

Yet another cardinal coefficients of analytic P-ideals

Pbn

Abstract

This is a working note. The note may (and, probably, does) contain mistakes and errors. Any comments, new ideas, old ideas and beer proposals warmly invited.

1 Introduction

This note contains mostly problems, of which the main one is Problem 2.1 which most of you already know. I tried to formulate it in a general form, for analytic P-ideals. Perhaps for some of them the presented problems are very easy. I am interested mostly in the ideal of asymptotic density sets (or, generally, in density ideals) and, I believe, for this one not every problem contained here is easy.

The last sections are about MAD families. Problems (Problem 2.2 and Problem 5.3) presented there seems to be easier than the above. I show that under MA (and CH) there are MAD's without \mathcal{I} -intersections. I guess, one does not need additional axioms for that, but I don't know how to show it.

2 Preliminaries

Recall that a set $P \subseteq \omega$ is a *pseudo-intersection* of a family $\mathcal{A} \subseteq P(\omega)$ if $P \subseteq^* A$ (which means that $P \setminus A$ is finite) for every $A \in \mathcal{A}$.

The *pseudo-intersection number* \mathfrak{p} is the minimal size of a family \mathcal{A} with *strong finite intersection property* (i.e. such that $A \cap B$ is infinite for every $A, B \in \mathcal{A}$) without a pseudo-intersection.

One can consider other ideals than *Fin* in the above definitions. It was done in various ways in many papers. For an ideal \mathcal{I} on ω one can define the analog of the pseudo-intersection number for \mathcal{I} as e.g.

- the minimal cardinality of the family from the dual filter \mathcal{I}^* which does not have a pseudo-intersection in \mathcal{I}^* ($\mathfrak{p}(\mathcal{I}^*)$, see [4]; $\text{add}^*(\mathcal{I})$, see [8]);
- the minimal cardinality of the family from the dual filter \mathcal{I}^* which does not have a pseudo-intersection ($\chi\mathfrak{p}(\mathcal{I}^*)$, see [4]; $\text{cov}^*(\mathcal{I})$, see [8]) ;
- the minimal cardinality of a family \mathcal{A} with the \mathcal{I} -strong finite intersection property (every finite subfamily has an intersection outside \mathcal{I}) without a set outside \mathcal{I} which is almost included (in the sense of \mathcal{I}) in every member of \mathcal{A} (see [6]).

The aim of this note is to present another way of generalizing the pseudo–intersection number, as well as other classical cardinal coefficients.

Let \mathcal{I} be an ideal on ω . For an infinite set $N = \{n_1 < n_2 < n_3 < \dots\} \subseteq \omega$ a copy of \mathcal{I} on N is the ideal consisting of subsets of N of the form $\{n_i: i \in \mathcal{I}\}$.

Denote by \mathcal{I}_N the ideal of **subsets of ω** A for which $A \cap N$ belongs to the copy of \mathcal{I} on N . So,

$$\mathcal{I}_N = \{A \subseteq \omega: \{i: n_i \in A \cap N\} \in \mathcal{I}\}.$$

For a family \mathcal{F} of subsets of ω say that $P \subseteq \omega$ is an \mathcal{I} –intersection of \mathcal{F} if $F^c \in \mathcal{I}_P$ for every $F \in \mathcal{F}$. Notice that for \mathcal{I} being a trivial ideal, an \mathcal{I} –intersection is just a subset of the intersection, and for \mathcal{I} being the ideal of finite subsets of ω an \mathcal{I} –intersection of \mathcal{F} is a pseudo–intersection of \mathcal{F} .

In this setting a natural way to define an \mathcal{I} –intersection number is of course the following:

$$\mathfrak{p}_{\mathcal{I}} = \min\{|\mathcal{F}|: \mathcal{F} \text{ has sfip and } \mathcal{F} \text{ does not have an } \mathcal{I} \text{ – intersection}\}.$$

We can consider analogs of other cardinal coefficients obtained in this way (for the definitions of the classical coefficients, confront e.g. [2]).

A family $\mathcal{T} \subseteq P(\omega)$ is a tower if for every $A, B \in \mathcal{T}$ either $A \subseteq^* B$ or $B \subseteq^* A$ and \mathcal{T} possess no pseudo–intersection. The \mathcal{I} –tower number $\mathfrak{t}_{\mathcal{I}}$ is the smallest cardinality of any tower without an \mathcal{I} –intersection, provided such a tower exists.

