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Existence of Measurable Versions of Stochastic Processes

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Abstract

Let $(X, \mathfrak{A}, P)$, $(Y, \mathfrak{B}, Q)$ be two arbitrary probability spaces and $\mathbb{P} := \{(\mathfrak{A}, P_y) : y \in Y\}$ be a regular conditional probability (rcp) on $\mathfrak{A}$ with respect to Q. Denote by R the skew product of P and Q determined by $\mathbb{P}$ on the product σ-algebra $\mathfrak{A} \otimes \mathfrak{B}$ and by $\widehat{R}$ its completion. I prove that if $(X, \mathfrak{A}, P)$ is separable in the Fréchet–Nikodým pseudo-metric, then the stochastic process $\{\xi_y : y \in Y\}$ has an equivalent measurable modification if and only if it is measurable with respect to a certain particular σ-algebra larger than $\mathfrak{A} \otimes \mathfrak{B}$. The theorem is a strong generalization of two earlier results of the author and coauthors, where it was only proved that a suitable class of liftings transfer a measurable process into a measurable process. It is known that not every process possesses an equivalent measurable modification. My approach is essentially different from the earlier trials. It reverts to an earlier paper of Talagrand, who proved the existence of an equivalent separable modification of a measurable process (in case of $R = P \times Q$), provided Y is endowed with a separable pseudo-metric.

Keywords: stochastic processes; measurable stochastic processes; measurable modifications; nil sets

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1. Preliminaries

Let $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ be fixed probability spaces. The algebra on $X \times Y$ generated by the rectangles $A \times B$, where $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, is denoted by $\mathfrak{A} \times \mathfrak{B}$. The σ-algebra generated by $\mathfrak{A} \times \mathfrak{B}$ is denoted by $\mathfrak{A} \otimes \mathfrak{B}$. If $(Z, \mathfrak{Z}, T)$ is a measure space, then $(Z, \widehat{\mathfrak{Z}}, \widehat{T})$ denotes its completion and $\mathfrak{Z}_0 := \{A \in \mathfrak{Z} : T(A) = 0\}$. $\mathcal{L}^\infty(T)$ is the collection of bounded T-measurable real functions. $\mathfrak{B}_\mathbb{R}$ denotes the Borel subsets of the real line $\mathbb{R}$. $\mathbb{N}$ is the set of natural numbers.

Assume that $\{P_y : y \in Y\}$ is a regular conditional probability on $\mathfrak{A}$ with respect to Q. The measure R on $(X \times Y, \mathfrak{A} \otimes \mathfrak{B})$ defined by the formula $R(E) = \int_Y P_y(E^y)\, dQ(y)$ ($E^y := \{x \in X : (x, y) \in E\}$) is called a skew product of P and Q. We will use the notation $R \in P \circledast Q$ ($P \circledast Q$ denotes the family of all measures on $\mathfrak{A} \otimes \mathfrak{B}$ with P and Q as marginal measures).

There are several papers concerning the existence of an equivalent measurable modification of a stochastic process defined on a measure space (cf. [1–5]). All are investigated for the direct product of marginal measures. In particular, Cohn [1] presented an example of a process, as well as a lifting (assuming the continuum hypothesis) producing a non-measurable modification of the measurable process (defined in case of the ordinary product measure). The construction is complex and its detailed description is beyond the scope of

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this article. A more subtle construction is presented in [5], where the preservation of stability under liftings is investigated. In [6] (Section 15.5) one can find an example of a process without any measurable modification. In [2], Cohn proved that the existence of a measurable modification is equivalent to the existence of a separable (in the classical sense) measurable modification, but he did not describe any general situation guaranteeing the existence of a measurable modification. However, he proved in [2] (Theorem 2) that each stochastically continuous process has a measurable separable modification. Hoffmann-Jørgensen in [4] presented a necessary and sufficient condition for the existence of equivalent measurable process in terms of two-dimensional distributions of the variables (ξ_s, ξ_t). It follows that the conditions from [4] and mine are equivalent in case of separable $(X, \mathfrak{A}, P)$, but they look completely different. It seems that a direct proof of their equivalence may be complicated. In [7], Talagrand proved that if $\{\xi_y : y \in Y\}$ is a stochastic process and Y is endowed with a separable pseudometric (no measure on Y is required), then the measurable process possesses a separable modification (produced with the help of a special lifting).

I present a necessary and sufficient condition for the existence of a measurable modification of a process in the situation, when $(X, \mathfrak{A}, P)$ is separable in the Fréchet–Nikodým pseudo-metric and R is defined by a regular conditional probability (Theorem 3). It is known that an rcp does not always exist, but it always exists if P is a compact probability (see [8]). Thus, the generalization is of two kinds. Firstly, the product measure $P \times Q$ is replaced by an arbitrary skew product of P and Q, determined by a regular conditional probability. Secondly, I prove that even if some liftings do not produce measurable modifications of the process (in spite of satisfying the necessary measurability assumptions required by Theorem 2), there is another lifting (or rather the large family of liftings) that modifies the initial process into a measurable one. The main result is a strong generalization of [9] (Theorem 5.5) and [10] (Theorem 6.1), where it was only proved that a suitable class of liftings transfer a measurable process into a measurable process.

