DST 4 2024

Zad. 1 Let X, Y be Polish spaces and let $f\colon X \to Y$ be Borel. Show that the graph of f is a Borel subset of $X \times Y$.

Zad. 2 Show that the family of analytic sets is closed under Borel images and pre-images, countable intersections and countable unions.

Zad. 3 Using the theorem about the existence of universal set for Σ^0_α parametrized by 2^ω (which was proved on the lecture) show the existence of universal set parametrized by an uncountable Polish space X. More precisely, for every uncountable Polish spaces X, Y and for each $\alpha < \omega_1$ there is a set $U \subseteq X \times Y$ which is Σ^0_α and such that for each $A \in \Sigma^0_\alpha(Y)$ there is $x \in X$ such that $U_x = A$.

Zad. 4 Is there a universal Borel set?

Zad. 5 Show that the following sets are Borel. Try to estimate its Borel complexity.

- the set of normal numbers in $[0,1]$ (a real is *normal* if its binary expansion contains arbitrary finite sequences of bits),
- the collection of sets $A \subseteq \omega \times \omega$ such that there is a function $f \in \omega \times \omega$ such that, for all $n \in \omega$, $A \cap (\{n\} \times \omega)$ has less than $f(n)$ elements,
- the collection of all $f \in 2^\omega$ such that $\displaystyle\sum_{i \in \omega, f(i)=1} \frac{1}{i} < \infty$,
- the collection of all nowhere dense sets in $\mathbb{Q}$.
- the collection of all subsets $R \subseteq \omega \times \omega$ such that R is a linear order of a subset of ω,
- the collection of all subsets $R \subseteq \omega \times \omega$ such that R is a well order of a subset of ω order isomorphic to an ordinal smaller or equal than ω,
- given any $\alpha < \omega_!$, the collection of all subsets $R \subseteq \omega \times \omega$ such that R is a well order of a subset ω order isomorphic to an ordinal smaller than α.

Zad. 6 Show that every Polish space is an injective continuous image of a closed subset of ω^ω. Conclude that every Borel set is an incjective continuous image of a closed subset of ω^ω.