It would be convenient to say that an ideal \mathcal{J} does not have an \mathcal{I} –intersection if the filter dual to \mathcal{J} does not have an \mathcal{I} –intersection. For an almost disjoint family \mathcal{A} of subset of ω we will say that it is \mathcal{I} –maximal if the ideal generated by \mathcal{A} does not have an \mathcal{I} –intersection. Alternatively, we can say that an AD family is \mathcal{I} –maximal if it cannot be extended (to an AD family) by a set X such that $A \in \mathcal{I}_X$ for every $A \in \mathcal{A}$. The \mathcal{I} –almost disjointness number $\mathfrak{a}_{\mathcal{I}}$ is the smallest cardinality of an almost disjoint family which is \mathcal{I} –maximal, provided such a family exists.

A family \mathcal{S} is splitting if it splits every infinite $A \subseteq \omega$, i.e. there is a set $S \in \mathcal{S}$ such that $S \cap A$ and $A \setminus S$ are infinite. The \mathcal{I} –splitting number $\mathfrak{s}_{\mathcal{I}}$ is the smallest cardinality of any splitting family such that for every $A \subseteq P(\omega)$ there is $S \in \mathcal{S}$ such that $S \notin \mathcal{I}_A$ and $S^c \notin \mathcal{I}_A$, provided such a family exists.

The schema of the above definitions is the following: *consider subfamilies of $P(\omega)$ satisfying certain condition expressed modulo the ideal \mathcal{I} and ask about the cardinality of the minimal among them with certain property expressed modulo the fixed ideal \mathcal{I} .*

One can consider in the same manner the analog of the reaping number \mathfrak{r} , the distributivity number \mathfrak{h} or the independent number \mathfrak{i} .

Clearly, if \mathcal{I} contains every finite set and \mathfrak{k} is one of the above cardinal coefficients, then $\mathfrak{k} \leq \mathfrak{k}_{\mathcal{I}}$, provided $\mathfrak{k}_{\mathcal{I}}$ is well defined.

Problem 2.1 *Let \mathcal{I} be a tall analytic P -ideal and let \mathfrak{k} be one of the above cardinal coefficients. Is it consistent to assume $\mathfrak{k} < \mathfrak{k}_{\mathcal{I}}$? In particular, is it consistent that $\mathfrak{p} < \mathfrak{p}_{\mathcal{I}}$, $\mathfrak{t} < \mathfrak{t}_{\mathcal{I}}$ or $\mathfrak{a} < \mathfrak{a}_{\mathcal{I}}$?*

To get a filter without an \mathcal{I} -intersection we can take any ultrafilter, so it is obvious that $\mathfrak{p}_{\mathcal{I}}$ is well-defined. However, it is not so obvious that any of the remaining coefficients presented above is well-defined in ZFC (and for every (reasonable) ideal).

Problem 2.2 *Let \mathcal{I} be a tall analytic P -ideal. Which of the above defined coefficients are well-defined?*

3 Analytic background of the problem

The idea of $\mathfrak{p}_{\mathcal{I}}$ came from certain analytic considerations.

Recall that the definition of the pseudo-intersection number can be reformulated in topological terms. Namely, one can see the pseudo-intersection number as *the smallest weight of a countable space which is not Frechet-Urysohn* (i.e. there is a set A and a point $x \in \overline{A}$ such that no sequence from A converges to x).

A compact space K can be seen as a closed subspace of the space of probability measures $P(K)$ on K with the weak* topology (the embedding is given by $x \rightarrow \delta_x$). Recall that in this topology μ_n converges to μ if and only if $\mu_n(A) \rightarrow \mu(A)$ for every clopen A . Notice that (x_n) converges to x if and only if (δ_{x_n}) converges to δ_x .

Of course $P(K)$ is usually much more complicated than K . One can distinguish different levels of complexity in $P(K)$. The simplest measures are the Dirac measures (and $\{\delta_x : x \in K\}$ is homeomorphic to K). Then we can consider space of all convex combinations of Dirac measures (which we will call $S_0(K)$ after [3]). Next, all limits of sequences from $S_0(K)$ (called $S_1(K)$), and so on. In this manner we obtain a sequence $((S_{\xi})_{\xi < \omega_1})$ (on limit stages setting union of all previously defined spaces).

Here, we list several known facts about S_n spaces.