Classically, rather, the direct product measure on $X \times Y$ is considered, but even in that case, my characterization is new. I require only a measurability of the investigated process with respect to some larger σ-algebra. It is the first necessary and sufficient purely measurable theoretical condition, guaranteeing the existence of measurable modifications of an arbitrary stochastic process, defined on an arbitrary separable probability space.That is the main novelty of the presented approach. Additionally, I also present one necessary and one sufficient condition for the existence of a measurable modification in case of an arbitrary $(X, \mathfrak{A}, P)$ (Theorem 2).

Definition 1. *The family $\mathbb{P} := \{(\mathfrak{A}, P_y) : y \in Y\}$ is called a regular conditional probability (rcp) on $\mathfrak{A}$ with respect to Q if for every $A \in \mathfrak{A}$ the function $y \longrightarrow P_y(A)$ is $\widehat{Q}$-measurable and*

$$\int_B P_y(A)\, dQ(y) = R(A \times B) \quad \text{for every } A \in \mathfrak{A} \text{ and } B \in \mathfrak{B}\,.$$

Definition 2. *Let $(X, \mathfrak{A}, P)$ and $(Y, \mathfrak{B}, Q)$ be arbitrary probability spaces and $\mathbb{P} := \{(\mathfrak{A}, P_y) : y \in Y\}$ be a regular conditional probability on $\mathfrak{A}$ with respect to Q. Moreover, let $\Xi := \{\xi_y : y \in Y\}$ and $\Theta := \{\theta_y : y \in Y\}$ be stochastic processes defined on $(X, \mathfrak{A}, \mathbb{P})$ (that means that each ξ_y and θ_y is a random variable defined on the P_y-completion of $(X, \mathfrak{A}, P_y)$). We say that the processes are $\mathbb{P}$-equivalent if $\widehat{P}_y\{x \in X : \xi_y(x) \neq \theta_y(x)\} = 0$ for every $y \in Y$. The process $\{\theta_y : y \in Y\}$ is called measurable if the function $(x, y) \longrightarrow \theta_y(x)$ is $\widehat{R}$-measurable. If Ξ and Θ are $\mathbb{P}$-equivalent and Θ is measurable, then Θ is called a measurable modification of Ξ.*

Let $M := \{(x, y) \in X \times Y : \xi_y(x) \neq \theta_y(x)\}$. One knows that in general, $M \notin \mathfrak{A} \widehat{\otimes} \mathfrak{B}$, which means that $R^*(M) > 0$.

If $\mathfrak{C} \subset \mathfrak{A}$ is a σ-algebra, then we define the following collection of sets:

$$\mathcal{M}_{\mathfrak{C}} := \{E \subset X \times Y : \exists\, N \in \mathfrak{B}_0\ \forall\, y \notin N\ \widehat{P_y|_{\mathfrak{C}}}(E^y) = 0\}.$$

$\mathcal{M}_{\mathfrak{C}}$ is called a family of R-left nil sets (compare with [11] (3.2.2, Satz 1, Satz 2)) with respect to $\mathfrak{C}$. Since for each $y \in Y$ the $\widehat{P_y|_{\mathfrak{C}}}$-zero sets form a σ-ideal in $\mathcal{P}(X)$, the elements of $\mathcal{M}_{\mathfrak{C}}$ forms a σ-ideal in $\mathcal{P}(X \times Y)$.

If $\mathfrak{D} \supset \mathfrak{C}$ is a σ-algebra, then $\mathfrak{C} \circledast_{\mathfrak{D}} \mathfrak{B} := \{W \triangle N : W \in \mathfrak{C} \otimes \mathfrak{B}\ \ \&\ N \in \mathcal{M}_{\mathfrak{D}}\}$. $\mathfrak{C} \circledast_{\mathfrak{D}} \mathfrak{B}$ is in general a σ-algebra larger than $\mathfrak{C} \otimes \mathfrak{B}$.

If $V \in \mathfrak{A} \circledast_{\mathfrak{A}} \mathfrak{B}$, then the formula

$$R_{\circledast}(V) = \int_Y \widehat{P}_y(V^y)\, d\widehat{Q}(y)$$

defines an extension of R. One can check that if $V = W \triangle N$ ($W \in \mathfrak{A} \otimes \mathfrak{B}$ and $N \in \mathcal{A}_{\mathfrak{A}}$), then $R_{\circledast}(W \triangle N) = R(W)$. The function $y \to V^y$ is $\widehat{Q}$-measurable, because the function $y \to W^y$ is $\widehat{Q}$-measurable.

