\alpha : \alpha < \gamma\} \cup \mathfrak{Z}_0)$. Moreover, for each limit ordinal $\gamma < \kappa$, let $\{\gamma_n : n \in \mathbb{N}\}$ be a fixed increasing sequence of ordinals cofinal with γ.

(1) *$\tau_1 \in \vartheta(T|\mathfrak{Z}_1)$ is the only existing density on $(Z, \mathfrak{Z}_1)$, i.e., $\tau_1(B) = \emptyset$ if $B \in \mathfrak{Z}_0$ and, $\tau_1(B) = Z$, if $B \notin \mathfrak{Z}_0$.*

(2) *If $\gamma < \kappa$ is a limit ordinal of uncountable cofinality, then $\mathfrak{Z}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{Z}_\alpha$ and we define $\tau_\gamma \in \vartheta(T|\mathfrak{Z}_\gamma)$ by setting*

$$\tau_\gamma(B) := \tau_\alpha(B) \quad \textit{if} \quad B \in \mathfrak{Z}_\alpha \quad \textit{and} \quad \alpha < \gamma\,.$$

(3) *Let now γ be a limit odinal of countable cofinality. For simplicity, put $\tau_n := \tau_{\gamma_n}$ and $\mathfrak{Z}_n := \mathfrak{Z}_{\gamma_n}$ for all $n \in \mathbb{N}$. Then, $\mathfrak{Z}_\gamma = \sigma(\cup_{n\in\mathbb{N}} \mathfrak{Z}_n)$ and we can define τ_γ by setting*

$$\tau_\gamma(B) := \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \tau_m(\{\mathbb{E}_{\mathfrak{Z}_m}(\chi_B) > 1 - 2^{-k}\}) \quad \textit{for} \quad B \in \mathfrak{Z}_\gamma\,. \tag{2}$$

It follows from Lemma 2 that $\tau_\gamma \in \vartheta(T|\mathfrak{Z}_\gamma)$ and $\tau_\gamma|\mathfrak{Z}_n = \tau_n$ for each $n \in \mathbb{N}$.

(4) *Let now $\gamma = \beta + 1$. It is well known that*

$$\mathfrak{Z}_\gamma = \{(G \cap M_\beta) \cup (H \cap M_\beta^c) : G,\, H \in \mathfrak{Z}_\beta\}\,.$$

Let $F_1 \supseteq M_\beta$ and $F_2 \supseteq M_\beta^c$ be $\mathfrak{Z}_\beta$-envelopes of M_β and M_β^c, respectively. We define a new density by

$$\begin{aligned}&\tau_\gamma\Big((G \cap M_\beta) \cup (H \cap M_\beta^c)\Big) := \\ &\Big(M_\beta \cap \tau_\beta\Big((G \cap F_1) \cup (H \cap F_1^c)\Big)\Big) \cup \Big(M_\beta^c \cap \tau_\beta\Big((H \cap F_2) \cup (G \cap F_2^c)\Big)\Big)\end{aligned}$$

for $G, H \in \mathfrak{Z}_\beta$. By Lemma 2 $\tau_\gamma \in \vartheta(T|\mathfrak{Z}_\gamma)$ and $\tau_\gamma|\mathfrak{Z}_\beta = \tau_\beta$.

(5) *We define $\tau \in \vartheta(T)$ by setting $\tau(E) = \tau_\kappa(E)$ for $E \in \mathfrak{Z}$.*

Remark 2. *We might have assumed in the above definition that $M_\gamma \notin \mathfrak{Z}_\gamma$ for all γ but in Theorem 3 we need a definition free of such an assumption.*

Definition 3. *Let $\tau \in \vartheta(\mu)$ be an admissible density. If $\rho \in \Lambda(\mu)$ is such that $\tau(A) \subset \rho(A)$ for every $A \in \mathfrak{Z}$, then ρ is called lifting admissibly generated by τ.*

Definition 4. *Let $(X, \mathfrak{A}, P)$ be a probability space and $\mathfrak{C}$ be a sub-σ-algebra of $\mathfrak{A}$. Moreover let $\mathbb{S} := \{(\mathfrak{A}_y, S_y) \colon y \in Y\}$ be a disintegration of R with respect to Q. Assume that $\mathfrak{C} := \sigma(\{M_\alpha \colon \alpha < \kappa\}) \subset \bigcap_{y\in Y} \mathfrak{A}_y$ is a σ-algebra and $\mathfrak{C}_\gamma := \sigma(\{M_\alpha : \alpha < \gamma\})$ for $\gamma \leq \kappa$. We assume that $M_0 = \emptyset$ and κ is the smallest ordinal such that $\mathfrak{C} := \sigma(\{M_\alpha \colon \alpha < \kappa\})$. Moreover, let*

$$\mathfrak{C}_{y0} := \{C \in \mathfrak{C} : S_y(C) = 0\}\,, \quad \textit{and} \quad \mathfrak{C}_{y\gamma} := \sigma(\mathfrak{C}_\gamma \cup \mathfrak{C}_{y0}) \quad \textit{for every } y \in Y.$$

For each $y \in Y$ we are given an admissible density τ_y on $\mathfrak{C}$, that is constructed along the transfinite sequence $\{M_\alpha \colon \alpha < \kappa\}$ with fixed cofinal sequences $\{\gamma_n : n \in \mathbb{N}\}$ and for the transfinite sequence of σ-algebras $\mathfrak{C}_{y\beta}$. It is important to notice that the construction of τ_y yields the inclusion $\tau_y(\mathfrak{C}_{y\beta}) = \tau_{y\beta}(\mathfrak{C}_{y\beta}) \subset \mathfrak{C}_{y\beta}$ for every $y \in Y$. Therefore, in the proofs, we will sometimes write

τ_y in place of $\tau_{y\beta}$. We call such a collection $\{\tau_y : y \in Y\}$ an equi-admissible family of densities determined by $\{M_\alpha : \alpha < \kappa\}$. Due to the density assumption, each τ_y is uniquely extendable to $\widehat{\mathfrak{A}_y}$.

