Ideals with bases of unbounded Borel hierarchy

Piotr Borodulin–Nadzieja, Szymon Głąb

Abstract

We present several naturally defined ideals which have a Borel base but, unlike the classical examples of such ideals, not of bounded Borel complexity. We investigate its basic properties. We show that they have property (M) and we compute some of their cardinal coefficients.

1 Introduction

The famous Fubini theorem says that a subset of plane is of measure 0 if and only if the set of its non–negligible sections is negligible. It is natural to ask about an ideal, consisting of subsets of plane of whose every section is negligible. It was done by Gabriel Mokobodzki (see [10]). He investigated ideal of sets which can be contained in an open set, whose sections are of measure less then ε for every $\varepsilon > 0$.

In [10] David Fremlin formulated a conjecture that under Martin's Axiom the additivity of this ideal equals continuum. It was verified negatively by Cichoń and Pawlikowski in [7]. Surprisingly, the additivity of Mokobodzki's ideal turned out to be equal ω_1, regardless of additional set theoretic axioms.

In the paper we will focus on ideals of subsets of $[0,1]^2$ which can be contained in a Borel sets whose every section is small (eg. of measure 0, meager). These ideals also have additivity ω_1. Cichoń and Pawlikowski proved that they have one more interesting property: their bases cannot be of bounded Borel complexity. In other words, $\mathcal{M}_\alpha \subsetneq \mathcal{M}$, where $\mathcal{M}$ is such an ideal and $\mathcal{M}_\alpha$ is the ideal generated by $\mathcal{M} \cap \Sigma^0_\alpha$.

In this paper we prove that in fact, using the above terminology, we have $\mathcal{M}_\alpha \subsetneq \mathcal{M}_{\alpha+1}$ for every $1 < \alpha < \omega_1$. We consider several similarly defined ideals and prove that their behavior is similar.

This phenomenon occurs when we set a topological condition globally and a *smallness* condition locally everywhere (e.g. on every section, in every direction) or almost everywhere (i.e. with respect to an ideal with property (M)).

The following fact due to Fremlin (for the proof see [4]) shows that this phenomenon vanishes when we set the topological condition on every section.

Theorem 1.1 *For every Borel set $A \subseteq [0,1]^2$ and every $\epsilon > 0$ we will find a Borel cover B of A such that B_x is open for every x and $\lambda((B \setminus A)_x) \leq \varepsilon$. Moreover if A_x is meager*

2000 *Mathematics Subject Classification*: Primary 54A35, 28A05.

Keywords: ideals on the plane, Mokobodzki ideal, Fubini theorem, Fubini products, uniformization, property (M).

for all x then there is a sequence of Borel sets B_n such that all sections of its elements are closed nowhere dense and $A \subseteq \bigcup B_n$.

2 Preliminaries

All terminology which is not explained here can be found e.g. in [12] (concerning descriptive set theory) and [6] (concerning cardinal coefficients).

By $\Sigma^0_\alpha(X)$, $\Pi^0_\alpha(X)$ and Borel(X) we mean the families of, respectively, Σ^0_α, Π^0_α and Borel subsets of X. Usually, X is clear from the context and we omit it.

Let $\mathcal{I}$ be an ideal on X and $\mathcal{J}$ - an ideal on Y. By Fubini product $\mathcal{I} \otimes \mathcal{J}$ we understand the ideal on $X \times Y$ generated by the family $\{B \in \text{Borel}(X \times Y)\colon \{x\colon B_x \notin \mathcal{I}\} \in \mathcal{J}\}$.

We say that an ideal $\mathcal{I}$ is Borel on Borel (Π^0_α on Π^0_α, Σ^0_α on Σ^0_α) if for every Borel (Π^0_α, Σ^0_α) set B

$$\{x\colon B_x \in \mathcal{I}\} \in \text{Borel } (\Pi^0_\alpha, \Sigma^0_\alpha).$$

For an ordinal number $\alpha \leq \omega_1$ and a σ–ideal $\mathcal{J}$ with a Borel base by $\mathcal{J}_\alpha$ we denote the σ–ideal generated by $\mathcal{J} \cap \Sigma^0_\alpha$.

Let $\mathcal{I}$ be an ideal of subsets of the unit interval. We say that $A \subseteq [0,1]^2$ is in the ideal $\mathcal{M}(\mathcal{I})$ if there is a Borel set $B \supseteq A$ such that $B_x \in \mathcal{I}$ for every $x \in [0,1]$. We write $\mathcal{M}_\alpha(\mathcal{I})$ instead of $(\mathcal{M}(\mathcal{I}))_\alpha$.

We can generalize the ideal $\mathcal{M}(\mathcal{I})$ in two ways.

We can consider ideals of subsets of $[0,1]^2$, which are *small* not only on vertical but also on horizontal sections or which are *small* in every direction.

On the other hand, we can think about ideals which are small not on every section but on *almost* every section.

Seeking natural generalizations covering all the above cases we were led to the following considerations.

Mokobodzki sub–ideals of $\mathcal{M}$.

For x, $y \in [0,1]$ let $x \oplus y = x + y - \lfloor x+y \rfloor$, where $\lfloor x \rfloor$ stands for the greatest integer smaller than x. For functions f, $g\colon [0,1] \to [0,1]$ define $f \oplus g(x) = f(x) \oplus g(x)$ and let $\ominus$ denote the operation inverse to $\oplus$.

