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Continuous logic as a tool to transfer model-theoretic dichotomies to functional analysis and conversely

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Abstract

This dissertation gives a full self-contained and detailed account of Ben Yaacov's results of [Ben13], where the 'fundamental theorem of stability' (i.e. the equivalence between various definitions of stable theory) in continuous logic is proved in a very simple and elegant fashion using a functional analysis theorem of Grothendieck. We structure this work in four parts: (1) We present stability theory in the classical case and prove the fundamental theorem in that setting. (2) We introduce the functional analysis notions needed to prove Grothendieck's theorem and indeed prove it; in doing so, we provide a wider characterization theorem for compactness of certain sets of continuous functions. (3) We give a detailed account of Continuous First Order Logic. (4) Finally, we prove the fundamental theorem in the continuous setting as done by Ben Yaacov in [Ben13]; we conclude with some meta-mathematical and philosophical comments, pointing out a possible interpretation of the nature of such a proof.

Keywords: continuous logic, functional analysis, model theory, stability, no-order property, definability of types, Grothendieck, weak compactness, pointwise compactness, sequential compactness.

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Contents

The life of science is as strong and carefree and glorious as a fairy tale. And Ulrich felt: people simply don't realize it, they have no idea how much thinking can be done already; if they could be taught to think a new way, they would change their lives.

— **Robert Musil**

Introduction

Modern model theory is widely believed to have began in 1962 with Morley's PhD dissertation,[1] where he proved his Categoricity Theorem, which provided a positive answer to a conjecture formulated by Łoś in 1954:

> *If a complete theory in a* countable *language is categorical for some uncountable cardinal, then so is for every uncountable cardinal.*

To prove this result Morley introduced new concepts and techniques that have become a standard part of the subject, e.g. ranks and total transcendence.

In the following years there were numerous attempts to answer problems left open in Morley's work and also to generalize it. During extensive investigations of this sort,[2] Shelah introduced in the 70s some dividing lines which separated theories into two classes: non-structured ones having as many models as possible and structured ones admitting a sort of dimension theory. One of such dividing lines is *stability*, a concept which generalizes Morley's total transcendence. Later on, other perspectives developed in which stability theory gained even deeper geometric meaning.[3,4]

There are various equivalent ways to define what it means for a theory T to be stable, which can be informally grouped into three ideas:

(i) The theory T cannot 'define' an infinite linear order.
(ii) All types over models of T are definable.
(iii) The number of types over models of T is 'small'.

All of these convey the idea that a stable theory is in some sense tame, suggesting some kind of structure underneath. The proof of their equivalence has a distinct *discrete* model-theoretic flavour. In particular, the direction (i)⇒(ii) is quite clunky, no matter which strategy is employed.[5]

[1] It was then published as the article [Mor65].
[2] They eventually culminated in the massive book [She78].
[3] See for instance [Pil96].
[4] This first part of the introduction is based on [Pil03, p. 1].
[5] Approaches include: heirs/coheirs [Poi00, §11.2] [Pil83, §1], binary trees [TZ12, §8.3] [Bal88, §3.1], ranks [Che17, §2.2] [Bal88, §3.1] [Bue96, §5.1], direct [Dri05, §8] [Pil96, §1.2].

Continuous logic was first developed[6] in 1966 with the intent to generalize results in two-valued model theory. However, proofs of the analogues in the new logic of classical results were far from similar to the old ones. Other generalizations[7] of classical model theory were carried out aiming to be able to deal with *continuous* structures using model-theoretic techniques, which up to that point had only been used for algebraic structures.

Continuous First Order Logic (henceforth CFO) has been developed by Ben Yaacov, Berenstein, Henson, Iovino, Usvyatsov and others in the 00s to combine both these features: suitability for applications to the study of metric structures with model-theoretic techniques and generalization of classical model theory. With applications in mind, these authors opted for a slightly less general version of continuous logic, which is however equivalent to both other generalizations mentioned above. The construction of the resulting logic is very close to the classical one, with the truth-values set $\{0, 1\}$ replaced with $[0, 1]$. One nice and important feature is that this time proofs are essentially the same as the classical ones.

In 2013, Ben Yaacov published the (quite sketchy) paper [Ben13] where he presented a new proof[8] of (i)$\Rightarrow$(ii) – i.e. how definability of types over models of T follows from the fact that T cannot 'define' an infinite linear order – for CFO which crucially involves the use of a functional analysis theorem[9] first proved by Grothendieck in 1952; this allows the proof to be beautifully simple and crystal clear. There are at least two surprising things: first, this new proof appears to be smoother than the (translation in CFO of the) one in the two-valued setting; second, it essentially shows how a certain kind of stability-like concept emerged in Grothendieck's work, which was originally totally unrelated to model theory. We try to address these topics in the final section of this work.

This dissertation was first and foremost intended to be a unified, detailed and self-contained account of this result of Ben Yaacov and of stability in continuous logic. While working on it, the goal got broader and has become trying to clarify and explain the reason why such a striking proof works, shifting the focus more on a meta-mathematical level. This has led to developing a wider framework where to collocate Grothendieck's theorem. In order to do all of this, we structure this dissertation as follows:

- In Chapter 1 we give a treatment of stability in the classical case leading to the 'fundamental theorem' of equivalence of the various definitions. This includes: a brief recall of some basic model-theoretic definitions and results, also to fix notation (the notion of satisfaction, the Compactness and Löwenheim–Skolem theorems, types and the

[6] KC66.
[7] E.g. the logic of *positive bounded formulae* or the setting of *compact abstract theories*.
[8] Ben13, Theorem 3.
[9] Gro52, Théorème 6.

monster model), the no-order property, size of type spaces, a combinatorial theorem of Erdős and Makkai,[10] the theorem of definability of types, the 'fundamental theorem of stability', both in the local and global versions and a double limit property equivalent to the no-order property.