$R_{\circledast}$ is a complete measure and $\mathcal{M}_{\mathfrak{A}}$ is the σ-ideal of $R_{\circledast}$-zero sets.

$\mathfrak{A} \widehat{\otimes} \mathfrak{B}$—the completion of $\mathfrak{A} \otimes \mathfrak{B}$ with respect to R—is contained in $\mathfrak{A} \circledast_{\mathfrak{A}} \mathfrak{B}$ and $R_{\circledast}$ is an extension of $\widehat{R}$.

Let

$$\mathcal{N}_{\mathfrak{C}} := \{E \subset X \times Y : \forall\, y \in Y\ \widehat{P_y|_{\mathfrak{C}}}(E^y) = 0\}.$$

The new family seems to illustrate better equivalent relation of the processes. One can see immediately that processes Ξ and Θ are equivalent if and only if there exists a set $N \in \mathcal{N}_{\mathfrak{A}}$ such that $\Xi = \Theta$ on N^c. Thus, the σ-ideal $\mathcal{M}_{\mathfrak{A}}$ is implicitly contained in the definition of the equivalence. Unfortunately, the σ-algebras defined by $\mathfrak{C} \diamond_{\mathfrak{C}} \mathfrak{B} := \{W \triangle N : W \in \mathfrak{C} \otimes \mathfrak{B}\ \ \&\ N \in \mathcal{N}_{\mathfrak{C}}\}$ are usually not complete with respect to $(R_{\circledast})|_{\mathfrak{C} \diamond_{\mathfrak{C}} \mathfrak{B}}$. That excludes them from Theorem 1, where the target σ-algebra must be complete. The proper choice remain σ-algebras $\mathfrak{C} \circledast_{\mathfrak{C}} \mathfrak{B}$.

The following theorem is the key result in our examination of measurability of stochastic processes. Lifting is defined in the classical way as in [12]. The collection of all liftings on $(X, \mathfrak{A}, P)$ is denoted by $\Lambda(P)$.

Theorem 1. *Assume that $\mathfrak{A}$ contains a countably generated σ-algebra which is dense in $\mathfrak{A}$ (in the Fréchet–Nikodým pseudo-metric) with respect to P. Moreover, let $\{(\mathfrak{A}, P_y) : y \in Y\}$ be an rcp on $\mathfrak{A}$ with respect to Q. Then, for every $y \in Y$ there exists $\sigma_y \in \Lambda(\widehat{P}_y)$ and there exists a lifting $\pi : \mathcal{L}^\infty(R_{\circledast}) \to \mathcal{L}^\infty(\widehat{R}) \subset \mathcal{L}^\infty(R_{\circledast})$ such that*

$$[\pi(f)]^y = \sigma_y\Big([\pi(f)]^y\Big) \qquad \text{for all} \quad y \in Y \quad \text{and} \quad f \in \mathcal{L}^\infty(R_{\circledast}). \tag{1}$$

Proof. According to [10] (Theorem 3.6), for every $y \in Y$, there exists $\sigma_y \in \Lambda(\widehat{P}_y)$ and there exists a lifting $\pi : \mathcal{L}^\infty(\widehat{R}) \to \mathcal{L}^\infty(\widehat{R})$ satisfying (1) with $R_{\circledast}$ replaced by $\widehat{R}$. The liftings are constructed step-by-step on an arbitrary (but fixed) countably generated σ-algebra that is dense in $\mathfrak{A}$ and then extended to the domains of $\widehat{P}_y$ and onto $\mathfrak{A} \widehat{\otimes} \mathfrak{B}$. Since in the whole lifting theory, the axiom of choice plays an important role, the construction is not unique.

$\mathfrak{A} \widehat{\otimes} \mathfrak{B}$ is $R_{\circledast}$-dense in $\mathfrak{A} \circledast_{\mathfrak{A}} \mathfrak{B}$ and so if $E \in \mathfrak{A} \circledast_{\mathfrak{A}} \mathfrak{B}$ and $F \in \mathfrak{A} \widehat{\otimes} \mathfrak{B}$ are such that $R_{\circledast}(E \triangle F) = 0$, then we may set $\widetilde{\pi}(E) := \pi(F)$. Then, $\widetilde{\pi} : \mathfrak{A} \circledast_{\mathfrak{A}} \mathfrak{B} \to \mathfrak{A} \widehat{\otimes} \mathfrak{B}$ is a lifting satisfying (1). We denote it also by π. □

Remark 1. *The thesis of Theorem 1 remains valid if the separability of $\mathfrak{A}$ is replaced by the absolute continuity of every P_y with respect to P or by the absolute continuity of R with respect to $P \times Q$*

(see [13]). But each of the last two assumptions (that are in some sense almost equivalent) would be too restrictive for our main result. Theorem 3 does not require any absolute continuity. The assumption of absolute continuity would exclude from our investigation a large class of processes with not absolutely continuous rcp. To the best of my knowledge, no necessary and sufficient condition guaranteeing validity of (1) is known.