Fact 3.1 *The following facts hold for a space K :*

- $S_0(K)$ is the space of finitely supported measures and $S_1(K)$ contains the space of purely atomic measures;
- $S_0(K)$ is dense in $P(K)$;
- $S_1(K) = P(K)$ if and only if every $\mu \in P(K)$ has a uniformly distributed sequence (x_k) , i.e. $\mu = \lim \frac{1}{n} \sum_1^n \delta_{x_k}$ (see [3]);
- for many spaces $S_1(K) = P(K)$, e.g. for all scattered spaces, but surprisingly also for 2^c (see **491Q** in [7])
- in [3] it is proved that $S_{\omega_1}(K) = P(K)$ if K is Koppelberg compact.

\mathcal{I} -intersection numbers allows us to look one step deeper in the structure of $P(K)$.

Recall that a set $A \subseteq \omega$ has asymptotic density zero if $\lim(|A \cap n|/n) = 0$ and denote by \mathcal{Z} the ideal of density zero sets. If $N = \{n_1, n_2, \dots\}$ is a \mathcal{Z} -intersection of \mathcal{F} , then

$$\frac{\{n_1, \dots, n_k\} \cap A}{k} \rightarrow 1$$

if and only if $A \in \mathcal{F}$. But this means that

$$\delta_{\mathcal{F}} = \lim \frac{\delta_{n_1} + \cdots + \delta_{n_k}}{k}.$$

We will say that a topological space K with a structure of linear space satisfies the *convex Frechet–Urysohn condition* if for every convex set $A \subseteq K$ and $x \in \overline{A}$, there is a sequence in A convergent to x

Fact 3.2 *The number $\mathfrak{p}_{\mathcal{Z}}$ is the smallest weight of a countable space K such that $S_0(K)$ does not satisfy the convex Frechet–Urysohn condition.*

Proof. Let \mathcal{A} be a Boolean algebra generated by a filter \mathcal{F} witnessing $\mathfrak{p}_{\mathcal{Z}}$ and let K be its Stone space. Let $L = \omega \cup \{\mathcal{F}\} \subseteq K$. Then $\delta_{\mathcal{F}} \in \overline{\{\delta_n : n \in \omega\}} \subseteq \overline{S_0(\omega)}$ (since $\overline{\omega} = K$), but there is no sequence from $S_0(\omega)$ converging to $\delta_{\mathcal{F}}$ and (by Fact 3.1(a)) L is a countable space with weight $\mathfrak{p}_{\mathcal{Z}}$, which is not Frechet–Urysohn.

If $\text{weight}(K) < \mathfrak{p}_{\mathcal{Z}}$, K is countable and $x \in K$ is in the closure of $S_0(A)$, then $\{A \cap U : U \text{ is a neighbourhood of } x\}$ forms a filter with a \mathcal{Z} -intersection $\{x_1, x_2, \dots\} \subseteq K$. Then

$$\delta_x = \lim \frac{\delta_{x_1} + \cdots + \delta_{x_k}}{k}.$$

■

In a sense we can call $\mathfrak{p}_{\mathcal{Z}}$ a convex pseudo–intersection number.

In [3] the following question loosely connected to the above considerations was raised.

Problem 3.3 *Is there a Boolean algebra $\mathcal{A} \subseteq P(\omega)$ such that no ultrafilter on \mathcal{A} has a pseudo–intersection but every ultrafilter has a \mathcal{Z} -intersection?*

In [3] it was showed that if there is such a Boolean algebra and this Boolean algebra is dense in $P(\omega)$, then there is a Gelfand–Philips space without Mazur property. It was proved that in the models of set theory, where $\mathfrak{p}_{\mathcal{I}} > \mathfrak{h}$, there is a Boolean algebra as described above.

4 \mathcal{I} -intersection of filters

One can think about the following ordering on ideals on ω : $\mathcal{I} \leq \mathcal{J}$ if there is a bijection $f: \omega \rightarrow \omega$ such that $f[I] \in \mathcal{J}$ for every $I \in \mathcal{I}$. So, $\mathcal{I} \leq \mathcal{J}$ if \mathcal{I} is included in \mathcal{J} modulo a permutation.

An inequality $\mathcal{I} \leq \mathcal{J}$ implies e.g. inequality between these ideals in the sense of Katetov.

Fact 4.1 *If a filter \mathcal{F} has an \mathcal{I} -intersection, then $\mathcal{F}^* \leq \mathcal{I}$.*

Proof. Fix a set $N = \{n_1 < n_2 < \dots\} \in \mathcal{I}^*$ and define f in such a way that $f(x_i) = n_i$, where $\{x_1 < x_2 < \dots\}$ is an \mathcal{I} -intersection of \mathcal{F} .

