Wrocław Forcing Notes

This is more less what we are talking about on the Spring 2023 course *Forcing. How to prove consistency results.* with many details omitted. I still have some time to write it but I don't have time to read it, so please inform me about all the mistakes, gaps, misprints and errors.

Pbn

...

University of Wrocław

Contents

1. Basics

The basics can be found e.g. in https://www.math.toronto.edu/sunger/ucla/LogicSummerSchool/ForcingNotes.pdf (although our notation is slightly different).

In what follows,

- V be our ground model (think about V as a countable transitive model of rich enough part of ZFC or think about V as our mathematical universe, a model of ZFC, I don't care),
- $\mathbb{P}$ will be a partial order (with 1, separative), an element of V,
- G will be a $\mathbb{P}$-generic over V. As $\mathbb{P}$ is separative, $G \notin \mathbb{P}$.

1.1. Names and their interpretations.

We are going to investigate $V[G]$ living in V. It's going to be a delicate matter as we don't have any *direct* access to elements of $V[G]$. Instead, we will use something called names.

A $\mathbb{P}$-name $\dot{x}$ is any collection of ordered pairs $\langle \dot{y}, p\rangle$, where $p \in \mathbb{P}$ and $\dot{y}$ is a $\mathbb{P}$-name. Sounds like too much self-reference? Let's see some examples:

- $\emptyset$ is definitely a $\mathbb{P}$-name regardless of $\mathbb{P}$.
- Once we have one $\mathbb{P}$-name, we can produce another, say $\dot{x} = \{\langle \emptyset, p\rangle\}$, where $p \in \mathbb{P}$ (we could consider here some concrete example of $\mathbb{P}$ and p but I don't think it would help now).
- Once we have $\dot{x}$, let's produce $\dot{y} = \{\langle \emptyset, r\rangle, \langle \emptyset, q\rangle, \langle \dot{x}, p\rangle\}$.
- OK, it becomes complicated, so we will not give go further now. But, don't panic: the formal definition of ordinal numbers is also not very handsome and I bet you haven't written 9 using only $\{,\}$ and $\emptyset$ in your life. But you do use 9 from time to time and you will be able to use names without noticing much of the above recursive non-sense.

So what is a name and what it is for? Without an interpretation it is just a piece of meaningless crap. It gains value only when interpreted by a generic filter:

A interpretation of a $\mathbb{P}$-name $\dot{x}$ with respect to a generic filter G is defined as

$$\dot{x}_G = \{\dot{y}_G \colon \exists p \in G\ \langle \dot{y}, p\rangle \in \dot{x}\}.$$

Again, the pain of recursion... Let interpret the examples above:

- $\emptyset_G = \emptyset$, no doubt.
- $\dot{x}_G = \ldots$ Well, it depends. It depends on the one fact: whether $p \in G$. If yes, then $\dot{x}_G = \{\emptyset\}$. If not, $\dot{x}_G = \emptyset$. For example if $p = 1$, then $\dot{x}_G = \{\emptyset\}$ regardless of what G is.
- $\dot{y}_G$ can be $\emptyset$ or $\{\emptyset\}$ or $\{\emptyset, \{\emptyset\}\}$ depending on p, q, r and whether they belong to G or not. For example, imagine that $\{p, q, r\}$ is a maximal antichain in $\mathbb{P}$. Then we don't know if $\dot{y}_G$ is $\{\emptyset\}$ or $\{\emptyset, \{\emptyset\}\}$ but we know that $V[G] \models \dot{y}_G \neq \emptyset$. So, we don't know how $\dot{y}_G$ does exactly look like, but we have some piece of information about $\dot{y}_G$. This is a common situation when you do forcing:

you cannot fully understand the objects you are talking about, but you know enough to derive some conclusions.

A name is an element of the ground model. An interpretation of a name is an element of the forcing extension.

By $V^{\mathbb{P}}$ we will denote the class of all $\mathbb{P}$-names.

$$V[G] = \{\dot{x}_G \colon \dot{x} \in V^{\mathbb{P}}\}.$$

There is a way of thinking about names: elements of a $\mathbb{P}$-name are of the form $\langle \dot{y}, p\rangle$. You can think about it as an element $\dot{y}$ belonging to $\dot{x}$ *with probability* p. It doesn't make much sense, since p is an element of $\mathbb{P}$, not a real number. But somehow it grasps the sense of names. The generic filter transforms those amorphous things into objects which are concrete (particularly for observers from $V[G]$) but still quite inaccessible for observers from V.

Big part of forcing praxis is the management of names. Fortunately in many cases we are able to look at names in a quite concrete way.

Remark 1.1. Notice that $V^{\mathbb{P}}$ is really full of rubbish. For example, think about the names which will be interpreted as $\{\emptyset\}$ by each generic G. How many of them are there? Consider $\dot{x} = \{\langle \dot{z}, p\rangle\}$ and $\dot{y} = \{\langle \dot{x}, q\rangle, \langle \emptyset, 1\rangle\}$. Suppose that $p \perp q$. Then, regardless of what $\dot{z}$ is, $\dot{y}_G = \{\emptyset\}$. But look, the name $\dot{z}$ can be *any* $\mathbb{P}$-name. So, there are class many names which have to interpreted as $\{\emptyset\}$!

Finally, we will present two particularly important types of names.

Standard name for an object in V: let $x \in V$. The standard name $\check{x}$ for x is of the form

$$\{\langle \check{y}, 1\rangle \colon y \in x\}.$$

The point is that $\check{x}_G = x$ regardless of G. In fact if you look closer $\check{x}$ looks exactly like x modulo those artificial 1's we glued to elements of x (and to elements of its elements, etc).

Name for the generic. Consider the name

$$\dot{G} = \{\langle \check{p}, p\rangle \colon p \in \mathbb{P}\}$$

Then $\dot{G}_G = G$. This is fun, isn't it?

1.2. **The forcing extension.** Here we will only list what we will need further on, without proving anything. All the proofs you can find in `https://www.math.toronto.edu/sunger/ucla/LogicSummerSchool/ForcingNotes.pdf`

- $V \subseteq V[G]$ and $G \in V[G]$ (just look at the standard names and the name for the generic),
- $V[G]$ and V have the same ordinal numbers,
- $V[G]$ satisfies all the ZFC axioms (satisfied by V).

We will use the following (Main Forcing) Theorem:

Theorem 1.2. *The relation* $\Vdash$ *defined by*

$$p \Vdash \varphi(\dot{x}) \iff V[G] \models \varphi(\dot{x}_G) \text{ for each generic } G \ni p$$

is definable in V. *If* $V[G] \models \varphi(\dot{x}_G)$, *then there is* $p \in G$ *such that* $p \Vdash \varphi(\dot{x})$.

2. Tools.

A chaotic sequence of basic facts which will be useful, usually without proof.

Fact 2.1.
- *$p \Vdash \varphi$ if and only if $\{q \leq p \colon q \Vdash \varphi\}$ is dense below p,*
- *if $p \Vdash \varphi$ and $q \leq p$, then $q \Vdash \varphi$,*
- *if p does not force φ, then there is $q \leq p$ such that $q \Vdash \neg p$.*
- *if $V[G] \models \varphi$ (for each G), then $1 \Vdash \varphi$.*

Proof. It follows directly from the definition of $\Vdash$ and forcing theorems. $\square$

Proposition 2.2. *$V[G]$ is the minimal model such that $V \subseteq V[G]$ and $G \in V[G]$.*

Proof. $V[G]$ consists of objects of the form $\dot{x}_G$. To define such object, it is enough to have V (to define $\dot{x}$) and G (to interpret $\dot{x}$). And that's it. $\square$

Theorem 2.3 (Maximum Principle). *If $p \Vdash \exists x \ \varphi(x)$, then there is $\dot{x}$ such that $p \Vdash \varphi(\dot{x})$.*

We will use the above theorem all the time. It follows from the definition of $\Vdash$: we have that $p \Vdash \exists x \ \varphi(x)$ iff the set

$$\{q \leq p \colon \text{ exists } \dot{x}_q \ q \Vdash \varphi(\dot{x}_q)\}$$

is dense below p. We just have to clean $\dot{x}_q$'s and amalgamate them into one name, see e.g.

3. Cohen forcing

This is a good moment to force something and see what it is all about. In this section we will kill CH.

3.1. **Adding single Cohen real.** Define

$$\mathbb{C} = \{p \colon p \colon \omega \to \{0,1\}, \mathrm{dom}(p) \text{ is finite } \}$$

and order it with reverse inclusion: $p \leq q$ iff $p \supseteq q$. In particular, $1_{\mathbb{C}}$ is the empty function.

This is the famous Cohen forcing, mother of all forcing notions.

Let G be a $\mathbb{C}$-generic. Let $c = \bigcup G$ (of course, writing it, we present ourselves in $V[G]$). In fact c, which we will call the *generic real* will be more important than G itself. Note that since G is a filter, c is a function.

Let's be slow. Let's try to devise a name for c. First of all, denote by p^i_n, $n \in \omega$, $i \in \{0,1\}$, the function with domain $\{n\}$ and such that $p^i_n(n) = i$. (In other words: $p^i_n = \{\langle n, i\rangle\}$.)

$$\dot{c} = \{\langle \check{\langle n, i\rangle}, p^i_n \rangle \colon n \in \omega, i \in \{0,1\}\}.$$

First, notice that $\dot{c}$ is really a $\mathbb{C}$-name: it is collection of appropriate ordered pairs. Then, notice that $\dot{c}$ is a $\mathbb{C}$-name for an element of 2^ω: just because the domain of $\dot{c}$ consists of elements of $\omega \times \{0,1\}$ (like elements of 2^ω usually do). Finally, evaluate $\dot{c}_G$. Suppose $p \in G$ and $n \in \mathrm{dom}(p)$. Then $\dot{c}_G(n) = p(n)$ (as G can choose either p^0_n or p^1_n and she can choose only something smaller than p and thus something which p extends).

Let's play a little bit.

Example 3.1. Suppose that $\mathrm{dom}(p) = \{0,1,2\}$ and $p(0) = 0$, $p(1) = 1$. What p forces about c? Well

$$p \Vdash \dot{c}(0) = 0.$$

Also,

$$p \Vdash \dot{c}(1) = 1,$$

and so whenever $p \in G$ we have $V[G] \models c(0) = 0 \wedge c(1) = 1$. But, p knows nothing about $c(4)$ or $c(17)$. So, what p knows (i.e. forces) is that $\dot{c}$ extends p and this is more less all of its knowledge. We will see that we can deduce from it surprisingly many properties of c.

At this point we can make an important observation. At the beginning you might wonder why partial orders are so important in the forcing theory. Why it is partial order who induce forcing extension, not anything else? Now, in the case of $\mathbb{C}$ you see that the partial order is used to construct certain object (in this case c) and elements of the partial order (conditions) can be seen as approximations of this object. For most forcing notions (at least which you will see at this course) this is the case: we want to approximate some new object using a partial order.

Proposition 3.2. *$V[G] \models \mathrm{dom}(c) = \omega$.*

Proof. For $n \in \omega$ let $D_n = \{p \in \mathbb{C} \colon n \in \mathrm{dom}(p)\}$. It is plain to check that D_n is dense in $\mathbb{C}$ for every n (take p; if $n \in \mathrm{dom}(p)$ do nothing; otherwise take $p \cup \{\langle n, 0\rangle\}$). So, if G is a $\mathbb{C}$-generic, then $G \cap D_n \neq \emptyset$. But this exactly means that $n \in \mathrm{dom}(c)$. $\square$

We have just performed the so called *density argument*. This is what you do more less all the time doing forcing. We know nothing about the concrete values of c (like we know nothing about $c(0)$ in $V[G]$ since it depends on G). But! Some properties of c does not depend on G. These are the properties for which the density argument works. It is not an accidents that the most important feature of G is generecity: the ability to intersect every dense subset.

Let's come back to Example 3.1 and the question: how much a single condition p knows about c. As we pointed out, p *only* knows that $p \subseteq c$. But actually p knows much more. E.g. it knows that $\mathrm{dom}(c) = \omega$!

By Proposition 3.2 and Fact 2.1, we have that $p \Vdash \mathrm{dom}(\dot{c}) = \omega$ for every $p \in \mathbb{C}$. So, p knows that c is a total function even though it seems to have only finite information. The point is that p 'inherits' the knowledge of its extensions, provided this knowledge is spread in a dense way.

What else we can deduce about c using density arguments? It will be convenient to look at c in a slightly different way: from c we can get $C \subseteq \omega$ in the usual way: by taking $c^{-1}[1]$.

Proposition 3.3. *For each $N \in [\omega]^\omega \cap V$ we have $V[G] \models |C \cap N| = \infty$ and $|N \setminus C| = \infty$. [The jargon formulation: the $\mathbb{C}$-generic real is splitting]*

Proof. Fix N as above. Let $n \in \omega$. The set

$$D_n = \{p \in \mathbb{C} \colon \exists m > n \; m \in A \wedge p(m) = 1\}$$

is dense. So, for each n we have $G \cap D_n \neq \emptyset$. And it means that there are infinitely many m's such that $m \in C \cap N$ and so $C \cap N$ is infinite. By taking

$$D_n = \{p \in \mathbb{C} \colon \exists m > n \; m \in A \wedge p(m) = 0\}$$

we obtain the second part of the proposition. $\square$

The conclusion is that C does not belong to V. This is not a surprise as it was defined using generic. But the above proposition says that it is something *really* new.