2. Measurability of Processes

There are several known examples of stochastic processes without equivalent measurable modifications and several conditions guaranteeing existence of measurable modifications, all in the case when $R = P \times Q$ (cf. [1,3–5]). Below I suggest a different purely measurable theoretical approach, based on existence of special liftings in product measure spaces endowed with a regular conditional probability.

Theorem 2. *Let $\Xi := \{\xi_y : y \in Y\}$ be a stochastic process defined on $(X, \mathfrak{A}, \mathbb{P})$, where $\mathbb{P} = \{(\mathfrak{A}, P_y) : y \in Y\}$ is an rcp on $\mathfrak{A}$ with respect to Q. Moreover, let $R \in P \circledast Q$ be determined by $\mathbb{P}$. Then, the following statements hold true:*

(M1) If the process $\{\xi_y : y \in Y\}$ possesses a $\mathbb{P}$-equivalent measurable modification, then there exists a countably generated σ-algebra $\mathfrak{C} \subset \mathfrak{A}$ such that Ξ is measurable with respect to $\mathfrak{C} \circledast_{\mathfrak{A}} \mathfrak{B}$;

(M2) If there exists a σ-algebra $\mathfrak{C} \subset \mathfrak{A}$ such that $\mathfrak{C}$ is separable in the Fréchet–Nikodým pseudometric and Ξ is measurable with respect to $\mathfrak{C} \circledast_{\mathfrak{C}} \mathfrak{B}$, then the process $\{\xi_y : y \in Y\}$ possesses a $\mathbb{P}$-equivalent measurable modification.

If the process Ξ is bounded and the measurable modification $\{\theta_y : y \in Y\}$ is built with the help of the liftings π and $\{\sigma_y : y \in Y\}$ taken from Theorem 1, then for every $y \in Y$, the equality $\theta_y = \sigma_y(\xi_y)$ is fulfilled.

Proof. $(M1)$ Let $\Theta(x, y) := \theta_y(x)$ be $\widehat{R}$-measurable and equivalent to $\{\xi_y : y \in Y\}$. Define the set N by the equality

$$N := \{(x, y) \in X \times Y \colon \theta_y(x) \neq \xi_y(x)\}.$$

According to the assumption $\widehat{P}_y(N^y) = 0$ for every $y \in Y$. Thus, $N \in \mathcal{M}$. If $B \subset \mathbb{R}$ is a Borel set, then

$$\Xi^{-1}(B) = \Xi^{-1}(B) \cap N^c \cup \Xi^{-1}(B) \cap N = \Theta^{-1}(B) \cap N^c \cup \Xi^{-1}(B) \cap N$$

and consequently, $\Xi^{-1}(B) \triangle \Theta^{-1}(B) \subset N \in \mathcal{M}$. This means that $\{\xi_y : y \in Y\}$ is measurable with respect to $\sigma(\Theta^{-1}(\mathfrak{B}_{\mathbb{R}}) \cup \mathcal{M})$. But $\Theta^{-1}(\mathfrak{B}_{\mathbb{R}})$ is a countably generated σ-algebra what immediately yields the existence of a countably generated σ-algebra $\mathfrak{C} \subset \mathfrak{A}$ satisfying the inclusion $\Theta^{-1}(\mathfrak{B}_{\mathbb{R}}) \subset \mathfrak{C} \otimes \mathfrak{B}$. Clearly, Ξ is measurable with respect to $\mathfrak{C} \circledast_{\mathfrak{A}} \mathfrak{B}$.

$(M2)$ Assume at the beginning that the process defined by $\Xi(x, y) := \xi_y(x)$ is bounded, i.e., there exists $\alpha \in (0, +\infty)$ such that $|\xi_y(x)| \leq \alpha$, for every $(x, y) \in X \times Y$. For simplicity of the notation, assume also that $\mathfrak{C} = \mathfrak{A}$.

According to Theorem 1 there exist a lifting $\pi : \mathcal{L}^\infty(R_\circledast) \to \mathcal{L}^\infty(\widehat{R})$ and liftings $\{\sigma_y \in \Lambda(\widehat{P}_y) : y \in Y\}$ such that $[\pi(f)]^y = \sigma_y([\pi(f)]^y)$ for every $f \in \mathcal{L}^\infty(R_\circledast)$ and every $y \in Y$.