■

(**BTW:** can we reverse the above implication? Is it true that if $\mathcal{I} \leq \mathcal{J}$, then \mathcal{I}^* has a \mathcal{J} -intersection?)

From Fact 4.1 and the theorem that every analytic ideal is feeble it follows that every filter with an \mathcal{I} -intersection is feeble. This fact together with Petr Simon's result that (in ZFC) there is a non-feeble filter generated by \mathfrak{b} many sets, give us the following upper bound for $p_{\mathcal{I}}$.

Fact 4.2 *If \mathcal{I} is an analytic P-ideal, then*

$$p_{\mathcal{I}} \leq \mathfrak{b}.$$

Recall that for a tall ideal \mathcal{I} the coefficient $\text{add}^*(\mathcal{I})$ denotes the minimal cardinality of a family in \mathcal{I}^* without a pseudo-intersection from \mathcal{I}^* . If $\mathfrak{p} < \text{add}^*(\mathcal{I})$ ($\mathfrak{t} < \text{add}^*(\mathcal{I})$), then the filter (tower) witnessing \mathfrak{p} (\mathfrak{t}) cannot have an \mathcal{I} -intersection. Therefore

Fact 4.3 *If $\mathfrak{p} < \text{add}^*(\mathcal{I})$, then $p_{\mathcal{I}} = \mathfrak{p}$. Also if $\mathfrak{t} < \text{add}^*(\mathcal{I})$, then $\mathfrak{t}_{\mathcal{I}} = \mathfrak{t}$.*

It follows that under $\mathfrak{p} < \text{add}^*(\mathcal{I})$ and $\mathfrak{p} < \mathfrak{b}$ there is a filter which is feeble but does not have an \mathcal{I} -intersection. In the next section we will come back to this subject.

If we start to believe that consistently $\mathfrak{p} < p_{\mathcal{I}}$ or $\mathfrak{t} < \mathfrak{t}_{\mathcal{I}}$, we can think about an iteration of forcing which adds \mathcal{I} -intersections without adding pseudo-intersections. Or, we can try to start with a model in which there is only one (modulo permutation) tower \mathcal{T} of height ω_1 (if there is such a model) and try to supply \mathcal{T} with an \mathcal{I} -intersection without diagonalizing it and without adding other towers, too (if it is possible).

We finish this section by an easy observation: if we change a little the notion of \mathcal{I} -intersection demanding it to be a member of our ideal, then the analogously defined \mathcal{I} -intersection number equals \mathfrak{p} .

Fact 4.4 *If \mathcal{I} is an analytic P-ideal, then*

$$\mathfrak{p} \geq \min\{|\mathcal{F}| : \mathcal{F} \text{ does not have an } \mathcal{I}\text{-intersection in } \mathcal{F}\}.$$

Proof. Consider a filter \mathcal{F} generated by κ smaller than the cardinal on the right side of the above inequality. We show that it has an \mathcal{I} -intersection.

Let φ be a submeasure associated to \mathcal{I} . Let (k_n) be a sequence as above, i.e. such that $\varphi([k_n, k_{n+1})) > \varepsilon$ for some ε and every n .

Assume that \mathcal{G} be a filter generated by the family

$$\left\{ \bigcup_{n \in F} [k_n, k_{n+1}) : F \in \mathcal{F} \right\}.$$

The filter \mathcal{G} is therefore generated by κ many sets so there is an \mathcal{I} -intersection $I \in \mathcal{G}$ of \mathcal{G} . But then the set $\{n \in \omega : [k_n, k_{n+1}) \subseteq I\}$ is a pseudo-intersection of \mathcal{F} . \diamond

5 Towers and MAD's without \mathcal{I} -intersection

As we have already mentioned it is not clear if the cardinal coefficients presented in Section 2 were defined properly. In particular, it is not clear if in ZFC we have towers and MADs without an \mathcal{I} -intersection.

