Forcing 4 2025

Zad. 1 Let $i\colon \mathbb{P} \to \mathbb{Q}$ be a dense embedding. Let H be a $\mathbb{Q}$-generic and let $G = i^{-1}[H]$. Show that G is a generic filter.

Zad. 2 Show that every countable (separative) forcing notion is forcing equivalent to the Cohen forcing.

Zad. 3 Suppose that we have $M \subseteq N$ and $N \models \exists a \in \mathbb{R} \setminus M$. Show that whenever $N \models \mathbb{R} \cap M$ is measurable, then $N \models \mathbb{R} \cap M$ is null. (Hint: use Steinhaus theorem)

Zad. 4 Check if

$$[\![\exists n \in \omega\ \varphi(n)]\!] = \bigvee_n [\![\varphi(n)]\!],$$

$$[\![\neg\varphi]\!] = [\![\varphi]\!]^c$$

Zad. 5 Let G be a $\mathbb{C}$-generic (where $\mathbb{C}$ is the Cohen forcing). Show an example of a new real in $V[G]$ which is not a Cohen real (i.e. which does not avoid all ground model meager sets).

Zad. 6 Let G be as above. Show an example of an open subsets of 2^ω, in $V[G]$, which is not coded in the ground model.

Zad. 7 Propose some coding for Borel sets.

Zad. 8 Let $\mathcal{A}$ be an infinite maximal pairwise almost disjoint family. Define a forcing which will add an infinite subset of ω which is almost disjoint with each element of $\mathcal{A}$. What kind of properties this forcing has?

Zad. 9 Consider the forcing by $\mathbb{P} = \mathcal{P}(\omega)/fin$ (so the Boolean algebra of equivalence classes of the relation $A \sim B \iff A \triangle B$ is finite). Let G be a $\mathbb{P}$-generic. Let $\mathcal{U}$ be an ultrafilter (in the extension) defined by $U \in \mathcal{U}$ is $[U]_\sim \in G$.

- Show that $\mathbb{P}$ is σ-closed.
- Show that $\mathcal{U}$ is a non-principal ultrafilter.
- Show that $\mathcal{U}$ is a selective ultrafilter, i.e. if we color $[\omega]^2$ by two colors, then there is $X \in \mathcal{U}$ which is homogenuous, i.e. all elements of $[X]^2$ have one color. (Hint: by σ-closedness every such coloring is in the ground model).

Zad. 10 Let G be as above and let c be the generic real. Let $A \subseteq 2^\omega$, $A \in \Sigma^0_\alpha$. Show that there is $A' \in V$, $A' \in \Sigma^0_\alpha$, $A' \subseteq 2^\omega \times 2^\omega$ such that

$$A'_c = A.$$

Hint: consider a universal Σ^0_α set U. In $V[G]$: let x be such that $U_x = A$. Find $f\colon D \to 2^\omega$ such that $f(c) = x$. Cook up A' using U and g.

Zad. 11 Show that Cohen forcing $\mathbb{C}$ adds $\mathfrak{c}$ many Cohen reals.

Zad. 12 Show that the random forcing $\mathbb{M} = \mathrm{Bor}(2^\omega)/\mathcal{N}$ is not σ-centered.

Zad. 13 Let $\mathbb{M}_{\omega_2} = \mathrm{Bor}(2^{\omega_2})_{/\lambda_{\omega_2}=0}$. Let G be $\mathbb{M}_{\omega_2}$-generic and assume that $V \models CH$. Show that $V[G] \models \mathfrak{c} = \omega_2$.

Zad. 14 Is the generic real of the random forcing splitting?