Let Ξ be $R_\circledast$-measurable. $\Xi : X \times Y \to \mathbb{R}$ is a function of two variables. In virtue of Theorem 1 the function $\pi(\Xi)$ is $\widehat{R}$-measurable and

$$\sigma_y([\pi(\Xi)]^y) = [\pi(\Xi)]^y \quad \text{for every } y \in Y.$$

It follows from the elementary properties of lifting that there exists a set $N_1 \in \mathfrak{B}_0$ such that for every $y \notin N_1$

$$\sigma_y\big([\pi(\Xi)]^y\big) = [\pi(\Xi)]^y = \sigma_y(\Xi^y) = \sigma_y(\xi_y) \quad \widehat{P}_y - a.e.$$

It follows that Θ defined by $\Theta(\cdot, y) := \sigma_y(\xi_y)$ for every $y \in Y$, is $\widehat{R}$-measurable and $\mathbb{P}$-equivalent to Ξ. More precisely, $\Theta|_{X\times N_1^c} = \pi|_{X\times N_1^c}$.

If the process Ξ is unbounded and Ξ is $\mathfrak{A} \circledast_{\mathfrak{A}} \mathfrak{B}$-measurable, then one can decompose the space $X \times Y$ into at most countably many pairwise disjoint sets $\Omega_n \in \mathfrak{A} \circledast_{\mathfrak{A}} \mathfrak{B}$ such that Ξ is bounded on each Ω_n, $n \in \mathbb{N}$. One may set for instance $\Omega_n := \{(x, y) : n \le |\Xi(x, y)| < n + 1\}$. Let for each $n \in \{0, 1, 2, \ldots\}$

$$\Xi_n(x, y) := \Xi(x, y)\, \chi_{\Omega_n}(x, y).$$

On the strength of the proof for bounded processes, for each n there exists an $\widehat{R}$-measurable

$$\Theta_n := \pi\big(\Xi_n\big) = \pi\big(\Xi_n\big)\chi_{\pi(\Omega_n)}$$

that is equivalent to Ξ_n. Since the supports of the functions Θ_n are pairwise disjoint (if $n \ne m$, then $\pi(\Omega_n) \cap \pi(\Omega_m) = \emptyset$) and $\widehat{R}\big(\bigcap_n \pi(\Omega_n^c)\big) = 0$, we may define the process Θ setting

$$\Theta(x, y) = \begin{cases} \Theta_n(x, y) & \text{if } (x, y) \in \pi(\Omega_n) \\ \xi_y(x) & \text{if } (x, y) \in \bigcap_n \pi(\Omega_n^c)\,. \end{cases}$$

Let $N := \{y : \widehat{P}_y\big(\big[\bigcap_n \pi(\Omega_n^c)\big]^y\big) = 0\}$. Then, $\widehat{Q}(N) = 1$ and $\Theta^y \equiv_{\widehat{P_y}} \xi_y$ for $y \in N$. If $y \notin N$ and $A := \big[\bigcap_n \pi(\Omega_n^c)\big]^y$, then $\Theta(x, y) = \Theta_n(x, y)$ if $x \notin A$ and $(x, y) \in \pi(\Omega_n)$ and, $\Theta(x, y) = \xi_y(x)$ in case of $x \in A$ and $(x, y) \in \pi(\Omega_n)$. All measurability requirements and consistency properties follow from the analogical properties of each Θ_n. $\square$

Theorem 3. *Let $\Xi := \{\xi_y : y \in Y\}$ be a stochastic process defined on $(X, \mathfrak{A}, \mathbb{P})$, where $\mathbb{P} = \{(\mathfrak{A}, P_y) : y \in Y\}$ is an rcp on $\mathfrak{A}$ with respect to Q. Moreover, let $R \in P \circledast Q$ be determined by $\mathbb{P}$. If $(X, \mathfrak{A}, P)$ is separable in the Fréchet–Nikodým pseudo-metric, then the process $\{\xi_y : y \in Y\}$ has an equivalent measurable modification if and only if it is measurable with respect to $\mathfrak{A} \circledast_{\mathfrak{A}} \mathfrak{B}$.*

Remark 2. *If the process $\{\xi_y : y \in Y\}$ has an equivalent measurable modification, then the modification is separable in the Fréchet–Nikodým pseudo-metric (in virtue of $(M1)$). But that does not mean the separability in the classical sense (cf. [7]). It follows that the separability condition—as it is formulated in (M1)—is indispensable. However, it is probably not sufficient in case of non-separable measure spaces. Thus, some kind of separability reflects a deeper property of the class of processes admitting measurable modifications. But, accidentally, it is also a technical assumption required by the lifting argument (without the separability assumption, Theorem 1 may be false).*

Remark 3. *At this point, it is important to note that the liftings σ_y are not quite arbitrary; they are chosen in a very special way. These are so-called admissibly generated liftings (for details, see [10]). Without such an assumption, the lifting modification of a measurable process may produce a non-measurable process (see [1,3]).*

In the most classical version $R = P \times Q$ and processes $\{\xi_y : y \in Y\}$ and $\{\theta_y : y \in Y\}$ are called equivalent if $P(\{\xi_y \ne \theta_y\}) = 0$ for every $y \in Y$. In that case, the rcp is almost uniquely defined ($P_y = P$ for Q-a.e. $y \in Y$) and one may assume that $P_y = P$ for every $y \in Y$. Hence, $\mathcal{M}_{\mathfrak{C}} = \{E \subset X \times Y : \exists\, N \in \mathfrak{B}_0\ \forall\, y \notin N\ \widehat{P|_{\mathfrak{C}}}(E^y) = 0\}$.