- The goal of Chapter 2 is proving a characterization theorem for compactness of certain families of continuous functions, which subsumes Grothendieck's theorem. To do so, some functional analysis notions are recalled, such as the weak topology and the pointwise convergence topology. Then we prove the main theorem, i.e. the equivalence between weak compactness, pointwise compactness and a double limit property for those continuous functions.
- In Chapter 3, we introduce from scratch continuous logic and all the main results that are needed in the following: its syntax (language, structures, formulae), semantics (interpretations, induced maps, theories), fundamental theorems (Compactness, Löwenheim-Skolem, existence of monsters), types, their logic and metric topologies, predicates and restricted formulae.
- In Chapter 4 we combine all the previous results to use Ben Yaacov's idea for proving the fundamental theorem of stability in the continuous case, again both in the local and global versions, trying to carefully parallel the exposition to that of the classical case.

We conclude with a final section which is meant to be the place where we draw some philosophical conclusions on the meaning of Ben Yaacov's proof.

[10] A nice proof for this result [ME66] is hard to find in the literature and thus we decided to include here a detailed one.

Chapter 1

Stability in the classical case

In this chapter we introduce various classical model-theoretic notions related to stability and eventually show that they are all equivalent. We base our exposition on [Che17] for §1.2 and §1.3 and on [Dri05] for §1.4.

1.1 Preliminary model theory

We recall some basic classical model theory notions and results, also to fix notation. As a reference the reader may browse any introductory model theory textbook, for instance [Mar02] or [TZ12].

1.1.1 First order logic

Convention 1.1. We adopt the following conventions:

- For every language L, we take by default its cardinality $|L|$ to be infinite. To say that ϕ is an L-formula, sometimes we write $\phi \in L$.
- When it does not create confusion, we use the same symbols $M, N, \ldots$ to denote both models and their universes.
- If M is a model and $A \subseteq M$ a set of parameters, often we write $a \in M$ to say that $a \in M^{|a|}$, where $|a|$ is the length of the tuple a. To stress the fact that a is a tuple, sometimes we write $\vec{a}$. We also use the notation M_x for the cartesian product $M^{|x|}$.
- If $F\colon M \to N$ and $a_1, \ldots, a_n \in M$, we use the shorthand $F(\vec{a})$ for $(F(a_1), \ldots, F(a_n))$.
- We write $M \preceq N$ to say that M is an elementary substructure of N.
- If M is an L-structure and $A \subseteq M$ a set of parameters, we use the notation M_A or $(M, a)_{a \in A}$ to denote the structure in the language $L_A \doteq L \cup \{a : a \in A\}$ consisting of M with an additional constant symbol for each $a \in A$. To simplify, we use the same notation for the symbol and its interpretation, i.e. $a^M = a$.

Definition 1.2. A formula $\phi(x) \in L$ is *realized* by $b \in M$ if $M \models \phi(b)$. If the model is clear, we may also write $b \models \phi(x)$. A set of formulae $\Phi \subseteq L_A$ is *realized* by b if it realizes all formulae in Φ. We say that Φ is *finitely satisfiable* if every finite subset of Φ is realized in some model, and that Φ is *satisfiable* if it is realized in some model. An L-*theory* is a set of L-sentences.

We list now some fundamental theorems that we will freely use multiple times and that are deeply related to the very essence of first order logic and its finitary nature. Fix a language L. We start from the main tool for proving that a theory is satisfiable.

Fact 1.3 (Compactness Theorem, [Mar02, Theorem 2.1.4])**.** *If an L-theory is finitely satisfiable, then it is satisfiable.*

The next theorem gives a way to produce non-standard models of any theory of almost any cardinality.

Fact 1.4 (Upward Löwenheim–Skolem Theorem, [Mar02, Theorem 2.3.4])**.** *If M is an infinite L-structure and $\kappa \geq |M| + |L|$, there is an L-structure $N \succeq M$ with $|N| = \kappa$.*

The following result goes in the opposite direction. From a model M with a specific subset A of sufficiently big size κ, we can always extract an elementary submodel $N \preceq M$ of size κ containing A.

Fact 1.5 (Downward Löwenheim–Skolem Theorem, [Mar02, Theorem 2.3.7])**.** *If M is an L-structure and $A \subseteq M$, there is an L-structure $A \subseteq N \preceq M$ such that $|N| \leq |A| + |L|$. In particular, if $|A| \geq |L|$ then $|N| = |A|$.*

1.1.2 Types

The next notion is fundamental in what follows. A type is a consistent set of formulae. Every type is realized in some big structure. In such a structure we can think of any type as the set of all possible first order properties of some element.

Definition 1.6. Fix an L-structure M and $A \subseteq M$.

- A *partial type over A* (in M) is a set Φ of L_A-formulae which is finitely satisfiable in M_A.
- A partial type p over A is *complete* if it is maximal finitely satisfiable. We denote the space of complete types over A with $S^M(A)$. For $b \in M$, the complete type of b over A is

$$\mathrm{tp}^M(b/A) \doteq \{\phi(x) \in L_A : M \models \phi(b)\}.$$

- An n-*type* is a type with n free variables. An x-*type* is a type such that every formula in it has free variables in the tuple x. We write $p(x)$ to stress that p is an x-type. The spaces of complete n-types and x-types over A are denoted respectively by $S^M_n(A)$ and $S^M_x(A)$.

When we just say *type*, we usually mean complete type. Notice that $|x|$ can even be infinite of any size. Often, if the context makes the model we are working in clear, we omit the superscripts indicating the model.

A complete type is equivalently a partial type which for every L-formula ϕ contains either ϕ or $\neg\phi$. This definition has the problem that it does not generalize nicely to the continuous logic case. We recall now two important properties that models we would like to work in should have.

Definition 1.7. Let κ be infinite and M be a structure. We say that:

- M is *κ-saturated* if for any $A \in [M]^{<\kappa}$, every partial type $\Phi(x)$ over A with $|x| < \kappa$ is realized in M_A.
- M is *κ-homogeneous* if every partial elementary map $F\colon M \to M$ with $|F| < \kappa$ can be extended to an automorphism of M.

It is always possible to find a model with these properties.