If $\mathfrak{t} = \mathfrak{c}$, then we can construct such a tower easily. Just enumerate $[\omega]^\omega = \{X_\alpha : \alpha < \mathfrak{c}\}$ and define the tower inductively ensuring at the α -th step, that X_α is not an \mathcal{I} -intersection. Recently, Barnabas Farkas and Joerg Brendle (private communication) showed that for every P-analytic ideal \mathcal{I} on ω there is a model of set theory in which the dual filter does not contain a tower. It means that in such a model there is even no tower with an \mathcal{I} -intersection (and $\mathfrak{t}_{\mathcal{I}} = \mathfrak{t}$). On the other hand e.g. in the standard Cohen model there are no towers of height \mathfrak{c} (so, the above argument cannot be repeated).

In case of MAD families, it is even more difficult. The reason for that is the febleness of AD's.

Fact 5.1 *Every ideal generated by an almost disjoint family is feeble.*

Proof. Let \mathcal{A} be the almost disjoint family and let \mathcal{I} be generated by \mathcal{A} . Let (A_n) be a partition of ω into elements of \mathcal{I} . Let $A_n = \{a_k^n : k \in \omega\}$. Define $f : \omega \rightarrow \omega$ by $f(a_j^i) = i + j$. Such f is finite-to-one and we have $f^{-1}[X] \notin \mathcal{I}$ for every infinite $X \subseteq \omega$. ■

So, every example of MAD without an \mathcal{I} -intersection, will be an example of a feeble ideal without an \mathcal{I} -intersection. This is not the case of towers. The example mentioned above is not feeble.

It will be convenient to make the following simple fact available.

Fact 5.2 *If \mathcal{J} is tall then there is a MAD family included in \mathcal{J} .*

Proof. Consider an almost disjoint family consisting only of members of \mathcal{J} , which cannot be extended by any member of \mathcal{J} . It is MAD. Indeed, if it can be extended by $B \subseteq \omega$, then it can be extended by any infinite subset of B , in particular by its infinite subset from \mathcal{J} (whose existence is guaranteed by the tallness). ■

We can translate the notion of tallness into \mathcal{I} -tallness in the following way: an ideal \mathcal{J} is \mathcal{I} -tall if in every infinite X there is $J \in \mathcal{J}$ such that $J \notin \mathcal{I}_X$. As in the case of tallness, \mathcal{J} is \mathcal{I} -tall if and only if \mathcal{J}^* does not have an \mathcal{I} -intersection.

Problem 5.3 *Is it true that for every \mathcal{I} -tall ideal \mathcal{J} there is an \mathcal{I} -MAD family included in \mathcal{J} ?*

Using Fact 5.2 we can see that if \mathcal{I} is tall, then there are MAD's which are not \mathcal{I} -maximal.

6 \mathcal{I} -maximal MAD under CH and MA

Now, we will show an example of a MAD family which is \mathcal{I} -maximal. First, let us prove the following.

Fact 6.1 *Assume \mathcal{I} is a tall analytic P-ideal. If $(A_n)_n$ is a countable almost disjoint family of sets from \mathcal{I} , then it can be extended by a set $B \subseteq \omega$ such that $B \in \mathcal{I}^+$.*

Proof. Let φ be a submeasure associated to \mathcal{I} . Assume without loss of generality that the family (A_n) is pairwise disjoint. Since \mathcal{I} is a proper P-ideal, there is $\varepsilon > 0$ such that for every natural numbers n and k if $F = (\bigcup_{i < n} A_i)^c$ then $\varphi(F \setminus k) > \varepsilon$.

Consider sequences (B_n) , (L_n) of finite subsets of ω and a strictly increasing sequence (k_n) of natural numbers such that

1. $B_0 = L_0 = \emptyset$ and $k_0 = 0$;
2. B_n intersects A_m only if $m \in L_n$;
3. $L_n \cap L_m = \emptyset$ if $n \neq m$;
4. $B_n \subseteq [k_n, k_{n+1})$;
5. $\varphi(B_n) > \varepsilon$ (so, $\varphi(B_n \setminus k_n) > \varepsilon$).

It is clear that such families can be constructed inductively. For k_{n+1} take $\max(B_n) + 1$. The set $B = \bigcup B_n$ intersects each A_n on a set included in L_n (so, on a finite set). The sequence (k_n) witnesses that $\varphi(B \setminus n)$ does not converge to 0 as $n \rightarrow \infty$, so $B \in \mathcal{I}^+$. ■

Under CH we can use the above result to construct an almost disjoint family without an \mathcal{I} -intersection.