A family of functions $\mathcal{F} \subseteq [0,1]^{[0,1]}$ is *ubiquitous with respect to an ideal $\mathcal{I}$* (or $\mathcal{I}$–ubiquitous) if for every Borel function $g : [0,1] \to [0,1]$ and every Borel $A \notin \mathcal{I}$ there is a Borel set $A \supseteq B \notin \mathcal{I}$ and a function $f \in \mathcal{F}$ such that $f|B = g|B$. The natural example of $\mathcal{N}ull$– and $\mathcal{M}eager$–ubiquitous family is the family of continuous functions.

There are many families of Borel functions which are closed under $\oplus$ but which are not ubiquitous neither with respect to $\mathcal{N}ull$ nor to $\mathcal{M}eager$ ideals: constant functions, linear functions, Lipschitz functions.

For an ideal $\mathcal{I}$ and a family of functions $\mathcal{F}$ define $\mathcal{M}(\mathcal{F},\mathcal{I})$ in the following way. A set $Y \subseteq [0,1]^2$ is in $\mathcal{M}(\mathcal{F},\mathcal{I})$ if $Y \in \mathcal{M}(\mathcal{I})$ and it can be covered by a Borel set $B \subseteq [0,1]^2$ such that $\{x\colon f(x) \in B_x\} \in \mathcal{I}$ for every $f \in \mathcal{F}$. As before, $\mathcal{M}_\alpha(\mathcal{F},\mathcal{I}) = (\mathcal{M}(\mathcal{F},\mathcal{I}))_\alpha$.

In Section 3 we discuss the properties of ideals of this form.

Mokobodzki ideals including $\mathcal{M}$.

We can see $\mathcal{M}(\mathcal{I})$ as a Fubini product of $\mathcal{I}$ and the ideal containing only the empty set. So, we can ask if $\{\emptyset\}$ can replaced here by some other ideals without destroying the property of having only unbounded bases.

In Section 4 we prove that if $\mathcal{J}$ resembles in a way the trivial ideal, then $\mathcal{I} \otimes \mathcal{J}$ has properties similar to $\mathcal{M}(\mathcal{I})$. More precisely, we show that if an ideal $\mathcal{J}$ has property (M), then $(\mathcal{I} \otimes \mathcal{J})_\alpha$ forms a strictly increasing sequence.

We will call all of the ideals presented above *Mokobodzki ideals* (for the precise definition and a discussion confront Section 4).

We will show that all Mokobodzki ideals do not have base of a bounded Borel hierarchy. In fact, we will show that if $\mathcal{M}$ is any of the above ideals, then $(\mathcal{M}_\alpha)$ forms a strictly increasing (at least, from some α on) sequence of ideals.

The proof is based on the following theorems of Pawlikowski & Cichoń (see [7]) and Holicky (see [11]).

One of the well–known uniformization theorems states that if every nonempty section of a Borel set $A \subseteq [0,1]^2$ is not in $\mathcal{I}$ (where $\mathcal{I}$ is a Borel on Borel ideal), then A has a Borel uniformization. Recently Petr Holicky proved a theorem which gives an information about the Borel class of this uniformization.

Theorem 2.1 (Holicky) *Let $\mathcal{I}$ be a Π^0_α on Π^0_α σ–ideal containing all singletons for $\alpha < \omega_1$. Let $A \subseteq [0,1]^2$ be such that $A_x \notin \mathcal{I}$ for every $x \in \pi_x[A]$. If A is of the class Π^0_α, then there is a function $F : \{0,1\}^\omega \times [0,1] \to [0,1]$ such that:*

$$\forall x \in \{0,1\}^\omega \ \ (y \mapsto F(x,y) \text{ is a } \Sigma^0_\alpha \text{ uniformization of } A),$$

$$\forall y \in [0,1] \ \ (x \mapsto F(x,y) \text{ is continuous and 1-1}).$$

The following theorem was proved in [7]. Since we will use a similar argument in the next sections we repeat the short proof of this theorem.

Theorem 2.2 (Cichoń, Pawlikowski) *Assume $\mathcal{I}$ is a σ–ideal of subsets of $[0,1]$ such that $[0,1] \notin \mathcal{I}$. For every $\alpha < \omega_1$ there is a Π^0_α set $A \subseteq [0,1]^2$ such that for every $M \in \mathcal{M}_\alpha(\mathcal{I})$ there is $x \in [0,1]$ such that $\emptyset \neq A_x \subseteq M^c_x$.*

Proof. Let $\alpha < \omega_1$. Let $U \subseteq [0,1] \times [0,1]^2$ be an universal set for $\Sigma^0_\alpha([0,1]^2)$, i.e. for every $E \subseteq [0,1]^2$ such that $E \in \Sigma^0_\alpha$ there is $x \in [0,1]$ with $E = U_x$. Put $A = \{(x,y) \in [0,1]^2 : (x,x,y) \notin U\}$.

Clearly, A is a Π^0_α subset of $[0,1]^2$. For each $E \in \Sigma^0_\alpha$ there exists $x \in [0,1]$ such that $E = U_x$. If all sections of E are in $\mathcal{I}$, then $E_x = (U_x)_x = U_{(x,x)} = [0,1] \setminus A_x$.