Fact 1.8 ([Mar02, Theorem 4.3.12])**.** *For any theory T and any cardinal κ, there exists a κ-saturated and κ-homogeneous model $M \models T$.*

For the rest of the following chapter, we fix a a language L and a complete L-theory T. We would like to always work in a huge model where every type we are interested in is realized and such that every model we encounter elementarily embeds into it. Due to the previous result, we can fix such a model

$$\mathbb{M}$$

which is $\kappa(\mathbb{M})$-saturated and $\kappa(\mathbb{M})$-homogeneous for some cardinal $\kappa(\mathbb{M})$ larger than the size of any set we will consider. We refer to $\mathbb{M}$ as the *monster model*. Whenever we talk about objects without specifying where they belong, they are implicitly assumed to live in the model $\mathbb{M}$. We use the adjective *small* to mean smaller than $\kappa(\mathbb{M})$. For $\phi(x) \in L(\mathbb{M})$ and $a \in \mathbb{M}$ we write $\models \phi(a)$ to mean $\mathbb{M} \models \phi(a)$.

1.2 No-order property

Convention 1.9. From now onwards in this chapter, we work with the following fixed data: a language L, a complete L-theory T and a monster model $\mathbb{M} \models T$. So here by model we mean a small L-structure $M \models T$.

The first concept we are interested in is a combinatorial property, called the *no-order property*. The idea is that a theory T has this property when it cannot 'define' an infinite linear order. When this happens T is in some sense sufficiently tame. We will see that it is equivalent to other properties which have a more prominent model-theoretic content, referring to definability and cardinalities of type spaces.

Definition 1.10. Let $\phi(x, y) \in L$ and $n \in \omega$. We say that:

- $\phi(x, y)$ has the *n-order property in $M \models T$* if there are two sequences $(a_i)_{i<n} \subseteq M_x$ and $(b_i)_{i<n} \subseteq M_y$ such that

  $$M \models \phi(a_i, b_j) \quad \iff \quad i < j.$$

  When we omit the model M it is assumed to be the monster $\mathbb{M}$.
- $\phi(x, y)$ has the *order property* (OP) in M if it has the n-order property in M for every $n \in \omega$. Otherwise, we say that $\phi(x, y)$ has NOP in M.
- T has NOP if *no* L-formula has the order property.
- T has OP if there is *an* L-formula with the order property.

Remark 1.11. If $M, N \models T$ and $\phi(x, y) \in L$, then

$$\phi \text{ has } n\text{-OP in } M \quad \iff \quad \phi \text{ has } n\text{-OP in } N.$$

Indeed, "ϕ has n-OP" can be expressed by the formula

$$\exists x_1, \ldots, x_n \ \exists y_1, \ldots, y_n \Big[\bigwedge_{i<j\leq n} \phi(x_i, y_j) \wedge \bigwedge_{j\leq i\leq n} \neg\phi(x_i, y_j) \Big].$$

Since T is complete, M and N have the same theory; the conclusion follows.

Hence, the same is true for OP. In other words, (N)OP is a property of the theory T. So specifying the model ϕ has (N)OP in is redundant; we will often omit it or say that "ϕ has (N)OP in T".

Lemma 1.12. *If an L-formula $\phi(x, y)$ has* OP*, then for every linear order I of any small size $\kappa \geq |L|$ there is a small set of parameters A, a model $N \models T$ of size κ and sequences $(a_i)_{i\in I} \subseteq N_x$ and $(b_i)_{i\in I} \subseteq N_y$ such that $N \models \phi(a_i, b_j) \Leftrightarrow i < j$.*

Proof. Let $\phi(x, y) \in L$ have OP. For every $i \in I$, add to the language two new symbols a_i, b_i for constants. The theory

$$\Sigma = \{\phi(a_i, b_j) : i < j,\ i, j \in I\} \cup \{\neg\phi(a_i, b_j) : i \geq j,\ i, j \in I\}$$

is finitely satisfiable by assumption and then it has a model N, that by the Löwenheim–Skolem Theorem 1.4 can be assumed of size κ. □

Recall the following fundamental combinatorial fact that we need to prove the next result.

Fact 1.13 (Infinite Ramsey Theorem, [Mar02, Theorem 5.1.1])**.** *For $n, k \in \omega$ we have $\aleph_0 \to (\aleph_0)^n_k$. More explicitly, for any colouring with k colours of sets in $[\mathbb{N}]^n$, there is an infinite $I \subseteq \mathbb{N}$ such that $[I]^n$ is homogeneous.*

The no-order property for formulae behaves well with respect to Boolean combinations and other simple operations.

Lemma 1.14. *If* $\phi(x,y)$ *and* $\psi(x,z)$ *have* NOP*, so do the following:*

(i) $\phi^*(y,x) \doteq \phi(x,y)$
(ii) $\neg\phi(x,y)$
(iii) $\theta(x,yz) \doteq \phi(x,y) \wedge \psi(x,z)$
(iv) $\theta(x,yz) \doteq \phi(x,y) \vee \psi(x,z)$
(v) $\theta(x,u) \doteq \phi(x,uc) \in L_{\{c\}}$*, where* $y = uv$ *and* $c \in \mathbb{M}_v$*.*

Proof. We prove the statements by contradiction.

(i): Suppose $\phi^*(y,x)$ has NOP. Let $\prec$ be the inverse ordering on ω. By Lemma 1.12, there are $b_j, a_i \in \omega$ such that $\models \phi^*(b_j, a_i) \Leftrightarrow i \prec j$. This means that $\models \phi(a_i, b_j) \Leftrightarrow j < i$, against the fact that $\phi(x,y)$ has NOP.
(ii): If $\models \neg\phi(a_i,b_j) \Leftrightarrow i < j$, then $\models \phi(a_i,b_j) \Leftrightarrow j \leq i$. If $a_i' \doteq a_{i+1}$, then $\models \phi^*(b_j, a_i') \Leftrightarrow j < i$, against (i).
(iii): Suppose that $\models \theta(a_i, b_j c_j) \Leftrightarrow i < j$. Then

$$(\models \phi(a_i,b_j) \text{ and } \models \psi(a_i,c_j)) \text{ iff } (i < j).$$

The fact that either $\phi(x,y)$ or $\psi(x,z)$ has OP follows from the fact that the following propositional sentence is a tautology

$$((A \wedge B) \leftrightarrow C) \rightarrow ((A \leftrightarrow C) \vee (B \leftrightarrow C)).$$