Let us recall that $\mathbb{E}^y_{\mathfrak{D}}$ denotes the conditional expectation of $f^y = f(\cdot, y)$ with respect to $\mathfrak{D}$ in the space $(X, \mathfrak{A}_y, S_y)$.

Proposition 3. *Let $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ be probability spaces and $R \in P \circledast Q$. Let $\mathbb{S} = \{(\mathfrak{A}_y, S_y) : y \in Y\}$ be a disintegration of R with respect to Q. Assume that $\mathfrak{C} \subset \bigcap_{y \in Y} \mathfrak{A}_y$ is such a σ-algebra that P and each S_y is inner regular with respect to $\mathfrak{C}$. Assume moreover that $\{\tau_y : y \in Y\}$ is an equi-admissible family of densities determined by $\{M_\alpha : \alpha < \kappa\}$ generating $\mathfrak{C}$. Then, there exists a density $\varphi : \mathfrak{A} \circledast \mathfrak{B} \to \mathfrak{C} \circledast_{\mathbb{S}} \mathfrak{B}$ such that for each $F \in \mathfrak{A} \circledast \mathfrak{B}$*

$$[\varphi(F)]^y = \tau_y([\varphi(F)]^y) \qquad \text{for all } \; y \in Y\,. \tag{3}$$

Proof. First we will construct a density $\varphi : \mathfrak{C} \otimes \mathfrak{B} \to \mathfrak{C} \circledast_{\mathbb{S}} \mathfrak{B}$. To do it, we will use the transfinite induction.

To simplify further notation, I denote each σ-algebra $\mathfrak{C}_\alpha \circledast_{\mathbb{S}} \mathfrak{B} := \sigma(\mathfrak{C}_\alpha \otimes \mathfrak{B} \cup \mathcal{N})$ by $\mathfrak{N}_\alpha$ and the σ-algebra $\sigma((\mathfrak{C}_\alpha \otimes \mathfrak{B}) \cup (\mathfrak{C} \otimes \mathfrak{B})_0)$ by $\mathfrak{M}_\alpha$. Let $\delta \in \vartheta(Q)$ be a fixed density. At the first step, we define $\varphi_1 \in \vartheta\Big(R_\circledast|_{\mathfrak{M}_1}\Big)$ by $\varphi_1(E) = \emptyset$ if $R_\circledast(E) = 0$ and $\varphi_1(E) = X \times \delta(B)$, if $E \equiv_{R_\circledast} X \times B$ with $B \in \mathfrak{B}$.

Let us fix now $\gamma < \kappa$ and assume that for each $\alpha < \gamma$ there exists a density $\varphi_\alpha \in \vartheta(R_*|_{\mathfrak{M}_\alpha}; R_*|_{\mathfrak{N}_\alpha})$ such that for each $F \in \mathfrak{M}_\alpha$

$$[\varphi_\alpha(F)]^y \in \mathfrak{C}_{y\alpha} \quad \text{and} \quad [\varphi_\alpha(F)]^y = \tau_y([\varphi_\alpha(F)]^y) \qquad \text{for every } y \in Y \tag{4}$$

and

$$\varphi_\beta(F) = \varphi_\alpha(F) \quad \text{if } \alpha < \beta < \gamma \text{ and } F \in \mathfrak{M}_\alpha. \tag{5}$$

We will split now the proof into three parts:

(A) γ is a limit ordinal of countable cofinality.

$(\gamma_m)_m$ be the increasing sequence of ordinals cofinal with γ. Applying (2), we define $\overline{\varphi}_\gamma \in \vartheta(R_*|_{\mathfrak{M}_\gamma}; R_*|_{\mathfrak{N}_\gamma})$ extending all φ_{γ_m}, setting for each $F \in \mathfrak{M}_\gamma$

$$\overline{\varphi}_\gamma(F) := \bigcap_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \bigcap_{m > n} \varphi_{\gamma_m}\Big(\Big\{\mathbb{E}_{\mathfrak{M}_{\gamma_m}}(\chi_F) > 1 - 2^{-k}\Big\}\Big).$$

Yet $\mathfrak{C}_{\gamma_m} \otimes \mathfrak{B}$ is $R_\circledast|_{\mathfrak{M}_\gamma}$-dense in $\mathfrak{M}_{\gamma_m}$, which gives

$$\mathbb{E}_{\mathfrak{M}_{\gamma_m}}(\chi_F) = \mathbb{E}_{\mathfrak{C}_{\gamma_m \otimes \mathfrak{B}}}(\chi_F) \quad R_\circledast|_{\mathfrak{M}_\gamma} - a.e.$$

and then

$$\varphi_{\gamma_m}\Big(\Big\{\mathbb{E}_{\mathfrak{M}_{\gamma_m}}(\chi_F) > 1 - 2^{-k}\Big\}\Big) = \varphi_{\gamma_m}\Big(\Big\{\mathbb{E}_{\mathfrak{C}_{\gamma_m} \otimes \mathfrak{B}}(\chi_F) > 1 - 2^{-k}\Big\}\Big)\,.$$

Then, for $F \in \mathfrak{M}_\gamma$ let $N_{m,F} \in \mathfrak{B}_0$ be such that (see Theorem 1) for each $y \notin N_{m,F}$.

$$[\mathbb{E}_{\mathfrak{C}_{\gamma_m} \otimes \mathfrak{B}}(\chi_F)]^y = \mathbb{E}^y_{\mathfrak{C}_{\gamma_m}}(\chi_{F^y}) \;\; S_y|_{\mathfrak{C}_{\gamma_m}} - a.e.$$

and (by the inductive assumption)

$$[\varphi_{\gamma_m}(\{\mathbb{E}_{\mathfrak{C}_{\gamma_m} \otimes \mathfrak{B}}(\chi_F) > 1 - 2^{-k}\})]^y = \tau_y([\{\mathbb{E}_{\mathfrak{C}_{\gamma_m} \otimes \mathfrak{B}}(\chi_F) > 1 - 2^{-k}\}]^y).$$

According to (2), we have also for each $A \in \mathfrak{C}_\gamma$

$$\tau_y(A) := \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m>n} \tau_y\Big(\Big\{\mathbb{E}^y_{\mathfrak{C}_{\gamma m}}(\chi_A) > 1 - 2^{-k}\Big\}\Big)$$