Fact 6.2 (CH) *If \mathcal{I} is a tall analytic P-ideal, then there is a MAD family without an \mathcal{I} -intersection.*

Proof. Let $(X_\xi)_{\xi \leq \omega_1}$ be an enumeration of infinite co-infinite subsets of ω . We will construct the desired MAD family inductively. Start with a disjoint partition (A_n) of ω into infinite sets and assume we have constructed all A_ξ 's for $\xi < \alpha < \omega_1$.

If for some $\xi < \alpha$ $A_\xi \notin \mathcal{I}_{X_\alpha}$, then take for A_α any infinite set almost disjoint with all A_ξ 's for $\xi < \alpha$.

If not, reenumerate $(A_\xi)_{\xi < \alpha}$ in the following way:

$$\{C_n : n < \beta\} = \{X_\alpha \cap A_\xi \in [X_\alpha]^\omega : \xi < \alpha\}.$$

Use Fact 6.1 to show that we can then extend $(C_n)_{n \in \omega}$ by a set $B \subseteq X_\alpha$ such that $B \notin \mathcal{I}_{X_\alpha}$. Put $A_\alpha = B$.

Let $\mathcal{A} = \{A_\xi : \xi < \omega_1\}$. If $X \subseteq \omega$, then $X = X_\alpha$ for some $\alpha < \omega_1$. Either there is $\xi < \alpha$ such that $A_\xi \notin \mathcal{I}_X$ or $A_\alpha \notin \mathcal{I}_X$, so X is not an \mathcal{I} -intersection of \mathcal{A} .

■

Notice that in fact we have something more here.

Remark 6.3 *Under CH every maximal ideal \mathcal{J} contains an \mathcal{I} -maximal MAD (where \mathcal{I} is as above).*

Indeed, we can divide B in the above proof into $B_0 \cup B_1 = B$ both with the property $B_i \notin \mathcal{I}_{X_\alpha}$ for $i \in 2$. One on these (say, B_0) is in \mathcal{J} and we can extend our AD by B_0 instead of B .

If \mathcal{I} have additionally the following property:

- if $A \notin \mathcal{I}_B$ and $B \notin \mathcal{I}_C$, then $A \notin \mathcal{I}_C$

then the above remark would work for every \mathcal{I} -tall ideal \mathcal{J} , not only for the maximal ones. I don't know if there are some interesting ideals with the above property. \mathcal{Z} does not possess it.

One can ask how to get rid of CH in the last fact. We would have to make Fact 6.1 working not only for countable families. We have to be careful. It is easy to construct an almost disjoint family, which can be extended only by a set from \mathcal{I} (we can divide $\omega = A \cup B$, where A is from \mathcal{I} , define a MAD family $\mathcal{B} = \{\mathcal{B}_\alpha : \alpha < \mathfrak{c}\}$ on B and a AD but not MAD family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ on A and consider the AD family defined by $\{A_\alpha \cup B_\alpha : \alpha < \mathfrak{c}\}$).

Here, we present how to repeat the above construction under a version of MA. Probably it is not the easiest way to prove it. Moreover, it is proved here only for $\mathcal{I} = \mathcal{Z}$. I believe that it can be easily rewritten in a general form but, as it can turn out to be obsolete (because of the existence of a better and simpler argument), I haven't done it.

An almost disjoint family \mathcal{A} is *tree-like* if there is a countable tree $T \subseteq \omega^\omega$ and a bijection $\rho: T \rightarrow \omega$ such that for every $A \in \mathcal{A}$ there is $s \in [T]$ such that $A =^* \{\rho(\sigma) : \sigma \sqsubseteq s\}$. Loosely speaking, an AD family is tree-like if it can be constructed similarly to one of the standard construction of uncountable almost disjoint family. It is worth mentioning here that every countable AD is tree-like.

For $\sigma \in T$ and $s \in [T]$ denote

$$[\omega]^{\leq \omega} \ni N_\sigma = \{\rho(\tau) : \tau \sqsubseteq \sigma\},$$

$$[\omega]^{\leq \omega} \ni N_s = \{\rho(\tau) : \tau \sqsubseteq s\},$$

$$[\omega]^{< \omega} \ni M_\sigma = \{\rho(\tau) : \sigma \sqsubseteq \tau\}.$$

Fact 6.4 *If an almost disjoint family $\mathcal{A} \subseteq \mathcal{Z}$ is tree-like, then there is a family $(D_n)_n$ of subsets of natural numbers such that*

- $d^*(\bigcup_{n < N} D_n) < 1/2$ for every $N \in \omega$;
- the set $\{n : D_n \cap A\}$ is finite for every $A \in \mathcal{A}$.