(iv): Suppose that $\models \phi(a_i,b_j) \vee \psi(a_i,c_j)$ iff $i < j$. For each $p \in [\mathbb{N}]^2$, let us name $p_1 \doteq \min p$ and $p_2 \doteq \max p$. By assumption, the sets

$$P \doteq \{p \in [\mathbb{N}]^2 : \phi(a_{p_1}, b_{p_2}) \text{ holds}\}, \qquad Q \doteq \{p \in [\mathbb{N}]^2 : \psi(a_{p_1}, c_{p_2}) \text{ holds}\}$$

form a colouring of $[\mathbb{N}]^2$. Therefore, by Ramsey Theorem, there exists an infinite $I \subseteq \mathbb{N}$ such that either $[I]^2 \subseteq P$ or $[I]^2 \subseteq Q$. In the first case, ϕ has OP, in the second, ψ has OP.
(v): If $\models \phi(a_i, b_j c) \Leftrightarrow i < j$, let $b_j' \doteq b_j c$. Then $\models \phi(a_i, b_j') \Leftrightarrow i < j$. □

We devote the following sections to relate the no-order property to the size of types spaces and definability of types.

1.3 Size of type spaces

In this section we see how the no-order property is related to the size of type spaces. We need two new ingredients: a cardinal function related to linear orders and a combinatorial result of Erdős and Makkai.

1.3.1 The Dedekind number

Recall the following concept, which for instance can be used to construct the set of real numbers from the rationals.

Definition 1.15. Let I be a linear order. A *Dedekind cut* of I is a non-empty subset $C \subset I$ which is closed downwards and without a greatest element.

Now we define a cardinal function $\operatorname{ded}\colon \mathrm{Card} \to \mathrm{Card}$ related to the number of cuts, which will be useful to measure the size of type spaces.

Definition 1.16. Let κ be a cardinal. Define the cardinal number

$$\operatorname{ded}\kappa \doteq \sup\{\lambda : \exists I \text{ linear order with } |I| = \lambda \text{ and a dense subset of size } \kappa\}.$$

We also give an alternative characterization that is sometimes useful and involves indeed the concept of Dedekind cuts.

Lemma 1.17. *For any cardinal κ, the following holds:*

$$\operatorname{ded}\kappa = \sup\{\lambda : \exists I \text{ linear order with } |I| = \kappa \text{ and } \lambda\text{-many cuts}\}.$$

Proof. Let μ be the cardinal on the right hand side of the equation.
$(\operatorname{ded}\kappa \geq \mu)$: Let I be a linear order with $|I| = \kappa$ and λ-many cuts. Without loss of generality, we can suppose I is dense (i.e. for any two points $a < b$ in I there is $c \in I$ such that $a < c < b$). Let J be the linearly ordered set of cuts of I. Clearly, I embeds in J in a dense way and then the order J of size λ has a dense subset of size κ, so $\operatorname{ded}\kappa \geq \lambda$. Since λ was arbitrary, it follows that $\operatorname{ded}\kappa \geq \mu$.
$(\mu \geq \operatorname{ded}\kappa)$: Let J be a linear order with $|J| = \lambda$ and a dense subset I of size κ. For every $j \in J$, the set $\{i \in I : i < j\}$ is a cut on I, and these cuts are all pairwise distinct by density of I in J. Therefore there are at least $|J|$-many (i.e. λ-many) of them, and so $\mu \geq \lambda$. Since λ was arbitrary, we obtain that $\mu \geq \operatorname{ded}\kappa$. $\square$

In general, the value of $\operatorname{ded}\kappa$ depends on set-theoretic assumptions. However, it has both lower and upper bounds.

Proposition 1.18. $\kappa < \operatorname{ded}\kappa \leq 2^\kappa$.

Proof. We show that there is a linear order I with $|I| > \kappa$ and a dense subset of size κ. Let λ be the smallest cardinal such that $\kappa^\lambda > \kappa$. Since $\kappa^\kappa > \kappa$, then $\lambda \leq \kappa$. Let I be the set of functions ${}^\lambda\kappa$ with the lexicographic order and $I_0 \subset I$ consist of those functions that are eventually zero; I_0 is clearly dense in I. Write $I_0 = \bigcup_{\alpha<\lambda} I_0^\alpha$, where I_0^α contains the functions that are zero for all $\beta \geq \alpha$. By minimality of λ, we have $|I_0| = |\bigcup_{\alpha<\lambda} I_0^\alpha| \leq \lambda \cdot \kappa = \kappa$.

To see that $\operatorname{ded}\kappa \leq 2^\kappa$, just observe that if $|I| = \kappa$, then there are at most $|\mathcal{P}(I)| = 2^\kappa$ many types. $\square$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, from the definition it follows that $\operatorname{ded}\aleph_0 = 2^{\aleph_0}$. Similarly, under GCH, $\operatorname{ded}\kappa = 2^\kappa$ for every κ.

1.3.2 Local types

Now we introduce *local types* with respect to a fixed formula. This is useful to formulate a local theory of stability, which can then be quite easily adapted for theories.

Definition 1.19. Let $\phi(x, y)$ be a fixed formula and M a model. A (complete) *ϕ-type over $B \subseteq M_y$* is a maximal finitely satisfiable set of formulae of the form $\phi(x, b)$ or $\neg\phi(x, b)$ for $b \in B$. By $\mathrm{tp}_\phi(a/B)$, for $a \in \mathbb{M}$, we mean the ϕ-type of over B realized by a. We denote by $S_\phi(B)$ the space of ϕ-types over B.

The next results show that a theory which can 'encode' an infinite linear order necessarily has 'many' types.

