Forcing 5 2023

Definition. Given a poset $\mathbb{P}$ add a name for a real $\dot{f}$ (as an element of 2^ω), a $\mathbb{P}$-*interpretation for* $\dot{f}$ is a pair $\langle g, \hat{P}\rangle$ where $g \in 2^\omega$ and $\hat{P}$ is a decreasing sequence $\langle p_n : n \in \omega\rangle \subseteq \mathbb{P}$ such that for all $n \in \omega, p_n \Vdash$ "$\dot{f}|_n = g|_n$".

Zad. 1 Prove that interpretations exist under any condition, i.e. that for every name for a real $\dot{f}$ and $p \in \mathbb{P}$ there is an interpretation $\langle g, \hat{P}\rangle$ for $\dot{f}$ such that, if $\hat{P} = \langle p_n : n \in \omega\rangle$, then for all $n \in \omega$, $p_n \leq p$.

Zad. 2 Assume that $\dot{f}$ is a $\mathbb{P}$-name for an element of 2^ω, $p \in \mathbb{P}$ and $A \subseteq \omega$ such that $p \Vdash A \cap \dot{f}^{-1}(1)$ is infinite. Prove that there is an interpretation $\langle g, \hat{P}\rangle$ for $\dot{f}$ such that $A \cap g^{-1}(1)$ is infinite.

Zad. 3 Prove that if $\mathbb{P}$ is a poset such that $|\mathbb{P}| < \mathfrak{d}$ then $\mathbb{P}$ does not add dominating reals (Hint: Use interpretations to try to guess the name of a potential function).

Zad. 4 Observe that the previous exercise is false if we change $\mathfrak{d}$ for $\mathfrak{b}$ and dominating real for unbounded real (i.e. the analogue of the previous exercise for $\mathfrak{b}$ is false).

Zad. 5 (Destructibility of MAD families) Let $\mathcal{A}$ be a MAD family extending $\{\{x|_n : n \in \omega\} : x \in 2^\omega\}$ and let $\mathbb{P}$ be a poset that adds a new real. Prove that $1 \Vdash$ "$\mathcal{A}$ is not a MAD family" (Hint: Use Zad. 2).

Definition (Kurilić) An AD family $\mathcal{A}$ is ω-MAD if for every sequence $\langle X_n : n \in \omega\rangle \subseteq \mathcal{P}(\omega)$ of sets that cannot be covered by finitely many elements of $\mathcal{A}$ there is an $A \in \mathcal{A}$ such that, for every $n \in \omega$, $|A \cap X_n|$ is infinite.

Zad. 6 Prove that ω-MAD families are MAD families.

Zad. 7 Prove that, under CH, there is an ω-MAD family.

Zad. 8 (ω-MAD families are Cohen indestructible) Prove that, if $\mathcal{A}$ is ω-MAD, then $\mathbb{C} \Vdash$ "$\mathcal{A}$ is MAD" (Hint: Use interpretations!!!)

Zad. 9 Extend the previous result to $\mathbb{C}_\kappa$.