Proof. Notice that the set $D_\varepsilon = \{\sigma \in T: d^*(M_\sigma) < \varepsilon\}$ is dense open for every $\varepsilon > 0$. Therefore, there is a maximal antichain $(\sigma_n)_n$ in T such that $d^*(M_{\sigma_n}) < 1/(2^{n+1})$.

Consider the purely atomic tree $T_0 \subseteq T$ defined by

$$T_0 = \{\tau: \text{there is } n \text{ such that } \tau \sqsubseteq \sigma_n\}.$$

Let $\{\tau_n: n \in \omega\}$ be an enumeration of T_0 . Put $C_n = \{\rho(\tau_n): n \in \omega\}$ for every $n \in \omega$.

We will define the sequence $(D_n)_n$ on even and odd numbers separately. Let $D_{2n} = C_n$ and $D_{2n+1} = N_{\sigma_n}$.

Since all C_n 's are singletons, we have $d^*(\bigcup_{n < N} D_n) < 1/2$. Every $A \in \mathcal{A}$ is of the form $A = N_s$ for some $s \in 2^\omega$. There is $n \in \omega$ such that $s \in [\sigma_n]$. Thus, almost all elements of A are included in $M_{\sigma_n} = D_{2n+1}$ and we are done. ■

Fact 6.5 *If $\mathcal{A} \subseteq \mathcal{Z}$ is a tree-like almost disjoint family, then it can be extended by a set B , such that $d^*(B) > 0$.*

Proof. Let (D_n) be the family whose existence is proved by Fact 6.4. Consider sequences (B_n) , (L_n) of finite subsets of ω and a strictly increasing sequence (k_n) of natural numbers such that:

1. $B_0 = L_0 = \emptyset$ and $k_0 = 0$;
2. B_n intersects D_m only if $m \in L_n$;
3. $L_n \cap L_m = \emptyset$ if $n \neq m$;
4. $B_n \subseteq [k_n, k_{n+1})$;
5. $|B_n \cap k_n|/k_n > 1/2$;

Construction is the same as in Fact 6.1. The set $B = \bigcup_n B_n$ intersects only finitely many D_m 's. A set $A \in \mathcal{A}$ intersects every D_m only on a finite set, so A intersects B only on a finite set. Additionally, (k_n) witnesses that $d^*(B) > 1/2$, so $B \notin \mathcal{Z}$. ■

We will strengthen a result due to Roitman & Soukup [9].

Theorem 6.6 *Assume $\mathfrak{MA}_{\text{precaliber}_{\omega_1}}$. Then every almost disjoint family $\mathcal{A} \subseteq \mathcal{Z}$ of size $< 2^\omega$ on ω has a subset $T \notin \mathcal{Z}$ such that $\{T \cap A: A \in \mathcal{A}\}$ is tree-like.*

Proof. We will follow the proof of Theorem 4.4 from [9]. The authors showed that every almost disjoint family \mathcal{A} of size $< 2^\omega$ on ω has an infinite subset $T \subseteq \omega$ such that $\{T \cap A: A \in \mathcal{A}\}$ is tree-like. We will show that it is easy to force this T to be outside \mathcal{Z} .

Define an order \mathcal{P} in the following way. A condition $p \in \mathcal{P}$ is of the form $(T_p, <_p, \mathcal{A}_p, h_p)$, where

- $T_p \in [\omega]^{<\omega}$, $(T_p, <_p)$ is a finite tree,
- $\mathcal{A}_p \subseteq \mathcal{A}$ is finite,
- $h_p: \mathcal{A}_p \rightarrow \omega$ is a function,
- each $(A \setminus h_p(A)) \cap T_p$ is linearly ordered by $<_p$,
- for every $A \in \mathcal{A}_p$ $\forall n \in A \cap (T_p \setminus h_p(A)) \forall k \in T_p \setminus h_p(A)$ if $k <_p n$, then $k \in A$.

We define $p \leq q$ iff

- $(T_q, <_q)$ is a initial subtree of $(T_p, <_p)$,
- $\mathcal{A}_q \subseteq \mathcal{A}_p$,
- $h_q \subseteq h_p$.

Roitman & Soukup proved that such \mathcal{P} has a precaliber ω_1 and for each $n \in \omega$ and $A \in \mathcal{A}$ the set

$$D_{A,n} = \{p \in \mathcal{P}: A \in \mathcal{A}_p \text{ and } A \cap (T_p \setminus n) \neq \emptyset\}$$

is dense in \mathcal{P} .