Proposition 1.20. *If an L-formula $\phi(x, y)$ has* OP*, then for all $\kappa \geq |L|$ there is a model $M \models T$ of size κ with $|S_\phi(M)| \geq \operatorname{ded}\kappa$.*

Proof. Let $\phi(x, y) \in L$ have OP and fix $\kappa \geq |L|$. By Lemma 1.12, let I be a dense linear order of size κ and M a model of size κ with $(a_i)_{i\in I} \subseteq M_x$ and $(b_i)_{i\in I} \subseteq M_y$ such that $M \models \phi(a_i, b_j) \Leftrightarrow i < j$. For every cut $C \subseteq I$, the set

$$\Phi_C \doteq \{\phi(x, b_j) : j \in C^\complement\} \cup \{\neg\phi(x, b_j) : j \in C\}$$

is by compactness and density a partial ϕ-type over M. Let p_C be a complete ϕ-type over M extending Φ_C. Clearly if $C_1 \neq C_2$ then $\Phi_{C_1} \neq \Phi_{C_2}$ and *a fortiori* $p_{C_1} \neq p_{C_2}$. So the map $C \mapsto p_C$ is injective. Since I was arbitrary, our thesis follows. □

We can translate the latter result to obtain an analogue for theories. To do so we introduce the following concept.

Definition 1.21. For a theory T, define the cardinal function

$$\mathrm{ntp}_T(\kappa) \doteq \sup\{|S_n(M)| : M \models T,\ |M| = \kappa,\ n \in \omega\}$$

where $S_n(M)$ is the set of n-types over M, as in Definition 1.6.

Corollary 1.22. *If T has* OP*, then $\mathrm{ntp}_T(\kappa) \geq \operatorname{ded}\kappa$ for all $\kappa \geq |L|$.*

Proof. Let $\phi(x, y) \in L$ have OP in $M \models T$, without loss of generality of size $\kappa \geq |T|$. Every ϕ-type over M_y can be extended to an x-type and to distinct ϕ-types correspond distinct extensions, hence $|S_x(M)| \geq |S_\phi(M)|$. So from Proposition 1.20 it follows that $\sup\{|S_x(M)| : M \models T,\ |M| = \kappa\} \geq \operatorname{ded}\kappa$. Since x is a variable in a formula, it is finite. Hence $\mathrm{ntp}_T(\kappa) \geq \operatorname{ded}\kappa$. □

As a simple corollary we get that if a countable theory T has OP then $\mathrm{ntp}_T(\aleph_0) = \operatorname{ded}\aleph_0 = 2^{\aleph_0}$, i.e. its number of types is the maximum possible.

1.3.3 The Erdős-Makkai lemma

We need another combinatorial fact in order to prove the next result.

Lemma 1.23 (Erdős-Makkai, [ME66])**.** *For any $n \in \omega$, any infinite set A and any family $\mathcal{F} \subseteq \mathcal{P}(A)$ such that $|\mathcal{F}| > |A|$, the following holds:*

$$(\exists a_1, \dots, a_n \in A)(\exists S_0, \dots, S_n \in \mathcal{F})(\forall i, j)\ [i \le j \iff a_i \in S_j]. \qquad (A, \mathcal{F})_n$$

Proof. The case $n = 0$ is obvious since the subformula starting with the universal quantifier is trivially true. Now suppose $(A, \mathcal{F})_n$ holds up to n for all A's and $\mathcal{F}$'s as in the statement. Fix A infinite and $\mathcal{F} \subseteq \mathcal{P}(A)$ with $|\mathcal{F}| > |A|$. Pick any $F \in \mathcal{F}$ and let $\kappa \doteq |A|$. We separate two cases.

<u>Case I.</u> $|\{S \cap F : S \in \mathcal{F}\}| > \kappa$.

Then there is some $f \in F$ such that the family $\mathcal{G} \doteq \{S \cap F : S \in \mathcal{F},\ f \notin S\}$ has size $|\mathcal{G}| > \kappa$, otherwise

$$\begin{aligned} |\{S \cap F : S \in \mathcal{F},\ F \neq S \cap F\}| &= |\{S \cap F : S \in \mathcal{F},\ \exists f \in F\ (f \notin S)\}| \\ &= |\textstyle\bigcup_{f \in F} \{S \cap F : S \in \mathcal{F},\ f \notin S\}| \\ &\le |F| \cdot \sup_{f \in F} |\{S \cap F : S \in \mathcal{F},\ f \notin S\}| \\ &\le \kappa \cdot \kappa = \kappa \end{aligned}$$

against the assumption of this case. Applying $(F, \mathcal{G})_n$ yields distinct elements $a_1, \dots, a_n \in F$ and $S_0, \dots, S_n \in \mathcal{F}$ such that for all i, j we have $f \notin S_i$ and

$$i \le j \quad \iff \quad a_i \in S_j \cap F \quad \iff \quad a_i \in S_j.$$

Hence, adding the elements $a_{n+1} \doteq f$ and $S_{n+1} \doteq F$ proves $(A, \mathcal{F})_{n+1}$.

<u>Case II.</u> $|\{S \cap F : S \in \mathcal{F}\}| \le \kappa$.

The family $\mathcal{H} \doteq \{S^{\complement} \cap F^{\complement} : S \in \mathcal{F}\}$ has size $|\mathcal{H}| > \kappa$, since the map $\mathcal{F} \ni S \mapsto (S \cap F, S \cap F^{\complement})$ is bijective, $|\mathcal{F}| > \kappa$ and $|\mathcal{H}| = |\{S \cap F^{\complement} : S \in \mathcal{F}\}|$. Now by repeating the same argument of Case I applied to $\mathcal{H} \subseteq \mathcal{P}(F^{\complement})$ we see that there must be some $f \in F^{\complement}$ such that the family

$$\mathcal{G} \doteq \{S^{\complement} \cap F^{\complement} : S \in \mathcal{F},\ f \in S\}$$

has size $|\mathcal{G}| > \kappa$. Now $(F^{\complement}, \mathcal{G})_n$ yields $a_1, \dots, a_n \in F^{\complement}$ and $S_0, \dots, S_n \in \mathcal{F}$ such that for all $1 \le i \le n$ and all $0 \le j \le n$ we have $f \in S_i \cap F^{\complement}$ and

$$i \le j \quad \iff \quad a_i \in S_j^{\complement} \cap F^{\complement} \quad \iff \quad a_i \in S_j^{\complement}. \tag{1.1}$$