3.2. **Adding many Cohen reals.** We have just added a new subset of ω (so, a new real). In fact adding this one, we added many more (can you see them?). But not too many. In particular, Continuum Hypothesis is not destroyed (provided it was modeled by V).

But if we know how to add one real, adding many of them is not so difficult Define

$$\mathbb{C}_{\omega_2} = \{p \colon p \colon \omega \cdot \omega_2 \to \{0,1\}, \mathrm{dom}(p) \text{ is finite } \}$$

and order it in the same way as in the case of $\mathbb{C}$.

Remark 3.4. Of course instead of ω_2 we could take any cardinal (or even ordinal) number κ.

Again, let $c = \bigcup G$ and again c is a function, $c\colon \omega \cdot \omega_2 \to \{0,1\}$. The proof that $\mathrm{dom}(c) = \omega \cdot \omega_2$ is the same as in the case of $\mathbb{C}$. Also, $p \Vdash \check{p} \subseteq \dot{c}$.

Now, for $\alpha < \omega_2$ define $c_\alpha \in 2^\omega$ by $c_\alpha(n) = c(\omega \cdot \alpha + n)$. The following is crucial:

Proposition 3.5. *For $\alpha < \beta < \omega_2$ we have*

$$V[G] \models c_\alpha \neq c_\beta.$$

Proof. Fix $\alpha < \beta < \omega_2$. Let

$$D^\alpha_\beta = \{p \in \mathbb{C}_{\omega_2}\colon \exists n\ p(\omega \cdot \alpha + n) \neq p(\omega \cdot \beta + n)\}.$$

It is clear that D^α_β is dense. Also, if $p \in D^\alpha_\beta$, then

$$p \Vdash \dot{c}_\alpha \neq \dot{c}_\beta.$$

For each $\mathbb{C}_{\omega_2}$-generic, $G \cap D^\alpha_\beta \neq \emptyset$ and we are done. □

It means that we have just added ω_2 new reals (elements of 2^ω) to V. But, Continuum Hypothesis is not yet completely dead. At this point all we know is that in $V[G]$ the value of $\mathfrak{c}$ is not smaller that ω_2 but ... which ω_2? The poset $\mathbb{C}_{\omega_2}$ is defined in V, so when I say "adding ω_2 reals" I really mean adding ω_2^V reals). How do we know that $V[G]$ will have the same opinion about ω_2? In other words, how do we know that adding all those new reals, we have not added a function $f\colon \omega_1 \to \omega_2$ which is a surjection and so which would witness that $\omega_2^V = \omega_1^{V[G]}$? (If it sounds absurd, notice that f is an element of $V[G]$ and ω_1 and ω_2 are elements of V. Also, see the next section for an example of a forcing notion which adds precisely such function).

Fortunately, there is a theorem we can use. But first recall that a partial order is *ccc* if it does not contain an uncountable antichain (where $A \subseteq \mathbb{P}$ is an antichain if $p \perp q$ whenever $p, q \in A$, $p \neq q$).

A piece of terminology: suppose A, B are elements of $V[G]$ and $f\colon A \to B$ is a function, again in $V[G]$. Suppose $\dot{f}$ is a $\mathbb{P}$-name for f and $\dot{a}$ ia $\mathbb{P}$-name for some $a \in A$. We say that p decides $f(a)$ if $p \Vdash \dot{f}(\dot{a}) = \dot{b}$ for some $\dot{b}$, a $\mathbb{P}$-name for an element of B.

Proposition 3.6. *For each f, a as above the set*

$$D = \{p \in \mathbb{P}\colon p \textit{ decides } f(a)\}$$

is dense.

Proof. Let $p \in \mathbb{P}$ and G be a $\mathbb{P}$-generic, such that $p \in G$. Of course there is $b \in B$ such that $V[G] \models f(a) = b$. So, there is $p' \in G$ forcing it, i.e. such that

$$p' \Vdash \dot{f}(\dot{a}) = \dot{b},$$

where $\dot{b}$ is a $\mathbb{P}$-name for b. Let $r \leq p, p'$. Then $r \leq p$ and

$$r \Vdash \dot{f}(\dot{a}) = \dot{b}.$$

□

And here is the tool we can put Continuum Hypothesis out of its misery.

Theorem 3.7. *If $\mathbb{P}$ is a ccc forcing notion, G is a $\mathbb{P}$-generic and κ is a cardinal number in V, then $V[G] \models \kappa^V = \kappa^{V[G]}$. (jargon formulation: ccc forcing notions do not collapse cardinals, in the lowersilesian dialect of Polish: forcingi ccc nie kolapsują kardynałów)*

Proof. Suppose in $V[G]$ there is a function $f\colon \lambda \to \kappa$, where $\lambda < \kappa$. Let $\dot{f}$ be a $\mathbb{P}$-name for such f. For each $\alpha < \lambda$ there is a set D_α of conditions deciding $f(\alpha)$. By the proposition above, each D_α is dense. For every α fix a maximal antichain $A_\alpha \subseteq D_\alpha$ (we can find such an antichain exactly because D_α is dense, by transfinite induction). For each $p \in A_\alpha$ let β^α_p be such that

$$p \Vdash \dot{f}(\check{\alpha}) = \check{\beta^\alpha_p}.$$

Now, since A_α is countable (by ccc), the set $K_\alpha = \{\beta^\alpha_p \colon p \in A_\alpha\}$ is countable. And notice that K_α lists *all* the possible values of $f(\alpha)$ (more precisely, if $\gamma \notin K_\alpha$, then there is no $p \in \mathbb{P}$ which can think that $f(\alpha) = \gamma$, by maximality of A_α). So, $K = \bigcup_{\alpha<\lambda} K_\alpha$ lists all the possible values f can ever attain, for any argument. But $|K| < \kappa$ and so there is no way f can be surjective. □

Of course $\mathbb{C}$ is ccc (as it is just countable). But what about $\mathbb{C}_{\omega_2}$?

Proposition 3.8. *$\mathbb{C}_\kappa$ is ccc (for any κ).*

Proof. Consider $(p_\alpha)_{\alpha<\omega_1} \subseteq \mathbb{C}_\kappa$. By Δ-system lemma, we may assume that $\mathrm{dom}(p_\alpha)$ forms a Δ-system, i.e. there is (a root) r such that $\mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta) = r$ for every $\alpha \neq \beta$ (just throw out anything which is not in the Δ-system). Again, without loss of generality we may assume that for each α, β, p_α agrees with p_β on r (as there are only finitely many possible shapes of a condition on r and we have ω_1 conditions on the table). But now, what we have is very far from being an antichain: just take α, β and let $r = p_\alpha \cup p_\beta$. Then r is an element of $\mathbb{C}_{\omega_2}$ (it is a function as p_α and p_β agrees on the intersection of their domains) and r witnesses that p_α is compatible with p_β. □

That's it. Travesting the classic: Das Continuum Hypothese ist tot.

4. Few further illuminating examples of forcing notions

From this point on we will assume that the Reader is already mature enough and we will skip writing 'checks' to indicate standard names.

4.1. **Resurrecting Continuum Hypothesis, the collapsing forcing.** This time we suppose that $V \models \mathfrak{c} = \omega_2$. Can we force the Continuum Hypothesis? Of course we cannot throw out real numbers from V. So, the only way is to add a surjection $f\colon \omega_1^V \to \omega_2^V$. This is exactly the function we wanted to avoid in the previous section, when we were killing CH.

How to add such a function? We will do it quite brutally: we will consider a partial order consisting of approximations of such function:

$$\mathbb{P} = \{p\colon p\colon \omega_1 \to \omega_2, \mathrm{dom}(p) \text{ is countable}\}.$$

Why we use functions of countable domains and not of finite domains? You will see in a moment...

Let G be a $\mathbb{P}$-generic and let $f = \bigcup G$. As in the case of Cohen forcing, f is a function (since G is a filter) and $\mathrm{dom}(f) = \omega_1$ (just use the same argument as in the case of the Cohen forcing). The analogous argument shows that $\mathrm{rng}(f) = \omega_2$. Just notice that $D_\beta = \{p\colon \beta \in \mathrm{rng}(p)\}$ is dense. Look, we didn't even try to impose surjectivity of the generic function in the definition of $\mathbb{P}$. It comes for free.

So, we managed to collapse ω_2 to ω_1. This is not the end. We have to check if we haven't added too many new real numbers. If we add ω_3 new reals, then it might happen that $\omega_3^V = \omega_2^{V[G]}$ and so the continuum is still big. Fortunately, $\mathbb{P}$ doesn't add *any* real numbers.

Definition 4.1. A forcing notion $\mathbb{P}$ is σ-closed if for every countable decreasing sequence $p_0 \geq p_1 \geq p_2 \geq \dots$ there is p such that $p \leq p_n$ for each n.

Theorem 4.2. *If $\mathbb{P}$ is σ-closed, then $\mathbb{P}$ does not add real numbers, i.e. for each $\mathbb{P}$-name $\dot{r}$ for a real number and each $p \in \mathbb{P}$ there is $q \leq p$ and $x \in 2^\omega \cap V$ such that $q \Vdash \dot{r} = x$.*

Proof. Let $\dot{r}$ be a name for an element of 2^ω and let $p \in \mathbb{P}$.

Let $p_0 \leq p$ and $i_0 \in \{0,1\}$ be such that $p_0 \Vdash \dot{r}(0) = i_0$ (recall, from the previous section that the set of conditions deciding a value of a function is dense). Let $p_1 \leq p_0$ be such that $p_1 \Vdash \dot{r}(1) = i_1$ and so on. As $\mathbb{P}$ is σ-closed, there is q such that $q \leq p_n$ for each n. Let $x \in 2^\omega \cap V$ be defined by $x(n) = i_n$. Then $q \Vdash \forall n\ \dot{r}(n) = x(n)$. □

It is plain to check that our $\mathbb{P}$ collapsing ω_2 is σ-closed. So, that's it. We have CH in $V[G]$ (where G is $\mathbb{P}$-generic).

Remark 4.3. Notice that if we take functions of finite domains in the definition of $\mathbb{P}$, then we would not have σ-closedness and we could not use the above theorem.

Remark 4.4. Such $\mathbb{P}$ as above is usually denoted by $\mathrm{Coll}(\omega_1, \omega_2)$. Of course you can define analogously $\mathrm{Coll}(\kappa, \lambda)$ for other cardinals κ, λ.

Actually, this is not the end of the story. The fascinating thing about forcing is that the partial order used to approximate some object (as in the case of $\mathbb{P}$) brings us not only the approximated object but the whole set theoretic universe. So, many times, after forcing something you obtain a lot of surprising bonuses. Here is an example. The poset $\mathbb{P}$ not only forces CH, but much more: the diamond principle.

Recall that the diamond principle ($\diamondsuit$) says that there is a sequence $(A_\alpha)_{\alpha<\omega_1}$ such that

- $A_\alpha \subseteq \alpha$ for each $\alpha < \omega_1$,
- for each $A \subseteq \omega_1$, the set $\{\alpha \colon A \cap \alpha = A_\alpha\}$ is stationary (i.e. it meets every closed unbounded (club) subset of ω_1).

$\diamondsuit$ is much stronger than CH (it holds under $V = L$). It is a strong guessing principle implying the existence of many pathological objects.

Theorem 4.5. *Let $\mathbb{P} = \mathrm{Coll}(\omega_1, \omega_2)$ and let G be $\mathbb{P}$-generic. Then $V[G] \models \diamondsuit$.*

Proof. First, for each $\omega \leq \alpha < \omega_1$ fix a bijection $g_\alpha \colon [\alpha, \alpha + \omega) \to \alpha$.

Now work in $V[G]$. Let f be the generic function. Let

$$A_\alpha = g_\alpha[f^{-1}[\{0\}] \cap [\alpha, \alpha + \omega)].$$

(Here, we just want to produce a 'generic' subset of α using f. Instead of $f^{-1}[\{0\}]$ we could take $f^{-1}[(17, 23)\}]$ or whatever.)

We will show that $(A_\alpha)_{\alpha<\omega_1}$ is a guessing sequence.

Now, we step down to V. Let $\dot{A}$ be a name for a subset of ω_1 and let $\dot{C}$ be a name for a club subset of ω_1. Start with $p_0 \in \mathbb{P}$. We may assume that $\mathrm{dom}(p) = \beta$ for some $\beta < \omega_1$ (density). In fact we will assume that domains of all the conditions below will be of this form (i.e. will be initial segments of ω_1).