Notice that for every $m \in \omega$ the set

$$D_m = \{p \in \mathcal{P}: \exists k > m \frac{|T_p \cap k|}{k} > 1/2\}$$

is dense. Indeed, consider $q \in \mathcal{P}$ and $m \in \omega$. It is clear that we can harmlessly extend T_q by natural numbers not belonging to $\bigcup \mathcal{A}_q$. But, since \mathcal{A}_q is finite and consists of density 0 sets, the set $(\bigcup \mathcal{A}_q)^c$ is of density 1 and we can find $k > m$ and a set $T_p \subseteq T_q$ such that $(T_p \setminus T_q) \cap \bigcup \mathcal{A}_q = \emptyset$ and

$$\frac{|T_p \cap k|}{k} > 1/2.$$

Extending $<_q$ to $<_p$ in such a way that $(T_q, <_q)$ is an initial subtree of $(T_p, <_p)$, and taking $\mathcal{A}_q = \mathcal{A}_p$ and $h_p = h_q$ we obtain a condition $p \leq q$ with the desired property.

Therefore we can assume that $d^*(T) > 0$. ■

Now we are ready to reprove Fact 6.2.

Theorem 6.7 *Assume $\mathbf{MA}_{\text{precaliber}\omega_1}$. There is a MAD family without a condenser.*

Proof. Recall that $\mathbf{MA}_{\text{precaliber}\omega_1}$ implies that $\mathfrak{a} = \mathfrak{c}$.

Let $(X_\xi)_{\xi \leq \mathfrak{c}}$ be an enumeration of infinite co-infinite subsets of ω . We will construct the desired MAD inductively. Start with a disjoint partition (A_n) of ω into infinite sets and assume we have constructed all A_ξ 's for $\xi < \alpha < \mathfrak{c}$.

If for some $\xi < \alpha$ we have $d_{X_\alpha}^*(A_\xi) > 0$, then take for A_α any infinite set almost disjoint with all A_ξ 's for $\xi < \alpha$.

If not, consider AD family

$$\mathcal{B} = \{A_\xi \cap \mathcal{X}_\alpha \in [\mathcal{X}_\alpha]^\omega : \xi < \alpha\}.$$

Use Theorem 6.6 and then Fact 6.5 to extend \mathcal{B} by a set $B \subseteq X_\alpha$ such that $d_{X_\alpha}^*(B) > 0$. Put $A_\alpha = B$. Conclusion is as in 6.2. ■

In fact, as before, we have something more.

Remark 6.8 Under $MA_{\text{precaliber}_{\omega_1}}$ every maximal ideal contains a \mathcal{Z} -maximal MAD.

7 Another question

Problem 7.1 Is there a nice ideal \mathcal{I} for which consistently $\mathfrak{p}_{\mathcal{I}} < \mathfrak{t}_{\mathcal{I}}$?

It may sound funny but nobody said that the ideal of finite subsets of ω is always the easiest one.

References

- [1] B. Balcar, J. Pelant, P. Simon, *The space of ultrafilters on N covered by nowhere dense sets*, Fund. Math. **110** (1980), 11–24;
- [2] A. Blass, *Combinatorial cardinal characteristics of the continuum*, to appear as a chapter in the *Handbook of Set Theory*;
- [3] P. Borodulin–Nadzieja, G. Plebanek *On sequential properties of Banach spaces, spaces of measures and densities*, accepted to Czechoslovak Mathematical Journal;
- [4] J. Brendle, S. Shelah, *Ultrafilters on ω — their ideals and their cardinal characteristics*, Transactions American Math Soc 351 (1999) 2643-2674
- [5] B. Farkas, L. Soukup, *More on cardinal invariants of analytic P -ideals*, to appear in CMUC
- [6] R. Filipów, N. Mrozek, I. Reclaw, P. Szuca, *Bolzano-Weierstrass property for ideals* J. Symb. Logic. Vol. 72, No. 2 (2007) 501-512;
- [7] D. Fremlin, *Measure Theory vol. 4*;
- [8] F. Hernandez-Hernandez, M. Hrusak *Cardinal invariants of P -ideals*, Canad. J. Math. 59 (2007), no 3, 575–595;
- [9] J. Roitman and L. Soukup, *Luzin and anti-Luzin almost disjoint families*, Fundamenta Mathematicae, 158 (1998), pp. 51-67.