DST 1

1 POLISH SPACES

Instead of examples ..

Thm Let X - Polish space. Then

$Y \subseteq X$ is Polish $\iff$ Y is G_δ

Proof. $\Leftarrow$ Suppose first that Y is open. WLOG

Let d be a complete metric on X, $d \leq 1$

Consider d' on Y:

$$d'(x,y) = d(x,y) + \left| \frac{1}{d(x,Y^c)} - \frac{1}{d(y,Y^c)} \right| .$$

- d' and d are equivalent:

$$x_n \to x \quad \text{iff} \quad d(x_n, Y^c) \to d(x, Y^c)$$

- d' is complete!

If $x_n \xrightarrow{d} x \in Y$, then by the above $x_n \xrightarrow{d'} x$

If $x_n \to x \notin Y$, then (x_n) is not Cauchy in d':

let $k \in \omega$. Then we can find $n > k$ such that $\left| \frac{1}{d(x_k, Y^c)} - \frac{1}{d(x_n, Y^c)} \right| > 1$.

$\downarrow n$ as ∞.

ε

Now, let $Y = \bigcap_n U_n$.

Fore each n fix d'_n as above.

Let $d'(x,y) = \sum \frac{d'_n(x,y)}{2^n}$.

d' is equivalent to d ("pointwise convergence")

$\Rightarrow$ Let ~~V_n~~ $\mathcal{V}_n$ consists of all open subsets U of X
such that
* $\mathrm{diam}_X(U) < \frac{1}{n}$
* ~~diam~~ $Y \cap U \neq \emptyset$ and $\mathrm{diam}_Y(U) < \frac{1}{n}$.

Let $V_n = \bigcup \mathcal{V}_n$

Then $V_n = Y$.

* let $x \in Y$

Then $\exists W \subseteq X$ $\mathrm{diam}_X(W) < \frac{1}{n}$ $x \in W$

$\exists U \subseteq Y$ $\mathrm{diam}_Y(U) < \frac{1}{n}$ $x \in U$

Let $U' \subseteq X$ $U' \cap Y = U$.

Then $U' \cap W \in \mathcal{V}_n$ and so $x \in V_n$

* let $x \in \bigcap V_n$

Then $\forall n$ $\exists U_n \in \mathcal{V}_n$ $x \in U_n$

WLOG (U_n) is decreasing.

because $x \in \overline{Y}$ and so any open nbhd of x has no empty intersection with Y.

Pick $x_n \in U_n \cap Y$

Clearly, $\underbrace{x_n \xrightarrow{d_X} x}$ in X

Also, x_n is d_Y-Cauchy.

Thus, x_n is convergent (wrt d_Y).

$x_n \xrightarrow[d_Y]{} x$, the only option.

$\blacksquare$

Thm X - Polish space iff $X \underset{G_\delta}{\hookrightarrow} [0,1]^\omega$.

Proof (sketch). Let d - complete, $d \leq 1$.

$D \subseteq X$ - dense $D = \{d_0, d_1, \dots\}$

$f: X \to [0,1]^\omega$ $\qquad f(x) = (d(x, d_0), d(x, d_1), \dots)$

f - homeomorphic embedding. $\qquad$ by the previous thm.

Thus $f[X]$ is Polish, and so $f[X]$ is G_δ. ∎

Thm X - compact Polish space iff $X \underset{\text{closed}}{\hookrightarrow} [0,1]^\omega$

Proof: $[0,1]^\omega$ is compact. ∎

Thm X - Polish space iff $X \underset{\text{closed}}{\hookrightarrow} \mathbb{R}^\omega$.

Proof $\Leftarrow$ clear

$\Rightarrow$ WLOG $X \underset{G_\delta}{\hookrightarrow} [0,1]^\omega \subseteq \mathbb{R}^\omega$

$X = \bigcap_n U_n$

$f: X \to \mathbb{R}^\omega \times \mathbb{R}^\omega$ $\qquad f(x) = \left\langle x, \frac{1}{d(x, U_0^c)}, \frac{1}{d(x, U_1^c)}, \dots \right) \in \mathbb{R}^\omega \times \mathbb{R}^\omega \cong \mathbb{R}^\omega$

f - 1-1 ✓

f - continuous ✓

f^{-1} - continuous ✓ $\leftarrow$ projection onto 0th coordinate

$f[X]$ - closed: $f(X) \ni x_n \to x \notin f[X]$

$X \ni x_n' \to x' \notin X$

$x' \notin U_n$ for almost all n

$1/d(x_n', U_m^c) \to \infty$

2 TREES.

A - alphabet.
We consider $A^{<\omega}$ (usually $A=\{0,1\}$ or $A=\omega$)

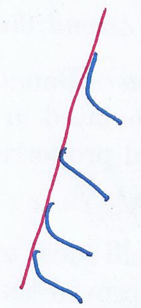
$T \subseteq A^{<\omega}$ - tree $\equiv$ well-ordered by $\subseteq$

* pruned $\equiv$ $\forall s \in T \ \exists t \in T \ s \subsetneq t$
* finitely branching $\equiv$ every node has only finitely many successors.

* $x \in A^{\omega}$ is a branch of T if $\forall n \ x \restriction n \in T$.
* $[T]$ - body of T, the set of all branches.

__Fact.__ $Y \subseteq A^{\omega}$ is closed iff $Y = [T]$ for some $T \subseteq A^{<\omega}$ tree.

__Proof.__
$\Leftarrow$ $[T]$ is closed:



$\Rightarrow$ $Y \subseteq A^{\omega}$ $\to$ $T = \{x \restriction n : x \in Y, n \in \omega\}$

* $[T] \supseteq Y$ - clear, * $Y \supseteq [T]$ - otherwise Y is not closed

$\square$

<u>Examples.</u> * 2^ω - the Cantor set.

the compact case.

* ω^ω - the Baire space.

very non-compact case:

<u>Prop</u> ω^ω is not ~~σ-closed~~ compact.

<u>Proof.</u>

1. If $Y \subseteq \omega^\omega$ is compact, then there is $f \in \omega^\omega$ $\forall y \in Y$ $y \leq f$.

   (otherwise, $\exists n$ s.t. $\{y(n) : y \in Y\}$ is unbounded.

   and we can produce a sequence witnessing non-compactness).

2. There is no unbounded, countable family in ω^ω.

$\blacksquare$

<u>Thm (Hurewicz)</u> If Y is Polish and not σ-compact, then

$$\omega^\omega \underset{\text{closed}}{\hookrightarrow} Y.$$

<u>Proof.</u> Later, later, ...

DST 2

Thm Every compact metric space is a continuous image of 2^ω. (So, every compact Polish space is a continuous image of 2^ω).

Before the proof.

DEF. Say that $f: X \xrightarrow{\text{continuous!}} D$, where $D \subseteq X$, is a retraction if $f(x) = x$ for each $x \in D$. If there is a retraction $f: X \to D$, then we say that D is a retract of X.

Example Let $X = \{x \in \mathbb{R}^2 : d_e(x, 0) \leq 1\}$.
Then $[-1, 1] \times \{0\}$ is a retract of X
but $S^1 = \{x \in \mathbb{R}^2 : d_e(x, 0) = 1\}$ is not (Brouwer fix point thm).

Thm Every closed $D \subseteq A^\omega$ is a retract of A^ω.

Proof. Ex. 4 on the list 1. ∎

<u>Proof.</u> (of the theorem).

First, notice that there is

$$f: 2^\omega \to [0,1] \text{, a continuous surjection.}$$

Consider

$$g: (2^\omega)^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$$

$$g(x_0, x_1, \dots) = (f(x_0), f(x_1), \dots)$$

Then g is a continuous surjection, too.

But $(2^\omega)^{\mathbb{N}} \cong 2^\omega$

$(|\omega \times \mathbb{N}| = \omega)$

(I have no idea why I wrote $\mathbb{N}$ instead of ω above).

So, we have WLOG $g: 2^\omega \longrightarrow [0,1]^{\mathbb{N}}$

Let X be a compact metric space.

WLOG $X \underset{\text{closed}}{\subseteq} [0,1]^{\mathbb{N}}$.

So $g^{-1}[X] \subseteq 2^\omega$ and $g^{-1}[X]$ is closed. Hence, it is a retract and so there is $h: 2^\omega \to g^{-1}[X]$, a retraction.