Now let p_{n+1} be such that p_{n+1} decides all the values of $\dot{A}$ and $\dot{C}$ on $\mathrm{dom}(p_n)$ (we can find one by density and σ-closedness). Moreover, we want to have $\alpha_n \in \mathrm{dom}(p_{n+1}) \setminus \mathrm{dom}(p_n)$ such that $p_{n+1} \Vdash \alpha_n \in \dot{C}$ (we use that C is unbounded and we extend p_{n+1} if necessary).

Now let $p = \bigcup p_n$ and let $\alpha = \mathrm{dom}(p)$. Then $p \in \mathbb{P}$, p knows everything about $\dot{A} \cap \alpha$ and $\dot{C} \cap \alpha$, and $p \Vdash \alpha \in \dot{C}$ (here we use the fact that C is closed and $\alpha = \lim_n \alpha_n$).

Now notice that we can extend p to q, where $\mathrm{dom}(q) = \alpha + \omega$. And we can do *everything* we want on $[\alpha, \alpha + \omega)$! In particular we can do the following: if n is such that $p \Vdash g_\alpha(n) \in \dot{A}$, then let $q(\alpha + n) = 0$. Otherwise, let $q(\alpha + n)$ be anything but 0. Then $q \Vdash \dot{A} \cap \alpha = \dot{A}_\alpha$. $\square$

4.2. **Mathias-Prikry forcing.** Now we will show an example of forcing notion with elements of slightly more complicated form.

We will consider filters on ω and we will assume they do not contain finite sets. First, recall that P is a pseudointersection of a filter $\mathcal{F}$ on ω if $P \subseteq^* F$ for each $F \in \mathcal{F}$. Examples of filters without infinite pseudointersections are: ultrafilters, the density filter, the summable filter (i.e. the filter dual to the summable ideal). In general, filters containing a finite set are those we consider completely non-interesting. Filters with an infinite pseudointersection are only slightly higher in the hierarchy. Those really interesting are filters without infinite pseudointersection (the dual ideals of such filters are called tight or dense).

The Mathias-Prikry forcing for a filter $\mathcal{F}$ is defined in the following way:

$$\mathbb{M}(\mathcal{F}) = \{\langle s, F\rangle \colon s \in [\omega]^{<\omega}, F \in \mathcal{F}, \max s < \min F\}.$$

We say that $\langle s', F'\rangle \leq \langle s, F\rangle$ if

- $s \subseteq s'$,
- $F' \subseteq F$,
- $s' \setminus s \subseteq F$.

Let G be a $\mathbb{M}(\mathcal{F})$-generic. Let $S = \bigcup\{s \colon \exists F \in \mathcal{F}\ \langle s, F\rangle \in G\}$.

Proposition 4.6. *$V[G] \models S$ is infinite.*

Proof. Exercise. □

Proposition 4.7. *$V[G] \models S$ is a pseudointersection of $\mathcal{F}$ (jargon formulation: $\mathbb{M}(\mathcal{F})$ destroys (or diagonalizes) $\mathcal{F}$).*

Proof. Let $p = \langle s, F\rangle \in \mathbb{M}(\mathcal{F})$. Suppose $F_0 \in \mathcal{F}$. Let $q = \langle s, F \cap F_0\rangle$. Notice that $q \in \mathbb{M}(\mathcal{F})$ and $q \leq p$. But

$$q \Vdash \dot{S} \setminus \max s \subseteq F_0.$$

So, by density argument for each $F_0 \in \mathcal{F}$, $1 \Vdash \dot{S} \subseteq^* F_0$ and we are done. □

So this is how it works: conditions of $\mathbb{M}(\mathcal{F})$ have two coordinates. The first one is a *working* part of the condition: it will be used to approximate the desired object. The second one is to guide the working part of the condition in the way we want.

What are the properties of $\mathbb{M}(\mathcal{F})$? It is usually nice to know that the forcing we are using does not too much damage to the universe (e.g. that it does not collapse ω_1). The forcing $\mathbb{M}(\mathcal{F})$ is ccc, so it won't. Actually it has even stronger property.

Definition 4.8. A partial order $\mathbb{P}$ is σ-centered if $\mathbb{P} = \bigcup_n \mathbb{P}_n$, where each $\mathbb{P}_n$ is centered (meaning: for every finite $p_0, \ldots, p_k \in \mathbb{P}_n$ there is $p \in \mathbb{P}_n$ such that $p \leq p_0, \ldots, p_k$).

It is easy to see that each σ-centered poset is ccc.

Proposition 4.9. *$\mathbb{M}(\mathcal{F})$ is σ-centered.*

Proof. Just let $\mathbb{P}_s = \{\langle s, F\rangle \colon F \in \mathcal{F}, \min F > \max s\}$ for $s \in [\omega]^{<\omega}$. It is centered since $\mathcal{F}$ is $\subseteq$-centered. □

4.3. **Adding dominating real.** We say that a function $f \in \omega^\omega \cap V[G]$ is *dominating* if for each $g \in \omega^\omega \cap V$ we have $g \leq^* f$. Before reading the definition below try to inspire yourself by the Mathias-Prikry forcing and to invent a forcing for adding a dominating function by yourself.

This is what we invented ad hoc during the lecture. It can be done in other ways (see e.g. Hechler forcing).

Define

$$\mathbb{P} = \{\langle s, g\rangle \colon s, g \colon \omega \to \omega, |\mathrm{dom}(s)| < \omega, |\omega \setminus \mathrm{dom}(g)| < \omega, \mathrm{dom}(s) \cap \mathrm{dom}(g) = \emptyset\}.$$

Let $\langle s', g'\rangle \leq \langle s, g\rangle$ iff

- $s \subseteq s'$,
- $g(n) \leq g'(n)$ if $n \in \mathrm{dom}(g')$
- $s'(n) \leq g(n)$ for $n \in \mathrm{dom}(g)$.

Let $f = \bigcup\{s \colon \exists g \in \omega^\omega, \langle s, g\rangle \in G\}$.

Proposition 4.10. $V[G] \models f$ *is dominating.*

Proof. Exercise. □

4.4. **Forcing with Suslin tree.** So you have seen already several examples of forcing notions. All of them were produced to construct some concrete object (new real number, surjection $f \colon \omega_1 \to \omega_2$, a pseudo-intersection of a filter, ...). But you can take any partial order and ask, what kind of universe it produces!

Let's take for example a Suslin tree. It forms a forcing notion (in this sense tree grows from up to down). Recall that a tree is Suslin if it is of height ω_1, it is ccc and it does not have a cofinal branch. It is a stronger version of Aronszajn tree. Suslin conjectured that there are no Suslin trees and for many years it was an open problem if the Suslin Hypothesis is true. Finally, it turned out to be independent. Btw, Suslin tree is one of the objects guaranteed by $\diamondsuit$ and in fact $\diamondsuit$ was invented by Ronald Jensen exactly to show that consistently there is a Suslin tree.

There are different Suslin trees, some of them have nodes which cannot be extended. We don't want to force with those (as then for some generic filters G, we would have $V = V[G]$). So, we take a Suslin tree and we add branches here and there to obtain a *normal* Suslin tree $\mathbb{S}$, so a tree which is separative and such that for each $p \in \mathbb{S}$ and $\alpha < \omega_1$ there is $q \leq s$ such that q is from the α's level. (That you can normalize a Suslin tree, you can read e.g. in Jech's book).

Now let G be a $\mathbb{S}$-generic. For $\alpha < \omega_1$ let $D_\alpha = \{p \in \mathbb{S} \colon \mathrm{level}(p) > \alpha\}$ and notice that D_α is dense. So $G \cap D_\alpha \neq \emptyset$ for each α and since G is a filter, it means that G produces a branch in $\mathbb{S}$ of length ω_1. In the jargon we called it *shooting a branch through* $\mathbb{S}$). In a sense many forcing notions is shooting branches through some structures.

Shooting a branch through a Suslin tree makes it no Suslin. Of course there may be many other Suslin trees around but we can kill them as well in the same manner.

This is the beginning of the story of Martin's Axiom and proving the consistency of Suslin Hypothesis. We will (hopefully) come back to it later.

5. Forcing equivalence and forcing with Boolean algebras

As you can guess, some partial orders may give the same forcing extensions (e.g. if they are isomorphic). We say that $\mathbb{P}$ and $\mathbb{Q}$ are *forcing equivalent* if for each $\mathbb{P}$-generic G there is a $\mathbb{Q}$-generic H such that $V[G] = V[H]$ and vice versa.

We say that $i\colon \mathbb{P} \to \mathbb{Q}$ is a *dense embedding* if

- $p \leq p' \iff i(p) \leq i(p')$,
- $p \perp p' \iff i(p) \perp i(p')$ and
- $i[\mathbb{P}]$ is dense in $\mathbb{Q}$.

Notice that I used "$\leq$" and "$\perp$" above in two different meanings (once this is $\leq$ in $\mathbb{P}$, once in $\mathbb{Q}$). I hope there is no confusion.

The main theorem here:

Theorem 5.1. *If $\mathbb{P}$ can be densely embedded in $\mathbb{Q}$, then $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent.*

The proof is rather simple (and boring). I will sketch the basic idea omitting proofs of simple claims.

Proof. Let G be a $\mathbb{P}$-generic. Define

$$H = \{q \in \mathbb{Q}\colon \exists p \in G\ i(p) \leq q\}.$$

We have to check that H is a filter (easy) and that is generic (easy).

Let's check that $V[H] \subseteq V[G]$. To do that, it is enough to show that $H \in V[G]$ (by Proposition 2.2). And, well, this is kind of obvious - H is defined by G. So, let's check that $G \in V[H]$ (and so we will have that $V[G] = V[H]$). So we have to define G by H. Here you are: $G = \{p \in \mathbb{P}\colon \exists q \in H\ q \leq i(p)\}$. That "$\subseteq$" part is obvious. So, suppose that p is such that $q \leq i(p)$ for $q \in H$. The set $D = \{r \in \mathbb{P}\colon r \leq p \vee r \perp p\}$ is dense in $\mathbb{P}$. So there is $r \in G \cap D$. But it cannot happen that $r \perp p$ (then $i(r) \perp i(p)$ but both $i(r) \in H$ and $i(p) \in H$). So $r \leq p$ and so $p \in G$.

Now, suppose that H is a $\mathbb{Q}$-generic. Define

$$G = i^{-1}[H].$$

Again, we have to show that it is a filter: fix $p_0, p_1 \in G$. Then $i(p_0), i(p_1) \in H$. Consider a maximal antichain A of conditions q with the property: $q \leq i(p_0), i(p_1)$, $q \in i[\mathbb{P}]$. The genericity implies that there is $q \in A$ such that $q \in H$. Then there is $p \in \mathbb{P}$ such that $i(p) = q$. Of course $p \in G$ and $p \leq p_0, p_1$.

The proofs of genericity of G and that $V[G] = V[H]$ are analogous to the above. $\square$

5.1. Boolean algebras of regular open sets.

Recall that each Boolean algebra $\mathbb{B}$ can be treated as a partial order. So, we can force with it (of course we rather force with $\mathbb{B} \setminus \{0\}$ than $\mathbb{B}$ itself, to avoid trivialities).

What is interesting is that every partial order has his own complete Boolean algebra which is forcing equivalent with it.

Let $\mathbb{P}$ be a partial order and let $p \in \mathbb{P}$. By $[p]$ we will denote the *cone* generated by p, i.e. $[p] = \{q \colon q \leq p\}$. For us the default topology on a partial order $\mathbb{P}$ is the topology generated by cones.

We say that $A \subseteq X$, where X is a topological space, is *regular open* if $\mathrm{Int}(\overline{A}) = A$. Of course every regular open set is open. An example is $(0,1)$ (as a subset of $\mathbb{R}$ with euclidean topology). An anti-example is $(0,1) \cup (1,2)$.

Now, let $RO(\mathbb{P})$ denote the set of all regular open subsets of $\mathbb{P}$ (with the default topology). We can endow $RO(\mathbb{P})$ with a Boolean structure as follows: $A \vee B = \mathrm{Int}(\overline{A \cup B})$, $A \wedge B = \mathrm{Int}(\overline{A \cap B})$, $-A = \mathrm{Int}(\overline{A^c})$. Actually, this is always a complete algebra: $\bigvee \mathcal{A} = \mathrm{Int}(\overline{\bigcup \mathcal{A}})$. (The proof of completeness is not very romantic. Fortunately you can find it in many places, e.g. [1, Theorem 2.3.10]). We are not going to check any of these statements, but note that we need $\mathbb{P}$ to be separative if we don't want to have problems with defining complement.

Consider now the function $i \colon \mathbb{P} \to RO(\mathbb{P})$ defined by $i(p) = [p]$. Does this definition make sense? This is the point, where we again need that $\mathbb{P}$ is separative. Imagine, that there is $p, q \in \mathbb{P}$ such that q is the only immediate successor of p (and so $\mathbb{P}$ is not separative). Then $p \in \overline{[q]}$ and also $p \in \mathrm{Int}(\overline{[q]}$ and thus $[q]$ is not regularly open. If $\mathbb{P}$ is separative, then this kind of problems disappear and i is well defined. It is clearly an embedding and since cones forms the base of topology it is dense.