■

3 Sub–ideals of $\mathcal{M}(\mathcal{I})$

In this section $\mathcal{I}$ is either the ideal of null sets or the ideal of meager sets. We assume it because we need $\mathcal{I}$ to have the following properties (although some lemmas can be stated in a more general form):

- $\mathcal{I}$ is Π^0_α on Π^0_α for $\alpha < \omega_1$,
- $\mathcal{I}$ has Fubini property, i.e.

$$\{x \in [0,1]\colon B_x \notin \mathcal{I}\} \in \mathcal{I} \text{ iff } \{y \in [0,1]\colon B^y \notin \mathcal{I}\} \in \mathcal{I},$$

- $\mathcal{I}$ is $\oplus$-invariant,
- $\mathcal{I}$ is ccc, i.e. the quotient Boolean algebra $Borel/\mathcal{I}$ is ccc.

By the result of Farah & Zapletal (see [9]) these properties implies that $\mathcal{I}$ is isomorphic either to the ideal of null sets or to the ideal of meager sets on $[0,1]$.
We will show that if a family $\mathcal{F}$ of Borel functions is closed under $\oplus$ and is not $\mathcal{I}$–ubiquitous, then Π^0_α sets has $\Pi^0_{\alpha+1}$ uniformizations which are not in $\mathcal{F}$ on any Borel set from $\mathcal{I}^+$.

Lemma 3.1 *Assume that A is a Borel subset of $[0,1]^2$ such that $A_x \notin \mathcal{I}$ for every $x \in \pi[A] \notin \mathcal{I}$. For each Borel $h\colon [0,1] \to [0,1]$ we can find a Borel set $\pi[A] \supseteq B \notin \mathcal{I}$ and $y \in [0,1]$ such that $h(x) \oplus y \in A_x$ for every $x \in B$.*

Proof. Define $\varphi\colon [0,1]^2 \to [0,1]^2$ by

$$\varphi(x,y) = (x, y \ominus h(x)).$$

Then $\varphi[A] = \{(x, y \ominus h(x))\colon (x,y) \in A\}$ and for every x

$$(\varphi[A])_x = \{y \ominus h(x)\colon y \in A_x\} = A_x \ominus h(x).$$

Since $\mathcal{I}$ is $\oplus$–invariant

$$(\varphi[A])_x \in \mathcal{I} \text{ iff } A_x \in \mathcal{I}.$$

and thus $(\varphi[A])_x \notin \mathcal{I}$ for every $x \in \pi[A] \notin \mathcal{I}$. The set $\varphi[A]$ is Borel because φ is a Borel injection. By the Fubini property of $\mathcal{I}$ there is $y \in [0,1]$ such that

$$B = (\varphi[A])^y \in Borel \setminus \mathcal{I}.$$

For $x \in B$ define

$$f(x) = h(x) \oplus y.$$

If $x \in B$, then $(x,y) \in \varphi[A]$. Therefore $(x, y \oplus h(x) \ominus h(x)) \in \varphi[A]$, but this means that $(x, y \oplus h(x)) \in A$. So, $h(x) \oplus y \in A_x$ for every $x \in B$.

■

Lemma 3.2 *Assume $A \subseteq [0,1]^2$ is a Π^0_α set such that $A_x \notin \mathcal{I}$ for every $x \in [0,1]$. Then, for a Borel function $h\colon [0,1] \to [0,1]$ there is a countable collection of pairwise disjoint Π^0_2 sets $\mathcal{B}$ and a uniformization $f\colon [0,1] \to [0,1]$ of A such that*

(i) $[0,1] \setminus \bigcup \mathcal{B} \in \mathcal{I}$,

(ii) for every $B \in \mathcal{B}$ *there is* $y_B \in [0,1]$ *such that* $f|B = h \oplus y_B$*;*

(iii) graph(f) *is* $\Sigma^0_{\alpha+1}$*;*

Proof. Fix $A \subseteq [0,1]^2$ and $h\colon [0,1] \to [0,1]$ as above. Use Theorem 2.1 to set a Π^0_α uniformization $g\colon [0,1] \to [0,1]$ of A. Use Lemma 3.1 to find B, f and y for the set A and a function h. Since Π^0_α functions are $\mathcal{I}$–ubiquitous, we can assume that $f|B$ is Π^0_α, shrinking B if needed.

Assume now that we have constructed the demanded B_ξ, f_ξ, y_ξ for every $\xi < \alpha$.

If $X = [0,1] \setminus \bigcup_{\xi<\alpha} B_\xi \notin \mathcal{I}$, then use Lemma 3.1 for $A \cap (X \times [0,1])$ and h to find B_α, f_α and y_α in the same way as above. Since $\mathcal{I}$ is ccc, there is a countable α such that $[0,1] \setminus \bigcup_{\xi<\alpha} B_\xi \in \mathcal{I}$. Let $\mathcal{B} = \{B_\xi\colon \xi < \alpha\}$. Define $f\colon [0,1] \to [0,1]$ in the following way. Let $f(x) = f_\xi(x)$ if $x \in B_\xi$ and $f(x) = g(x)$ for these x which are not contained in $\bigcup \mathcal{B}$. The graph of such defined f is $\Sigma^0_{\alpha+1}$ since it is a countable union of graphs of Π^0_α and Σ^0_α functions and since we can shrink $\bigcup \mathcal{B}$ to a Π^0_2 set if needed.

■

Notice that $\mathcal{I}$ is Σ^0_α on Σ^0_α, so we can assume that all sections of A are not in $\mathcal{I}$, taking for the set U in the proof a set universal for $\Sigma^0_\alpha \setminus \mathcal{I}$. This set was used in [7] to prove that $\mathcal{M}$ is of additivity ω_1. In [7] it was mentioned that this set can be used to prove that the hierarchy of $\mathcal{M}_\alpha$ is non–trivial. Holicky's result (Theorem 2.1) allows us to be more precise.