$2^\omega \xrightarrow{h} g^{-1}[X] \xrightarrow{g} X$ continuous, onto.

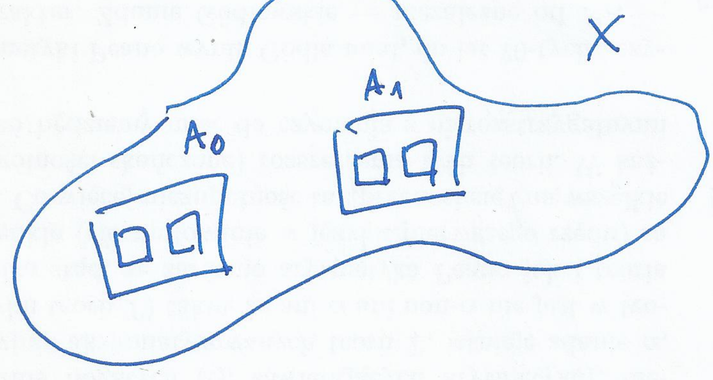
(X, d) - Polish.

subsets of X

DEF. CANTOR SCHEME : $(A_s)_{s \in 2^{<\omega}}$.

such that

* $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$ for $s \in 2^{<\omega}$
* $A_{s^\frown i} \subseteq A_s$ for $s \in 2^{<\omega}$, $i \in 2$
* $\operatorname{diam}(A_{x \upharpoonright n}) \xrightarrow[n]{} 0$ for $x \in 2^\omega$.



LUZIN SCHEME: $(A_s)_{s \in \omega^{<\omega}}$

with analogous conditions:

Thm (Perfect Set Theorem).

X - Polish space.

If X is perfect (i.e. no isolated points), then it contains a copy of 2^ω.

Proof.

We will define a Cantor scheme $(A_s)_{s \in 2^{<\omega}}$ such that

- A_s is open and $\overline{A_{s^\frown i}} \subseteq A_s$ for any $s \in 2^{<\omega}$, $i \in 2$.

$A_\emptyset = X$.

Suppose we have A_s.

X is perfect $\Rightarrow$ $\exists\, x \neq y \in A_s$.

X is Hausdorff $\Rightarrow$ $\exists\, U \ni x$, $V \ni y$ $U \cap V = \emptyset$.

Shrink U and V if needed (so that $\overline{U}, \overline{V} \subseteq A_s$).

Let $f : 2^\omega \to X$ be defined by

$$f(x) = t \text{ iff } \{t\} = \bigcap_n \overline{A_{x \restriction n}}.$$

Since X is complete (Cantor's thm), f is well defined.

Clearly, f is injective.

Also, f is continuous.

So $f : 2^\omega \to f[2^\omega]$ is a homeomorphism.

$\blacksquare$

DEF. $A \subseteq X$, closed.

Cantor-Bendixson derivative:

$$A' = \{x \in A : x \text{ is a limit point of } A\}$$

$\equiv$ there is a sequence of elements of A different then x ~~such~~ which converges to x.

We can iterate it:

$$A^{(\alpha+1)} = \left(A^{(\alpha)}\right)'$$

$$A^{(\lambda)} = \bigcap_{\alpha<\lambda} A^{(\alpha)} \quad \text{for } \lambda\text{-limit ordinal.}$$

Fact. $A^{(\alpha)}$ is closed for every α.

Fact. If X is Polish, $A \subseteq X$, then

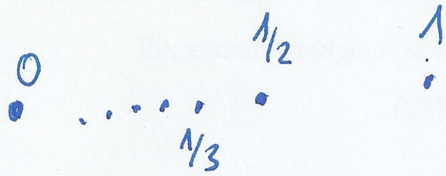
$$\exists \alpha < \omega_1 \quad A^{(\alpha)} = A^{(\alpha+1)} = \dots$$

Proof. Ex. 7, list no. 0 ∎

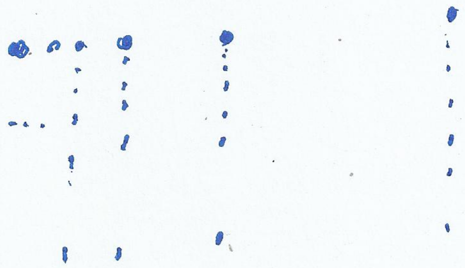
~~**Fact.**~~ **Def.** Cantor-Bendixson rank of A: the least α such $A^{(\alpha)} = A^{(\alpha+1)}$.

Fact. $A^{(\alpha)}$ is perfect if $\alpha \geq$ rank of A.

Examples in $\mathbb{R}^2$

① $A =$ 

$A' = \{0\}$

② A 

$A' =$ 

ETC.

Thm If X is Polish, then

$$X = P \,\dot\cup\, C, \text{ where } P\text{- perfect, } C\text{- ctbl.}$$

Proof. Consider $X^{(\alpha)}$, $\alpha \leq \mathrm{rank}(X) = \beta$

Let $P = \cancel{X^{(\mathrm{rank}(X))}}\ X^{(\beta)}$ - perfect.

For $\alpha < \beta$ $\quad X^{(\alpha+1)} \setminus X^{(\alpha)}$ is countable (X has countable base).

So $C = \bigcup_{\alpha<\beta} X^{(\alpha)}$ is countable, too ∎

DST 3

<u>Thm</u> Every non-empty Polish space is a continuous image of ω^ω.

<u>Proof.</u> List no. 2, ex. 3 — 6. $\blacksquare$

<u>Remark</u> Every unctbl Polish space has a homeomorphic copy of ω^ω.
(Just because it has a copy of 2^ω, and $\omega^\omega \hookrightarrow 2^\omega$).

Note that typically this copy is <u>not closed.</u>
If it is, then the space is not σ-compact (Hurewicz thm)

BAIRE MEASURABILITY

Recall: $A \subseteq X$ is <u>nowhere dense</u> if $\operatorname{Int} \bar{A} = \emptyset$.

$A \subseteq X$ is <u>meager</u> (1st Baire category) if

$A \subseteq \bigcup_{n \in \mathbb{N}} A_n$ for some (A_n) - sequence of nowhere dense sets.

MEAGER $\equiv$ SMALL IN THE TOPOLOGICAL SENSE

It makes sense only for some, fortunately quite wide, class of spaces:

<u>DEF.</u> A space X is Baire if every $U \neq \emptyset$, open, is not meager.

BAIRE THEOREM $\equiv$ Completely metrizable spaces are Baire.

<u>Fact</u> In Baire spaces co-meager sets are exactly those, which contain a dense G_δ.

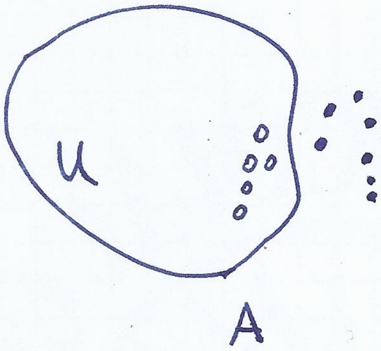
<u>Example.</u>
- $\mathbb{Q}$ is not a Polish space
- $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$.

$\uparrow$

Consequences of the Baire theorem.

DEF. $A \subseteq X$ is Baire measurable (has BP $\equiv$ Baire Property)

if $A = U \triangle M$ for some U-open and M-meager



Fact. Sets with BP form a σ-algebra.
(i.e. it is closed under complements, σ-unions, $\phi \in BP$).

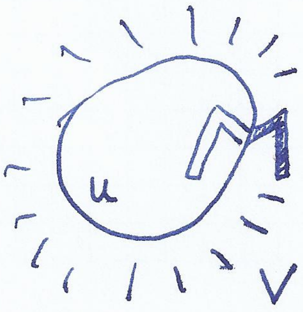
Proof.
Suppose B has BP.

Then $B = U \triangle M$.
We want to show that B^c is also of this form.

Let $V = \overline{U}^c$

Let $N = M \cup \mathrm{Bd}\, U$

Then $B^c = V \triangle N'$, where
$N' \subseteq N$ (and so is meager)



Typically Bd does not have to have empty interior. But if taken on open set, then it is nowhere dense.