Theorem 5.2. *Forcing notions $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent iff $RO(\mathbb{P})$ is isomorphic to $RO(\mathbb{Q})$.*

Proof. The reverse implication is a direct corollary of Theorem 5.1. □

5.2. ***Forcing is just the Baire theorem.*** There is an urban legend in Wrocław about Ryll-Nardzewski: when the manuscript with Cohen's proof of independence of Continuum Hypothesis came to Wrocław, Ryll-Nardzewski took it home for the evening. Next day he came to the Institute and commented "Well, it is just the Baire theorem".

Let us comment on his comment. Let M be a countable transitive model of ZFC. Consider the Cohen forcing $\mathbb{C} \in M$. This time we may look at it as the Boolean algebra of clopen subsets of 2^ω since the 'usual' Cohen forcing as defined before embeds densely in $\mathbb{C}$:

$$i(p) = \{x \in 2^\omega \colon x \text{ extends } p\}.$$

If $D \subseteq \mathbb{C}$ is a dense set, then $D' = \bigcup D$ is a dense open subset of 2^ω.

By Baire theorem, there is $x \in 2^\omega$ such that $x \in \bigcap \{D' \colon D \in M \text{ is dense}\}$. Such x is $\mathbb{C}$-generic over M, when viewed as an ultrafilter on the Cantor algebra. So, Rasiowa-Sikowski Lemma is really just the Baire theorem.

5.3. **Truth values.** We will force with a Boolean algebra $\mathbb{B}$ (and you already understand that we are abusing notation here and indeed we force with $\mathbb{B} \setminus \{0\}$).

One of the biggest advantage of forcing with a Boolean algebra is the existence of so called truth values.

Definition 5.3. For a sentence φ, the truth value of φ is defined by

$$[\![\varphi]\!] = \bigvee \{p \in \mathbb{B} \colon p \Vdash \varphi\}.$$

Recalling the analogy with probability, the truth value of φ refers to the 'probability' of the fact that φ holds (with all the reservations I mentioned when I introduced the definition of the generic interpretations of names). As we will see in a moment using truth values makes our life easier. First, notice the following fact.

Proposition 5.4.

$$p \Vdash \varphi \qquad \iff p \leq [\![\varphi]\!].$$

Proof. ($\Longrightarrow$) is clear.

To see the reverse implication, just notice that $[\![\varphi]\!] \Vdash \varphi$ (then clearly $p \Vdash \varphi$). Indeed, if not, then (by Proposition 2.1) we would find $q \leq [\![\varphi]\!]$ such that $q \Vdash \neg\varphi$. But then $[\![\varphi]\!]$ would not be the smallest upper bound for $\{p \colon p \Vdash \varphi\}$, a contradiction. □

Using the above fact may reduce the need to use the symbol "$\Vdash$". In general, the truth values enable more brainless calculations, like:

Proposition 5.5.

$$[\![\exists n \in \omega \ \ \varphi(n)]\!] = \bigvee_n [\![\varphi(n)]\!],$$

$$[\![\neg\varphi]\!] = [\![\varphi]\!]^c$$

Proof. Exercise. □

A nice application of truth values is the form of names. Consider a name $\dot{N}$ for a subset of ω. Define

$$\overline{N} = \{\langle n, [\![n \in \dot{N}]\!]\rangle \colon n \in \omega\}.$$

Notice that $\overline{N}$ is a $\mathbb{B}$-name for a subsets of ω. In fact:

Proposition 5.6.

$$1 \Vdash \dot{N} = \overline{N}.$$

Proof. Take $n \in \omega$, $p \in \mathbb{B}$ and suppose that $p \Vdash n \in \dot{N}$. Then, by the definition of truth value, $p \leq [\![n \in \dot{N}]\!]$. So, whenever a generic G chooses p (and so he thinks that $n \in \dot{N}$), it has to choose $[\![n \in \dot{N}]\!]$ (and so he has to think that $n \in \overline{N}$).

Similarly, you can prove that $1 \Vdash \overline{N} \subseteq \dot{N}$. □

By the above proposition, when you consider a name for a subset of ω, you are allowed to think that it is in the above form. Quite soon we will see a concrete example of such names.

5.4. **Cohen forcing once again: Cohen algebra.** In Subsection 5.2 we have seen that forcing with the (traditional) Cohen forcing $\mathbb{C}$ is equivalent to forcing with the Cantor algebra. But the Cantor algebra is very far from being complete, so it is not $RO(\mathbb{C})$. So what is $RO(\mathbb{C})$?

There are many ways of approaching this question.

First, notice that $RO(\mathbb{C})$ is isomorphic to $RO(2^\omega)$, the Boolean algebra of regular open subsets of 2^ω. First, use i defined in Subsection 5.2 and then extend it to isomorphism (by using e.g. Sikorski theorem or by noticing that there is no moral difference between open sets in $\mathcal{C}$ and open sets in 2^ω as both are unions of cones, although formally those cones are very different).

To discover what is $RO(2^\omega)$ recall the following theorem.

Theorem 5.7. *For every Borel $B \subseteq 2^\omega$ there is a regular open set U and a meager set M such that $B = U \triangle M$.*

You may know the above theorem formulated for open sets instead of regular open (it says that every Borel set has the Baire property). But if you look at the proof of it, in fact you have regularity for free. (Or you can notice that every open set is regular open modulo a meager set).

Using the above, and a little glance at what is happening with the operations, we have:

Theorem 5.8. *$RO(\mathbb{C})$ is isomorphic to $\mathrm{Bor}(2^\omega)/\mathcal{M}$.*

Also, we have the following corollary.

Corollary 5.9. *$\mathbb{C}$ is forcing equivalent to $\mathrm{Bor}(2^\omega) \setminus \mathcal{M}$.*

Proof. The natural embedding of

$$i \colon \mathrm{Bor}(2^\omega) \setminus \mathcal{M} \to .\mathrm{Bor}(2^\omega)/\mathcal{M}$$

is dense. □

So, if you want to use Cohen forcing, you may look at it as $\mathbb{C}$, $\mathrm{Bor}(2^\omega) \setminus \mathcal{M}$, $\mathrm{Bor}(2^\omega)/\mathcal{M}$ or e.g. any separative countable partial order.

Now we can take a second look at Proposition 5.6. If you want to consider a $\mathbb{C}$-name for a subsets of ω, you can think about the sequences of Borel subsets of 2^ω! It is a good exercise to consider various sequences of Borel subsets and check what kind of names it gives us.

6. Coding things and more about names

Let's contemplate subsets of 2^ω. Usually in $V[G]$ we have many new subsets of 2^ω. E.g. if $\mathbb{P}$ adds reals, then 2^ω is not an element of V. But on the other hand it is difficult to treat 2^ω as something really new. This is *just the set of all real numbers*. It is new but it can be *coded* in the ground model. Usually, the codes are real numbers, in this or that form, treated as a recipe (for a set or a function).

6.1. **Codes for sets and functions.** If we want to start our adventure with codes for subsets of 2^ω, it is better to start with the simplest case: open sets. Start with enumerating all the basic clopen subsets of 2^ω (and let it be induced by the simplest possible enumeration $e\colon 2^{<\omega} \to \omega$). Let $U \subseteq 2^\omega$ be open. Then $U = \bigcup\{[s]\colon s \in S\}$ for certain $S \subseteq 2^{<\omega}$. Then $c = e[S]$ is a subset of ω (so a real). We will treat it as a code for U. Any model (containing c) can take it and decode it (knowing e and knowing the procedure).

In this way we can speak about open subsets of 2^ω *coded in* V. These are exactly those open sets whose codes are elements of V. E.g. 2^ω is definitely coded in V (as it is coded e.g. by ω).

Now, try to code a continuous function $f\colon 2^\omega \to 2^\omega$. If $s \in 2^{<\omega}$, then $f^{-1}[[s]]$ is an open set, so it can be coded by a real, as above. Using the enumeration e we can produce a sequence of reals coding all the preimages of basic clopen sets. Of course you can encode this sequence in a single real. This real will have full information about f and you can decode it in every model containing this real.

You see, it is not so important *how* exactly you encode things, as long as you use computable methods.

Once you can encode open sets, you can encode the closed ones. Also, you can encode G_δ sets: these are just countable intersections of open sets. At this point you may think that you can encode all the Borel sets and you would be right. However, this is a more delicate and descriptive set theoretic matter as the Borel hierarchy is longer than ω.

There are several approaches to Borel codes. You can proceed as above and try to code Borel set inductively. Or you may try to use the universal Π^1_1 set. See e.g. [2, Section 29], https://arxiv.org/pdf/math/9401202.pdf.

A piece of notations: if $c \in 2^\omega$ is a code for an object, then by $c^{\#M}$ we will mean this object decoded in a model M. Actually, in the literature usually the difference between Borel sets and their codes is ignored, but for some time I will try to remember about $\#$'s.

What is important is that the set of Borel codes is Π^1_1 and so (by Schoenfiled absolutness theorem) it is absolute. This means that whatever we can say about the decoded set in the ground model, provided we say it in the Π^1_1 way, it will be true in the extension. For example

- $z^{\#V} = x^{\#V} \cup y^{\#V}$ if and only if $z^{\#V[G]} = x^{\#V[G]} \cup y^{\#V[G]}$,

- $V \models x^{\#V}$ is meager if and only if $V[G] \models x^{\#V[G]}$ is meager,

and so on...

6.2. **Continuous reading of names for the Cohen forcing.** We will treat $\mathbb{C}$ as $\mathrm{Bor}(2^\omega)/\mathcal{M}$. Let G be a generic. Consider the following $\mathbb{C}$-name:

$$\dot{c} = \{\langle\langle n, i\rangle, [\{x \in 2^\omega : x(n) = i\}]_{\mathcal{M}}\rangle : n \in \omega, i \in 2\}.$$

Perhaps you recognize echoes of the definition of the generic real from Section 3.1. Now, we have the following:

Proposition 6.1. *Suppose $b \in V$ is a code for a Borel set. Then*

$$[\![\dot{c} \in b^{\#V[G]}]\!] = [b^{\#V}]_{\mathcal{M}}.$$

Proof. Suppose b is a code for a basic clopen set. Then the above holds because of the form of $\dot{c}$ (see Example 3.1) The rest follows from the fact that basic clopen sets generate (in the sense of σ-algebra) all Borel sets, Proposition 5.5 or something like this, and the absoluteness of codes. □

The above proposition has the following bone-shaking corollary:

Proposition 6.2. $1 \Vdash \dot{c}$ *omits all the meager subsets of* 2^ω *coded in* V*.*

Proof. If b codes a meager subset of 2^ω, then by Proposition 6.1 $[\![\dot{c} \in b^{\#V[G]}]\!] = 0$. It means that $1 \Vdash \dot{c} \notin b^{\#V[G]}$. □

Remark 6.3. In fact the above can be proved also in the setting of Section 3.1, see the problem list.

We are ready for the definition of the Cohen real:

Definition 6.4. A real $r \in M$ is a Cohen real over $N \subseteq M$ if it omits all the meager sets coded in N (or, equivalently, it belongs to all the comeager sets coded in N).

(If you think about $\mathbb{C}$ as of $\mathrm{Bor}(2^\omega) \setminus \mathcal{M}$ you may define the generic real in one more way:

$$\{c\} = \bigcap\{b^{\#V[G]} : b \text{ is a code for a Borel set such that } b^{\#V} \in G\}.$$

This intersection is nonempty because certain Cohen real (this which 'follows' G) will be there and that it cannot have more than 1 element follows from genericity of G).

A moral comment: if you have understood shooting a branch through the Suslin tree, you may look at the above in the similar way. Forcing with $\mathrm{Bor}(2^\omega)/\mathcal{M}$ is like shooting a branch through all the comeager subsets of 2^ω. And then this branch induces a real.

6.3. **An application: consistency of $\mathfrak{t} < \mathfrak{c}$.** We have already seen that looking at the Cohen forcing as $\mathrm{Bor}(2^\omega) \setminus \mathcal{M}$ provided a nice names for subsets of ω. Now we will see yet another approach to Cohen names for reals, which is even nicer.

Theorem 6.5. *Suppose that $\dot{x}$ is a $\mathbb{C}$-name for an element of 2^ω. Then there is a comeager $D \subseteq 2^\omega$ and a continuous function $f \colon D \to 2^\omega$ (and $f, D \in V$) such that $V[G] \models f(\dot{c}) = \dot{x}$.*

Literally, the above does not make sense. If $f \in V$, then we cannot evaluate it on $\dot{c}$. So, what we really mean is that there is a ground model continuous function coded as f (and a Borel set coded as D) then decoding it in $V[G]$ we got $f^{\#V[G]}(\dot{c}) = \dot{x}$.

Proof. We treat $\mathbb{C}$ as $\mathrm{Bor}(2^\omega) \setminus \mathcal{M}$. Let $U^i_n = [\![\dot{x}(n) = i]\!]$. We may assume that U^i_n is open (by Theorem 5.7) and let $D = \bigcap_n (U^0_n \cup U^1_n)$. For every n $U^0_n \cup U^1_n$ is dense (otherwise the generic would not decide the value of $\dot{x}(0)$. Then D is a dense G_δ.

