Forcing **5** 2025

Zad. 1 Consider the forcing $\mathbb{C}^\kappa_\lambda$ of partial functions $f\colon \kappa \times \lambda \to 2$ with $|\mathrm{dom}(f)| < \lambda$ (here, $\kappa > \lambda$ are regular cardinals). Show that $\mathbb{C}^\kappa_\lambda$ does not collapse cardinals and that $M[G] \models 2^\lambda = \kappa$.

Zad. 2 Prove that it is consistent that $2^{\aleph_{67}} = \aleph_{69}$ and $2^{\aleph_{420}} = \aleph_{666}$ at the same time. Hint: use the previous exercise and product.

Zad. 3 Show that $\mathbb{P} \times \mathbb{Q}$ is forcing equivalent to $\mathbb{Q} \times \mathbb{P}$.

Zad. 4 Show that product of two ccc forcing notions is not necessarily ccc (Hint: think about Suslin trees).

Zad. 5 Show that the product of two σ-closed forcing notions is σ-closed.

Zad. 6 Show that the product of two ccc forcing notions $\mathbb{P}$ and $\mathbb{Q}$ is ccc iff $1 \Vdash_{\mathbb{P}} \mathbb{Q}$is ccc.

Zad. 7 Show that $\mathbb{C}_D = \mathbb{C}_A \times \mathbb{C}_B$, where $D = A \cup B$, A, B - disjoint and $\mathbb{C}_X$ consists of functions $f\colon X \to 2$ of finite domain.

Zad. 8 Let $\langle \mathbb{P}_n \rangle$ be a sequence of separative forcings. Let $\mathbb{P} = \{p \in \prod_{n\in\omega} \mathbb{P}_n : \forall^\infty n \in \omega\ (p(n) = 1_{\mathbb{P}_n})\}$ with the order $p \leq p'$ if and only if $p(n) \leq p'(n)$ for all $n \in \omega$.

a) Prove that, if each $\prod_{i<n} \mathbb{P}_i$ is c.c.c., then $\mathbb{P}$ is c.c.c.

b) Prove that $\mathbb{P}$ adds a Cohen real.

Zad. 9 Consider ω with the probability measure defined by $\mu(A) = \sum_{n\in A} \frac{1}{2^{n+1}}$, and let λ be the product measure on ω^ω. Let $M \subseteq N$ be two transitive models of ZFC and let $f \in \omega^\omega \cap N$ be a random real over M.

a) Prove that then there is $g \in M \cap \omega^\omega$ such that $f \leq g$.

b) Prove that, for all $g \in M \cap \omega^\omega$, $f \cap g$ is finite (i.e. f and g only agree on finitely many values).

Zad. 10 Let $\mathbb{P}$ be a forcing, $n \in \omega$ and $\dot{x}$ be a $\mathbb{P}$-name for an element of n. Let $P \subseteq \mathbb{P}$ be centered, and consider the set $A = \{k \in n : \exists p \in P(p \vdash \text{“}\dot{x} \neq k\text{”})\}$. Show that $A \neq n$.

Zad. 11 Use the previous exercise to prove that, if $\mathbb{P}$ is a σ-centered forcing, $f \in \omega^\omega$ and $\dot{g}$ is a $\mathbb{P}$-name for a function such that $\mathbb{P} \vdash \text{“}\dot{g} \leq f\text{”}$, then there is $h \in \omega^\omega$ such that $\mathbb{P} \vdash \text{“}\dot{g} \cap h$ is infinite”.

Zad. 12 Prove that σ-centered forcings cannot add random reals (as defined in **Zad 9**).

Zad. 13 Let M, N be transitive models such that $M \subseteq N$ and that there is $f \in 2^\omega \cap N \setminus M$. Show that there is a MAD family $\mathcal{A}$ in M such that is no longer MAD in

N. (Hint: Consider any MAD that extends the almost disjoint family of size $\mathfrak{c}$ that you know how to construct)

Zad. 14 Let M, N be transitive models such that N contains a dominating real over M. Prove that there is no MAD families in M that are MAD in N.

Zad. 15 Let $\mathcal{F}$ be a free filter. Consider the forcing

$$\begin{aligned}\mathbb{L}_{\mathcal{F}} = \{T \subseteq \omega^{<\omega} : \ & T \text{ is a tree,} \\ & \forall s \in T(\text{stem}(T) \subseteq s \rightarrow \text{suc}_T(s) \in \mathcal{F})\}\end{aligned}$$

with the order $T \leq T'$ iff $T \subseteq T'$, where $\text{suc}_T(s) = \{n \in \omega : s \cup \langle |s|, n\rangle \in T\}$ and $\text{stem}(T)$ is the largest node in T that is $\subseteq$-comparable with every other node in T.

a) Prove that $\mathbb{L}_{\mathcal{F}}$ is c.c.c.

b) Prove that $\mathbb{L}_{\mathcal{F}}$ does not add a random real (as defined in **Zad 9**).

c) Prove that $\mathbb{L}_{\mathcal{F}}$ adds a dominating real.