Now, let (B_n) - sequence of meager sets.

$$B_n = U_n \triangle M_n$$

$$V = \bigcup_n U_n$$

meager

$$\bigcup B_n = V \triangle N, \text{ where } N \subseteq \bigcup_n M_n$$

■

Fact TFAE:

a. A has BP

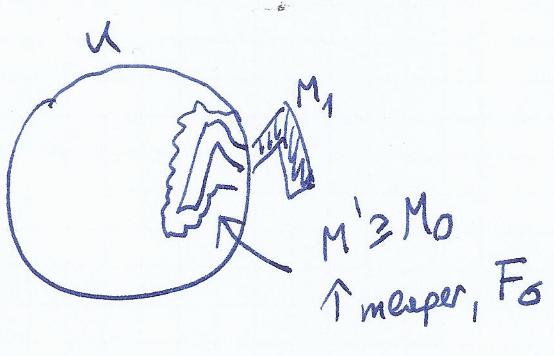
b. $A = G \cup N$, where G is G_δ (and N - meager)

c. $A = F \setminus N$, where F is F_σ (-"-)

Proof. If M - meager, then $\exists M'$ - meager F_σ s.t. $M \subseteq M'$.

$$A = U \triangle M = U \setminus M_0 \cup M_1$$

a $\Rightarrow$ b



$G = U \setminus M'$ - G_δ

$N = M_1 \cup ((M' \setminus M_0) \cap U)$ - meager

The rest of the proof is analogous. ■

* BANACH-MAZUR GAME (on a topological space X)

$G(B)$

$$\begin{array}{lccccc} \text{Adam} & U_0 & & U_1 & & \\ & \cup\!\!\backslash & \supseteq & \cup\!\!\backslash & \supseteq & \dots \\ \text{Eve} & & V_0 & & V_1 & \end{array} \quad \leftarrow BM(B)$$

B — fixed subset of X

Each U_n, V_n - open.

Eve wins if $\bigcap V_n \cap B \neq \emptyset$.

- A winning strategy $\equiv$ winning strategy.

Theorem (Oxtoby) X - Polish space. $B \subseteq X$ has BP

Then

- B is comeager $\Longrightarrow$ Eve has a winning strategy in $BM(B)$
- B is meager in a nonempty open set $\Longrightarrow$ Adam —"—

Proof. $\odot$ B is comeager.

$B^c \subseteq \bigcup F_n$, where F_n - closed nowhere dense.

A. plays U_0. Then Eve plays V_0 such that $U_0 \setminus F_0 \supseteq \overline{V_0}$

At round n Eve plays V_n $U_n \setminus (F_0 \cup \dots \cup F_n) \supseteq \overline{V_n}$, $\operatorname{diam}(V_n) < \frac{1}{n}$

clearly $\bigcap \overline{V_n} = \{x\}$, and $x \notin \bigcup F_n$.

$\bigcap \overline{V_n} = \bigcap V_n$. So $x \in B$.

So, Eve has a winning strategy.

$\odot\odot$ Suppose $U \cap B$ is meager, for a nonempty open U.
Let Adam play U in his first move.
Then he can follow Eve's strategy from $\odot$.

$\blacksquare$

<u>DEF.</u> We say that a game is <u>determined</u> if either Adam or Eve has a winning strategy.

<u>Corollary.</u> If B has BP, then $G(B)$ is determined
(in a Polish space)

<u>Proof.</u> We use something which Anush calls 100% lemma:
If B has BP, then either
it is meager or it is co-meager in an non-empty open subset.
(the proof is clear). $\blacksquare$

<u>Remark</u> Actually one can prove that
B has BP $\iff G^{BM}(B)$ is determined.
The proof is slightly more complicated.

Example A set $A \subseteq 2^\omega$ which does not have the Baire Property.

Let A have the following property:

* if $x \in A$ and we change 1 bit in x, ~~then~~ (and obtain x') then $x' \notin A$
* if $x \notin A$ and we change 1 bit (and obtain x'), then $x' \in A$.

This is a version of a Vitaly set and its existence can be proved in a similar way.

Claim $BM(A)$ is **NOT** determined.

Proof. Suppose Adam has a winning strategy. (We may assume that Adam and Eve play with basic open sets and so they play sequences of 0's and 1's).

$$\underbrace{0010}_{\text{1st move of Adam}},\ \underbrace{1\,\overbrace{\boxed{110111}}^{\text{Adam's response for 0}}}_{\text{1st move of Eve}},\ \underbrace{0010}_{\text{2nd move of Adam}},\ \underbrace{\qquad\qquad}$$

Eve plays ~~something~~ 1 but ~~she~~ then she checks Adam's response (in his book of strategy) for 0 and plays it.

Eve plays Adam's response to the game

$$\underbrace{0010,\ 0,\ 110111,\ 0010,}$$

In this way Eve use Adam's strategy to obtain something NOT in A.

Corollary A does not have Baire Property.

Axiom of Determinacy (Mycielski & Steinhaus)

Let G be a game in which we fix $A \subseteq 2^\omega$, Adam and Eve play finite sequences of 0 and 1's.

A	0010		010		1111	
E		11		01000		...

and Eve wins if the infinite sequence they produce:

$$0010\ 11\ 010\ 01000\ 1111\ \ldots \in A.$$

AD: every game is determined.

Remark AD ↯ AC

Thm (AD) Every subset of 2^ω has Baire Property.

DST 4 BOREL SETS

<u>DEF.</u> Let X be a topological space.

Then $Bor(X)$, the family of Borel sets, is the smallest σ-algebra containing all the open subsets of X.

Remark: Borel sets have the Baire Property.

- THE HIERARCHY OF BOREL SETS:

$$\begin{cases} \Sigma^0_\alpha(X) = \{ \bigcup_n A_n : \forall n \ A_n \in \Pi^0_{\alpha_n}, \ \alpha_n < \alpha \} \\ \Pi^0_\alpha(X) = \{ A^c : A \in \Sigma^0_\alpha(X) \} \\ \Sigma^0_1(X) \text{ - the family of open subsets of } X. \end{cases}$$

$\Pi^0_1(X) \equiv$ Closed sets, $\Sigma^0_2(X) \equiv F_\sigma$ subsets of X.

$\Pi^0_2(X)$ - G_δ subsets of X.

$\Delta^0_\alpha(X) = \Sigma^0_\alpha(X) \cap \Pi^0_\alpha(X)$. (e.g. $\Delta^0_1(X) \equiv$ Clopens).

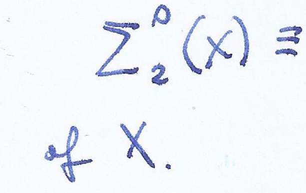
WE FIX X - Polish space.

<u>Fact</u> $\Sigma^0_\alpha \cup \Pi^0_\alpha \subseteq \Delta^0_\beta$ if $\alpha < \beta$.

<u>Proof.</u> It is enough to show that $\Sigma^0_\alpha \subseteq \Sigma^0_{\alpha+1}$.

Induction on α

For $\alpha = 1$:



We may see U as union of closed balls centered at elements of a dense countable set, with rational radia.

②

For $\alpha = \alpha' + 1$

$\cancel{\Sigma^0_{\alpha'} \subseteq \Sigma^0_{\alpha'+1}}$

We have $\Pi^0_{\alpha'} \subseteq \Pi^0_{\alpha'+1}$

If $B \in \Sigma^0_{\alpha'}$, then it is a countable union of sets from $\Pi^0_{\alpha'}$ and so also from $\Pi^0_{\alpha'+1} \rightarrow \in \Sigma^0_{\alpha+1}$

For α being a limit ordinal

$\cancel{B \in \Sigma^0_\alpha}$

$\forall \beta < \alpha \qquad \Pi^0_\beta \subseteq \Sigma^0_{\beta+1} \subseteq \Pi^0_\alpha$

So, ~~every~~ $\forall B \in \Sigma^0_\alpha \quad B = \bigcup_n B_n \quad B_n \in \Pi^0_\alpha$

and so $\quad B \in \Sigma^0_{\alpha+1}$

■

$$\begin{array}{ccccccc} & \Sigma^0_1 & & \Sigma^0_2 & & & \\ \Delta^0_1 \subseteq & & \subseteq \Delta^0_2 \subseteq & & \subseteq \ \dots & & \omega \\ & \Pi^0_1 & & \Pi^0_2 & & & \end{array}$$

↑

Note, this doesn't need to stop at ω:

If the hierarchy does not stabilize till ω, then $\forall n \ \exists B_n \in \Sigma^0_{n+1} \setminus \Sigma^0_n$ and we may cook up a set $B = \bigcup B_n$ which is not in $\Sigma^0_n \ \forall n$.