Theorem 3.3 *Let* $\mathcal{F} \subseteq [0,1]^{[0,1]}$ *be a family of functions which is not* $\mathcal{I}$*-ubiquitous. Assume that* $\mathcal{F}$ *is closed under adding constant functions, i.e. for any* $f \in \mathcal{F}$ *and* $y \in [0,1]$ *a function* $x \mapsto f(x) \oplus y$ *belongs to* $\mathcal{F}$*. Then* $\mathcal{M}_{\alpha+1}(\mathcal{F},\mathcal{I}) \setminus \mathcal{M}_\alpha(\mathcal{I}) \neq \emptyset$ *for every* $\alpha < \omega_1$*.*

Proof. Let $\alpha < \omega_1$. Let A be Π^0_α set whose existence is guaranteed by Theorem 2.2.

Let h be a Borel function witnessing that $\mathcal{F}$ is not $\mathcal{I}$-ubiquitous, i.e. there are no Borel set $B \notin \mathcal{I}$ and $g \in \mathcal{F}$ with $g|B = h|B$.

Use Lemma 3.2 for A and h to find a Π^0_α uniformization $f\colon [0,1] \to [0,1]$ of A and countable collection $\mathcal{B}$ of pairwise disjoint Borel sets satisfying conditions (i)–(iii) of Lemma 3.2.

Suppose that graph$(f) \notin \mathcal{M}_{\alpha+1}(\mathcal{F},\mathcal{I})$. Then, by the definition of $\mathcal{M}_{\alpha+1}(\mathcal{F},\mathcal{I})$, for every $C \in \Sigma^0_{\alpha+1}$ with $C \supseteq$ graph(f) there is $g \in \mathcal{F}$ such that $\{x : g(x) \in C_x\} \notin \mathcal{I}$. Since graph$(f) \in \Sigma^0_{\alpha+1}$, there is $g \in \mathcal{F}$ with $\{x : g(x) \in \text{graph}(f)_x\} = \{x : g(x) = f(x)\} \notin \mathcal{I}$. Let $B' = \{x : g(x) = f(x)\}$. There is $B \in \mathcal{B}$ with $B' \cap B \notin \mathcal{I}$. Since $f|B = (h \oplus y_B)|B$, we have $g|(B' \cap B) = (h \oplus y_B)|(B' \cap B)$. As $\mathcal{F}$ is closed under adding constant functions, $h \oplus y_B \in \mathcal{F}$, but this contradicts our assumption on h. Hence graph$(f) \in \mathcal{M}_{\alpha+1}(\mathcal{F},\mathcal{I})$.

By Theorem 2.2 for every $M \in \mathcal{M}_\alpha(\mathcal{I})$ there is $x \in [0,1]$ such that $f(x) \notin M_x$. So, the graph of f is in $\mathcal{M}_{\alpha+1}(\mathcal{F},\mathcal{I}) \setminus \mathcal{M}_\alpha(\mathcal{I})$.

■

Since $\mathcal{M}_\alpha(\mathcal{F},\mathcal{I}) \subseteq \mathcal{M}_\alpha(\mathcal{I})$ for every $\alpha < \omega_1$, the following theorem holds.

Conclusion 3.4 *Consider* $\mathcal{I}$ *and* $\mathcal{F}$ *as in the above theorem. Then for every* $1 < \alpha < \omega_1$ *we have* $\mathcal{M}_\alpha(\mathcal{F},\mathcal{I}) \subsetneq \mathcal{M}_{\alpha+1}(\mathcal{F},\mathcal{I})$*. In particular,* $\mathcal{M}_\alpha(\mathcal{I}) \subsetneq \mathcal{M}_{\alpha+1}(\mathcal{I})$*.*

In fact, one can prove that for every $\alpha<\omega_1$ we have $\mathcal{M}_\alpha(\mathcal{I})\subsetneq\mathcal{M}_{\alpha+1}(\mathcal{I})$ assuming only that $\mathcal{I}$ is a σ–ideal with a Borel base, which is Π^0_ξ on Π^0_ξ for every $\xi<\omega_1$.

Conclusion 3.5 *The following families of ideals form a strictly increasing hierarchy (with respect to α) for $\alpha>1$:*

- *ideals of sets which can be covered by Σ^0_α subsets whose all vertical sections are null (meager);*
- *ideals of sets which can be covered by Σ^0_α sets whose all vertical and horizontal sections are null (meager);*
- *ideals of sets which can be covered by Σ^0_α sets which are null (meager) in every direction;*
- *ideals of sets which can be covered by Σ^0_α sets which are null (meager) on vertical sections and on graphs of Lipschitz functions.*

4 Fubini products

In the previous section we were looking for nice sub–ideals of $\mathcal{M}(\mathcal{I})$, with the property of generating strictly increasing sequence of ideals. Now, we will try to find this property among ideals containing $\mathcal{M}(\mathcal{I})$.

Of course $\mathcal{M}(\mathcal{I})$ can be viewed as $\mathcal{I}\otimes\{\emptyset\}$. Usually Fubini products of null or meager ideals have base of bounded Borel hierarchy (e.g. $\mathcal{N}ull\otimes\mathcal{M}eager$, $\mathcal{N}ull\otimes\mathcal{N}ull$), so we cannot replace $\{\emptyset\}$ by $\mathcal{N}ull$ or $\mathcal{M}eager$ if we want obtain an ideal with a base of unbounded Borel hierarchy. However, we will show that the ideal of the form $\mathcal{N}ull\otimes\mathcal{J}$ ($\mathcal{M}eager\otimes\mathcal{J}$) does not have a base of bounded Borel hierarchy if $\mathcal{J}$ in a way resembles the trivial ideal. Namely, if it has the following property.