Let $a_{n+1} \doteq f$ and $S_{n+1} \doteq F$. Then $a_1, \dots, a_{n+1}$ and $S_0^{\complement}, \dots, S_{n+1}^{\complement}$ still satisfy (1.1), since $f \in F^{\complement} = S_{n+1}^{\complement}$ and $f \notin S_j^{\complement}$ for all $j \le n$, whereas $a_i \in S_{n+1}^{\complement} = F^{\complement}$ for all $1 \le i \le n$. Now we switch the order of the elements. Formally, let

$$\begin{cases} a'_i \doteq a_{n-i+2} \\ \text{for } 1 \le i \le n+1 \end{cases} \qquad \text{and} \qquad \begin{cases} S'_j \doteq S_{n-j+1} \\ \text{for } 0 \le j \le n+1. \end{cases}$$

Thus we get $a'_1, \dots, a'_{n+1} \in A$ and $S'_0, \dots, S'_{n+1} \in \mathcal{F}$ which, by (1.1), satisfy

$$\begin{aligned} a'_i \in S'_j \quad &\iff \quad a_{n-i+2} \in S_{n-j+1} \quad \iff \quad n-i+2 > n-j+1 \\ &\iff \quad i < j+1 \quad \iff \quad i \leq j \end{aligned}$$

which proves $(A, \mathcal{F})_{n+1}$. All cases are thus covered. $\square$

To get a result which is more suitable for our applications we introduce the following nomenclature.

Definition 1.24. We say that a sequence of pairs of sets $(a_i, S_i)_{i<n}$ is

- of *$\leq$-type* if for all $i, j < n$ we have $a_i \in S_j \Leftrightarrow i \leq j$;
- of *$<$-type* if for all $i, j < n$ we have $a_i \in S_j \Leftrightarrow i < j$.

Remark 1.25. Notice that if $(a_i, S_i)_{i<n}$ is of $\leq$-type, then $(a_{i+1}, S_i)_{i<n-1}$ is of $<$-type; whereas if it is of $<$-type, then $(a_i, S_{i+1})_{i<n-1}$ is of $\leq$-type.

This allows for a simpler rephrasing of Lemma 1.23:

Corollary 1.26. *Let A be an infinite set and $\mathcal{F} \subseteq \mathcal{P}(A)$ a family such that $|\mathcal{F}| > |A|$. Then for every $n \in \omega$ there are $(a_i)_{i<n} \subseteq A$ and $(S_i)_{i<n} \subseteq \mathcal{F}$ such that $(a_i, S_i)_{i<n}$ is a sequence of $\leq$-type. Moreover, the same holds for sequences of $<$-type.*

Proof. By Lemma 1.23, for all $n \in \omega$ there are elements $a_1, \dots, a_n \in A$ and $S_0, \dots, S_n \in \mathcal{F}$ such that $(a_i, S_i)_{1 \leq i < n}$ is a sequence of $\leq$-type of length $n-1$. By Remark 1.25, we get also a sequence of $<$-type of length $n-2$. This is enough. $\square$

In Proposition 1.20 we saw that the presence of a formula with OP automatically forces the size of type spaces over models to be big. The following result goes in the other direction, using local types: if the ϕ-types space over an infinite set of parameters is big, then the formula ϕ must have the order property.

Proposition 1.27. *Let $\phi(x, y) \in L$. The following are equivalent:*

(i) *The formula $\phi(x, y)$ has* OP*.*

(ii) *There is an infinite (small) set B such that $|S_\phi(B)| > |B|$.*

Proof. (i)⇒(ii): Let $\phi(x, y)$ have OP in a small model $M \models T$, that by Löwenheim–Skolem can be assumed of size $|M| \geq |L|$. From Proposition 1.20 it follows that $|S_\phi(M)| > |M|$.
(ii)⇒(i): Since B is small, every ϕ-type over B in realized in the monster model $\mathbb{M}$, so it is of the form $\mathrm{tp}_\phi(a/B)$ for some $a \in \mathbb{M}$. Moreover, there is a bijection

$$\mathrm{tp}_\phi(a/B) \longleftrightarrow \{b \in B : \phi(a, b) \text{ holds}\} \doteq S_a.$$

Define the family $\mathcal{F} \doteq \{S_a : a \in \mathbb{M}_x\} \subseteq \mathcal{P}(B)$. By assumption, $|\mathcal{F}| > |B|$, hence we can apply Corollary 1.26 to obtain for every $n \in \omega$ sequences $(b_i)_{i<n} \subseteq B$ and $(a_i)_{i<n} \subseteq \mathbb{M}_x$ of $<$-type such that for all $i, j < n$ we have

$$\models \phi^*(b_i, a_j) \quad\iff\quad \models \phi(a_j, b_i) \quad\iff\quad b_i \in S_{a_j} \quad\iff\quad i < j.$$

Hence ϕ^* has n-OP for every $n \in \omega$, and by Lemma 1.14(i) so does ϕ. This means that they have OP. □

1.4 Definability of types

It turns out that in the presence of NOP all types are definable. We devote this section to prove this statement, which for trivial reasons implies that there are not too many types. We already saw that this latter fact implies NOP (actually the contrapositive of this implication). Therefore we shall get a circular equivalence between these three properties.

Definition 1.28 (Definability of types)**.**

- Fix $\phi(x, y) \in L$. We say that a type $p(x) \in S_\phi(A)$ is *definable over* B if there is $\psi(y) \in L_B$ such that for all $a \in A$

$$\phi(x, a) \in p \quad\iff\quad \models \psi(a).$$

- A type $p \in S_x(A)$ is *definable over* B if for every $\phi(x, y) \in L$ the ϕ-type $p|_\phi$ is definable over B, where

$$p|_\phi \doteq \{\phi(x, a) \in p : a \in A\}.$$

- A type is *definable* if it is definable in its domain.