But

<u>PROP</u> $\quad \text{Borel} = \bigcup_{\alpha < \omega_1} \Sigma^0_\alpha$

This is a natural occurence of ω_1 in the Nature! :)

<u>Proof.</u> $\mathcal{A} = \bigcup_{\alpha<\omega_1} \Sigma^0_\alpha$ is a σ-algebra.

(i) $\mathcal{A}$ is closed under complements as

$$\mathcal{A} = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha \text{ as well.}$$

(ii) $\mathcal{A}$ is closed under countable unions:

Take $(A_n) \in \mathcal{A}$

$\forall n \quad \exists \alpha_n < \omega_1$ s.t. $A_n \in \Sigma_{\alpha_n}$.

Take $\sup_n \alpha_n = \alpha$. Then $\alpha < \omega_1$.
(regularity of ω_1).

Then $\{A_n : n \in \omega\} \subseteq \Sigma^0_\alpha$ and so

$\bigcup A_n \in \Sigma^0_{\alpha+1} \subseteq \mathcal{A}$. ■

(+ $\forall \alpha$
$\Sigma^0_\alpha \subseteq \mathrm{Bor}$)

So, this is how the BOREL SETS look like

$$\begin{array}{ccccccccccc}
 & & \Sigma^0_1 & & & & & & \Sigma^0_{\omega_1} & & \\
 & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \cdots & & \| & & \\
\Delta^0_1 & & & & \Delta^0_2 & & \cdots\cdots & & & = & \Delta^0_{\omega_1} \\
 & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \cdots & & & & \| \\
 & & \Pi^0_1 & & & & & & \Pi^0_{\omega_1} & & \mathrm{Bor}
\end{array}$$

In this picture there is still one unclear thing:

Is this hierarchy non-trivial till the end $= \omega_1$?

To answer it, we need a notion of UNIVERSAL SET, which is important on its own.

DEF. Let $\mathcal{A}$ be a ~~class~~ family of sets on a Polish space, ~~We say~~ ~~(like~~ $\mathcal{A} = \Sigma^0_\alpha$ for $\alpha < \omega_1$.

We say that $\mathcal{U} \subseteq 2^\omega \times X$ is $\mathcal{A}$-universal if

$\mathcal{U} \in \mathcal{A}$ and
$$\{\mathcal{U}_x : x \in 2^\omega\} = \mathcal{A}(X).$$

[In fact, we will replace 2^ω with other ~~sets~~ convenient sets].

Thm. There is a universal open set.

Proof. The whole idea is to code open sets in reals.
We will work in ~~...~~ an unctbl Polish space X.
Fix a countable base of open sets in X ~~...~~ and enumerate it.
$$\{B_n : n \in \omega\}.$$
We can code an open set $V \subseteq X$ in the following way:

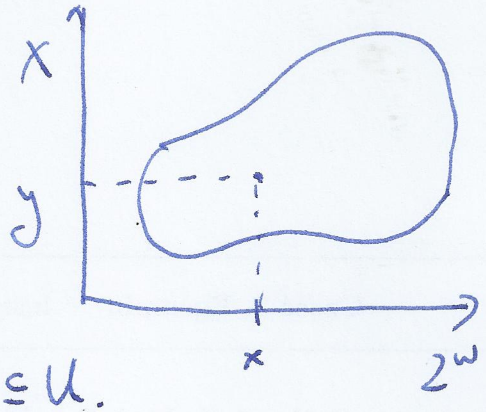
$x \in 2^\omega$ $\quad x(n) = 1$ iff $B_n \subseteq V$.

For $s \in 2^{<\omega}$ let $V_s = \bigcup \{B_n : n \in dom(s),\ s(n) = 1\}$ too complicated

Let $\mathcal{U} = \{\langle x, y\rangle \in 2^\omega \times \cancel{X} : y \in V_{x \restriction n}\ \forall n\}$

- $\mathcal{U}$ - open

- U - open



$\langle x, y\rangle \in U$

Let n s.t. $x(n)=1$.

Then $C^1_n \times B_n \subseteq U$.

- Let V - open in X.

Then $V = \bigcup_{n\in N} B_n$.

Take $x = \chi_N$.

■

<u>Corollary</u> There is a universal closed set!

<u>Thm</u> There is a Σ^0_α-universal set for every $\alpha<\omega_1$.

<u>Proof.</u> Induction.

Suppose we are done for ~~all~~ everybody below α.

Or, just let us take a look at $\alpha=2$. :)

We have U_1 - universal closed.

Define

$$U = \{\langle x, y\rangle \in \underbrace{(2^\omega)^\omega}_{\text{homeomorphic to } 2^\omega} \times X : \exists n \ \langle x(n), y\rangle \in U_1\}$$

- U is Σ^0_2:

$$U = \bigcup_n \{\langle x, y\rangle : \underbrace{\langle x(n), y\rangle \in U_1}_{\text{closed}}\}$$

- Let $B \in \Sigma^0_2(X)$.

$B = \bigcup_n F_n$ $\quad x = \langle x_1, x_2, \dots\rangle$, where x_i codes F_i.

For the general case: Let $\alpha < \omega_1$.

Suppose that we are done $\forall \beta < \alpha$.

Fix β_n - increasing such that $\beta_n + 1 \to \alpha$.

And do the same thing for:

$$\mathcal{U} = \{ \langle x, y \rangle \in (2^\omega)^\omega \times X : \exists n \ \langle x(n), y \rangle \in \mathcal{U}_{\beta_n} \}$$

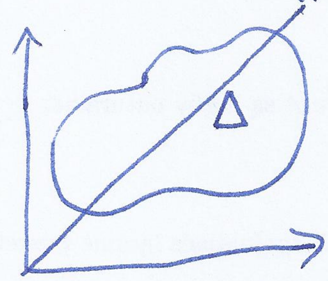
∎

Why it has something to do with the Borel hierarchy?

<u>Prop.</u> $\Sigma^0_\alpha \neq \Pi^0_\alpha$

$\mathcal{U}$ - universal for Σ^0_α

<u>Proof</u>



Let $A = \{ x : \langle x, x \rangle \notin \mathcal{U} \}$

Then
- $A \notin \Sigma^0_\alpha$.
- $A \in \Pi^0_\alpha$ as $\Delta \cap \mathcal{U}^c$ is Π^0_α and we have clear homeomorphism between A and $\Delta \cap \mathcal{U}^c$.

∎

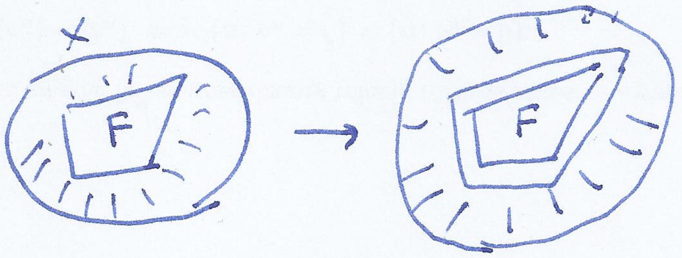
So $\Sigma^0_\alpha \subsetneq \Sigma^0_{\alpha+1}$.

<u>Remark.</u> There is no universal set for Borel sets. The procedure as above will give us a contradiction

~~DEF.~~ REMARK In DST we are often more interested in the σ-algebra of Borel sets than in the topology itself. In particular, we can modify the topology of the space and don't care about as long as the Borel sets are the same . . .

Prop. Let (X,τ) Polish and F - closed $\subseteq X$.
Then the topology τ' generated by $\tau \cup \{F\}$ is still Polish.
and $Bor(X,\tau) = Bor(X,\tau')$.

Proof.



(X,τ') = coproduct of F and F^c.

Polish subspaces.

$\blacksquare$

Prop. (*) (X,τ) - Polish
τ_n - sequence of topologies on X s.t.
$\tau \subseteq \ldots \subseteq \tau_n \subseteq \tau_{n+1} \subseteq \ldots$, and they are Polish + $Bor(X,\tau_n) = Bor$
Then the topology τ' generated by $\bigcup \tau_n$ is Polish
and $Bor(X,\tau') = Bor(X,\tau)$.