We will say that an ideal $\mathcal{J}$ of subsets of X has *property (M)* if there is a Borel function $f\colon X\to[0,1]$ such that $f^{-1}(x)\notin\mathcal{J}$ for every $x\in[0,1]$.

Theorem 4.1 *Assume that $\mathcal{I}$ is Π^0_α on Π^0_α for $\alpha<\omega_1$. If $\mathcal{J}$ has property (M) then there is $\beta<\omega_1$ such that the sequence $(\mathcal{I}\otimes\mathcal{J})_\alpha$ is strictly increasing for $\alpha>\beta$.*

Proof. Let $f\colon[0,1]\to[0,1]$ be a Borel function witnessing (M) for $\mathcal{J}$, i.e. such that

$$f^{-1}[\{x\}]\notin\mathcal{J}\text{ for every }x\in[0,1].$$

Let β be such that $f\in\Sigma^0_\beta$. Fix $\beta<\alpha<\omega_1$ and let $U\subseteq[0,1]\times[0,1]^2$ be universal for $\Sigma^0_\alpha([0,1]^2)$. Define $H\colon[0,1]^3\to[0,1]^3$ by

$$H(x,y,z)=(f(x),y,z),$$

let $V=H^{-1}(U)$ and consider $A=\{(x,y)\in[0,1]^2\colon(x,x,y)\notin V\}$.

Notice that H is Σ^0_β and thus, since $\alpha>\beta$, A is Π^0_α. By Theorem 2.1 the set A has a uniformization C of class $\Sigma^0_{\alpha+1}$. We proceed as in Theorem 3.3. Of course $C\in\mathcal{M}_{\alpha+1}(\mathcal{I})$.

We will show that $C \notin (\mathcal{I} \times \mathcal{J})_\alpha$. Indeed, if $E \in \Sigma^0_\alpha([0,1]^2)$, then $E = U_x$ for some $x \in [0,1]$ and $E = V_t$ for each $t \in f^{-1}(x)$. Thus,

$$E_t = (V_t)_t = V_{(t,t)} = [0,1] \setminus A_t.$$

and $C_t \cap E_t = \emptyset$. Since $\mathcal{M}_{\alpha+1}(\mathcal{I}) \subseteq (\mathcal{I} \otimes \mathcal{J})_\alpha$, the set C witnesses that $(\mathcal{I} \otimes \mathcal{J})_{\alpha+1} \setminus (\mathcal{I} \otimes \mathcal{J})_\alpha \neq \emptyset$.

■

There are many ideals with property (M), e.g. the ideal of countable sets, ideals generated by an analytic equivalence relation with uncountable many equivalence classes (i.e. ideals of sets which can be covered by countably many equivalence classes), the ideal of subsets of the plane which can be covered by countably many lines (see [8]), ideals on Polish spaces with a Σ^0_2 base which are not ccc (see [13]), some ideals defined by translations (see [3] and [14]). For the further discussion on property (M) see [1], [3] and [5]. The following proposition give a next sort of examples of ideals with property (M):

Proposition 4.2 *The Mokobodzki ideals have property (M).*

Proof. For the ideals of the form $\mathcal{M}(\mathcal{F},\mathcal{I})$ consider the function $f(x,y) = x$. Then $f^{-1}(\{x\}) = \{x\} \times [0,1] \notin \mathcal{M}(\mathcal{F},\mathcal{I})$. Moreover, if a σ–ideal $\mathcal{J}$ has property (M), then $\mathcal{I} \otimes \mathcal{J}$ has (M). To see this consider $g = f \circ \pi_x$, where f is a witness that $\mathcal{J}$ has (M). Hence, every Mokobodzki ideal has property (M).

■

Notice, that this means that Mokobodzki ideals are very far from being ccc.

Conclusion 4.3 *The Fubini product of Mokobodzki ideals is isomorphic to a Mokobodzki ideal. The Fubini product of $\mathcal{N}ull$ ($\mathcal{M}eager$) with a Mokobodzki ideal is isomorphic to a Mokobodzki ideal. For any σ–ideal $\mathcal{K}$ and a σ–ideal with property (M) the ideal $\mathcal{N}ull \otimes (\mathcal{K} \otimes \mathcal{J})$ is isomorphic to a Mokobodzki ideal.*

We finish this section with the following observation. We defined Mokobodzki ideal as an ideal of the one of the two forms: either $\mathcal{M}(\mathcal{F},\mathcal{I})$ (as in Section 3) or $\mathcal{I} \otimes \mathcal{J}$ (as in this Section), for appropriate $\mathcal{I}$, $\mathcal{F}$ and $\mathcal{J}$. This may seem quite artificial. In fact, it is possible to combine these two definitions.

For an ideal $\mathcal{I}$ of null or meager sets, a family $\mathcal{F} \subseteq [0,1]^{[0,1]}$ which is not $\mathcal{I}$–ubiquitous and a σ–ideal $\mathcal{J}$ with property (M) say that $Y \in \mathcal{M}(\mathcal{F},\mathcal{I},\mathcal{J})$ if $Y \in \mathcal{I} \otimes \mathcal{J}$ and for every $f \in \mathcal{F}$ the set

$$\{y\colon \{x\colon f(x) \oplus y\} \notin \mathcal{I}\} \in \mathcal{J}.$$

Loosely speaking, this *generalized* Mokobodzki ideal is a σ–ideal with a Borel base which is null (meager) *locally* (e.g. on sections, along linear functions, along Lipschitz functions) everywhere, or everywhere modulo a *very small* (e.g. countable) set.