Define $f \colon D \to 2^\omega$ by $f(x)(n) = i$ if $x(n) \in U^i_n$. Then f is continuous (just notice that $f^{-1}[\{x \in 2^\omega \colon x(n) = i\}] = U^i_n$). Now, jump to $V[G]$. Let c be the generic real and let $n \in \omega$. Then $f(c)(n) = i$ iff $U^i_n \in G$. But $U^i_n \in G$ iff $1 \Vdash \dot{x}(n) = i$. Therefore $1 \Vdash f(\dot{c}) = \dot{x}$. □

The above theorem says that instead of thinking about $\mathbb{C}$-names for a reals you may think about ... continuous functions. Let's play a little bit. Consider the function $f \colon 2^\omega \to 2^\omega$ defined by $f(x) = r$. What is the name associated with f? Just evaluate this function on the generic real c and you get nothing else but r. What if $f = r\chi_C + s\chi_{C^c}$ (where C is a clopen subset of 2^ω)? Then it depends on the generic, but whatever it is, it is either r or s. What if f is the identity? Well, the evaluation is quite easy.

Ok, so now let's prove something more serious. Recall that $(T_\alpha)_{\alpha<\kappa}$ is a tower if it is $\subseteq^*$-decreasing and it has no (infinite) pseudo-intersection.

Theorem 6.6. *Let G be a $\mathbb{C}$-generic. Suppose that $(T_\alpha)_{\alpha<\kappa}$ is a tower (in V). Then $V[G] \models (T_\alpha)_{\alpha<\kappa}$ is still a tower.*

In other words, $\mathbb{C}$ does not destroy towers. The following proof is due to James Hirschorn.

Proof. Suppose not, i.e. suppose that

$$V[G] \models \exists T \in [\omega]^\omega \ \forall \alpha < \ T \subseteq^* T_\alpha.$$

Then

$$1 \Vdash \exists \dot{T} \in [\omega]^\omega \ \forall \alpha < \kappa \ \dot{T} \subseteq^* T_\alpha.$$

By Maximum Principle (see ...) there is a name $\dot{T}$ for an infinite subset of ω such that

$$1 \Vdash \forall \alpha < \kappa \ \dot{T} \subseteq^* T_\alpha.$$

Now, by Theorem 6.5, there are f, D coded in V such that $f \colon D \to [\omega]^\omega$ and $f(\dot{c}) = \dot{T}$.

Then for each $\alpha < \kappa$ the set

$$B_\alpha = \{x \in D \colon f(x) \subseteq^* T_\alpha\}$$

is comeager. Indeed,

$$[\![\dot{c} \in B_\alpha]\!] = [B_\alpha]_{\mathcal{M}}.$$

And $[\![\dot{c} \in B_\alpha]\!] = 1$ since $1 \Vdash f(\dot{c}) = \dot{T}$ and $1 \Vdash \dot{T} \subseteq^* T_\alpha$.

It is an important moment. The fact that B_α is comeager is a groundmodel statement and the rest of the proof will also done from the groundmodel view.

Notice that for every n and $\alpha < \kappa$ the set

$$B^n_\alpha = \{x \in D \colon f(x) \setminus n \subseteq T_\alpha\}$$

is closed (because of continuity of f and the fact that $\subseteq$ is a closed relation). By the Baire theorem there is a nonempty basic open U_α and n_α such that $U_\alpha \cap D \subseteq B^n_\alpha$.

Notice that κ is regular (or at least we can assume it it regular) and uncountable. So, there is a set $\Gamma \subseteq \kappa$, cofinal in κ, n and U such that for every $\alpha \in \Gamma$ we have $n = n_\alpha$, $U = U_\alpha$.

But then if you take $x \in U \cap D$, you have that $f(x) \setminus n \subseteq T_\alpha$ for every $\alpha \in \Gamma$, and so $f(x) \subseteq^* T_\alpha$ for $\alpha \in \Gamma$. Since Γ is cofinal in κ we have $f(x) \subseteq^* T_\alpha$ for every $\alpha < \kappa$. So, we have defined, in the groundmodel, an infinite set which is a pseudointersection of $(T_\alpha)_{\alpha<\kappa}$! A contradiction. $\square$

In the above proof you can see what is nice in Theorem 6.5: of course we love forcing but we also like when the statements about the extension are being translated into statements expressed in the 'human language'.

The *tower number* $\mathfrak{t}$ is defined as the minimal cardinality of a tower. Clearly, $\omega_1 \leq \mathfrak{t} \leq \mathfrak{c}$.

Theorem 6.7. *Consistently,* $\mathfrak{t} = \omega_1 < \mathfrak{c}$.

Proof. Start with a model V with CH. Then there is a tower $(T_\alpha)_{\alpha<\omega_1}$ in V. Let G be a $\mathbb{C}_{\omega_2}$-generic. Then $V[G] \models \mathfrak{c} = \omega_2$. We claim that $V[G]$ thinks that $(T_\alpha)_{\alpha<\omega_1}$ is still a tower. Suppose not. Then there is a name $\dot{T}$ for a subset of ω such that $1 \Vdash \dot{T}$ is a pseudo-intersection of $(T_\alpha)_{\alpha<\omega_1}$. (Btw: here we really take elements of the ground model, we don't use any codes here).

By the fact from the Exercise List, a $\mathbb{C}_{\omega_2}$-name for a real is added by a forcing with $\mathbb{C}$. But, Theorem 6.6, says that $\mathbb{C}$ does not destroy towers. A contradiction. $\square$

7. Random forcing

If you can force with $\mathrm{Bor}(2^\omega)/\mathcal{M}$, then you have to be able to force with $\mathbb{M} = \mathrm{Bor}(2^\omega)/\mathcal{N}$, where $\mathcal{N}$ is the ideal of the (Lebesgue) measure zero sets. Well, this is a prominent forcing notion called random forcing (or, sometimes, Solovay forcing).

If you see a partial order (or a Boolean algebra) and you want to force with it, the first question should be: is it separative? Yes, the random forcing is clearly separative. The next question: does it collapse cardinals? No, the random forcing is ccc (this is very clear from the definition: you cannot have uncountably many pairwise disjoint sets of positive measure), so it collapses nothing.

So, consider $\mathbb{M}$ and let G be $\mathbb{M}$-generic. Define the generic real:

$$\dot{r} = \{\langle\langle n, i\rangle, [\{x \in 2^\omega : x(n) = i\}]_{\mathcal{N}}\rangle : n \in \omega, i \in 2\}.$$

Notice that this is almost the same definition as for the generic Cohen real. The only difference is that now me mod out null sets.

Also, as in the case of the Cohen real we have:

Proposition 7.1. *If $B \in V$ is a Borel subset of 2^ω, then*

$$[\![\dot{r} \in B]\!] = [B]_{\mathcal{N}}.$$

Notice that this time I already skipped using notation using codes. But the codes are still there: by $B \in V$ I mean a Borel set coded in the ground model (and then B in $[\![\cdot]\!]$ means the set decoded in $V[G]$ and B at the right side means the set decoded in V).

And again, the analogous thing:

Proposition 7.2. $1 \Vdash \dot{r}$ *omits all the meager subsets of 2^ω coded in V.*

Definition 7.3. A real r is a *random real* over a model M if r omits all the null sets coded in M.

The fact that our forcing is defined in terms of measure is sometimes very convenient.

Proposition 7.4. *TFAE:*

- $1 \Vdash \varphi$,
- $\forall \varepsilon > 0\ \exists B\ \lambda(B) > 1 - \varepsilon\ \wedge\ B \Vdash \varphi$

Proof. Exercise (of course only one implication is not completely trivial). □

Definition 7.5. A forcing $\mathbb{P}$ is ω^ω-bounding if

$$1 \Vdash \forall \dot{f} \in \omega^\omega\ \exists g \in V \cap [\omega]^\omega\ \dot{f} \leq g.$$

In other words, a forcing $\mathbb{P}$ is ω^ω-bounding if it does not add an unbounded real.

Theorem 7.6. *The random forcing is ω^ω-bounding.*

Proof. Let $\dot{f}$ be a name for an element of ω^ω. Let $B_n^k = [\![\dot{f}(n) = k]\!]$.

Fix n. Then there is l_n big enough so that

$$\lambda(\bigcup_{k\leq l_n} B_n^k) > 1 - \varepsilon/2^{n+1}.$$

Let

$$B = \bigcap_n \bigcup_{k\leq l_n} B_n^k.$$

Clearly, $\lambda(B) > 1-\varepsilon$ and $B \Vdash \dot{f}(n) \leq l_n$.

So, the function $g\colon \omega \to \omega$ defined by $g(n) = l_n$ is such that $B \Vdash \dot{f} \leq g$.

So, by Proposition 7.4, we have $1 \Vdash \exists g \in V \cap [\omega]^\omega \ \dot{f} \leq g$. □

Now, we will define higher dimension analogous of $\mathbb{M}$. Recall that for each cardinal κ there is a (unique) measure λ_κ on 2^κ such that $\lambda_\kappa(\{x \in 2^\kappa\colon x(\alpha) = 0\}) = 1/2$ (for any $\alpha < \kappa$). Let $\mathcal{N}_\kappa$ be the ideal of λ_κ-null sets and let

$$\mathbb{M}_\kappa = \mathrm{Bor}(2^\kappa)/\mathcal{N}_\kappa.$$

In the jargon we say that $\mathbb{M}_\kappa$ is the forcing adding κ random reals (for $\kappa \geq \omega$).

Proposition 7.7. *Let $\kappa \geq \omega$. Then*

- $\mathbb{M}_\kappa$ *is ccc,*
- $\mathbb{M}_\kappa$ *is ω^ω-bounding.*

Proof. Just notice that the proofs for $\kappa = \omega$ work for the general case. □

Proposition 7.8. $\mathbb{M}_\kappa$ *adds at least κ reals.*

Proof. Exercise. Confront the proof for Cohen forcing in case of problems. □

Recall that the *dominating number* $\mathfrak{d}$ is the smallest cardinality of a family $\mathcal{A} \subseteq \omega^\omega$ which is dominating, i.e. such that for each $f \in \omega^\omega$ there is $g \in \mathcal{A}$ such that $f \leq g$.

Proposition 7.9. *Let $V \models CH$ and let G be a $\mathbb{M}_{\omega_2}$-generic. Then $V[G] \models \mathfrak{d} = \omega_1 < \mathfrak{c}$.*

Proof. Consider $\mathcal{A} = \omega^\omega$ in V. In $V[G]$ the family $\mathcal{A}$ is still dominating, as $\mathbb{M}_{\omega_2}$ is ω^ω-bounding. □

7.1. **Magic world of Cohen and random reals.** Note that if you have a generic real c and you forgot the generic filter which induced it, then you can recover it from the real c. Consider $V[c]$ (so, the smallest model containing V and c) and just take all the Borel non-meager sets B in V such that $c \in B$ (or, more formally we take all $B^{\#V}$ such that $c \in B^{\#V[c]}$). The same applies to the random forcing. This is the reason why we will call $V[c]$ the model obtained by forcing with the Cohen forcing, and $V[r]$, the model obtained by forcing with the random algebra.

Let $\varphi(x)$ be a predicate, where $x \in 2^\omega$. We say that a *typical* real number is φ if the set $\{x \in 2^\omega \colon \varphi(x)\}$ is comeager. Analogously, a *random* real numbers has φ if $\{x \in 2^\omega \colon \varphi(x)\}$ is co-null.

Now, the Cohen reals have, by its very definition, all the typical properties and random reals have all the random properties of real numbers.

Let's see an example. Treat a random real R as a subset of ω. You may ask about the asymptotic density of R. What is it? By Weak Law of Large Numbers we have $\lambda(\{X \subseteq \omega \colon d(X) = 1/2\}) = 1$. The set $\{X \subseteq \omega \colon d(X) = 1/2\}$ is coded in V, so R has to belong to it. Hence, $d(R) = 1/2$. On the other side, the family of subsets of ω which does not have density is comeager. So, if $C \subseteq \omega$ is Cohen, then C does not have asymptotic density.

Now I will formulate several nice facts about names for reals and for Borel sets in the Cohen and random extensions.

Theorem 7.10. *Suppose $\dot{x}$ is an $\mathbb{M}$-name for a real. Then there is a Borel function f coded in V such that $f(\dot{r}) = \dot{x}$.*

This is analogous to Theorem ...and the proof is also analogous (but this time you have to deal with codes for Borel sets, not only open sets). We have the following nice corollary:

Theorem 7.11. *Suppose $\dot{B}$ is an $\mathbb{M}$-name for a Borel (Σ^0_α) subset of 2^ω. Then, there is $\tilde{B} \subseteq 2^\omega \times 2^\omega$, a Borel ($\Sigma^0_\alpha$) set coded in V such that $\tilde{B}_{\dot{r}} = \dot{B}$.*

The similar theorem holds for the Cohen forcing. The proofs are analogous. See the Exercise List.