Proof $X_n = (X,\tau_n)$. Consider $\prod X_n$. Let X_∞ be (X,τ')

Let ~~X~~ $\Delta : X_\infty \to \prod X_n$ be the map $\Delta(x) = (x,x,\ldots)$
$\Delta : X_\infty \to Diag(\prod X_n)$ is a homeo. (this is not automatic, but it is not difficult either)
And so X_∞ is homeomorphic to a closed subspace of $\prod X_n \Rightarrow$ Polish.

Theorem P (X, τ) - Polish, $B \subseteq X$ - Borel.

Then there is a richer topology τ' s.t.

$$Bor(X, \tau) = Bor(X, \tau') \text{ and } B \text{ is clopen.}$$

Proof. Let $\mathcal{A}$ be the family of sets ~~which work~~ for which the above works.

- $\mathcal{A}$ contains closed sets (proposition)
- $\mathcal{A}$ is closed under complement (clearly) and countable unions (proposition) $\square$

Corollary (X, τ) - Polish.
Then there is a richer Polish topology τ' which is zero-dimensional and $Bor(X, \tau) = Bor(X, \tau')$

Corollary Borel subsets of Polish spaces have Perfect Set Property.

Proof. B - Borel, uncountable

Make it clopen. $\to \tau'$

So $(B, \tau'\restriction_B)$ is Polish.

So, by the theorem from previous lectures, there is a copy of 2^ω in $(B, \tau'\restriction_B)$.

Formally there is $f: 2^\omega \xrightarrow[\text{continuous}]{1-1} (B, \tau'\restriction_B)$.

But $f: 2^\omega \xrightarrow{1-1} (B, \tau_B)$ is still continuous as $\tau_B \subseteq \tau'_B$.

+ 2^ω is compact and we are done $\square$

The proof of Corollary J.

Let (X, τ) be Polish.

Fix a countable base $\mathcal{B}_0$.

Using Theorem P subsequently (and the previous proposition at the end) we can make all elements of $\mathcal{B}_0$ clopen, obtaining a topology τ_0.

Notice! $\mathcal{B}_0$ is no longer the base for τ_0

(sober remark of Mikołaj: if two spaces have the same base, then they have the same topology :)

So, we have to iterate the above process, taking ~~obtaining~~ $\mathcal{B}_{n+1}$ - base of (X, τ_n) and obtaining τ_{n+1} as above.

At the end, we use Proposition * one more time to get τ_∞.

Clearly, $\bigcup_n \mathcal{B}_n$ is a base for (X, τ_∞). $\boxtimes$

DST 5

ANALYTIC SETS

The biggest mistake in the history of set theory:
(Lebesgue) The continuous images of Borel sets are Borel.

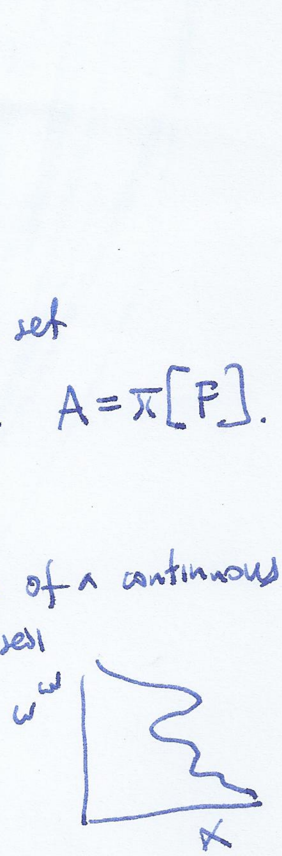
Suslin: Not necessarily!

DEF. X - Polish.
We say that $A \subseteq X$ is analytic if it a continuous image of ω^ω.

THM. X - Polish, $A \subseteq X$. TFAE

1. • A - analytic
2. • A is a continuous image of a Borel set
3. • there is a closed $F \subseteq X \times \omega^\omega$ s.t. $A = \pi[F]$.

Proof. 3 $\Rightarrow$ 2 - clear
1 $\Rightarrow$ 3 - clear, since the graph of a continuous function is closed



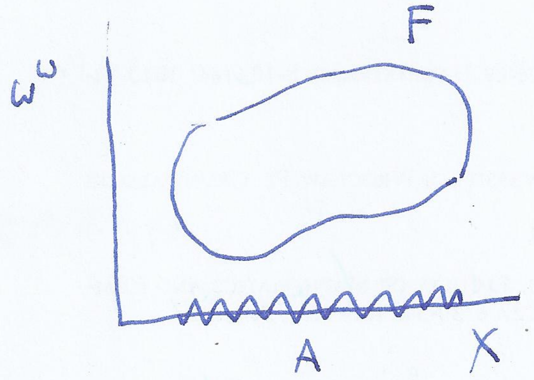
2 $\Rightarrow$ 1

• Every uncountable Polish space is a continuous image of ω^ω

• $\Downarrow$
Every Borel subset of a Polish space is a continuous image of ω^ω
(trick with enriching the topology)

Notation: $\Sigma^1_1(X)$ - family of analytic subsets of X.

$\Pi^1_1(X) = \{A^c : A \in \Sigma^1_1(X)\}$ - co-analytic sets.



$\leftarrow$ unfolding of A.

A is analytic iff

$$A = \{x : \underbrace{\exists y \in \omega^\omega}_{} \langle x, y\rangle \in F\}$$

In general, A is analytic if we can ~~describe~~ define it via formula which has a quantifier $\exists$ over reals and then is Borel.

Simiraly, A is co-analytic iff

$$A = \{x : \underbrace{\forall y \in \omega^\omega}_{} \langle x, y\rangle \notin F\}$$

Let $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$

and notice that Borel $\subseteq \Delta^1_1$.

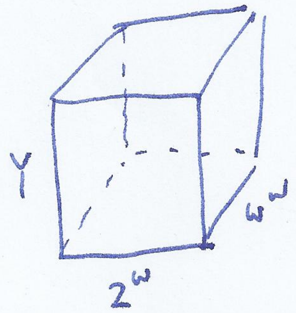
Fact. Σ^1_1 is closed under Borel images and preimages, countable intersections and unions (but not complements!)

Proof: Exercise.

Thm There is a universal analytic set.

Proof. There is a (2^ω-) universal ~~set~~ closed set
$$F \subseteq 2^\omega \times (Y \times \omega^\omega).$$

$$U = \{\langle x, y\rangle \in 2^\omega \times Y : \exists z \in \omega^\omega \ \langle x, y, z\rangle \in F\}$$

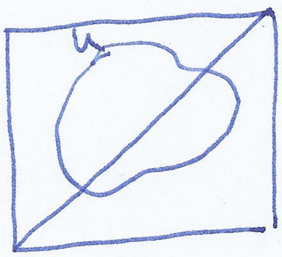


- U is analytic: just because it is the projection of $F \subseteq (2^\omega \times Y) \times \omega^\omega$.

- U is universal: Let $A \subseteq Y$ be analytic.
  ~~Then there is a closet~~ ~~Let $C \subseteq 2^\omega \times Y$ be the c~~
  Then there is $C \subseteq Y \times \omega^\omega$, closed, such that $\pi[C] = A$.
  Find $x \in 2^\omega$ s.t. $C = F_x$.
  Then $U_x = \{y \in Y : \exists z \in \omega^\omega \ \langle x, y, z\rangle \in F\} =$
  $= \{y \in Y : \exists z \in \omega^\omega \ \langle y, z\rangle \in F_x\}$
  $= \pi[F_x] = \pi[C] = A.$ ∎

Corollary:
$$\Sigma^1_1 \neq \Pi^1_1$$

Proof.



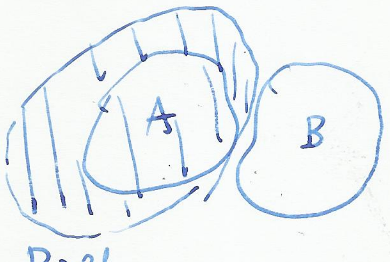
Let U be universal for Σ^1_1.
Let $A = \{x : \langle x, x\rangle \notin U\}$
$x \mapsto \langle x, x\rangle$ is homeomorphism
so A is co-analytic.
But it cannot be analytic, by universality.