By combining proofs of Theorem 3.3 and Theorem 4.1 it is possible to show that generalized Mokobodzki ideals also forms an eventually strictly increasing sequence of ideals. We decided to prove it in the two cases separately for the sake of clarity.

5 Set-theoretic properties

In [7] it was proved that $\mathcal{M}(\mathcal{I})$ has additivity ω_1. The proof uses Theorem 2.2, Kondo–Adisson and Sierpiński theorems.

In fact, it seems that Theorem 2.2 is the only tool we need here and that the essential reason for the equality $\mathrm{add}(\mathcal{M}(\mathcal{I})) = \omega_1$ lies in the fact that $\mathcal{M}(\mathcal{I})$ is a union of sequence of ω_1 strictly increasing ideals. More generally, the following fact holds.

Proposition 5.1 *Suppose that $\mathcal{I}$ has a Borel base and there are ideals $\mathcal{I}_\alpha$ with $\mathcal{I} = \bigcup_{\alpha<\kappa} \mathcal{I}_\alpha$ and $\mathcal{I}_\alpha \subsetneq \mathcal{I}_{\alpha+1}$ for every $\alpha < \kappa$. Then $\mathrm{add}(\mathcal{I}) \leq \kappa$. In particular, if $\mathcal{I}$ has a Borel base which is unbounded, i.e. for any $\alpha < \omega_1$ there is $A \in \mathcal{I}$ which is not contained in any set $B \in \mathcal{I} \cap \Sigma^0_\alpha$. Then $\mathrm{add}(\mathcal{I}) = \omega_1$.*

Proof. For every $\alpha < \kappa$ take $A_\alpha \in \mathcal{I}_{\alpha+1} \setminus \mathcal{I}_\alpha$ and consider a set $A = \bigcup_{\alpha<\omega_1} A_\alpha$. If A is in $\mathcal{I}$, then we could find Borel $B \in \mathcal{I}$ with $B \supseteq A$, but there would be $\alpha < \kappa$ with $B \in \mathcal{I}_\alpha$. Then $B \in \mathcal{I}_\alpha$ which contradicts the fact that $A_\alpha \subseteq B$ and $A_\alpha \notin \mathcal{I}_\alpha$.

The second part of the statement follows easily. ∎

Conclusion 5.2 *If $\mathcal{M}$ is a Mokobodzki ideal, then $\mathrm{add}(\mathcal{M}) = \omega_1$.*

Some cardinal coefficients of $\mathcal{M}(\mathcal{I})$ are inherited from the ideal $\mathcal{I}$.

Proposition 5.3 *If $\mathcal{I}$ is an ideal and $\mathcal{F} \subseteq [0,1]^{[0,1]}$. Then*

$$\mathrm{cov}(\mathcal{M}(\mathcal{F},\mathcal{I})) = \mathrm{cov}(\mathcal{I}) \text{ and}$$

$$\mathrm{non}(\mathcal{M}(\mathcal{F},\mathcal{I})) = \mathrm{non}(\mathcal{I}).$$

Proof. If the family $\{A_\xi \colon \xi < \kappa\} \subseteq \mathcal{I}$ covers $[0,1]$ then $\{A_\xi \times A_\eta \colon \xi, \eta < \kappa\} \subseteq \mathcal{M}(\mathcal{F},\mathcal{I})$ and it covers $[0,1]^2$. On the other hand if $\{Z_\xi \colon \xi < \kappa\} \subseteq \mathcal{M}(\mathcal{F},\mathcal{I})$ covers $[0,1]^2$, then $\{(Z_\xi)_0 \colon \xi < \kappa\} \subseteq \mathcal{I}$ and it covers $[0,1]$.

If $X \notin \mathcal{I}$, then $\{0\} \times X \notin \mathcal{M}(\mathcal{F},\mathcal{I})$ and $|\{0\} \times X| = |X|$. On the other hand, if $Z \notin \mathcal{M}(\mathcal{F},\mathcal{I})$, then there is $f \in \mathcal{F}$ such that $\{x \colon f(x) \in Z_x\} \notin \mathcal{I}$, so $|Z| \geq \mathrm{non}(\mathcal{I})$. ∎

In the case of Mokobodzki ideals defined in Section 4 we have the following fact.

Proposition 5.4 (folklore) *If $\mathcal{I}$ and $\mathcal{J}$ are ideals on $[0,1]$, then*

$$\mathrm{non}(\mathcal{I} \otimes \mathcal{J}) = \max\{\mathrm{non}(\mathcal{I}), \mathrm{non}(\mathcal{J})\} \text{ and}$$

$$\mathrm{cov}(\mathcal{I} \otimes \mathcal{J}) = \min\{\mathrm{cov}(\mathcal{I}), \mathrm{cov}(\mathcal{J})\}.$$

Proof. Let X, Y witness respectively $\mathrm{non}(\mathcal{I})$ and $\mathrm{non}(\mathcal{J})$. Then $X \times Y \notin \mathcal{I} \otimes \mathcal{J}$ and $|X \times Y| = \max\{\mathrm{non}(\mathcal{I}), \mathrm{non}(\mathcal{J})\}$. If Z witnesses $\mathrm{non}(\mathcal{I} \otimes \mathcal{J})$, then $|\pi_x(Z)| \geq \mathrm{non}(\mathcal{J})$ and $|\pi^y(Z)| \geq \mathrm{non}(\mathcal{I})$. Therefore, $\mathrm{non}(\mathcal{I} \otimes \mathcal{J}) = \max\{\mathrm{non}(\mathcal{I}), \mathrm{non}(\mathcal{J})\}$.