In the language of functions of two variables, Theorem 3 can be formulated in the following way:

Claim 1. *Let $(X, \mathfrak{A}, P)$ be separable in the Fréchet–Nikodým pseudo-metric. A function $f : X \times Y \to \mathbb{R}$ with measurable Y-sections has an equivalent $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$-measurable modification if and only if f is measurable with respect to $\mathfrak{C} \circledast_{\mathfrak{A}} \mathfrak{B}$.*

Example 1. *Let $(X, \mathfrak{A}, P) = (Y, \mathfrak{B}, Q) = ([0,1], \mathfrak{L} \cap [0,1], \lambda)$, where λ is the Lebesgue measure on the σ-algebra $\mathfrak{L}$ of Lebesque measurable sets. Moreover, let $V \subset Y$ be a Vitali set. For each $y \in V$ choose a non-trivial λ-zero set $A_y \in \mathfrak{B}_{\mathbb{R}}$. Define the set $E \subset [0,1]^2$ in the following way: if $y \in V$, then $E^y := A_y$ and if $y \notin V$, then $E^y = \emptyset$. The function χ_E is $\mathfrak{L} \circledast \mathfrak{L}$ measurable but not necessarily plane-measurable. It depends on the position of the sets A_y. The zero function is the measurable function that is equivalent to χ_E. The concrete non-trivial example of E can be found in [14], where—under the continuum hypothesis—Sierpiński constructed a set $E \subset [0,1] \times [0,1]$ with the following properties:*

(1) All vertical sections E^c_x are at most countable;
(2) All horizontal sections E^y are at most countable.

$E \in \mathfrak{L} \circledast_{\mathfrak{L}} \mathfrak{L}$ (more precisely, $E \in \mathcal{M}_{\mathfrak{L}}$) but $E \notin \mathfrak{L}\widehat{\otimes}\mathfrak{L}$.

In a more general case, assume that there exists a function $h : X \to Y$ such that its graph G is not in $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$. Then, $G \in \mathfrak{A} \circledast_{\mathfrak{A}} \mathfrak{B}$. If $\{\xi_y : y \in Y\}$ is a measurable process, then Θ defined by $\Theta(x,y) = \xi_y(x)$ for $(x,y) \notin G$ and $\Theta(x,y) = 0$ for $(x,y) \in G$ is the process that is $\mathfrak{A} \circledast_{\mathfrak{A}} \mathfrak{B}$-measurable and admits a measurable modification.

Example 2. *Cohn [1], assuming the continuum hypothesis, constructed a measurable process Ξ such that for some lifting ρ on the completion of $(X, \mathfrak{A}, P)$ the process Θ defined by $\Theta(x,y) := \rho[\Xi(\cdot,y)](x)$ is not measurable. The construction is too complicated even for a short description. It is not based on any product lifting. The process Θ is a non-trivial example of a non-measurable process possessing a measurable modification. It follows from the condition (M1) of Theorem 2 that Θ is measurable with respect to some $\mathfrak{C} \circledast_{\mathfrak{A}} \mathfrak{B}$ with separable $\mathfrak{C}$.*

Example 3. *Applying Theorem 3, we present a simple example of a stochastic process that has no measurable modification. The example is copied from [6] (Section 19.5). Let $(X, \mathfrak{A}, P)$ be an arbitrary separable probability space, $(Y, \mathfrak{B}, P) = ([0,1], \mathfrak{L}, \lambda)$ and $R = P \times \lambda$. Let Ξ be the process defined by $\Xi(x,y) = \xi_y(x)$, where the random variables $\{\xi_y : y \in Y\}$ are mutually independent, bounded, and such that*

$$\mathbb{E}(\xi_y) = 0 \quad \text{and} \quad \mathbb{E}(\xi_y^2) = 1 \quad \text{for every } y \in Y. \tag{2}$$

Suppose that Ξ is $\mathfrak{A} \circledast \mathfrak{L}$-measurable. Then, according to Theorem 3, there exists a lifting $\pi : \mathcal{L}^\infty(R_\circledast) \to \mathcal{L}^\infty(\widehat{R})$ such that Θ defined by $\Theta := \pi(\Xi)$ is a measurable modification of Ξ. But the new variables $\theta := \sigma_y(\xi_y)$ are also mutually independent and satisfy (2) with ξ_y replaced by θ_y. Applying the Fubini theorem, we obtain for each closed interval $I \subset [0,1]$ with rational endpoints

$$\mathbb{E}\left\{\left(\int_I \theta_y(x)\,dy\right)^2\right\} = \mathbb{E}\left\{\int_I \int_I \theta_y(x)\theta_z(x)\,dy\,dz\right\} = \int_I \int_I \mathbb{E}(\theta_y\theta_z)\,dy\,dz = 0.$$