7.2. **Application:** $\mathrm{non}(\mathcal{N}) < \mathfrak{c}$.

Theorem 7.12. *Suppose $\dot{B}$ is an $\mathbb{M}$-name for a Borel null subset of 2^ω. Then there is $H \in V$, a null Borel set such that $\dot{B} \cap V \subseteq H$.*

Note that in the above theorem we are not talking about any code for H, we deal we an actual set from the ground model.

Proof. Let $\dot{B}$ be an $\mathbb{M}$-name for a Borel null subset of 2^ω. Use Theorem 7.11 to get $\tilde{B}$. Now, define, in V

$$H = \{y \in 2^\omega \colon \tilde{B}^y \notin \mathcal{N}\}.$$

We claim that H is as desired.

First, note that H is null. Otherwise, by Fubini theorem, we would have that the set $X = \{x \in 2^\omega \colon \tilde{B}_x \notin \mathbb{N}\}$ is not null. But then, there is a generic filter containing X and then the generic real $r \in X$. But then $\tilde{B}_r \notin \mathcal{N}$ (Here we use the fact that that fact that if b is a code for a Borel set, then $b^{\#V}$ is null iff $b^{\#V[G]}$ is null.) A contradiction as $\dot{B}$ is a name for a null set.

Now, let $y \in 2^\omega \cap V$. Suppose that $V[r] \models y \in \dot{B}$. It means that the set $\{x \in 2^\omega \colon \langle x, y\rangle \in \tilde{B}\}$ is not null (otherwise r would omit it as a Borel set coded in the ground model). And so $y \in H$. $\square$

Again, analogously (using Kuratowiski-Ulam theorem instead of Fubini theorem) we can prove:

Theorem 7.13. *Suppose $\dot{B}$ is an $\mathbb{C}$-name for a Borel meager subset of 2^ω. Then there is $H \in V$, a meager Borel set such that $\dot{B} \cap V \subseteq H$.*

Corollary 7.14.
- *$V[r] \models \mathbb{R} \cap V$ is non-measurable.*
- *$V[c] \models \mathbb{R} \cap V$ does not have the Baire property.*

Proof. We will prove it only for the case of the random forcing. We will first prove that $\mathbb{R} \cap V$ is not null. Suppose $V[r] \models \mathbb{R} \cap V \in \mathcal{N}$. Then, there is, in $V[G]$, a G_δ set F such that $\mathbb{R} \cap V \subseteq F$. But, by Theorem 7.12, there is $H \in V$, a null set such that $V[r] \models F \cap V \subseteq H$. A contradiction. That $\mathbb{R} \cap V$ cannot have positive measure follows from the Steinhaus theorem ($(\mathbb{R} \cap V) - (\mathbb{R} \cap V)$ contains an interval, but each interval has to contain a new real, see also Problem List ...). $\square$

To derive a nice corollary out of the above, we will need the following general theorem.

Theorem 7.15. *If $\dot{x}$ is a $\mathbb{M}_\kappa$-name for a real (for a Borel set), then there is a countable $X \subseteq \kappa$ such that $\dot{x}$ is a $\mathbb{M}_X$-name. In particular, every real added by $\mathbb{M}_\kappa$ can be added by a single random forcing $\mathbb{M}$.*

The analogous fact is true for Cohen (see Problem list). We will just sketch a proof of the above.

Proof. (Sketch) Let $\dot{x}$ be a name for a real. Without loss of generality (see Proposition 5.6) we may assume that it is a nice name, e.g. of the form

$$\dot{x} = \{\langle\langle n, i\rangle, [\![\dot{x}(n) = i]\!]\rangle \colon n \in \omega, i \in 2\}$$

Now, we need a fact from pure measure theory. For every Borel set $B \subseteq 2^\kappa$ there is a set $B' \subseteq 2^\kappa$ such that $\lambda_\kappa(B \triangle B') = 0$ and B' depends on countably many coordinates (i.e. there is a countable set $I \subseteq \kappa$ such that if $x \in B'$ and $y \in 2^\kappa$ and x, y agree on I, then $y \in B'$).

So, for each $n \in \omega$, $i \in 2$ we may assume that $[\![\dot{x}(n) = i]\!]$ depends on countably many coordinates X^i_n (here, formally we make a jump from $\mathrm{Bor}(2^\kappa)/\lambda_\kappa = 0$ to $\mathrm{Bor}(2^\kappa)$). So the whole name $\dot{x}$ depends only on $X = \bigcup_{n,i} X^i_n$ which is a countable set of coordinates.

Now, if $\dot{B}$ is an $\mathbb{M}_\kappa$-name for a Borel subset of 2^ω, then instead of $\dot{B}$ we may think about its code, which is again a real number (which has to be added by a single random forcing). $\square$

Corollary 7.16. *Let G be an $\mathbb{M}_{\omega_2}$-generic and let H be a $\mathbb{C}_{\omega_2}$-generic. Then*

- $V[G] \models \mathbb{R} \cap V$ *is non-measurable.*
- $V[H] \models \mathbb{R} \cap V$ *does not have the Baire property.*

Proof. Upgrade Theorem 7.12 and 7.13 using Theorem 7.15 (and the analogous fact for Cohen forcing). Then proceed as in Corollary 7.14. □

Recall that $\mathrm{non}(\mathcal{I})$ (where $\mathcal{I}$ is an ideal) is the minimal cardinality of a set outside of $\mathcal{I}$.

Corollary 7.17. *Let G be an $\mathbb{M}_{\omega_2}$-generic and let H be a $\mathbb{C}_{\omega_2}$-generic. Suppose that $V \models CH$. Then*

- $V[G] \models \mathrm{non}(\mathcal{N}) = \omega_1$
- $V[H] \models \mathrm{non}(\mathcal{M}) = \omega_1$.

Proof. Clearly $\mathbb{R} \cap V$ has cardinality ω_1. Use Corollary 7.16. □

7.3. **More on ground model objects in the extension.** In the previous subsection we studied properties of the set of old reals in the extension. It often turns out that this kind of the objects have peculiar properties. E.g. as we have seen $\mathbb{R} \cap V$ in the random extension is non-measurable.

Theorem 7.18. $V[c] \models \mathbb{R} \cap V \in \mathcal{N}$.

Proof. Let $\varepsilon > 0$. Define $\mathbb{P}_\varepsilon$ as the family of all finite unions of rational intervals $\bigcup_{k \in F} I_k$ such that $\lambda(\bigcup_{k \in F} I_k) < \varepsilon$. Define $p \leq q$ as $p \supseteq q$. This is a separative partial order and we can force with it.

Let G be a $\mathbb{P}_\varepsilon$-generic. Then $U = \bigcup G$ is an open set such that $\lambda(U) \leq \varepsilon$. We claim that $\mathbb{R} \cap V \subseteq U$. Indeed, if $x \in 2^\omega \cap V$, then the set $D_x = \{p \in \mathbb{P}_\varepsilon \colon x \in p\}$ is dense in $\mathbb{P}_\varepsilon$. So $G \cap D_x \neq \emptyset$ and so $x \in U$.

OK, you may ask, but what it has to do with the Cohen forcing? Well, $\mathbb{P}_\varepsilon$ is a countable separative forcing and so (Exercise List) it is forcing equivalent to the Cohen forcing!

So, for each $\varepsilon > 0$, the Cohen forces that $\lambda(\mathbb{R} \cap V) \leq \varepsilon$. So $V[c] \models \lambda(\mathbb{R} \cap V) = 0$. □

I think the above proof is due to Cichoń. The similar fact holds for the random forcing. This time the proof will be due to Kunen, but I will only sketch it.

Theorem 7.19. $V[r] \models \mathbb{R} \cap V \in \mathcal{M}$.

Proof. (Sketch) Fix $x \in 2^\omega$. Let

$$A_x = \{y \in 2^\omega \colon \exists^\infty n \ y_{|[n,2n)} = z_{|[n,2n)}\}.$$

Check that A_x is a dense G_δ. Consider A_r, where r is a random real. Then $\mathbb{R} \cap V \cap A_r = \emptyset$. □

Now, I would like to mention something not so standard.

Suppose that we force with a complete Boolean algebra $\mathbb{B}$ over a model V. Let $\mathbb{A}$ be a Boolean algebra in V. Then, in $V[G]$, where G is $\mathbb{B}$-generic, $\mathbb{A}$ also forms a Boolean algebra. Typically, in $V[G]$, there are new ultrafilters on $\mathbb{A}$.

A priori the access to those 'new' ultrafilters is quite remote. You know, ultrafilters in the ground model are rather complicated objects and names for objects are usually more complicated that the objects itself. So if you want to consider forcing names for ultrafilters, it should end up with a complete mess. Surprisingly, to the contrary: there are very elegant names for ultrafilters.

First, recall that there is a natural correspondence between ultrafilters on $\mathbb{A}$ and Boolean homomorphisms $h\colon \mathbb{A} \to \{0,1\}$ (in fact, in some parts of mathematics the Stone spaces are not spaces of ultrafilters but of such homomorphisms, and it makes sense).

The point is that if you force with a complete Boolean algebra $\mathbb{B}$ you can think about the names for ultrafilters on $\mathbb{A}$ (as an object from the ground model) as ... Boolean homomorphisms $h\colon \mathbb{A} \to \mathbb{B}$! It is as easy as this.

Indeed, assume $\dot{\mathcal{U}}$ is a $\mathbb{B}$-name for an ultrafilter on a Boolean algebra $\mathbb{A}$. Define $\varphi\colon \mathbb{A} \to \mathbb{B}$ by $\varphi(A) = [\![A \in \dot{\mathcal{U}}]\!]$. It is plain to check that φ is a Boolean homomorphism. On the other hand, fix a homomorphism $\varphi\colon \mathbb{A} \to \mathbb{B}$ and let

$$\dot{\mathcal{U}} = \{\langle A, \varphi(A)\rangle \colon A \in \mathbb{A}\}.$$

Then $\dot{\mathcal{U}}$ is a $\mathbb{B}$-name for an ultrafilter on $\mathbb{A}$.

So, every homomorphism $\varphi\colon \mathbb{A} \to \mathbb{B}$ can be interpreted as an $\mathbb{B}$-name for an ultrafilter on $\mathbb{A}$. Also, if $\dot{\mathcal{U}}$ is an $\mathbb{B}$-name for an ultrafilter on $\mathbb{A}$, then we may assume that it is of the above form for homomorphism φ defined by $\varphi(A) = [\![A \in \dot{U}]\!]$).

I have mentioned it is not standard. In fact for some time I thought that it is my invention. However, I found the above remark in my old notes from the forcing course by Anastasis Kamburelis.

Using the above, we will prove the following classical result: forcing with $\mathbb{M}$ makes $\mathbb{M}$ a σ-centered Boolean algebra. First, recall that a Boolean algebra $\mathbb{A}$ is σ-centered if $\mathbb{A} \setminus \{0\}$ is a countable union of ultrafilters on $\mathbb{A}$. Also, notice that $\mathbb{M}$ is not σ-centered (see Problem List ...).

There will be a little surprise in the following proof. We will use a fact from dynamical systems.

Theorem 7.20. *Let G be an $\mathbb{M}$-generic. Then*

$$V[G] \models \mathbb{M} \cap V \textit{ is } \sigma\textit{-centered.}$$

Proof. We will see $\mathbb{M}$ as $\mathrm{Bor}(S^1)/\mathcal{N}$, where S^1 is just the unit circle. Recall that there is $f\colon S^1 \to S^1$ which is ergodic, i.e. whenever $A, B \subseteq S^1$ are of positive measure, then there is n such that $f^{(n)}[A] \cap B$ is of positive measure. (Just take an irrational rotation).

Such f induces naturally a Boolean homomorphism $\varphi\colon \mathbb{M} \to \mathbb{M}$. Let $\dot{\mathcal{U}}_n$ be the $\mathbb{M}$-name for an ultrafilter induced by φ^n in the way described above.. We claim that

$$1 \Vdash \mathbb{M}^+ = \bigcup_n \dot{\mathcal{U}}_n.$$

Indeed, let $A \in \mathbb{M}^+$ and let $p \in \mathbb{M}^+$. There is n such that $q = (\varphi^n(A) \cap p) \neq 0$. But then $q \Vdash A \in \dot{\mathcal{U}}_n$. By the density argument we are done. $\square$

8. Sacks forcing: an example of a tree forcing

Tree forcings are important class of forcing notions, particularly for the set theory of the real line. Tree forcing means that the *conditions* are trees (usually subtrees of $2^{<\omega}$ or $\omega^{<\omega}$) not that the forcing is tree itself (as in the case of forcing with Suslin trees).

The standard example here is Sacks forcing $\mathbb{S}$. It consists of subtrees of $2^{<\omega}$ which are perfect, i.e. such that it splits above every node. The ordering is just inclusion.

The partial order $\mathbb{S}$ is separative so it is decent enough to consider it. But it is not ccc. So, a priori, it may collaps cardinals.