Assume that $\{A_\xi \colon \xi < \kappa\} \subseteq \mathcal{I}$ covers $[0,1]$. Then, $\{A_\xi \times [0,1] \colon \xi < \kappa\} \subseteq \mathcal{I} \otimes \mathcal{J}$ and it covers $[0,1]^2$. Similar argument for $\mathcal{J}$ shows that $\mathrm{cov}(\mathcal{I} \otimes \mathcal{J}) \leq \min\{\mathrm{cov}(\mathcal{I}), \mathrm{cov}(\mathcal{J})\}$. On the other hand, if $\mathrm{cov}(\mathcal{I}) \leq \mathrm{cov}(\mathcal{J})$ and $\{Z_\xi \colon \xi < \mathrm{cov}(\mathcal{I})\}$ covers $[0,1]^2$, then there is $x \in [0,1]$ such that $(Z_\xi)_x \in \mathcal{I}$ for every $\xi < \kappa$. As $(Z_\xi)_x$ covers $[0,1]$, we have $\mathrm{cov}(\mathcal{I} \otimes \mathcal{J}) \geq \min\{\mathrm{cov}(\mathcal{I}), \mathrm{cov}(\mathcal{J})\}$

■

Pawlikowski proved that $\mathrm{cof}(\mathcal{M}(\mathcal{I})) = \mathfrak{c}$. We do not know if this equality holds also for other Mokobodzki ideals.

To show that $\mathrm{add}(\mathcal{M}) = \omega_1$ we used the fact that $\mathcal{M}$ can be seen as a union of strictly increasing sequence of ω_1 many ideals. In fact, every σ–ideal is a union of strictly increasing κ–sequence of σ–ideals, where κ is a regular cardinal. Indeed, consider a base $\{B_\alpha \colon \alpha < \kappa\}$ of an ideal $\mathcal{I}$. Define $\mathcal{I}_\alpha$ as a smallest σ–ideal containing $\{B_\xi \colon \xi < \alpha\}$. Of course $\mathcal{I}_\kappa = \mathcal{I}$. We can assume without loss of generality that there is $\beta \leq \kappa$ such that the sequence $(\mathcal{I}_\alpha)_{\alpha<\beta}$ is strictly increasing (considering a subsequence if necessary) and that β is a limit ordinal (re-enumerating the base if necessary). Then β has to be of uncountable cofinality, as $\mathcal{I}$ is a σ–ideal.

We can define a coefficient indicating what is the minimal size of such a sequence:

$$\mathrm{cofin}(\mathcal{I}) = \min\{\kappa > \omega \colon \mathcal{I} = \bigcup_{\alpha<\kappa} \mathcal{I}_\alpha \text{ where } (\mathcal{I}_\alpha) \text{ is a strictly increasing}\}$$

The above remark and Proposition 5.1 imply that

$$\mathrm{add}(\mathcal{I}) \leq \mathrm{cofin}(\mathcal{I}) \leq \mathrm{cof}(\mathcal{I}).$$

In fact,

$$\mathrm{cofin}(\mathcal{I}) \leq \min\{|\mathcal{A}| \colon \mathcal{A} \text{ generates } \mathcal{I}\}.$$

Under the negation of CH we have

$$\omega_1 = \mathrm{add}(\mathcal{M}) = \mathrm{cofin}(\mathcal{M}) < \mathrm{cof}(\mathcal{M}) = \mathfrak{c}.$$

Problem 5.5 *Is it provable in* ZFC *that for every σ–ideal $\mathcal{I}$ we have* $\mathrm{add}(\mathcal{I}) = \mathrm{cofin}(\mathcal{I})$*?*

Recall that an ideal has a *hull property* if there is $X \subseteq [0,1]^2$ such that there is no Borel set $B \supseteq X$ such that $X = B$ modulo $\mathcal{M}(\mathcal{I})$. Classical examples of ideals with hull property have the hulls of bounded Borel complexity. It would be interesting to find a natural example of an ideal without the above property. Unfortunately, Mokobodzki ideals do not have the hull property at all.

Proposition 5.6 *If $\mathcal{I}$ is Borel on Borel, then $\mathcal{M}(\mathcal{I})$ does not possess the hull property.*

Proof. Consider $X = V \times [0,1]$, where $V \subseteq [0,1]$ is a Bernstein set. Assume $B \supseteq X$ is a Borel set and consider $L = \{x \in [0,1] \colon B_x \notin \mathcal{I}\}$. Notice that L is Borel. Since $V \subseteq L$, it has to be co–countable. So, there is no $M \in \mathcal{M}(\mathcal{I})$ such that $X = B \setminus M$.