If $N_F := \bigcup_{m\in\mathbb{N}} N_{m,F}$, then $N_F \in \mathfrak{B}_0$ and it follows from Theorem 1, that for all $y \in Y \setminus N_F$

$$\begin{aligned}
[\overline{\varphi}_\gamma(F)]^y &= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \tau_y\Big(\Big[\Big\{\mathbb{E}_{\mathfrak{C}_{\gamma m}\otimes\mathfrak{B}}(\chi_F) > 1 - 2^{-k}\Big\}\Big]^y\Big) \\
&= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \tau_y\Big(\Big\{[\mathbb{E}_{\mathfrak{C}_{\gamma m}\otimes\mathfrak{B}}(\chi_F)]^y > 1 - 2^{-k}\Big\}\Big) \\
&= \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \tau_y\Big(\Big\{\mathbb{E}^y_{\mathfrak{C}_{\gamma m}}(\chi_{F^y}) > 1 - 2^{-k}\Big\}\Big) = \tau_y(F^y).
\end{aligned}$$

It follows that $\tau_y([\overline{\varphi}_\gamma(F)]^y) = [\overline{\varphi}_\gamma(F)]^y$, for all $y \notin M_F \in \mathfrak{B}_0$, where $N_F \subset M_F$. Thus, $D := \{(x,y) : \tau_y([\overline{\varphi}_\gamma(F)]^y) \neq [\overline{\varphi}_\gamma(F)]^y\} \in \mathcal{N}$, because if $y \in M_F$, then $[\overline{\varphi}_\gamma(F)]^y \in \mathfrak{C}_{y\gamma}$. It follows that we may define φ_γ by setting $[\varphi_\gamma(F)]^y := \tau_y([\overline{\varphi}_\gamma(F)]^y)$ for every $y \in Y$. Obviously, $\varphi_\gamma(F) \in \mathfrak{N}_\gamma$. One can also see that if $F \in \mathfrak{M}_\alpha$ for some $\alpha < \gamma$, then there is $\gamma_m > \alpha$ such that $F \in \mathfrak{M}_{\gamma_m}$, what yields $\overline{\varphi}_\gamma(F) = \varphi_{\gamma_m}(F) = \varphi_\alpha(F)$. Hence, $[\varphi_\gamma(F)]^y = \tau_{y\gamma}\big([\overline{\varphi}_\gamma(F)]^y\big) = \tau_{y\gamma}\big([\varphi_\alpha(F)]^y\big) = [\varphi_\alpha(F)]^y$.

(B)

Let $W_1 \supset M_\beta \times Y$ and $W_2 \supset M^c_\beta \times Y$ be, respectively, $\mathfrak{M}_\beta$-envelopes of $M_\beta \times Y$ and $M^c_\beta \times Y$ with respect to $R_\divideontimes|_{\mathfrak{M}_\beta}$. We define an extension $\overline{\varphi}_\gamma : \mathfrak{M}_\gamma \to \mathfrak{N}_\gamma$ of φ_β by the formula (see Lemma 2)

$$\begin{aligned}
&\overline{\varphi}_\gamma\Big[\big(G \cap (M_\beta \times Y)\big) \cup \big(H \cap (M^c_\beta \times Y)\big)\Big] := \\
&\Big[(M_\beta \times Y) \cap \varphi_\beta\Big((G \cap W_1) \cup (H \cap W^c_1)\Big)\Big] \cup \Big[(M^c_\beta \times Y) \cap \varphi_\beta\Big((H \cap W_2) \cup (G \cap W^c_2)\Big)\Big]
\end{aligned}$$

if $G, H \in \mathfrak{M}_\beta$.

It follows from the basic properties of densities that for each $F \in \mathfrak{M}_\gamma$, the equality $R_\divideontimes(\overline{\varphi}_\gamma(F) \triangle F) = 0$ is valid. In virtue of Proposition 2, there exists a set $N \in \mathfrak{B}_0$ such that for every $y \notin N$, we have $\tau_{y\gamma}([\overline{\varphi}_\gamma(F)]^y) = \tau_{y\gamma}(F^y)$. Consequently, if the set N is chosen for $W := \big(G \cap (M_\beta \times Y)\big) \cup \big(H \cap (M^c_\beta \times Y)\big)$, $(G \cap W_1) \cup (H \cap W^c_1)$ and $(H \cap W_2) \cup (G \cap W^c_2)$, then we have for $y \notin N$

$$\mathfrak{C}_{y\gamma} \ni [\overline{\varphi}_\gamma(W)]^y \equiv_{S_y} \tau_{y\gamma}\big([\overline{\varphi}_\gamma(W)]^y\big) \in \mathfrak{C}_{y\gamma}.$$

If $y \in N$, then $[\overline{\varphi}_\gamma(W)]^y \in \mathfrak{C}_{y\gamma}$ as well. As a result, if $D^y := [\overline{\varphi}_\gamma(W)]^y \triangle \tau_{y\gamma}\big([\overline{\varphi}_\gamma(W)]^y\big)$, then $D \in \mathcal{N}$ and so we may define $\varphi_\gamma : \mathfrak{M}_\gamma \to \mathfrak{N}_\gamma$ setting for every $y \in Y$ $[\varphi_\gamma(W)]^y := \tau_{y\gamma}([\overline{\varphi}_\gamma(W)]^y)$. One can easily check that $\overline{\varphi}_\gamma|_{\mathfrak{M}_\beta} = \varphi_\beta$ and so if $E \in \mathfrak{M}_\beta$, then $[\varphi_\gamma(E)]^y = \tau_{y\gamma}([\overline{\varphi}_\gamma(E)]^y) = \tau_{y\gamma}([\varphi_\beta(E)]^y) = [\varphi_\beta(E)]^y$. It follows that $\varphi_\gamma|_{\mathfrak{M}_\beta} = \varphi_\beta$.

(C) Assume now that γ is of uncountable cofinality.

In such a case $\mathfrak{C}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{C}_\alpha$ and for each $F \in \mathfrak{C}_\gamma$ there exists $\alpha < \gamma$ with $F \in \mathfrak{C}_\alpha$. We simply set $\varphi_\gamma(F) = \varphi_\alpha(F)$.

