DST 0 2024

Zad. 1 Recall some basic examples and non-examples of F_σ and G_δ sets in Polish spaces.

Zad. 2 Show that in Polish spaces every closed set is G_δ.

Zad. 3 Let (X, d) be a metric space. Then d' defined by $d'(x, y) = \min(d(x, y), 1)$ induces the same topology. Show that!

Zad. 4 Show that a separable metric space has a countable base. Conclude that any subspace of a separable metric space is separable.

Zad. 5 Show that 2^ω is homeomorphic to $2^\omega \times 2^\omega$ and $\mathbb{R}^\omega$ is homeomorphic to $\mathbb{R}^\omega \times \mathbb{R}^\omega$. Find continuum many pairwise disjoint homeomorphic copies of 2^ω in 2^ω.

Zad. 6 Let (X, d) be a separable metric space, and let $D = \{y_0, y_1, \ldots\} \subseteq X$ be a countable dense set. Show that the mapping $f\colon X \to [0, 1]^\omega$ given by

$$f(x) = (d(x, y_0), d(x, y_1), \ldots)$$

is a homeomorphic embedding.

Zad. 7 Suppose $(F_\alpha)_{\alpha<\kappa}$ is a strictly decreasing sequence of closed subsets of a Polish space X. Show that κ is countable.

Zad. 8 Show that a metric space X is complete if and only if every decreasing sequence of closed sets in X with vanishing diameters has one-point intersection. Show an example of a Polish space and a descreasing sequence of its closed subsets whose intersection is empty.

Zad. 9 Show that ω^ω is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$. (This one is more difficult. One can think about continuous fractions. . . .)