This means that for each $x \notin N_I \in \mathfrak{A}_0$, we have $\int_I \theta_y(x)\,dy = 0$. If $N = \bigcup_I N_I$, then $P(N) = 0$ and $\int_J \theta_y(x)\,dy = 0$ for every interval $J \subset [0,1]$ and every $x \notin N$. This means that for each $x \notin N$, we have $\theta_y(x) = 0$ λ-a.e. Applying the Fubini theorem, we have

$$\int_X \int_0^1 \theta_y(x)^2\,dP(x)\,dy = 0.$$

On the other hand,

$$\int_X \int_0^1 \theta_y(x)^2 \, dP(x) \, dy = \int_0^1 \mathbb{E}(\theta_y^2) \, dy = 1.$$

Thus, Θ is not measurable. It follows from Theorem 3 that Ξ is not measurable with respect to $\mathfrak{A} \circledast_{\mathfrak{A}} \mathfrak{B}$.

3. Iterated Integrals

In case of the direct product, the coordinate measure spaces and non-measurable process applying the Fubini theorem is impossible. Instead, following [15], I remind the reader of a generalization of the Fubini theorem covering the case of separately integrable functions.

In case of the direct product, the coordinate measure spaces are equally important. From that point of view, it is suitable to introduce the σ-ideals

$$\mathcal{M}_{\mathfrak{A}}^{\perp} := \{E \subset X \times Y \colon \exists\, N \in \mathfrak{A}_0 \; \forall\, x \notin N \; \widehat{Q}(E_x) = 0\},$$

$$\mathcal{M}^{\bigstar} := \mathcal{M}_{\mathfrak{A}} \cap \mathcal{M}_{\mathfrak{A}}^{\perp} = \left\{ E \subset X \times Y \;\middle|\; \begin{array}{l} \exists\, N \in \mathfrak{A}_0 \; \exists\, M \in \mathfrak{B}_0 \\ \forall\, x \notin N \; \widehat{Q}(E_x) = 0 \; \forall\, y \notin M \; \widehat{P}(E^y) = 0 \end{array} \right\}.$$

and the σ-algebras $\mathfrak{A} \bigstar \mathfrak{B} := \sigma\big((\mathfrak{A} \otimes \mathfrak{B}) \cup M^{\bigstar}\big)$ and $\mathfrak{A} \circledast^{\perp} \mathfrak{B} := \sigma\big((\mathfrak{A} \otimes \mathfrak{B}) \cup \mathcal{M}^{\perp}\big)$. The σ-ideal $\mathcal{M}^{\bigstar}$ was introduced in [15] and was called the σ-ideal of nil sets. Let $R^{\circledast} : \mathfrak{A} \circledast^{\perp} \mathfrak{B} \to [0,1]$ be defined by $R^{\circledast}(F) = \int_X \widehat{Q}(F_x) \, d\widehat{P}(x)$. If $F = E \triangle M$ with $E \in \mathfrak{A} \widehat{\otimes} \mathfrak{B}$ and $M \in \mathcal{M}^{\bigstar}$, then

$$\begin{aligned} R_{\circledast}(F) &= \int_Y \widehat{P}(F^y) \, d\widehat{Q}(y) = \int_Y \widehat{P}(E^y) \, d\widehat{Q}(y) = (P \widehat{\otimes} Q)(E) \\ &= \int_X \widehat{Q}(E_x) \, d\widehat{P}(x) = \int_X \widehat{Q}(F_x) \, d\widehat{P}(x) = R^{\circledast}(E) \end{aligned}$$

It follows that the measure $P \widehat{\times} Q$ can be extended to $\mathfrak{A} \bigstar \mathfrak{B}$ by setting $R_{\bigstar}(F) = (P \widehat{\times} Q)(E)$. Moreover, $R_{\bigstar}(F) = R_{\circledast}(F) = R^{\circledast}(F)$ for every $F \in \mathfrak{A} \bigstar \mathfrak{B}$. The above equalities yield immediately the following interesting generalization of the Fubini theorem, that can be found in [15] but now seems to be forgotten. Let us say that f is nearly separately measurable (integrable) if f^y is $\mathfrak{A}$-measurable (P-integrable) for Q-a.e. $y \in Y$ and f_x is $\mathfrak{B}$-measurable (Q-integrable) for P-a.e. $x \in X$.