(D) When $\gamma = \kappa$, then we get a density $\varphi_\kappa \in \vartheta(R_*|_{\mathfrak{M}_\kappa}; R_*|_{\mathfrak{N}_\kappa})$ and the inductive part is finished.

(E) Now we are going to extend φ_κ to a density $\varphi : \mathfrak{A} \divideontimes \mathfrak{B} \to \mathfrak{C} \divideontimes_{\mathbb{S}} \mathfrak{B} = \mathfrak{N}_\kappa$. As it has been mentioned earlier, $\mathfrak{C} \otimes \mathfrak{B}$ is $R_\divideontimes$-dense in $\mathfrak{A} \divideontimes \mathfrak{B}$. If $W \in \mathfrak{A} \divideontimes \mathfrak{B} = \sigma\big((\mathfrak{A} \otimes \mathfrak{B}) \cup \mathcal{M}\big)$ and $V \in \mathfrak{C} \otimes \mathfrak{B}$ are such that $R_\divideontimes(W \triangle V) = 0$, then we set $\varphi(W) := \varphi_\kappa(V)$. $\varphi : \mathfrak{A} \divideontimes \mathfrak{B} \to \mathfrak{C} \divideontimes_{\mathbb{S}} \mathfrak{B}$ and the equality (3) holds true. $\square$

Proposition 4. *Under the assumptions of Theorem 3, there exists a density $\psi : \mathfrak{A} \circledast \mathfrak{B} \longrightarrow \mathfrak{C} \circledast_{\widehat{\mathbb{S}}} \mathfrak{B}$ satisfying for all $E \in \mathfrak{A} \circledast \mathfrak{B}$ the following conditions:*

(1) $\widehat{S}_y\Big([\psi(E)]^y \cup [\psi(E^c)]^y\Big) = 1$ *for all* $y \in Y$;

(2) $[\psi(E)]^y = \tau_y([\psi(E)]^y)$ *for all* $y \in Y$;

Proof. For each $y \in Y$ the measure S_y is inner regular with respect to $\mathfrak{C}$ and consequently the density $\tau_y \in \vartheta(S_y|_{\mathfrak{C}})$ can be uniquely extended to $\widehat{S}_y|_{\mathfrak{C}_y}$. We denote the extension also by τ_y. The density $\varphi : \mathfrak{A} \circledast \mathfrak{B} \to \mathfrak{C} \circledast_{\mathbb{S}} \mathfrak{B}$ satisfying the thesis of Proposition 3 can be treated as an element of $\vartheta(R_{\circledast})$. Let

$$\Psi := \left\{ \widetilde{\varphi} \in \vartheta(R_{\circledast}) \;\middle|\; \begin{array}{l} \forall E \in \mathfrak{A} \circledast \mathfrak{B}\ \varphi(E) \subseteq \widetilde{\varphi}(E)\ \&\ \widetilde{\varphi}(E) \in \mathfrak{C} \circledast_{\widehat{\mathbb{S}}} \mathfrak{B} \\ \&\ \forall y \in Y\ [\widetilde{\varphi}(E)]^y \subseteq \tau_y([\widetilde{\varphi}(E)]^y) \end{array} \right\}.$$

We consider Ψ with inclusion as the partial order: $\widetilde{\varphi}_1 \le \widetilde{\varphi}_2$ if $\widetilde{\varphi}_1(E) \subseteq \widetilde{\varphi}_2(E)$ for each $E \in \mathfrak{A} \circledast \mathfrak{B}$.

(I) *There exists a maximal element in* Ψ.

The only fact we have to prove is showing that each chain $\{\varphi_\alpha\}_{\alpha\in A} \subseteq \Psi$ has a dominating element in Ψ. The obvious candidate is $\widetilde{\varphi}$ given for each $E \in \mathfrak{A} \circledast \mathfrak{B}$ by

$$\widetilde{\varphi}(E) = \bigcup_{\alpha\in A} \varphi_\alpha(E)\,. \tag{6}$$

One can easily check that $\widetilde{\varphi}(E) \subseteq [\widetilde{\varphi}(E^c)]^c$ and so if an $\alpha \in A$ is fixed, then

$$\varphi_\alpha(E) \subseteq \widetilde{\varphi}(E) \subseteq [\widetilde{\varphi}(E^c)]^c \subseteq [\varphi_\alpha(E^c)]^c$$

for each $E \in \mathfrak{A} \circledast \mathfrak{B}$. Since R_* is complete and $\varphi_\alpha \in \vartheta(R_*)$, this proves that $\widetilde{\varphi}(E) \in \mathfrak{A} \circledast \mathfrak{B}$ and $\widetilde{\varphi}(E) \overset{R_*}{=} E$. Consider now the section properties of $\widetilde{\varphi}(E)$. For fixed $y \in Y$

$$[\widetilde{\varphi}(E)]^y = \bigcup_{\alpha\in A} [\varphi_\alpha(E)]^y \subseteq \bigcup_{\alpha\in A} \tau_y([\varphi_\alpha(E)]^y)$$

and so, in virtue of Lemma 1, the set $[\widetilde{\varphi}(E)]^y$ is $\widehat{S}_y$-measurable. As a result, $\widetilde{\varphi}(E) \in \mathfrak{C} \circledast_{\widehat{\mathbb{S}}} \mathfrak{B}$. Setting in the above inclusions $\widetilde{\varphi}(E)$ instead of $\varphi_\alpha(E)$, we see that the inclusion $[\widetilde{\varphi}(E)]^y \subseteq \tau_y([\widetilde{\varphi}(E)]^y)$ is satisfied also.

