DST 1 2024

Zad. 1 Show that for every alphabet A the space A^ω is zero-dimensional, i.e. it has a base of clopen sets.

Zad. 2 Try to characterize those trees $T \subseteq \omega^{<\omega}$ for which

a) $[T]$ does not have an isolated point,

b) $[T]$ has empty interior,

c) $[T]$ is compact.

Zad. 3 Consider $X \subseteq 2^{<\omega}$ (note that X need not be a tree). The set

$$\{x \in 2^\omega \colon \exists^\infty n\ x|n \in X\}$$

is called a G_δ closure of X. Show that it deserves its name.

Zad. 4 Show that for every closed $F \subseteq 2^\omega$ there is a continuous function $f\colon 2^\omega \to F$ such that $f(x) = x$ for each $x \in F$ (such a function is called a *retraction*). Hint: use the fact that $F = [T]$ for some tree $T \subseteq 2^{<\omega}$ and try to define, level by level, a mapping $g\colon 2^{<\omega} \to T$ from which you can cook up f.

Zad. 5 Show that for each function $f\colon \mathbb{R} \to \mathbb{R}$ the set of continuity points of f is a G_δ set. Hint: the *oscillation* of f at x is defined as

$$osc_f(x) = \inf\{diam f[U] \colon x \in U, U \text{ - open}\}.$$

Notice that x is a continuity point of f iff $osc_f(x) = 0$.

Zad. 6 Use the above fact to show that under Continuum Hypothesis there is an outrageously non-continuous function $f\colon \mathbb{R} \to \mathbb{R}$, i.e. such that $f|C$ is not continuous for each $C \subseteq \mathbb{R}$ of size $\mathfrak{c}$.

Zad. 7 Show that 2^ω is a continuous bijective image of ω^ω.