Remark 1.29. If $\phi(x, y)$ has NOP, then there is a minimal $N \in \omega$ such that there is no sequence $(a_i, b_i)_{i<N} \subseteq \mathbb{M}_x \times \mathbb{M}_y$ such that

$$\models \phi(a_i, b_j) \Leftrightarrow i \leq j \qquad \text{for all } i, j < N.$$

In fact, by Remark 1.25, taking $\leq$ instead of $<$ is not a problem, and it will come handy in the next proof. Note that trivially $N > 0$. Let $N(\phi)$ be such a number.

Convention 1.30. We stipulate that empty conjunctions such as $\bigwedge_{i<0} \phi_i(x)$ are tautologies, i.e. true in every model for any interpretation. This is common practice and it makes sense if we think that a conjunction should be true when all of its members are true: here we have none so it is trivially the case.

The next result is crucial to show that if a theory has NOP, then all types are definable over *any* model. The key for the proof is that all models elementarily embed in the monster model.

Lemma 1.31. *Let $\phi(x, y)$ have* NOP*, $a \in \mathbb{M}_x$ and M be an arbitrary model. Then there are $I, J \in \omega$ and $a^i_j \in M_x$ for all $i \leq I$ and $j \leq J$ such that for every $b \in M_y$*

$$\models \phi(a, b) \leftrightarrow \bigvee_{i \leq I} \bigwedge_{j \leq J} \phi(a^i_j, b).$$

Proof. Let $J \doteq N(\phi) - 1$ with $N(\phi)$ as in Remark 1.29.

Claim. *Let $\psi(x) \in \mathrm{tp}(a/M)$. Then there are $a_0, \ldots, a_n \in M_x$ for $n \leq J$ realizing $\psi(x)$ and such that for all $b \in M_y$*

$$\models \Big(\bigwedge_{i \leq n} \phi(a_i, b) \Big) \to \phi(a, b). \tag{1.2}$$

Proof. For $n \in \omega$, we say that $(a_i, b_i)_{i<n} \subseteq M_x \times M_y$ is a (ψ, n)*-sequence* if for all $i, j < n$ we have

(i) $\models \phi(a_i, b_j) \Leftrightarrow i \leq j$

(ii) $\models \psi(a) \wedge \neg\phi(a, b_i)$.

Trivially there is a $(\psi, 0)$-sequence, since it is the empty set. Now suppose that $(a_i, b_i)_{i<n}$ is a (ψ, n)-sequence. We try to extend it. The L_M-formula $\psi(x) \wedge \bigwedge_{i<n} \neg\phi(x, b_i)$ is realized by a by assumption (by Convention 1.30 also in the case $n = 0$). Since $\mathbb{M}$ is a monster, $M \preceq \mathbb{M}$, and so the formula is realized also by some element $a_n \in M_x$. If $\models (\bigwedge_{i \leq n} \phi(a_i, b)) \to \phi(a, b)$ for all $b \in M_y$, then $a_0, \ldots, a_n$ witness that the Claim holds and thus we are done. If not, there is $b_n \in M_y$ such that $\models \bigwedge_{i \leq n} \phi(a_i, b_n) \wedge \neg\phi(a, b_n)$ and we can extend our sequence by adding a_n and b_n. By definition of $N(\phi)$, there cannot be $(\psi, N(\phi))$-sequences, hence at some stage $n \leq J$ this process must stop and the Claim must hold. $\Diamond$

Note that if $a_0, \ldots, a_n$ satisfy the Claim, we can always add duplicates to obtain $a_0, \ldots, a_J$ still satisfying it. Observe that the formula

$$\phi_J(\vec{x}, y) = \phi_J(x_0, \ldots, x_J, y) \doteq \bigwedge_{i \leq J} \phi(x_i, y)$$

has NOP by Lemma 1.14(iii) and so does $\neg\phi_J(\vec{x}, y)$ by Lemma 1.14(ii). Let $I \doteq N(\neg\phi) - 1$. We proceed now in a similar way.

For $n \in \omega$, we say that $(\vec{a}^i, b_i)_{i<n} \subseteq M_{\vec{x}} \times M_y$ is a (ϕ, n)*-sequence* if for all $i, j < n$ we have

(iii) $\models \phi_J(\vec{a}^i, b) \to \phi(a, b)$ for all $b \in M_y$

(iv) $\models \neg\phi_J(\vec{a}^i, b_j) \Leftrightarrow i \leq j$

(v) $\models \phi(a, b_j)$.

As before, there is a (ϕ, n)-sequence, since it is the empty set. Now, suppose $(\vec{a}^i, b_i)_{i<n}$ is a (ϕ, n)-sequence. We try to extend it. Again, by Convention 1.30 what we are going to say makes sense also in the case $n = 0$. Applying the Claim to the formula $\psi(x) = \bigwedge_{j<n} \phi(x, b_j)$ yields $a_0, \ldots, a_J \in M_x$ such that

(vi) $\models \bigwedge_{j<n} \phi(a_i, b_j)$ for all $i \leq J$

(vii) $\models (\bigwedge_{i \leq J} \phi(a_i, b)) \to \phi(a, b)$ for all $b \in M_y$.

If we let $\vec{a}^n \doteq (a_0, \ldots, a_J)$, by definition of ϕ_J conditions (vi) and (vii) can be restated as follows:

(viii) $\models \bigwedge_{j<n} \phi_J(\vec{a}^n, b_j)$

(ix) $\models \phi_J(\vec{a}^n, b) \to \phi(a, b)$ for all $b \in M_y$.

Now we proceed in a similar way to what we did in the proof of the Claim. If there is $b \in M_y$ such that

(x) $\models \phi(a, b)$

(xi) $\models \neg\phi_J(\vec{a}^i, b)$ for all $i \leq n$,

by letting b_n be such an element, $(\vec{a}^i, b_i)_{i \leq n}$ is a $(\phi, n+1)$-sequence. Then we can proceed in the same way trying to extend it further. Since there cannot be $(\phi, N(\neg\phi))$-sequences, at some stage $n \leq I$ we must have a (ϕ, n)-sequence $(\vec{a}^i, b_i)_{i<n}$ and $\vec{a}^n$ satisfying (viii) and (ix) and such that for all $b \in M_y$ either $\models \neg\phi(a, b)$ or there is $i \leq n$ such that $\models \phi_J(\vec{a}^i, b)$, i.e.

$$\models \phi(a, b) \to \bigvee_{i \leq n} \phi_J(\vec{a}^i, b). \tag{1.3}$$

By (iii) and (ix), we have also the converse of (1.3), obtaining that

$$\models \phi(a, b) \leftrightarrow \bigvee_{i \leq n} \bigwedge_{k \leq J} \phi(\vec{a}^i_k, b).$$

Again by adding for $n < i \leq I$ duplicates $\vec{a}^i_k \doteq \vec{a}^n_k$ if needed, we conclude our proof. □

It is useful to introduce the notion of 'predicate', which will be generalized in the continuous logic setting. The idea is that a predicate is a 'combination' of instances of formulae.