<u>Thm</u> (Luzin Separation Theorem)

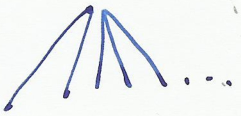
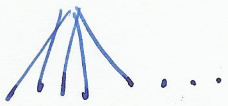
X - Polish space.

Let A, B - analytic, $A \cap B = \emptyset$.

Then there is a Borel set separating A and B.



<u>Proof.</u> Fix a continuous (A contains) $f : \omega^\omega \to A$, $g : \omega^\omega \to B$

f  g 

Suppose that $\forall s \in \omega^1$ we have that $f[[s]]$ and $g[[s]]$ can be separated by a Borel set.

Then $\forall s \in \omega^1$ $f[\omega^\omega]$ can be separated from $g[[s]]$ (taking ~~unions~~ intersection of Borel sets separating $f[[t]]$ from $g[[s]]$, $t \in \omega^1$).

But then $f[\omega^\omega]$ can be separated from $g[\omega^\omega]$; by the same trick. taking union this time

Going along the tree, we can choose a branch $x \in \omega^\omega$ such that

$\forall n$ $f[[x \upharpoonright n]]$ cannot be separated from $g[[x \upharpoonright n]]$.

But $f(x) \neq g(x)$ and $\{f(x)\}, \{g(x)\}$ can be separated by open U, V. But then, by continuity $\exists n$
$f[[x \upharpoonright n]] \subseteq U$, $g[[x \upharpoonright n]] \subseteq V$ $\lightning$

Corollary. $\Delta^1_1 = \text{BOREL}.$

Corollary X, Y - Polish, $f: X \to Y$.
TFAE

1) f is Borel
2) the graph of f is Borel
3) the graph of f is analytic.

Proof. 1) $\Rightarrow$ 2) Exercise

3) $\Rightarrow$ 1) Let $\underset{\text{open}}{U \subseteq Y}$.

$$x \in f^{-1}[U] \equiv \exists y \in Y \; f(x) = y \text{ and } y \in U$$
$$\equiv \forall y \in Y \; f(x) = y \Rightarrow y \in U$$
and so $f^{-1}[U] \in \Sigma^1_1 \cap \Pi^1_1 = \text{Borel}$

$\blacksquare$

6 Suslin Operation

$T \subseteq \omega^{<\omega}$ tree, X - Polish space

Let $(P_s)_{s \in T}$ be a <u>Suslin scheme</u> on X

(something like Luzin scheme but we don't assume anything ;) , just a family of sets indexed by T)

For a Suslin scheme $(P_s)_{s \in T}$ define the Suslin Operation by

$$\mathcal{A}(P_s)_{s \in T} = \bigcup_{y \in [T]} \bigcap_{n \in \omega} P_{y \restriction n}.$$

Typically (but not always) we want to think that $(P_s)_{s \in T}$ is such that

$(\blacktriangledown)$
- A. $P_s \subseteq P_t$ if s extends t and
- B. $P_{s^\frown i} \cap P_{s^\frown j} = \emptyset$ if $i \neq j$.

So that it looks like that:

Fact. If $(P_s)_{s \in T}$ satisfies $(\blacktriangledown)$, then

$$\mathcal{A}(P_s)_{s \in T} = \bigcap_{n \in \omega} \bigcup_{\substack{s \in T \\ |s| = n}} P_s.$$

Proof. $\subseteq$ - clear

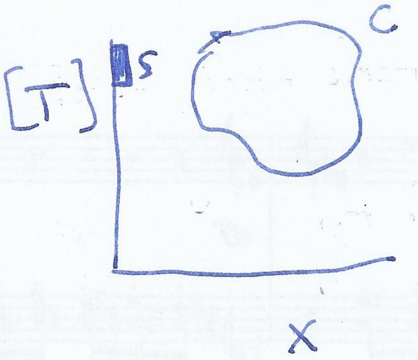
$\supseteq$ If $x \in \bigcap_{n \in \omega} \bigcup_{\substack{s \in T \\ |s| = n}} P_s$, then for each n it has a witness s_n of length n. And those witnesses have to form a branch $\blacksquare$

In fact $\mathcal{A}(P_s)_{s\in T}$ is like taking projection ~~of T~~ along $[T]$:

$$\mathcal{A}(P_s)_{s\in T} = \pi[C], \text{ where } C \subseteq X \times [T]$$

is defined:

$$C = \{\langle x, y\rangle : \forall n\in\omega\ \exists s\in T\ y\restriction_n = s \land x \in P_s\}$$



<u>Thm</u> X - Polish space, $A \subseteq X$. TFAE

1. A is analytic
2. $A = \mathcal{A}(F_s)_{s\in\omega^{<\omega}}$, where F_s's are closed, satisfy $\blacktriangledown$A and have vanishing diameters
3. $A = \mathcal{A}(P_s)_{s\in\omega^{<\omega}}$, where P_s's are analytic.

<u>Proof.</u> 1 $\Rightarrow$ 2 $\quad A = f[\omega^\omega]$ for some continuous f.

Take $\boxed{P_s = f[[s]]}$ and $F_s = \overline{P_s}$

First, notice that $f[\omega^\omega] = \mathcal{A}(P_s)_{s\in\omega^{<\omega}}$. Indeed, $y \in f[\omega^\omega] \equiv$

$$\equiv \exists x\in\omega^\omega\ f(x) = y \equiv$$

$$\equiv \exists x\in\omega^\omega\ \forall n\ y \in \underbrace{f[[x\restriction_n]]}_{P_s}$$

Now, we will show that $f[\omega^\omega] = \mathcal{A}(\overline{P_s})_{s \in \omega^{<\omega}}$.

Let $x \in \omega^\omega$ and suppose that

$$y \in \bigcap_n \overline{P_{x \restriction n}}.$$

Then $f(x) = y$.

If not, then there is an open $U \ni f(x)$ such that $y \notin \overline{U}$.

But, as f is continuous, $\exists n$

$$P_{x \restriction n} = f[[x \restriction n]] \subseteq U.$$

So $y \notin \overline{P_{x \restriction n}}$ ↯.

$\blacksquare$

(2) ⇒ (3) obvious

(3) ⇒ (1) We already know that $\mathcal{A}(P_s)_{s \in \omega^{<\omega}}$ is a projection of an analytic set, and analytic sets are closed under continuous functions.

$\blacksquare$

So, analytic sets = sets generated from closed sets by Suslin operation.

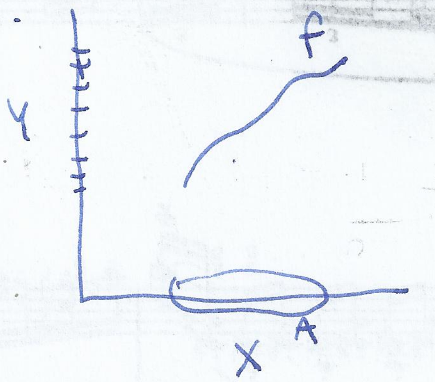
In particular, analytic sets are closed under taking Suslin operation.

Thm X, Y - Polish spaces. $f: X \to Y$ Borel and 1-1.

Then $f[B]$ is Borel if B - Borel.

Proof. We may assume that f - continuous!

The trick.



graph(f) is ~~analytic~~ Borel

and

$\pi^Y[\text{graph}(f)] = f[A]$

and

π^Y is continuous.

WLOG we may assume that $X = \omega^\omega$ and $A \subseteq \omega^\omega$ is closed.

(since every Borel set is a continuous 1-1 image of a closed subset of ω^ω)

So $A = [T]$.

$$f[[T]] = \mathcal{A}(P_s) = \mathcal{A}(\overline{P_s}), \text{ where } P_s = f[[s]].$$

Since f is 1-1, (P_s) satisfies and so

$$f[[T]] = \bigcap_{n \in \omega} \bigcup_{s \in T, |s| = n} P_s .$$

$\forall s$ we can find $P_s \subseteq B_s \subseteq \overline{P_s}$ - Borel

such that $B_s \subseteq B_t$ and $B_{s^\frown i} \cap B_{s^\frown j} = \emptyset$.