Proposition 8.1. *There is an antichain of size $\mathfrak{c}$ in $\mathbb{S}$.*

Proof. Take a pairwise almost disjoint family $\{A_\alpha\colon \alpha < \mathfrak{c}\} \subseteq \mathcal{P}(\omega)$. For $\alpha < \mathfrak{c}$ let T_α be such that $s \in T_\alpha$ iff $s(i) = 0$ for each $i \in A_\alpha$. So, T_α is a tree splits exactly on levels from A_α and whose nodes chooses 0 on the levels outside A_α. If $A_\alpha \cap A_\beta = \emptyset$, then the only tree included in $T_\alpha \cap T_\beta$ consists of only one branch. If $A_\alpha \cap A_\beta$ is finite, then $T_\alpha \cap T_\beta$ does not contain a perfect tree. So $(T_\alpha)_\alpha$ forms an antichain. $\square$

Nevertheless, under CH, Sacks forcing does not collaps cardinals. The reason lies in the following definition.

Definition 8.2. A forcing notion $\mathbb{P}$ has *the Sacks property* if for every $p \in \mathbb{P}$ and every $\mathbb{P}$-name for a function $f\colon \omega \to V$, there is $q \leq p$ and $S\colon \omega \to [\omega]^{<\omega}$, $S \in V$ such that $|S(n)| \leq 2^n$ and $q \Vdash \forall n\ \dot{f}(n) \in S(n)$.

The function S as above is called *slalom*.

(see ... for the nice picture of slaloms, the name was invented by Tomek Bartoszyń ski in this paper). Sacks property means that we cant localise function from the extension by small slaloms from the ground model.

Sacks property is quite strong. It implies that the forcing in question is ω^ω-bounding (just consider the 'up-most branch' of the slalom). Also, it implies that the forcing does not collapse ω_1.

Proposition 8.3. *Suppose that $\mathbb{P}$ has Sacks property. Then $\mathbb{P}$ does not collapse ω_1. If, additionally $|\mathbb{P}| = \omega_1$, then $\mathbb{P}$ does not collapse cardinals.*

Proof. Indeed, suppose that $\dot{f}$ is a $\mathbb{P}$-name for a function $f\colon \omega \to \omega_1$. We want to show that it is not onto. Let $p \in \mathbb{P}$. Use Sacks property to find an appropriate slalom S and $q \leq p$ forcing that S catches $\dot{f}$. Then $q \Vdash \dot{f}[\omega] \subseteq \bigcup_n S(n)$. But $\bigcup_n S(n)$ is countable and we are done.

The second part follows from the fact that κ-cc implies that no cardinals above κ are destroyed. $\square$

So, we have that the Sacks forcing does not collapse cardinals. As Sacks forcing does have the Sacks property, right? Well, we have to prove it. This proof is important, since it uses something called *fusion*.

DEFINITION. Let H be any countable set. Every element

$$\varphi \in \prod_n [H]^{<\omega} \stackrel{\text{df}}{=} \mathrm{ST}^H$$

is called a *slalom*.

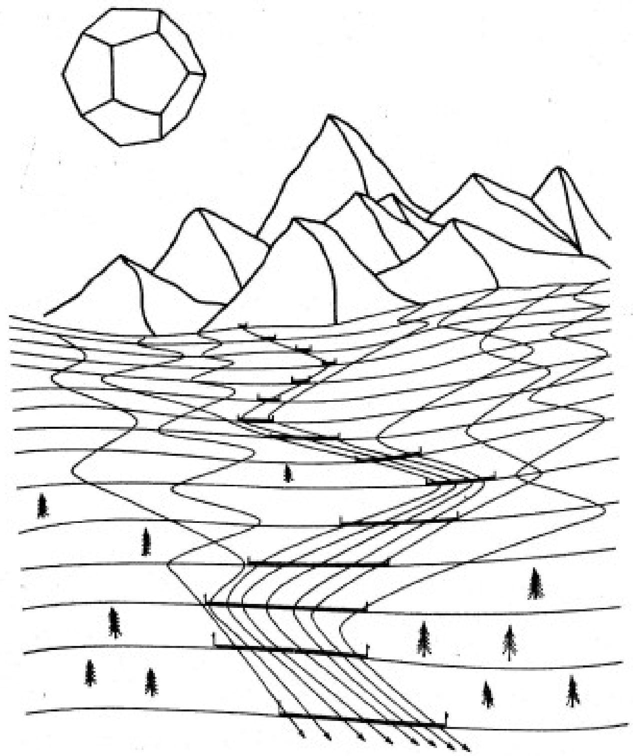

FIGURE 1. A picture of slalom in Bartoszynski's paper Combinatorial aspects of measure and category, where the slaloms were defined for the first time.

Theorem 8.4. *Sacks forcing has the Sacks property.*

Proof. Let $p \in \mathbb{P}$ and let $f\colon \omega \to V$. Someone below p, say $p_\emptyset$, decides $\dot{f}(0)$ to be (some) $k_\emptyset$. Let $S(0) = \{k_\emptyset\}$.

Now, consider $s = \mathrm{stem}(p_\emptyset)$. Let $q_0 = p_\emptyset \cap [s^\frown 0]$ and $q_1 = p_\emptyset \cap [s^\frown 1]$. There is $p_0 \le q_0$ deciding $\dot{f}(1)$ (to be k_0) and $p_1 \le q_1$ deciding $\dot{f}(1)$ (to be k_1; of course it may happen that $k_0 = k_1$). Let $S(1) = \{k_0, k_1\}$.

Now, proceed as above, obtaining p_s, k_s for $s \in 2^{<\omega}$ in such a way that

- each p_s decides $\dot{f}(n)$ (to be k_s), where n is the length of s.
- for each $s \in 2^{<\omega}$ and $i \in 2$ we have $p_{s^\frown i} \le p_s \cap [\mathrm{stem}(p_s)^\frown i]$.

Now define

$$q_n = \bigcup\{p_s \colon s \in 2^n\}.$$

Notice that q_n is a perfect tree and for each n we have $q_{n+1} \le q_n$.

If we let $S(n) = \{k_s \colon s \in 2^n\}$, then each q_n forces that $\dot{f}(n) \in S(n)$. The problem is that we need *one* condition which forces it for each n. It would be nice if there is q which is below each q_n but we don't have σ-closedness here.

Nevertheless, and this is the important moment, such q does exist. The point is that q_{n+1} is not only stronger than q_n: it looks like q_n on its initial fragment. More precisely, q_{n+1} and q_n has the same set of nodes which have been split n times (this relation is denoted by $\leq_n$). It means that each q_m, for $m > n$, will also have the same set of nodes split n times. So, the q can be defined as the condition whose set of nodes split n times looks like the set of nodes of q_n split n times. If you draw a picture, it makes sense.

Clearly, q forces what we want. □

A sequence as in the above theorem, i.e. such that $q_{n+1} \leq_n q_n$ is called a *fusion sequence* and the condition q is called the *fusion* of this sequence. Most of the proofs concerning Sacks forcing (and also some other tree forcings) uses fusion.

8.1. **Sacks preserves P-points.** Let's see one other example of the fusion argument. We will show that Sacks forcing preserves P-points. It means that whenever an ultrafilter $\mathcal{U}$ is a P-point, and we look at $\mathcal{U}$ in the Sacks extension, then the resulting family will generate a filter which is still an ultrafilter. Notice that if a forcing adds a splitting real (like the Cohen forcing), then it destroys all the ultrafilters: the splitting real (seen as a subset of ω cannot be automatically decided to be in or out of a filter generated by a ground model family).

So, when we say that a forcing preserves P-points, the accent should be rather on 'point' (=ultrafilter), not on 'P'.

I will present a proof which I learned on Martin Goldstern's tutorial on one of the Young Set Theory Workshops.

We need to know Laflamme's P-point game. Suppose $\mathcal{U}$ is an ultrafilter. Consider the following game: Adam plays an element X_0 of $\mathcal{U}$ and Eve responds with a finite subset F_0 of X_0. Then Adam plays X_1 (not necessarily connected neither to X_0 neither to F_0) and so on. Eve wins if $\bigcup_n F_n \in \mathcal{U}$. The theorem is: $\mathcal{U}$ is a P-point if and only if Adam does not have a winning strategy.

Another handy stuff is the notion of *interpretation*. This is not the interpretation by generic we were talking about at the beginning. Let $\dot{A}$ be a name for a subsets of ω. For each n we can find q_n deciding $A \cap n$. We say that $A^\star$ is an interpretation of $\dot{A}$ if there is a descending sequence (q_n) such that $q_n \Vdash \dot{A} \cap n = A^\star \cap n$. Notice that usually no condition forces that $\dot{A} = A^\star$ (that would mean that $\dot{A}$ is from the ground model). So, sincerely speaking, $A^\star$ has not much in common with $\dot{A}$ and even the name 'interpretation' is not the best choice. But I don't have any better idea (I thought about 'approximation' but $A^\star$ is hopelessly bad approximation of $\dot{A}$). And interpretations are useful here and there (see Problem List 5).

Theorem 8.5. *Sacks forcing preserves P-points.*

Proof. Let $\mathcal{U}$ be a P-point. Let $\dot{A}$ be a Sacks name for a subset of ω and let $p \in \mathbb{S}$. We are going to show that there is $q \leq p$ such that

$$q \Vdash \exists U \in \mathcal{U}\ U \subseteq \dot{A} \vee U \subseteq \omega \setminus \dot{A}.$$

Clearly, it is enough.

Now, for every $r \leq p$ choose an interpretation $A^\star_r$ of $\dot{A}$ such that $A^\star_r \in \mathcal{U}$ if possible (and choose any other interpretation if it is not possible). Then either every $A^\star_r \in \mathcal{U}$ or there is $r' \leq p$ such that every $\omega \setminus A^\star_r \in \mathcal{U}$ for $r \leq r'$ (since if there is no interpretation in $\mathcal{U}$ under r', then each interpretation under each $r \leq r'$ has to be outside $\mathcal{U}$).

Suppose, without loss of generality, that $A^\star_r \in \mathcal{U}$ for each $r \leq p$ and let's start the game.

Let s be the stem of p. For $i \in 2$ find an interpretation $A^\star_i \in \mathcal{U}$ of $\dot{A}$ below $p \cap [s^\frown i]$. Let's Adam play $A^\star_0 \cap A^\star_1$ in his first move. Let's F_0 be Eve's response. Now, find $p_i \leq p \cap [s^\frown i]$ forcing that F_0 is a subset of $\dot{A}$ (using the fact that $A^\star_i$ is an interpretation of $\dot{A}$).

Then, we continue, in a similar manner as in the proof of Theorem 8.4. We consider stems of p_0 its left an right choice, obtaining $A^\star_{00}$ and $A^\star_{01}$, and similarly with p_1 obtaining $A^\star_{10}$ $A^\star_{11}$. Then we let Adam play with $A^\star_{00} \cap A^\star_{01} \cap A^\star_{10} \cap A^\star_{11}$. Let F_1 be Eve's response and let p_s, $s \in 2^2$ be conditions forcing that $F_1 \subseteq \dot{A}$.

Continue and let $q_n = \bigcup\{p_s \colon s \in 2^n\}$. Then (q_n) is a fusion sequence and for each n we have $q_n \Vdash F_n \subseteq \dot{A}$. Take a fusion q of this sequence. Then $q \Vdash \bigcup F_n \subseteq \dot{A}$. But Adam does not have a winning strategy in the game, so we can take a sequence (F_n) witnessing that. Then $\bigcup_n F_n \in \mathcal{U}$ and we are done. □

You may wonder, why the above fact is interesting. Why do we care that something does not destroy an ultrafilter. Imagine, that we start with a model of CH. THen we force with a Sacks forcing. And then again, and we repeat it ω_2 times. Then, the above fact + a couple of other facts, imply that in the resulting model we have an ultrafilter which is generated by less than $\mathfrak{c}$ many sets.

This is easy to say but not so easy to perform. What does it mean to repeat a forcing ω_2 times? Well, this is the point when we have to think about learning iterated forcing. Iterated forcing is something which allows us to repeat forcings without really doing anything many times.

The model obtained by 'repeating' (whatever it means) Sacks forcing ω_2 times is called Sacks model. It behaves in an anti-Martin axiom manner: it makes all the cardinal coefficients small. In fact, there is an axiom which tries to capture all the combinatorial features of the Sacks model: it is called CPA (Covering Property Axiom or Ciesielski-Pawlikowski Axiom).

There is a very nice survey article on the Sacks forcing: [?] https://www.researchgate.net/publication/226615377_On_Sacks_forcing_and_the_Sacks_property,

9. Complete embeddings

In this small section I will define complete embeddings. In the category of forcing notions, the complete embeddings are 'homomorphisms' (whereas dense embeddings are like 'isomorphisms'). In particular, if a forcing $\mathbb{P}$ can be embedded completely to $\mathbb{Q}$, then $\mathbb{Q}$ is *reacher* than $\mathbb{P}$.

There is a problem with complete embeddings: find at random two set theorists and ask them about the definition of complete embedding. Very probably you get two different answers (though usually equivalent) and very probably they will be quite technical.

I think the easiest way to understand what complete embedding is, is to look at Boolean algebras.