We have to prove yet that $\widetilde{\varphi}$ is a density. To do it, take $E, F \in \mathfrak{A} \circledast \mathfrak{B}$. We have

$$\begin{aligned}
\widetilde{\varphi}(E) \cap \widetilde{\varphi}(F) &= \bigcup_{\alpha\in A} \varphi_\alpha(E) \cap \bigcup_{\alpha\in A} \varphi_\alpha(F) = \bigcup_{\alpha\in A} \Big[\varphi_\alpha(E) \cap \bigcup_{\beta\in A} \varphi_\beta(F)\Big] \\
&= \bigcup_{\alpha\in A} \Big[\bigcup_{\beta\in A} \varphi_\alpha(E) \cap \varphi_\beta(F)\Big] = \bigcup_{\alpha\in A} \Big[\bigcup_{\alpha\le\beta} \varphi_\alpha(E) \cap \varphi_\beta(F)\Big] \\
&\subseteq \bigcup_{\alpha\in A} \Big[\bigcup_{\alpha\le\beta} \varphi_\beta(E) \cap \varphi_\beta(F)\Big] = \bigcup_{\alpha\in A} \Big[\bigcup_{\alpha\le\beta} \varphi_\beta(E \cap F)\Big] \subseteq \widetilde{\varphi}(E \cap F)\,.
\end{aligned}$$

The reverse inclusion and other properties are clear and so $\widetilde{\varphi} \in \vartheta(R_*)$ and $\varphi : \mathfrak{A} \circledast \mathfrak{B} \longrightarrow \mathfrak{A} \circledast_{\widehat{\mathbb{S}}} \mathfrak{B}$. That proves that $\widetilde{\varphi}$ dominates the whole chain.

According to the Zorn–Kuratowski Lemma the set Ψ possesses a maximal element $\psi : \mathfrak{A} \circledast \mathfrak{B} \longrightarrow \mathfrak{C} \circledast_{\widehat{\mathbb{S}}} \mathfrak{B}$.

(II) *For each $y \in Y$ and $E \in \mathfrak{A} \circledast \mathfrak{B}$ $\widehat{S}_y([\psi(E)]^y \cup [\psi(E^c)]^y) = 1$.*

Notice first that as $\varphi \in \Psi$ all sections $[\psi(E)]^y$ are $\widehat{S}_y$-measurable. Suppose now that there exists $H \in \mathfrak{A} \circledast \mathfrak{B}$ and $y_0 \in Y$ such that $\widehat{S}_{y_0}([\psi(H)]^{y_0} \cup [\psi(H^c)]^{y_0}) < 1$. Let

$$W := \tau_{y_0}\left[\left([\psi(H)]^{y_0} \cup [\psi(H^c)]^{y_0}\right)^c\right]$$

and

$$[\widehat{\psi}(E)]^y = \begin{cases} [\psi(E)]^y & \text{if } y \neq y_0 \\ [\psi(E)]^{y_0} \cup (W \cap [\psi(H \cup E)]^{y_0}) & \text{if } y = y_0 \end{cases}$$

It is clear that $\psi(E) \subseteq \widehat{\psi}(E)$ for each $E \in \mathfrak{A} \circledast \mathfrak{B}$. In particular, $\widehat{\psi}(X \times Y) = X \times Y$. Also, the other density properties are fulfilled.

It follows directly from the definition that $\widehat{\psi}$ and ψ are different densities. In order to get a contradiction with our hypothesis it is enough to show that $[\widehat{\psi}(E)]^{y_0} \subseteq \tau_{y_0}\left([\widehat{\psi}(E)]^{y_0}\right)$, but this is immediate. If $E \in \mathfrak{A} \circledast \mathfrak{B}$, then

$$\begin{aligned}
\tau_{y_0}\left([\widehat{\psi}(E)]^{y_0}\right) &\supseteq \tau_{y_0}\left([\psi(E)]^{y_0}\right) \cup \tau_{y_0}\left(W \cap [\psi(H \cup E)]^{y_0}\right) \\
&\supseteq [\psi(E)]^{y_0} \cup \left[\tau_{y_0}(W) \cap \left([\psi(H \cup E)]^{y_0}\right)\right] \\
&\supseteq [\psi(E)]^{y_0} \cup \left(W \cap [\psi(H \cup E)]^{y_0}\right) = [\widehat{\psi}(E)]^{y_0} .
\end{aligned}$$

This completes the proof.

(III) *For each $y \in Y$ and $E \in \mathfrak{A} \circledast \mathfrak{B}$ $[\psi(E)]^y = \tau_y\left([\psi(E)]^y\right)$.*

Set for each $y \in Y$ and $E \in \mathfrak{A} \circledast \mathfrak{B}$

$$[\widetilde{\psi}(E)]^y = \tau_y\left([\psi(E)]^y\right) .$$

Clearly, $\psi(F) \subseteq \widetilde{\psi}(F)$ for each F. Moreover the equality $\psi(E) \cap \psi(E^c) = \emptyset$ yields for each y the relation $\tau_y\left([\psi(E)]^y\right) \cap \tau_y\left([\psi(E^c)]^y\right) = \emptyset$. As a consequence, we get $\widetilde{\psi}(E) \cap \widetilde{\psi}(E^c) = \emptyset$, and then $\widetilde{\psi}(E^c) \subseteq (\widetilde{\psi}(E))^c$. Hence

$$\psi(E^c) \subseteq \widetilde{\psi}(E^c) \subseteq [\widetilde{\psi}(E)]^c \subseteq [\psi(E)]^c .$$

Since ψ is a density, we have $R_{\circledast}([\psi(E)]^c) = R_{\circledast}[\psi(E^c)]$ and so $\widetilde{\psi}(E)$ is $\mathfrak{A} \circledast \mathfrak{B}$-measurable. Yet at the same time, $\widehat{S}_y([\psi(E)]^y) = \widehat{S}_y([\psi(E^c)]^y)$. Hence, $\widetilde{\psi}(E) \in \mathfrak{C} \circledast_{\widehat{\mathbb{S}}} \mathfrak{B}$. It follows that $\widetilde{\psi} \in \Psi$ and so $\psi = \widetilde{\psi}$, due to the maximality of ψ. $\square$

