

Zad. 1 Show that $V[G] \models \mathfrak{c} = \omega_2$ if G is \mathbb{C}_{ω_2} -generic. (Hint: use exercise 3 from the previous list.) What is the value of \mathfrak{c} in $V[G]$ if G is a $\mathbb{C}_{\omega_{2023}}$ -generic? What about $\mathbb{C}_{\omega_\omega}$?

Zad. 2 Consider $\mathbb{P} = \{f: \omega \rightarrow \omega, \text{dom}(f) \text{ is finite}\}$. Let G be a \mathbb{P} -generic and let $c \in \omega^\omega \cap V[G]$ be the generic real. Show that

- a) c is unbounded, i.e. for each $g \in \omega^\omega \cap V$ the set $\{n: c(n) > g(n)\}$ is infinite,
- b) c is not dominating, i.e. there is $g \in \omega^\omega \cap V$ such that $\{n: c(n) \leq g(n)\}$ is infinite.

Zad. 3 Show that Cohen forcing does not add a dominating real. Hint: Let \dot{f} be a \mathbb{C} -name for an element ω^ω . Enumerate $\mathbb{C} = \{p_i: i \in \omega\}$. For each i fix $q_i \leq p_i$ deciding the value of $\dot{f}(i)$, i.e. such that there is k_i such that $q_i \Vdash \dot{f}(i) = \check{k}_i$. Let $g(i) = k_i$ and show that this is what we are looking for.

Zad. 4 Let κ, λ be ordinal numbers. Let \mathbb{P} be a ccc forcing notion. Suppose that

$$1_{\mathbb{P}} \Vdash \dot{f}: \check{\kappa} \rightarrow \check{\lambda} \text{ is a function.}$$

Then there is $F \subseteq \kappa \times \lambda$, $F \in V$, such that F_ξ is countable for each $\xi < \kappa$ such that

$$1_{\mathbb{P}} \Vdash \dot{f} \subseteq \check{F}.$$

Conclude that whenever $f: \kappa \rightarrow \lambda$, $f \in V[G]$ (where G is a \mathbb{P} -generic and \mathbb{P} is ccc), there is $F \in V$ like above such that $V[G] \models f \subseteq F$.

Zad. 5 (Jensen's covering lemma) Let \mathbb{P} be a ccc forcing notion and let λ be an infinite cardinal number. If

$$1_{\mathbb{P}} \Vdash \dot{A} \subseteq \lambda, \dot{A} \text{ is infinite}$$

then there is $B \subseteq \lambda$ such that

$$1_{\mathbb{P}} \Vdash \dot{A} \subseteq B \wedge |\dot{A}| = |B|.$$

Hint, use the previous exercise.

Zad. 6 Show that if r is a real added by \mathbb{C}_{ω_2} , then it is added by forcing with a single Cohen real (i.e. there is a countable $X \subseteq \omega_2$ such that if

$$\mathbb{C}_X = \{p: p: X \rightarrow \{0, 1\}, \text{dom}(p) \text{ is finite}\}$$

and G is a \mathbb{C}_X -generic, then $f \in V[G]$).

Zad. 7 (Maximal Principle) Suppose $p \Vdash \exists \dot{x} \varphi(\dot{x})$. Then there is a name \dot{x} such that $p \Vdash \varphi(\dot{x})$.

Zad. 8 Let $X \in V$. Suppose that $p \Vdash \exists x \in \check{X} \varphi(x)$. Notice that it may happen that there is no $r \in X$ such that $p \Vdash \varphi(\check{r})$. However, there is $q \leq p$ and $r \in X$ such that $q \Vdash \varphi(\check{r})$.