(Luzin separation theorem)

Then $\mathcal{A}(P_s) \subseteq \mathcal{A}(B_s) \subseteq \mathcal{A}(\overline{P_s}) = f[[T]]$ and so it is Borel ∎

So

ANALYTIC SETS $\equiv$ CONTINUOUS IMAGES OF ω^ω

BOREL SETS $\equiv$ CONTINUOUS INJECTIVE IMAGES OF CLOSED SUBSETS OF ω^ω

<u>Thm</u> (BOREL CANTOR-BERNSTEIN)

X, Y - Polish. If there are

- $f: X \to Y$ Borel 1-1
- $g: Y \to X$ Borel 1-1,

then X and Y are Borel isomorphic.

<u>COROLLARY</u> EVERY 2 UNCTBL POLISH SPACES ARE BOREL ISOMORPHIC.

<u>Proof.</u> X - unctbl Polish (U_n) - fixed base

There is Borel $f: X \xrightarrow{1-1} 2^\omega$ $(f(x)(n) = 1 \text{ iff } x \in U_n)$

There is Borel $f: 2^\omega \xrightarrow{1-1} X$ (Perfect Set Theorem)

So X is Borel isomorphic to 2^ω $\blacksquare$

REGULARITY OF ANALYTIC SETS

We will begin with another example of a game. Let's call it Perfect Set Property game.

Fix X - perfect Polish space and $B \subseteq X$.

The PSP-game $\Gamma(B)$ is defined as follows

$$\begin{array}{llll} \text{Adam} & (U_0^0, U_1^0) & & (U_0^1, U_1^1) & \dots \\ \text{Eve} & & i_0 & & i_1 & \dots \end{array}$$

where $*$ U_i^n - basic ! open, $U_0^n \cap U_1^n = \emptyset$, $\overline{U_0^{n+1} \cup U_1^{n+1}} \subseteq U_{i_n}^n$

For some fixed countable base.

$\operatorname{diam}(U_i^n) < 1/n$.

$*$ $i_n \in \{0,1\}$ (One can think that Adam proposes Eve two disjoint open sets and she chooses one of them).

Adam wins if $\bigcap_n U_{i_n}^n \subseteq B$.

↑ this will be a singleton (Cantor thm).

<u>Thm</u>

① Adam has a winning strategy in $\Gamma(B)$ $\Leftrightarrow$ B contains a Cantor set

② Eve has a winning strategy in $\Gamma(B)$ $\Leftrightarrow$ B is countable

(So, if $\Gamma(B)$ is determined then B has PSP).

Proof. 1 $\Leftarrow$ Adam should pick sets with nonempty intersection with the ~~cop~~ fixed copy of 2^ω.

$\Rightarrow$ The winning strategy for Adam gives us a Cantor scheme.

2 $\Leftarrow$ if $B = \{b_0, b_1, \dots\}$, then Eve should avoid b_n at nth step.

$\Rightarrow$ Suppose σ is a winning strategy for Eve.

Fix $x \in A$. There is a maximal ~~s~~ run of the game $\hat{s}$ according to σ such that $x \in U^n_{i_n}$

$$s = ((U^0_0, U^0_1), i_0; \dots; (U^n_0, U^n_1), i_n)$$

(and i_n's are choosen according to σ).

s is maximal in the sense that in every further step played according to σ ~~with would~~ ~~$i_{n+1} \notin$~~ $x \notin U^{n+1}_{i_{n+1}}$.

If there is x without such a maximal s_x, then Adam would win a run of a game played according to σ.

Notice that if $x \neq y$, then $s_x \neq s_y$! Just use Hausdorff.

But it means that A is countable (as there are only countably many finite runs of the game; ~~we~~ Adam plays sets from a fixed countable base)

Corollary: Under Axiom of Determinacy every subset of $\mathbb{R}$ has PSP.

We aim at showing that the analytic sets have PSP. To show it we have to change the game a little bit.

Let $A \subseteq X$ - analytic.
We can ufold it: there is $F \subseteq X \times \omega^\omega$ closed such that

$$\pi[F] = A.$$

The game $\Gamma'(F)$:

- Adam $(U_0^0, U_1^0), \underline{\underline{k_0}}$ $\qquad$ $(U_0^1, U_1^1), \underline{\underline{k_1}}$
- Eve $\qquad i_0 \qquad i_1$

U_i^n are as before, i.e. open subsets of $\underline{\underline{A}}$.
$k_n \in \omega \quad \forall n$.
Now, let x be defined as before, and let $y = (k_0, k_1, \ldots) \in \omega^\omega$.
This time Adam wins if $\langle x, y \rangle \in F$.

<u>Thm</u>
1. If Adam ~~wins~~ has w.s. $\Gamma'(F)$, then A contains a copy of 2^ω.
2. If Eve has a winning strategy in $\Gamma'(F)$, then $|A| \le \omega$.

Proof ① Clear. (just forget K_n's and enjoy your Cantor scheme given by Adam's strategy).

② Suppose σ is a winning strategy for Eve. Let $x \in A$. <u>Fix</u> $y \in \omega^\omega$ such that $\langle x, y \rangle \in F$.

<u>CLAIM</u> There is a finite run of the game ~~in~~ s_x such that ~~which Adam plays $k_n = y(n)$~~

- s_x is compatible with σ
- s_x is "compatible" with y (i.e. Adam plays $k_i = y(i)$ at his "natural" moves)
- $x \in U^n_{i_n}$, where $U^n_{0,1}$, i_n are the last moves of s_x.
- for every Adam's move $((U^{n+1}_0, U^{n+1}_1), y(n+1))$ we have $x \notin U^{n+1}_{i_{n+1}}$ ← according to σ.

<u>PROOF</u>. Otherwise, there is a (full) run of the game ~~for~~ which Eve plays σ and Adam wins. ↯

Now, how many $x \in A$ can have the same s_x?

Let $x_0 \neq x_1$. ~~Suppo~~ Let y_0, y_1 be the fixed elements of ω^ω such that $\langle x_0, y_0 \rangle, \langle x_1, y_1 \rangle \in F$ (so that s_{x_0} and s_{x_1} are defined using y_0 and y_1).

Suppose that $y_0(n+1) = y_1(n+1)$, where n is the length of s_x.

Then $x_0 = x_1$ by Hausdorffness $\cdot x_0$ $\cdot y_1$

As there are only countably many choices for $y(n+1)$, there are only countably many x's ~~which for~~ with the same s_x ∎

OK but what does it prove?

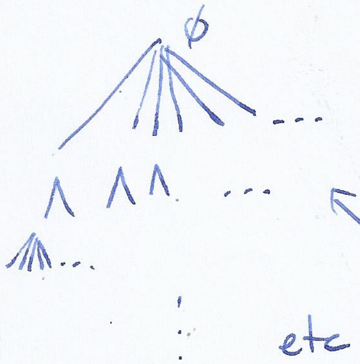
How do we know that this game is determined?

A little bit of a more general theory.

We can think that this game is connected to a countable pruned tree.

Every branch is interpreted as a run of the game →

T



← countable choice for the 1st move of Adam

← choice for Eve

etc

Now, we are thinking about the general setting, not connected to the particular game →

Let $F \subseteq [T]$ be the set of all the runs of the game in which Adam wins.

<u>Thm.</u> (Gale-Stewart) If F is closed or open, then the game is determined.

<u>Proof.</u> We can call players Closed and Open.

$s \in T$ is winning for Open if she has a winning strategy "starting" at s.

Remark: if s is not winning for Open, then Closed can answer in such a way that $s^\frown \underset{\text{answer}}{a}$ is still not winning.

Using this remark we can show that if Open does not have a winning strategy, then Closed can play is such a way that he wins.

If he plays in an obvious way, then x - the final run is in F. If not, then $\exists \underset{\text{open basic}}{U} \ni x$ such that $U \cap F = \emptyset$ and so $\exists n$ $T \cap [x|n] \subseteq U$

Thm Analytic subsets have PSP.

Proof. ~~We encode $\Gamma'(A)$, A a~~

We play $\Gamma'(F)$, where $F \subseteq X \times \omega^\omega$, $\pi[F] = A$.
We encode it in a tree T like above.

The function $g : [T] \to X \times \omega^\omega$

$g(x) = y$, where $\{y\}$ is the intersection of U^n_{in}, encoded in x.

is continuous (Exercise).