Theorem 2. *Let $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ be probability spaces. Assume that $R \in P \circledast Q$ has a disintegration $\mathbb{S} = \{(\mathfrak{A}_y, S_y) : y \in Y\}$ with respect to Q and $\mathfrak{C} \subset \bigcap_{y \in Y} \mathfrak{A}_y$ is such a σ-algebra that P and each S_y are inner regular with respect to $\mathfrak{C}$. Let $\{\tau_y \in \vartheta(\widehat{S}_y) : y \in Y\}$ be the family from Proposition 4. Then, there exist liftings $\sigma_y \in \Lambda(\widehat{S}_y)$ that are admissibly generated by τ_y for all $y \in Y$ and there exists a lifting $\pi : \mathfrak{A} \circledast \mathfrak{B} \to \mathfrak{C} \circledast_{\widehat{\mathbb{S}}} \mathfrak{B}$ such that the condition is satisfied that follows:*

$$[\pi(E)]^y = \sigma_y\left([\pi(E)]^y\right) \qquad \textit{for every } E \in \mathfrak{A} \circledast \mathfrak{B} \textit{ and } y \in Y;$$

$$[\pi(f)]^y = \sigma_y\left([\pi(f)]^y\right) \quad \textit{for every } f \in \mathcal{L}^\infty(R_{\circledast}) \textit{ and } y \in Y.$$

Proof. We take ψ from Proposition 4 and liftings $\sigma_y \in \Lambda(\widehat{S}_y)$ generated by τ_y. Then, we define $\pi : \mathfrak{A} \circledast \mathfrak{B} \to \mathfrak{C} \circledast_{\widehat{\mathbb{S}}} \mathfrak{B} = \sigma\left((\mathfrak{C} \otimes \mathfrak{B}) \cup \widehat{\mathcal{N}}\right)$ by setting for each $E \in \mathfrak{A} \circledast \mathfrak{B}$ and each $y \in Y$

$$[\pi(E)]^y = \sigma_y\left([\psi(E)]^y\right) .$$

□

The next theorem is a direct consequence of Theorem 2 and it is a strong generalization of [3] Theorem 3.6, where the σ-algebra $\mathfrak{A}$ was assumed to be separable in the Frechet–Nikodým pseudometric.

Theorem 3. *Let $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ be probability spaces and $R \in P \circledast Q$ be arbitrary. Assume that $\{(\mathfrak{A}, S_y) : y \in Y\}$ is a regular conditional probability on $\mathfrak{A}$ with respect to Q and $\{\tau_y \in \vartheta(S_y) : y \in Y\}$ is an equi-admissible family of densities. Then, for each $y \in Y$, there exists $\sigma_y \in \Lambda(\widehat{S_y})$ admissibly generated by τ_y and there exists a lifting $\pi : \mathfrak{A} \ast \mathfrak{B} \to \mathfrak{A} \ast_{\widehat{\mathbb{S}}} \mathfrak{B}$ such that the conditions are satisfied as follows:*

$$[\pi(E)]^y = \sigma_y\Big([\pi(E)]^y\Big) \quad \text{for every } E \in \mathfrak{A} \ast \mathfrak{B} \text{ and } y \in Y;$$
$$[\pi(f)]^y = \sigma_y\Big([\pi(f)]^y\Big) \quad \text{for every } f \in \mathcal{L}^\infty(R_\ast) \text{ and } y \in Y.$$

Remark 3. *If ξ is an arbitrary lifting on $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ or on $\mathfrak{A} \ast \mathfrak{B}$, then one can say only that for each $E \in \mathfrak{A}\widehat{\otimes}\mathfrak{B}$ or $E \in \mathfrak{A} \ast \mathfrak{B}$ Q—almost all Y sections $[\xi(E)]^y$ are measurable with respect to $\widehat{S}_y$. Some of the Y-section might be non-measurable. Theorem 3 shows that among a huge family of liftings one can find such lifting, that not only all Y-sections are properly measurable but they are even lifting invariant with respect to a family of a priori given coordinate liftings.*

Theorem 3 is also a partial generalization of the main results from [1,2]. In [2], the measure R was a direct product of P and Q whereas in [1] the measure R was assumed to be absolutely continuous with respect to the direct product $P \times Q$.

4. Connections with Earlier Results

4.1. Absolutely Continuous Disintegrations

The topic investigated in this paper was earlier undertaken in [1–3,5]. In [2], the measure R was simply the direct product of P and Q. In [3], the algebra $\mathfrak{A}$ was assumed to be separable in the Frechet–Nikodým metric. In [1] the measure R was assumed to be absolutely continuous with respect to $P \times Q$. In all the above works, the final lifting π was an element of $\Lambda(\widehat{R})$ but $\mathbb{S}$ was assumed to be a regular conditional probability. In [5], R was arbitrary and $\mathbb{S}$ was a disintegration, but the proof of the inclusion $\pi \in \Lambda(\widehat{R})$ was erroneus. Theorem 2 corrects the mistake, proving that in general, $\pi \in \Lambda(R_\ast)$.

Thus, in investigating densities and liftings in the context of disintegration or regular conditional probabilities, one immediately meets the following problem:

The density φ in Proposition 3 may take values in $\mathfrak{A} \otimes \mathfrak{B}$ and the lifting π from Theorem 2 or Theorem 3 may take values in $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$? It is so if $R = P \times Q$, because then all τ_y's may coincide with a fixed density on $\mathfrak{A}$.

It follows from the proof of Proposition 3 that, in general, such a relation is impossible. Let us recall that $\mathbb{S} := \{(\mathfrak{A}_y, S_y) : y \in Y\}$ is a Q-disintegration of $R \in P \circledast Q$ and $\mathfrak{M}_\beta = \sigma\big((\mathfrak{C}_\beta \otimes \mathfrak{B}) \cup (\mathfrak{A} \otimes \mathfrak{B})_0\big)$.

Below, I prove that if $R \ll P \times Q$ in case of the regular conditional probability (or $S_y \ll P|_{\mathfrak{A}_y}$ in case of a disintegration), then the density φ in Proposition 3 and the lifting π from Theorem 2 or Theorem 3 may take values in $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$. That is an alternate proof of the corresponding part of the proof of [1] Theorem 3.3.

