Forcing **1** 2023

Zad. 1 Prove that if V satisfies Axiom of Choice, then $V[G]$ satisfies Axiom of Choice.

Zad. 2 Discuss the form of $\mathbb{C}$-names for elements of 2^ω, $\mathcal{P}(\omega)$, ω^ω.

Zad. 3 Show that a filter G is $\mathbb{P}$-generic iff it intersects every maximal antichain.

Zad. 4 Suppose that $\dot{C}$ is a $\mathbb{C}$-name such that $1 \Vdash \dot{C} \subseteq \omega$. For each n let A_n be a maximal antichain of conditions deciding if $n \in C$.

$$\dot{B} = \{\langle n, p\rangle \colon n \in \omega, p \in A_n, p \Vdash n \in \dot{A}\}.$$

Show that

$$1 \Vdash \dot{A} = \dot{B}.$$

Zad. 5 Show that $V[G] \models \mathfrak{c} = \omega_2$ if G is $\mathbb{C}_{\omega_2}$-generic. (Hint: use the previous exercise.) What is the value of $\mathfrak{c}$ in $V[G]$ if G is a $\mathbb{C}_{\omega_{2023}}$-generic? What about $\mathbb{C}_{\omega_\omega}$?

Zad. 6 Consider $\mathbb{P} = \{f \colon f \colon \omega \to \omega, \mathrm{dom}(f) \text{ is finite}\}$. Let G be a $\mathbb{P}$-generic and let $c \in \omega^\omega \cap V[G]$ be the generic real. Show that

a) c is unbounded, i.e. for each $g \in \omega^\omega \cap V$ the set $\{n \colon c(n) > g(n)\}$ is infinite,

b) c is not dominating, i.e. there is $g \in \omega^\omega \cap V$ such that $\{n \colon c(n) \leq g(n)\}$ is infinite.

Zad. 7 Show that Cohen forcing does not add a dominating real. Hint: Let $\dot{f}$ be a $\mathbb{C}$-name for an element ω^ω. Enumerate $\mathbb{C} = \{p_i \colon i \in \omega\}$. For each i fix $q_i \leq p_i$ deciding the value of $\dot{f}(i)$, i.e. such that there is k_i such that $q_i \Vdash \dot{f}(i) = \check{k}_i$. Let $g(i) = k_i$ and show that this is what we are looking for.

Zad. 8 Let κ, λ be ordinal numbers. Let $\mathbb{P}$ be a ccc forcing notion. Suppose that

$$1_{\mathbb{P}} \Vdash \dot{f} \colon \check{\kappa} \to \check{\lambda} \text{ is a function.}$$

Then there is $F \subseteq \kappa \times \lambda$, $F \in V$, such that F_ξ is countable for each $\xi < \kappa$ such that

$$1_{\mathbb{P}} \Vdash \dot{f} \subseteq \check{F}.$$

Conclude that whenever $f \colon \kappa \to \lambda$, $f \in V[G]$ (where G is a $\mathbb{P}$-generic and $\mathbb{P}$ is ccc), there is‘$F \in V$ like above such that $V[G] \models f \subseteq F$.

Zad. 9 (Jensen's covering lemma) Let $\mathbb{P}$ be a ccc forcing notion and let λ be an infinite cardinal number. If

$$1_{\mathbb{P}} \Vdash \dot{A} \subseteq \lambda, \dot{A} \text{ is infinite}$$

then there is $B \subseteq \lambda$ such that

$$1_{\mathbb{P}} \Vdash A \subseteq B \wedge |A| = |B|.$$

Hint, use the previous exercise.

Zad. 10 Show that if r is a real added by $\mathbb{C}_{\omega_2}$, then it is added by forcing with a single Cohen real (i.e. there is a countable $X \subseteq \omega_2$ such that if

$$\mathbb{C}_X = \{p \colon p \colon X \to \{0,1\}, \mathrm{dom}(p) \text{ is finite}\}$$

and G is a $\mathbb{C}_X$-generic, then $f \in V[G]$).

Zad. 11 Let G be a $\mathbb{C}$-generic. Show that the generic real $c \in 2^\omega \cap V[G]$ omits every nowhere dense set *coded in the ground model* (first, try to make sense out of this statement).