■

6 Property (M) and thin ideals

It is well–known that every Borel function on $[0,1]^2$ is nice on a big domain (Σ^0_2 on a set of measure 1, continuous on a co–meager set). However, in [2] it is proved that this is not the case if instead of $\mathcal{N}ull$ or $\mathcal{M}eager$ ideals we would consider Mokobodzki ideals. Precisely, for every $\alpha > 2$ there is a Borel function $g\colon [0,1]^2 \to [0,1]$ such that for every Borel set $G \in \mathcal{M}(\mathcal{N}ull)$ (or $\mathcal{M}(\mathcal{M}eager)$) we can find $x \in [0,1]$ such that the function $g_x|G^c_x$ is not Σ^0_α (here $g_{x_0}(y) = g(x_0, y)$).
On the other hand (see [2]) these pathological points can be covered by null (or meager) sets, i.e. for every G as above the set

$$\{x\colon g_x|G^c_x \text{ is not } \Sigma^0_\alpha\} \in \mathcal{N}ull \cap \mathcal{M}eager$$

The following notion is connected to the above considerations.
Let $\mathcal{J}$ be a σ-ideal of subsets of $[0,1]$. We say that $\mathcal{J}$ is *thin* if for any $2 \le \alpha < \omega_1$ there is a Borel function $g\colon [0,1]^2 \to [0,1]$ such that for each $G \in \mathcal{M}_\alpha(\mathcal{N}ull) \cup \mathcal{M}_\alpha(\mathcal{M}eager)$ and each $C \in \mathcal{J}$ there is $x_0 \in [0,1] \setminus C$ with $g_{x_0}|([0,1] \setminus G_{x_0})$ is not of the class Σ^0_α.
So, the above theorems can be stated in this form: $\{\emptyset\}$ is thin, whereas $\mathcal{N}ull$ and $\mathcal{M}eager$ are not. We show that all (M) ideals are thin.

Proposition 6.1 *Every σ–ideal with property (M) is thin.*

Proof. Let $A \subseteq [0,1]^2$ be a set defined in the proof of Theorem 4.1. Let $F : \{0,1\} \times [0,1] \to [0,1]$ be a function from Theorem 2.1 for the set A. Let D be a Borel subset of $\{0,1\}^\omega$ with $D \notin \Sigma^0_\alpha(\{0,1\}^\omega)$ and let $\tilde{D} = F(D \times [0,1])$. Since F is one-to-one and Borel, the set $\tilde{D}$ is Borel. Clearly $\tilde{D} \subset A$. Let $g : [0,1]^2 \to [0,1]$ be a characteristic function of $\tilde{D}$.
Let $G \in \mathcal{M}_\alpha$ and let $f\colon [0,1] \to [0,1]$ be a Borel function witnessing (M) for $\mathcal{J}$. There is $x \in [0,1]$ such that for any $t \in f^{-1}[\{x\}]$

$$A_t \cap G_t = \emptyset.$$

Hence for such t we have

$$(g_t|([0,1] \setminus G_t))^{-1}(\{1\}) = F[D \times \{t\}].$$

Since $x \mapsto F(x,t)$ is continuous and 1-1, the set $F[D \times \{t\}]$ is homeomorphic to D. Finally, we obtain that $g_t|([0,1] \setminus G_t)$ is not of the class Σ^0_α.
■

References

[1] M. Balcerzak, *Can ideals without ccc be interesting?* Topology Appl. **55** (1994), no. 3, 251–260.

[2] M. Balcerzak and Sz. Głąb *Measure-category properties of Borel plane sets and Borel functions of two variables*, Acta Math. Hungar., to appear. ;

[3] M. Balcerzak, A. Rosłanowski and S. Shelah *Ideals without ccc* J Symbolic Logic **63** (1998), 128–147;

[4] T. Bartoszynski and H. Judah, *Borel Images of sets of reals*, Real Analysis Exchange **20** (1994/95), no. 2, 1–23;

[5] H. Becker, *Ideals without ccc and without property (M)*, Proc. Amer. Math. Soc. **128** (2000), no. 10, 3031?3034.

[6] A. Blass, *Combinatorial cardinal characteristics of the continuum*, to appear as a chapter in the *Handbook of Set Theory*;

[7] J. Cichoń and J. Pawlikowski, *On ideals of subsets of the plane and on Cohen reals*, The Journal of Symbolic Logic **51** (1986), no. 3, 561–569;

[8] F. van Engelen, K. Kunen, A. W. Miller, *Two remarks about analytic sets*, Set theory and its applications (Toronto, ON, 1987), 68–72, Lecture Notes in Math., 1401, Springer, Berlin, 1989.

[9] I. Farah and J. Zapletal, *Between Maharam and von Neumann's problems*, Math. Reseach Letters **11** (2004), no 5–6, 673–684;

[10] D. Fremlin, *Consequences of Martin's Axiom*, Cambridge Tracts in Math. 84, Cambridge University Press (1984);

[11] P. Holicky, *Borel classes of uniformizations of sets with large sections*, preprint;

[12] A. Kechris, *Classical descriptive set theory*, Springer-Verlag, New York, 1995;

[13] I. Recław, P. Zakrzewski, *Fubini properties of ideals*, Real Anal. Exchange **25** (1999/00), no. 2, 565–578.

[14] S. Solecki, *A Fubini theorem*, Topology Appl. **154** (2007), no. 12, 2462–2464.

Mathematical Institute, University of Wrocław
plac Grunwaldzki 2/4, 50-384 Wrocław
PBOROD@MATH.UNI.WROC.PL
http://www.math.uni.wroc.pl/~pborod

Institute of Mathematics, Technical University of Łódź
ul. Wólczańska 215, 90-924 Łódź
SZYMON2377@O2.PL
http://im0.p.lodz.pl/~sglab/indexeng.html