Definition 9.1. We say that $\mathbb{A}$ is a complete subalgebra of $\mathbb{B}$ if $\mathbb{A}$ is a subalgebra of $\mathbb{B}$ and for every family $\mathcal{A} \subseteq \mathbb{A}$ its supremum taken in $\mathbb{B}$ belongs to $\mathbb{A}$.

This means that $\mathbb{A}$ and $\mathbb{B}$ agree on suprema (of subsets of $\mathbb{A}$). A complete embedding $e \colon \mathbb{A} \to \mathbb{B}$ is an embedding whose image is a complete subalgebra.

Proposition 9.2. *Suppose that $\mathbb{A}$ is a complete subalgebra of $\mathbb{B}$ and let G be a $\mathbb{B}$-generic. Then $H = G \cap \mathbb{A}$ is an $\mathbb{A}$-generic.*

Proof. Clearly H is a filter on $\mathbb{A}$. Instead of showing that it intersects dense sets, we will show that it intersects maximal antichains. So let $\mathcal{A}$ be a maximal antichain in $\mathbb{A}$. Then it is an antichain in $\mathbb{B}$ and, by the fact that $\mathbb{A}$ is a complete subalgebra, its supremum has to be 1, which means that it is maximal in $\mathbb{B}$. So G intersects it and so $G \cap \mathbb{A} = H$ intersects it. $\square$

The above fact says that if we force with $\mathbb{B}$, then we have all the objects which can be achieved forcing with $\mathbb{A}$. For example, if $\mathbb{A}$ adds a dominating real, then $\mathbb{B}$ too. If $\mathbb{A}$ adds a measure zero sets containing old reals, then $\mathbb{B}$ too.

Example 9.3. The Cohen algebra $\mathbb{C} = \mathrm{Bor}(2^\omega)/\mathcal{M}$ is a subalgebra of the random algebra $\mathbb{M}$. Just take an equivalence class of $\mathbb{C}$, choose a regular open representative, and send it to the equivalence class with respect to measure.

But, $\mathbb{C}$ is not a complete subalgebra of $\mathbb{M}$. Forcing reason: $\mathbb{C}$ adds an unbounded real, but $\mathbb{M}$ is ω^ω-bounding. Simpler reason: take $A_n = (q_n - 1/4^n, q_n + 1/4^n)$, where $\mathbb{Q} = \{q_n \colon n \in \omega\}$. The supremum of $\{A_n \colon n \in \omega\}$ in $\mathbb{C}$ is 1. But in $\mathbb{M}$ it is much smaller.

The above approach to complete embeddings can be extended for partial orders: $\mathbb{P}$ can be completely embedded in $\mathbb{Q}$ if $RO(\mathbb{P})$ can be embedded completely into $RO(\mathbb{Q})$. The problem with this definition is that it is difficult to check it (many times it is difficult to handle $RO(\mathbb{P})$). This is the reason why we have to invent other, more technical, definitions of complete embeddings.

Instead of presenting any of the those definitions (you can find them in many places) I will define something 'contravariant'.

Definition 9.4. We say that $\pi\colon \mathbb{Q} \to \mathbb{P}$ is a *projection* if

- π is onto,
- if $q' \leq q$, then $\pi(q') \leq \pi(q)$,
- if $p \leq \pi(q)$, then there is $q' \leq q$ such that $\pi(q') = p$.

The connection with complete embeddings is explained by the following:

Proposition 9.5. *If $\pi\colon \mathbb{Q} \to \mathbb{P}$ is a projection, then $e\colon RO(\mathbb{P}) \to RO(\mathbb{Q}))$ defined by $e(A) = \pi^{-1}[A]$ is a complete embedding.*

Proof. Exercise. □

So, combining the above with Proposition 9.2, if $\mathbb{Q}$ can be projected onto $\mathbb{P}$, then whatever can be added by $\mathbb{P}$, can be added by $\mathbb{Q}$ as well.

Example 9.6. Consider the Hechler forcing, i.e. $\mathbb{H} = \{(n, f)\colon n \in \omega, f \in \omega^\omega\}$ with the ordering: $(n', f') \leq (n, f)$ if $f'_{|}n = f|n$ and $f'(i) \geq f(i)$ for each $i \leq n$. This is the simplest forcing for adding a dominating real. We will show that it can be projected onto the Cohen forcing (seen as $\bigcup_n s^n$). Let $\pi\colon \mathbb{H} \to \mathbb{C}$ be defined by $\pi(n, f) = s \in 2^n$ such that $s(i) = f(i) mod 2$ for each $i < n$. It is plain to check that this is a projection. So, in particular, the Hechler forcing adds a Cohen real. But Cohen real does not add a dominating real, so the Hechler forcing does not embed completely into the Cohen forcing.

10. Products

Suppose $\mathbb{P}$ and $\mathbb{Q}$ are forcing notions. The product forcing $\mathbb{P} \times \mathbb{Q}$ is what you expect it to be: $\mathbb{P} \times \mathbb{Q}$ with the ordering defined coordinatewise: $\langle p', q' \rangle \leq \langle p, q \rangle$ if $p' \leq p$ and $q' \leq q$ (of course in the above the sign '$\leq$' typically means three different orderings but I believe that you will not be confused).

Proposition 10.1. *Filters on $\mathbb{P} \times \mathbb{Q}$ are rectangles, i.e. if G is a filter on $\mathbb{P} \times \mathbb{Q}$, then $G = \pi_{\mathbb{P}}[G] \times \pi^{\mathbb{Q}}[G]$.*

Proof. Plain. □

So, every $\mathbb{P} \times \mathbb{Q}$-generic is of the form $G \times H$. What is less expectable is that it is not enough that G is $\mathbb{P}$-generic and H is $\mathbb{Q}$-generic!

Theorem 10.2 (Product Lemma)**.** *TFAE*

- *$G \times H$ is a $\mathbb{P} \times \mathbb{Q}$-generic.*
- *G is $\mathbb{P}$-generic (over V) and H is $\mathbb{Q}$-generic over $V[G]$.*
- *H is $\mathbb{Q}$-generic (over V) and G is $\mathbb{P}$-generic over $V[H]$.*

So, the intermediate model $V[G]$ (or $V[H]$) comes into the picture automatically. Notice that the above means that whenever we take $\mathbb{P} \times \mathbb{P}$ and G is a $\mathbb{P}$-generic, usually $G \times G$ is not $\mathbb{P} \times \mathbb{P}$-generic!

I will not prove the above. The reason is that you can find excellent notes about products (and iterations) by Itay Neeman:
https://www.math.ucla.edu/~ineeman/223s.1.11s/223s-spring11-lecture-notes-6-5.pdf

Just read the proof there.

Proposition 10.3. *$\mathbb{M} \times \mathbb{M}$ adds a Cohen real (here $\mathbb{M}$ is of course the random forcing..*

Proof. Let $\dot{r_0}$ be the generic real added by $\mathbb{M}$ (over V) and let $\dot{r_1}$ be the generic real added by $\mathbb{M}$ over $V[r_0]$.

Let $\langle A, B \rangle \in \mathbb{M} \to \mathbb{M}$. By Steinhaus theorem, $A + B$ contains an interval I.

Let F be a closed nowhere dense set (in V). Then, by continuity of the addition, we can find $A' \subseteq A$ and $B' \subseteq B$ of positive measure such that $A' + B' \subseteq I \setminus F$. Then $\langle A', B' \rangle \Vdash \dot{r_0} + \dot{r_1} \notin F$.

It means that $1 \Vdash \dot{r_0} + \dot{r_1}$ omits all the nowhere dense sets coded in V (and so it is a Cohen real over V). □

11. Iterations

11.1. Finite support iterations and the consistency of Martin's Axiom. Just read Itay Neeman's notes. I can't do any better.
https://www.math.ucla.edu/~ineeman/223s.1.11s/223s-spring11-lecture-notes-6-5.pdf

In particular you will need to read

- the definition of iteration,
- the proof of the iteration lemma (on how the generics looks like),
- the proof that finite support iteration of ccc forcing notions is ccc,
- the proof of the consistency of Martin's Axiom.

11.2. Few short examples of finite support iterations. The proof of consistency of Martin's Axiom is important for two reasons: 1) Martin's Axiom is important, 2) it is complicated enough. In fact many uses of finite support iterations are easier than that one. Many times you force with the same forcing all the time.

Nice way to see how the iterations work is to see how they can be used to settle the values of cardinal coefficient. Some of the cardinal coefficients already appeared on this course. In general, I strongly recommend the chapter by Andreas Blass: https://dept.math.lsa.umich.edu/~ablass/hbk.pdf

Martin's Axiom makes 'all' coefficients from the Cichoń's diagram and van Douwen diagram big. (In fact it is a good exercise to force $\mathfrak{p} = \mathfrak{c}$ in a direct way, without using Martin's Axiom. Hint: instead of forcing with all possible (and not too big) ccc forcing notions, use only Mathias-Prikry forcings for all the possible filters, with appropriate bookkeeping).

Here is an easy example of how we can make a coefficient small.

Proposition 11.1. *It is consistent that $\mathfrak{d} = \omega_1 < \mathfrak{c}$.*

Proof. Start with a model with a big continuum (let's say a model of Martin's Axiom + non CH). Now, apply finite support iteration of length ω_1 of Hechler's forcings. We claim that in the resulting model, $\mathfrak{d}$ and $\mathfrak{c}$ are as desired.

First of all, notice that Hechler's forcing is ccc (it is even σ-centered, the argument is analogous to the proof that the Mathias-Prikry forcing is σ-centered). So, finite support iteration of Hechler forcings does not collapse any cardinals. In particular, $\mathfrak{c}$ is still big in the final extension.

Now, each round of the iteration adds a function which dominates all the elements of ω^ω from the actual model. As objects which are added on the initial steps of the iterations stays with us to the very end of the iteration, we obtain in this way a family $\{f_\alpha \colon \alpha < \omega_1\}$ of functions. Suppose that $f \in \omega^\omega \cap V[G]$, where G is the generic for the iteration. Then, f will appear in $V[G_\alpha]$ (where G_α is the generic at α-s step of the iteration). The reason is that you can consider a nice name for f (which bears only 'countable' information). But then in the α's step of the iteration the Hechler forcings adds f_α which dominates everybody from $V[G_\alpha]$, including f. So $\{f_\alpha \colon \alpha < \omega_1\}$ is a dominating family and so it witnesses $\mathfrak{d} = \omega_1$. $\square$

Another example of a similar sort.

Proposition 11.2. *It is consistent that $\mathfrak{s} < \mathfrak{b}$.*

Proof. Again, start with a model of MA + non CH. Then $\mathfrak{s} = \mathfrak{b} = \mathfrak{c}$. Now, apply the finite support iteration of random forcings, of length ω_1. Random forcing is ccc, so the iteration does not collapse cardinals.

Then $\mathfrak{b}$ will stay big, because the random forcing is ω^ω-bounding. So, if we have a family of elements of ω^ω, of size ω_1, in the final model, then we could find a family of elements of ω^ω, of the same size, from the ground model which dominates it. But, as $\mathfrak{b} > \omega_1$ in the ground model, this family can be dominated.

On the other hand the set of the random reals $R = \{r_\alpha \colon \alpha < \omega_1\}$ forms a set which is not null. Otherwise, we could cover R by a Borel null set. But this Borel set could be coded as a real number. This real number would appear at some intermediate (say, α's) step of the iteration (nice names, 'countable' information). But then r_α would omit this set, a contradiction.

This R seen as a family of subsets of ω is splitting: this is because every family which is not splitting is null (just write what does it mean that a family does not split a particular set). So, we have a splitting family of size ω_1. $\square$

11.3. **A small sip of proper forcing.** The above examples are quite easy. Usually, to distinguish some cardinal coefficients, you need much more work and often finite support iterations are not enough. The reason is that finite support iterations will always add Cohen reals (so, for example, it is impossible to make $\mathrm{cov}(\mathcal{M})$ small).

The problem is that if we force with countable support iterations, then we can't preserve ccc. This is the reason why mankind had to invent some other condition than ccc, strong enough to preserve cardinalities but weaker than ccc. And mankind has invented it (unless you consider Shelah as a representative of another specie).

On the lecture, I gave a definition of the proper forcing and I showed that it preserves stationary subsets of ω_1 (and so it does not collapse ω_1). This is written in a very nice way in Joerg Brendle's *Bogota lectures*:
https://www.math.uni-hamburg.de/personen/khomskii/ST2013/bogotalecture.pdf

Proper forcing is a subject for another course (or seminar) but it is worth you get a flavour of it. As you can see this notion is slightly more involved than ccc.

References

[1] Viale M. Notes on forcing;Available from: https://math.i-learn.unito.it/pluginfile.php/97222/mod_resource/content/2/dispenseTI2014.pdf.

[2] Miller AW. Descriptive set theory and forcing. vol. 4 of Lecture Notes in Logic. Springer-Verlag, Berlin; 1995. How to prove theorems about Borel sets the hard way. Available from: https://doi.org/10.1007/978-3-662-21773-3.