Proposition 1. *Let $(X, \mathfrak{A}, P)$, $(Y, \mathfrak{B}, Q)$ be complete probability spaces and $f : X \times Y \to \mathbb{R}$ be a function. If f is $R_{\bigstar}$-integrable or non-negative, then it is nearly separately integrable and*

$$\begin{aligned} &\int_{X \times Y} f \, dR_{\bigstar} = \\ &= \int_Y \Big(\int_X f(x,y) \, dP(x) \Big) \, dQ(y) = \int_X \Big(\int_Y f(x,y) \, dQ(y) \Big) \, dP(x) \, . \end{aligned}$$

Proof. Assume that f is bounded and a uniform limit of a sequence of simple functions $f_n = \sum_{i=1}^{k_n} a_{i,n} \chi_{E_{k,n}}$, where $E_{k,n} \in \mathfrak{A} \bigstar \mathfrak{B}$. For every $n \in \mathbb{N}$, we have the equality

$$\begin{aligned} &\int_{X \times Y} f_n \, dR_{\bigstar} = \sum_{i=1}^{k_n} a_{i,n} \int_{X \times Y} R_{\bigstar}(E_{k,n}) \\ &= \sum_{i=1}^{k_n} a_{i,n} R_{\circledast}(E_{k,n}) = \int_Y \Big(\int_X f_n(x,y) \, dP(x) \Big) \, dQ(y) \\ &= \int_X \Big(\int_Y f_n(x,y) \, dQ(y) \Big) \, dP(x) \end{aligned}$$

Applying the Lebesgue Dominated Convergence Theorem, we obtain the equality

$$\int_{X\times Y} f\,dR_\bigstar = \lim_n \int_{X\times Y} f_n\,dR_\bigstar$$
$$= \lim_n \int_Y \Big(\int_X f_n(x,y)\,dP(x)\Big)\,dQ(y) = \int_Y \Big(\int_X f(x,y)\,dP(x)\Big)\,dQ(y)$$

This is similar for the reversed order of P and Q. □

Remark 4. *Assume that $R = P \times Q$ and $\Xi = \{\xi_y : y \in Y\}$ is a process with measurable paths $x \to \xi_y(x)$. If Ξ is not measurable, then applying the Fubini theorem is impossible. In such a case, Proposition 1 may be useful.*

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References

1. Cohn, D.L. Liftings and the construction of stochastic processes. *Trans. Amer. Math. Soc.* **1978**, *246*, 429–438. [CrossRef]
2. Cohn, D.L. Measurable Choice of Limit Points and the Existence of Separable and Measurable Processes. *Z. Wahrscheinlichkeitstheorie Verw. Geb.* **1972**, *22*, 161–165. [CrossRef]
3. Dudley, R.M. A counterexample on measurable processes. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*; University of California Press: Berkeley, CA, USA, 1972; Volume 2, pp. 57–66. Correction in *Ann. Prob.* **1973**, *1*, 191–192.
4. Hoffmann-Jørgensen, J. Existence of Measurable Modifications of Stochastic Processes. *Z. Wahrscheinlichkeitstheorie Verv. Geb.* **1973**, *25*, 205–207. [CrossRef]
5. Talagrand, M. On Liftings and the Regularization of Stochastic Processes. *Probab. Theory Relat. Fields* **1988**, *78*, 127–134. [CrossRef]
6. Stoyanov, J.M. *Counterexamples in Probability*; Dover Publications: Garden City, NY, USA, 2014.
7. Talagrand, M. Measurability problems for empirical processes. *Ann. Prob.* **1987**, *15*, 204–212. [CrossRef]
8. Pachl, J.K. Disintegration and compact measures. *Math. Scand.* **1978**, *43*, 157–168. [CrossRef]
9. Musiał, K.; Strauss, W.; Macheras, N.D. Product liftings and densities with lifting invariant and density invariant sections. *Fund. Math.* **2000**, *166*, 281–303. [CrossRef]
10. Strauss, W.; Macheras, N.D.; Musiał, K. Splitting of liftings in products of probability spaces. *Ann. Prob.* **2004**, *32*, 2389–2408. [CrossRef]
11. Haupt, O.; Pauc, C.Y. Ueber Erweiterungen von Inhalten durch Adjunktion von Nullsomen. In *Abhandlungen der Mathematisch-Naturwissenschaftlichen Klasse*; Akademie der Wissenschaften und der Literatur in Mainz: Mainz, Germany, 1957; Volume 8, pp. 3–20.
12. Ionescu Tulcea, A.; Ionescu Tulcea, C. *Topics in the Theory of Lifting*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1969.
13. Macheras, N.D.; Musiał, K.; Strauss, W. Splitting of liftings in products of probability spaces II. *JMAA* **2007**, *335*, 213–224. [CrossRef]
14. Sierpiński, W. *Hypothese du Continu*; Monografie Matematyczne; Towarzystwo Naukowe Warszawskie: Warsaw, Poland, 1934.
15. Bledsoe, W.W.; Morse, A.P. Product measures. *Trans. Amer. Math. Soc.* **1955**, *79*, 173–215. [CrossRef]