Forcing **6** 2023

Zad. 1 Show that if $\pi\colon \mathbb{Q} \to \mathbb{P}$ is a projection, then $e\colon \mathrm{RO}(\mathbb{P}) \to \mathrm{RO}(\mathbb{Q})$ defined by

$$e(A) = \pi^{-1}[A]$$

is a complete embedding.

Zad. 2 Silver $\mathbb{SI}$ consists of all functions $f\colon \omega \to 2$ with co-infinite domains. Show that $\mathbb{SI}$ has the Sacks property.

Zad. 3 Show that product of two ccc forcing notions is not necessarily ccc (Hint: think about Suslin trees).

Zad. 4 Prove that, for ccc forcings $\mathbb{P}$ and $\mathbb{Q}$, $\mathbb{P}\times\mathbb{Q}$ is ccc if and only if $1_{\mathbb{P}} \Vdash$ "$\mathbb{Q}$ is ccc".

Zad. 5 Prove that Sacks forcing is forcing equivalent to $\mathrm{Bor}(2^\omega)/[(2^\omega)]^{<\omega}$ with the order $[A] \leq [B]$ if and only if $A \subseteq B$. (*Hint: Recall that all uncountable Borel sets contain an uncountable perfect set)*

Zad. 6 Show that product of two σ-closed forcing notions is σ-closed.

Zad. 7 Let $\mathbb{P}$, $\mathbb{Q}$ be forcing notions. Show that $\mathbb{P}$ completely embeds into $\mathbb{P} \times \mathbb{Q}$ and that $\mathbb{P} \times \mathbb{Q}$ is forcing equivalent to $\mathbb{Q} \times \mathbb{P}$.

Zad. 8 Show that $\mathbb{C}_D = \mathbb{C}_A \times \mathbb{C}_B$, where $D = A \cup B$, A, B - disjoint and $\mathbb{C}_X$ consists of functions $f\colon X \to 2$ of finite domain.

Definition. A tree $T \subseteq \omega^{<\omega}$ is *super perfect* if the set of infinitely splitting nodes is dense. The *Miller forcing* $\mathbb{M}$ is the set of all super perfect trees, ordered by contention ($T \leq S$ iff $T \subseteq S$).

Zad. 9 Find a way to define a generic real for Miller forcing.

Zad. 10 Prove that the generic real for Miller forcing is unbounded, but not dominating. Compare this with the generic real for Sacks forcing.

Zad. 11 Prove that Miller forcing does not collapse ω_1. (Hint: Look at the proof for Sacks forcing).

Zad. 12 Prove that, assuming CH, Miller forcing does not collapse cardinals.