In section (B) of the proof of Proposition 3, the density $\overline{\varphi}_\gamma$ was defined by the formula

$$\overline{\varphi}_\gamma\Big[\big(G\cap(M_\beta\times Y)\big)\cup\big(H\cap(M_\beta^c\times Y)\big)\Big]:=$$
$$\Big[(M_\beta\times Y)\cap\varphi_\beta\Big((G\cap W_1)\cup(H\cap W_1^c)\Big)\Big]\cup\Big[(M_\beta^c\times Y)\cap\varphi_\beta\Big((H\cap W_2)\cup(G\cap W_2^c)\Big)\Big],$$

where $G,H\in\mathfrak{M}_\beta$, $\gamma=\beta+1$ and $W_1\supset M_\beta\times Y$ and $W_2\supset M_\beta^c\times Y$ are, respectively, $\mathfrak{C}_\beta\circledast_{\mathbb{S}}\mathfrak{B}$-envelopes of $M_\beta\times Y$ and $M_\beta^c\times Y$ with respect to $R_\circledast$. In particular,

$$\overline{\varphi}_\gamma(M_\beta\times Y)=\Big[(M_\beta\times Y)\cap\varphi_\beta(W_1)\Big]\cup\Big[(M_\beta^c\times Y)\cap\varphi_\beta(W_2^c)\Big]$$

If $\varphi_\beta:\mathfrak{M}_\beta\longrightarrow\mathfrak{A}\widehat{\otimes}\mathfrak{B}$, then we have also $\overline{\varphi}_\gamma:\mathfrak{M}_\gamma\longrightarrow\mathfrak{A}\widehat{\otimes}\mathfrak{B}$.

Let $V_{1y}\supset M_\beta$ and $V_{2y}\supset M_\beta^c$ be, respectively, $\mathfrak{C}_{y\beta}$-envelopes of M_β and M_β^c with respect to S_y. Without loss of generality, we may assume that

$$V_{1y}\subset W_1^y\,,\quad V_{2y}\subset W_2^y\quad\text{for all } y\in Y\,.$$

The idea of the proof is the following: to prove that for every $y\in Y$, one may assume that $V_{1y}=W_1^y$ and $V_{2y}=W_2^y$.

Assume that $S_y\ll P|_{\mathfrak{A}_y}$ for every $y\in Y$ (If $R\ll P\times Q$, then such absolutely continuous $\mathbb{S}$ exists). Let $T_1\supset M_\beta$ and $T_2\supset M_\beta^c$ be, respectively, $\mathfrak{C}_\beta$-envelopes of M_β and M_β^c with respect to P. We may assume then that $W_i\subset T_i\times Y,\ i=1,2$. But the equality $(P|_{\mathfrak{C}_\beta})_*(T_1\setminus M_\beta)=0$ implies $R_\circledast|_{\mathfrak{M}_\beta}((T_1\times Y)\setminus W_1)=0$ (due to the absolute continuity), and so we may assume that $W_1=T_1\times Y$. Similarly $W_2=T_2\times Y$.

Let $y\in Y$ be fixed and $\mathfrak{C}_{y\beta}\ni A\subset T_1\setminus M_\beta$. Since $(P|_{\mathfrak{C}_\beta})_*(T_1\setminus M_\beta)=0$ and $S_y\ll P|_{\mathfrak{A}_y}$, we have $S_y(A)=0$. That means that $(S_y|_{\mathfrak{C}_\beta})_*(T_1\setminus M_\beta)=0$; equivalently $(S_y|_{\mathfrak{C}_\beta})^*(M_\beta)=S_y(T_1)$ for every $y\in Y$. Thus, one may assume the equality $V_{1y}=W_1^y=T_1$, for every $y\in Y$, similarly for W_2.

The construction of the densities τ_y guarantees the equality

$$\tau_{y\gamma}\Big[(A\cap M_\beta)\cup(B\cap M_\beta^c)\Big]:= \tag{7}$$
$$\Big[M_\beta\cap\tau_{y\beta}\Big((A\cap V_{1y})\cup(B\cap V_{1y}^c)\Big)\Big]\cup\Big[M_\beta^c\cap\tau_{y\beta}\Big((B\cap V_{2y})\cup(A\cap V_{2y}^c)\Big)\Big]$$

if $A,B\in\mathfrak{C}_\beta$. It follows from the basic properties of densities that for each $F\in\mathfrak{M}_\beta$, there exists a set $N_1\in\mathfrak{B}_0$ such that for every $y\notin N_1$, we have $\tau_y([\varphi_\beta(F)]^y)=\tau_y(F^y)$. Similarly, for each $F\in\mathfrak{M}_\gamma$ there exists a set $N_2\in\mathfrak{B}_0$ such that for every $y\notin N_2$ we have $\tau_y([\overline{\varphi}_\gamma(F)]^y)=\tau_y(F^y)$. Consequently, it follows from (4) and (7) that if $N\in\mathfrak{B}_0$ is chosen for $(G\cap W_1)\cup(H\cap W_1^c)$, $(H\cap W_2)\cup(G\cap W_2^c)$ and $\big(G\cap(M_\beta\times Y)\big)\cup\big(H\cap(M_\beta^c\times Y)\big)$, then for $y\notin N$

$$\begin{aligned}
&\Big[\overline{\varphi}_\gamma\Big[\big(G\cap(M_\beta\times Y)\big)\cup\big(H\cap(M_\beta^c\times Y)\big)\Big]\Big]^y\\
&=\Big[M_\beta\cap\tau_y\Big((G^y\cap W_1^y)\cup(H^y\cap(W_1^y)^c)\Big)\Big]\cup\Big[M_\beta^c\cap\tau_y\Big((H^y\cap W_2^y)\cup(G^y\cap(W_2^y)^c)\Big)\Big]\\
&=\Big[M_\beta\cap\tau_y\Big((G^y\cap V_{1y})\cup(H^y\cap V_{1y}^c)\Big)\Big]\cup\Big[M_\beta^c\cap\tau_y\Big((H^y\cap V_{2y})\cup(G^y\cap V_{2y}^c)\Big)\Big]\\
&=\tau_y\Big((G^y\cap M_\beta)\cup(H^y\cap M_\beta^c)\Big)=\tau_y\Big[\Big[\big(G\cap(M_\beta\times Y)\big)\cup\big(H\cap(M_\beta^c\times Y)\big)\Big]^y\Big]\\
&=\tau_y\Big(\Big[\overline{\varphi}_\gamma\Big[\big(G\cap(M_\beta\times Y)\big)\cup\big(H\cap(M_\beta^c\times Y)\big)\Big]\Big]^y\Big).
\end{aligned}$$