Then $g^{-1}[F] \subseteq [T]$ is closed and so, by Gale-Stewart theorem it is determined.
So A contains a copy of 2^ω or is countable ∎

In a simmilar way, using Banach-Mazur game unfolded we can show that every analytic subset is Baire measurable.
(Luzin-Sierpiński thm)

Remark or rather

THEOREM (Martin) Every Borel game is determined
(i.e. if the payoff set for Adam is Borel, then the game is determined)

The question about determinacy of other classes leads to a surprising results involving large and often very LARGE cardinals...

PROJECTIVE CLASSES

$$\Pi^1_n = \{A : A \in \Sigma^1_n\}$$

$$\Sigma^1_{n+1} = \{\pi[A] : A \subseteq X \times \omega^\omega,\ A \in \Pi^1_n\}$$

$$\Delta_n = \Sigma_n \cap \Pi_n$$

<u>Remark</u> $\forall n \quad \Sigma^1_n \subseteq \Sigma^1_{n+1}$ ($\Sigma^1_1 \equiv$ projections of Borel sets $+ \underbrace{\text{Borel} \subseteq \Pi^1_1}$ + induction)

So

$$\text{Bor} = \Delta^1_1 \begin{smallmatrix}\subseteq \Sigma^1_1 \\ \subseteq \Pi^1_1\end{smallmatrix} \subseteq \Delta^1_2 \begin{smallmatrix}\subseteq \Sigma^1_2 \\ \subseteq \Pi^1_2\end{smallmatrix} \subseteq \Delta^1_3 \ \ldots \qquad \boxed{P} = \bigcup_n \Sigma^1_n$$

PROJECTIVE SETS

The inclusions are strict $\Leftarrow \exists$ universal sets (Exercise)

Σ^1_n, Π^1_n are closed under countable $\cup, \cap$ and Borel preimages. (Exercise)

Σ^1_n ~~is~~ are closed under Borel images

<u>Remark</u> Σ^1_{n+1} sets are those which can be defined by formula

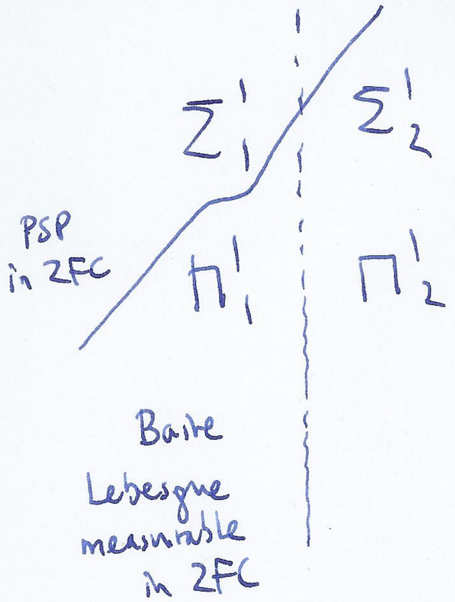
$$\exists x \in \omega^\omega \ \underbrace{\varphi(x)}_{\Pi^1_n \text{ formula}}$$

(and, of course, analytic sets are those, where φ is Borel)

Similarly Π^1_{n+1} sets are those defined by

$$\forall x \in \omega^\omega \ \underbrace{\varphi(x)}_{\Sigma^1_n \text{ formula}}$$

<u>Notice</u> Instead of ω^ω we can take any uncountable Polish space (since it is Borel isomorphic to ω^ω and Π^1_n sets are closed under Borel preimages).



The rest is a mess (but quite interesting).

<u>Thm</u> (Goedel) If $V=L$, then there is a Π^1_1 set without PSP and there is a non-measurable Σ^1_2 set

In both senses

<u>Thm</u> (Solovay) Suppose there is a strongly inaccessible cardinal. Then all the sets in the projective hierarchy are measurable.

<u>Thm.</u> AD $\Rightarrow$ measurability and PSP of every subset of $\mathbb{R}$.
(Mycielski, Świerczkowski).

Different kind of determinacy axioms:

Projective Determinacy (implies that all the projective sets are determined, and thus measurable, PSP).

Projective sets have an "ultimate" extension: $L(\mathbb{R})$.

One can have $AD^{L(\mathbb{R})}$ without loosing AC.

REDUCTIONS

All spaces below will be 0-dimensional.

<u>DEF.</u> X, Y - Polish

$A \subseteq X$

$B \subseteq Y$

We say that A is <u>Wadge-reducible</u> to B, $A \leq_W B$, if there is a continuous $f: X \to Y$ such that

$f^{-1}[B] = A$.

<u>Remark</u> It means that $\forall x \in X \quad x \in A \iff f(x) \in B$.

<u>DEF.</u> Let Γ be a class of sets (like Borel, Σ^1_1, ...)

We say that B is Γ-complete if

- $B \in \Gamma$ and
- $\forall A \in \Gamma \quad A \leq_W B$.

<u>Thm.</u> B is Σ^0_α-complete iff $B \in \Sigma^0_\alpha \setminus \Pi^0_\alpha$.

<u>Proof.</u> <u>Claim</u> Suppose A, B - Borel ($A \subseteq X$, $B \subseteq Y$) (X, Y - 0-dim)

Then $A \leq_W B$ or $B \leq_W A^c$

<u>Proof.</u> WLOG X, Y closed subsets of ω^ω and so $X = [S]$, $Y = [T]$.

We play the following game

Adam	a_0	a_1	a_2	...
Eve	e_0	e_1	e_2	

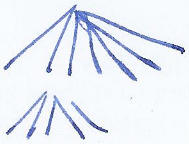
they have to play in such a way that

$\langle a_0, \ldots, a_n \rangle \in S$

$\langle e_0, \ldots, e_n \rangle \in T$.

Eve wins iff $\left((a_0\, a_1 \ldots) \in A \iff (e_0\, e_1 \ldots) \in B\right)$.

- Notice that this is a Borel game:



If B - Borel $\subseteq \omega^\omega$, then $B^{odd} = \{y \in \omega^\omega : (y(1), y(3), y(5), \ldots) \in B\}$ is Borel:

$B^{odd} = f^{-1}[B]$, where $f : \omega^\omega \to \omega^\omega$ is given by $f(x)(n) = x(2n+1)$ is continuous.

Then the set of winning runs for Eve =

$$= \left(A^{Even} \cap B^{odd}\right) \cup \left((A^{Even})^c \cap (B^{odd})^c\right).$$

- Suppose that Eve has a winning strategy σ.

Define a function $f : [S] \longrightarrow [T]$

$f(x) = y$, where $y \restriction n$ is the sequence of Eve strategic responses to $x \restriction n$.

This is continuous: take $t \in T$, a sequence of strategic responses. $f^{-1}[[t]]$ is open. (a union of basic clopens induced by sequence of length $|t|$)

Since this is Eve's winning strategy: $x \in A \iff f(x) \in B$.

~~$x A$~~ So $A = f^{-1}[B]$ and $A \leq_W B$.

- If Adam has a winning strategy, then we can repeat the above and get $B \leq_W A^c$. ◻ Claim

Continuing the proof of the theorem:

- If B is Σ^0_α-complete, then $B \notin \Pi^0_\alpha$

  Otherwise there is $A \in \Sigma^0_\alpha \setminus \Pi^0_\alpha$

  $A \leq_W B$ but

  $f^{-1}[B]$ is Π^0_α if

  f is continuous.

- If $B \in \Sigma^0_\alpha \setminus \Pi^0_\alpha$, then let $A \in \Sigma^0_\alpha$.

  Either, by the claim, $A \leq_W B$. Cool.

  Or $B \leq_W A^c$ but then B is Π^0_α

  (because of the argument as above) $\lightning$.

Now, what about analytic complete sets.

- Analytic sets $\equiv$ ~~give~~ projections of trees on $\omega \times \omega$.

  If $A \subseteq \omega^\omega$ is analytic, then it is a projection of a closed subset of $\omega^\omega \times \omega^\omega$.

  Closed subset $\equiv$ ~~given by~~ body of a tree S.

  Where this tree lives? On $\omega \times \omega$.

  So ~~iff~~ $A \subseteq \omega^\omega$ analytic iff $A = \pi[[S]]$, where S is a tree on $\omega \times \omega$