SMALL CANCELATION: EXERCISE SHEET 4

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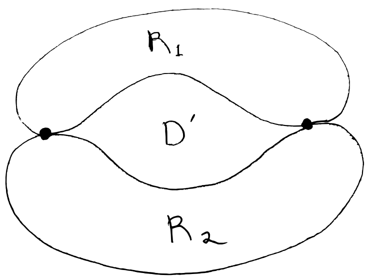
(1) Compute the turning angles of positive turns of C(6)-T(3) disk diagrams for which the 2-cells are hexagons and all pieces have length 1. Verify that these turning angles coincide with $2\pi \cdot \kappa(v_R, R)$.

(2) Extend Greendlinger's Lemma to singular disk diagrams.

(3) Let $X = \langle S \mid R \rangle$ be a $C(p)$-$T(q)$ group presentation with $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$. Show that if $w \in R$ is of the form u^n for some $n \in \mathbb{N}$ then $[u]$ has order n in $\pi_1(X)$. Hint: use the proposition proved in class that 2-cells embed in simply connected $C(p)$-$T(q)$ complexes.

(4) Let $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$. Prove that a disk diagram of the form



cannot be C(p)-T(q). Use this to prove that 2-cells have connected intersections in simply connected C(p)-T(q) complexes. Consequently 2-cells have connected intersections in reduced disk diagrams of C(p)-T(q) complexes.

(5) A finite presentation $\langle S \mid R \rangle$ is *Dehn* if every trivial word w has a cyclic subword u such that the following two conditions hold.
   (a) u is also a cyclic subword of some relator $r \in R$
   (b) $|u| < \frac{1}{2}|r|$

   Give a disk diagrammatic characterization of the Dehn property. Use Greendlinger's Lemma and the classification of positively curved turns to prove that C'($\frac{1}{6}$)-$T(3)$ and C'($\frac{1}{4}$)-$T(4)$ presentations are Dehn.

   Dehn presentations are of interest since they have solvable word problem via Dehn's algorithm, which Dehn first applied to fundamental groups of surfaces in the early 20th century. This algorithm reduces its input word w and then searches for u as above in w. Upon finding such u it is replaced with the negation of the complement of u in r. The resulting word w' is equivalent to w but is shorter. This process is repeated until either no u may be found, in which case the word is not trivial, or we end up with the trivial word. A more sophisticated algorithm exists that has linear time complexity [1].

References

[1] B. Domanski and M. Anshel. The complexity of Dehn's algorithm for word problems in groups. *J. Algorithms*, 6(4):543–549, 1985.

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