If $y \in N$, then $\left[\overline{\varphi}_\gamma\left[\left(G \cap (M_\beta \times Y)\right) \cup \left(H \cap (M_\beta^c \times Y)\right)\right]\right]^y \in \mathfrak{C}_{y\gamma}$ and so we may define $\varphi_\gamma : \mathfrak{M}_\gamma \longrightarrow \sigma(\mathfrak{C}_\gamma \otimes \mathfrak{B} \cup (\mathfrak{A}\widehat{\otimes}\mathfrak{B})_0)$ setting for every $y \in Y$ $[\varphi_\gamma(W)]^y = \tau_y([\overline{\varphi}_\gamma(W)]^y)$ for every $W \in \mathfrak{M}_\gamma$. The family $(\mathfrak{A}\widehat{\otimes}\mathfrak{B})_0$ is necessary, because $X \times N \in (\mathfrak{A} \otimes \mathfrak{B})_0$ and $\varphi_\gamma(W) \cap (X \times N) \neq \emptyset$ for some W. One can easily check that $\overline{\varphi}_\gamma|_{\mathfrak{M}_\beta} = \overline{\varphi}_\beta$ (defined one step earlier by limit or non-limit procedure) and so $\varphi_\gamma|_{\mathfrak{M}_\beta} = \varphi_\beta$.

If γ is a limit ordinal of countable cofinality, then the proof is exactly as in Proposition 3. The same concerns the rest of the proof.

Remark 4. *If the conditional probability is not absolutely continuous with respect to P, then the situation is similar but not exactly. With the above notation, if $\mathfrak{A}_\beta \ni A \subset T_1 \setminus M_\beta$ is such that $S_{y_0}(A) > 0$, then again $P(A) = 0$ and $S_y(A) = 0$ for Q-a.e. $y \in Y$, but not necessarily for y_0. The unit square with Borel sets and measure R concentrated on a graph of a measurable function (e.g., $y = x^2$) can serve as a good example of non absolutely continuous case. However here the lifting property can be achieved due to separability of measures (see [3]).*

The above considerations can be formulated as

Theorem 4. *Let $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ be probability spaces. Assume that $R \in P \circledast Q$ has a disintegration $\mathbb{S} = \{(\mathfrak{A}_y, S_y) : y \in Y\}$ with respect to Q that is absolutely continuous (i.e., $S_y \ll P|_{\mathfrak{A}_y}$ for every $y \in Y$) and $\mathfrak{C} \subset \bigcap_{y \in Y} \mathfrak{A}_y$ is such a σ-algebra that P and each S_y are inner regular with respect to $\mathfrak{C}$. Let $\{\tau_y \in \vartheta(\widehat{S_y}) : y \in Y\}$ be the family from Proposition 4. Then, there exist liftings $\sigma_y \in \Lambda(\widehat{S_y})$ that are admissibly generated by τ_y for all $y \in Y$ and there exists a lifting $\pi : \mathfrak{A} \circledast \mathfrak{B} \to \mathfrak{C}\widehat{\otimes}\mathfrak{B} \subset \mathfrak{A}\widehat{\otimes}\mathfrak{B}$ such that the condition is satisfied as follows:*

$$[\pi(f)]^y = \sigma_y\Big([\pi(f)]^y\Big) \quad \textit{for every } f \in \mathcal{L}^\infty(R_\circledast) \textit{ and every } y \in Y.$$

4.2. Measurable Modifications of Stochastic Processes

Let $(X, \mathfrak{A}, P)$, $(Y, \mathfrak{B}, Q)$, R, $R_\circledast$ and a regular conditional probability $\{(\mathfrak{A}, S_y),\ y \in Y\}$ on $\mathfrak{A}$ with respect to Q be given. Moreover, let $\{\xi_y : y \in Y\}$ and $\{\theta_y : y \in Y\}$ be stochastic processes defined on $(X, \mathfrak{A}, P)$. We say that the processes are equivalent if $\widehat{S_y}\{x \in X : \xi_y(x) \neq \theta_y(x)\} = 0$ for every $y \in Y$. The process $\{\theta_y : y \in Y\}$ is called measurable if the function $(x, y) \longrightarrow \theta_y(x)$ is $\widehat{R}$-measurable.

There are known several examples of stochastic processes without equivalent measurable version (cf. [16–18]). The following theorem describes the stochastic processes possessing such a measurable modification. It is an essential strengthening of [3] Theorem 5.1 and [1] Theorem 6.1, where only measurable processes were modified.

Theorem 5. *Let $\{\xi_y : y \in Y\}$ be a stochastic process defined on $(X, \mathfrak{A}, P)$. Let $R \in P \circledast Q$ be determined by the regular conditional probability $\{(\mathfrak{A}, S_y) : y \in Y\}$. Assume one of the following conditions:*

(α) *$(X, \mathfrak{A}, P)$ is separable in the Frechet–Nikodým pseudometric;*

(β) *$R \ll P \times Q$.*

Then, the process $\{\xi_y : y \in Y\}$ possesses an equivalent measurable version if and only if the process is $R_\circledast$-measurable. Every modification of a measurable process with the help of a family of liftings generated by an equi-admissibly generated collection of densities is a measurable process.

Proof. Assume that (α) is fulfilled. According to [3] Theorem 3.6 and Lemma 3, there exists a lifting $\pi : \mathfrak{A} \circledast \mathfrak{B} \to \mathfrak{A}\widehat{\otimes}\mathfrak{B}$ and liftings $\{\sigma_y \in \Lambda(\widehat{S_y}) : y \in Y\}$ such that $[\pi(f)]^y = \sigma_y([\pi(f)]^y)$ for every $f \in \mathcal{L}^\infty(R_\circledast)$ and every $y \in Y$. The same follows from Theorem 4 in case of (β). The rest of the proof is a copy of the proof of [5] Proposition 4.1, but now Theorem 4 should be applied. $\square$

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