Definition 1.32. Let $\phi(x, y) \in L$ and $B \subseteq \mathbb{M}$. By *ϕ-predicate over B* we mean a formula $\psi(x) \in L_B$ which is a Boolean combination of formulae of the form $\phi(x, b)$ for some $b \in B_y$.

With this terminology, we can rephrase Lemma 1.31 in a simpler way.

Corollary 1.33. *Let $\phi(x, y)$ have* NOP*, $a \in \mathbb{M}_x$ and M be an arbitrary model. Then there is a ϕ^*-predicate $\psi_a(y)$ over M such that for all $b \in M_y$*

$$\models \phi(a, b) \iff \models \psi_a(b).$$

1.5 The fundamental theorem of stability

It turns out that the order property, definability of types and the size of the types space are deeply connected, namely equivalent.

1.5.1 Local fundamental theorem in FOL

The next result states precisely what this means. To prove it, we just have to combine the results obtained so far.

Theorem 1.34. *For an L-formula $\phi(x, y)$, the following are equivalent:*

(i) *The formula ϕ has* NOP *(in T).*

(ii) *All ϕ-types over any $M \models T$ are definable (by a ϕ^*-predicate over M).*

(iii) *For all $\kappa \geq |L|$ and $M \models T$, if $|M| = \kappa$ then $|S_\phi(M)| \leq \kappa$.*

(iv) *There exists some $\kappa \geq |L|$ such that for all $M \models T$, if $|M| = \kappa$ then $|S_\phi(M)| < \operatorname{ded} \kappa$.*

Proof. (i)⇒(ii): Fix $M \models T$, let $\phi(x, y)$ have NOP and let $p(x) \in S_\phi(M)$. By saturation, $p = \operatorname{tp}_\phi(a/M)$ for some $a \in \mathbb{M}_x$. Then Corollary 1.33 yields a ϕ^*-predicate $\psi(y) \in L_M$ defining p. (ii)⇒(iii): Over a model of size κ, there are at most $\kappa + |L| = \kappa$ many L_M-formulae, thus at most κ-many definable types. (iii)⇒(iv): Obvious, since $\kappa < \operatorname{ded} \kappa$. (iv)⇒(i): By Proposition 1.20. □

Remark 1.35. Note that the implication (i)⇒(iii) can also be proved directly from Proposition 1.27, without referring to definable types. This is in fact the 'hardest' implication in this result, since both possible ways to prove it include some Lemma whose proof took us a bit long, namely Erdős-Makkai Lemma 1.23 or Lemma 1.31. We think it is instructive to see how one can obtain (iii) by means of these two seemingly different approaches: combinatorial the first and model-theoretic the second. Truth be told, the latter has some combinatorics going on in its proof, which indeed plays a big role in many model-theoretic arguments, e.g. ranks, indiscernibles, etc. Actually, Lemma 1.31 can also be proved using a notion of rank for stable formulae, as can be seen for instance in [Che17, Proposition 2.2] or [TZ12, Theorem 8.3.1].

1.5.2 Global fundamental theorem in FOL

We can translate the local result we have obtained into a global one.

Corollary 1.36. *Let T be a complete L-theory. The following are equivalent:*

(i) *The theory T has* NOP*.*

(ii) *All types over any model of T are definable.*

(iii) *For all $\kappa \geq |L|$, we have $\operatorname{ntp}_T(\kappa) \leq \kappa^{|L|}$.*

(iv) *There is $\kappa \geq |L|$ such that $\operatorname{ntp}_T(\kappa) < \operatorname{ded} \kappa$.*

Proof. We start with the following simple observation.

Claim. (ii) $\iff$ *For each* $\phi \in L$ *all* ϕ*-types are definable over any model.*

Proof. All $\phi(x, y)$-types over A are of the form $p|_\phi$ for some $p \in S_x(A)$, since each ϕ-type over A can be extended to an x-type over A. A type $p \in S(M)$ is definable (by definition) if for every $\phi(x, y) \in L$ the ϕ-type $p|_\phi$ is definable. The thesis follows by combining these observations. $\Diamond$

(i)$\Rightarrow$(ii): If T has NOP, then by definition each L-formula ϕ has NOP and by Theorem 1.34 all ϕ-types over any model are definable. By the Claim we conclude.
(ii)$\Rightarrow$(iii): Let $\kappa \geq |L|$ and $M \models T$ be a model of size κ. By the Claim and Theorem 1.34 it follows that for all $\phi(x, y) \in L$ we have $|S_\phi(M)| \leq \kappa$.

Claim. *For all finite variables* x*, the map*

$$S_x(M) \ni p \mapsto f_p \qquad \textit{where} \qquad f_p \colon L \to \textstyle\bigcup_{\phi \in L} S_\phi(M), \ \phi(x, y) \mapsto p|_\phi$$

is injective.

Proof. If $p, q \in S_x(M)$ are distinct, then there is $\phi(x, y) \in L$ and $a \in M$ such that $\phi(x, a) \in p \setminus q$. So $\phi(x, a) \in p|_\phi \setminus q|_\phi$ and then $f_p(\phi) \neq f_q(\phi)$. $\Diamond$

It follows that $|S_x(M)| \leq \kappa^{|L|}$ for all x and thus $\mathrm{ntp}_T(\kappa) \leq \aleph_0 \cdot \kappa^{|L|} = \kappa^{|L|}$.
(iii)$\Rightarrow$(iv): Take any $\kappa \geq |L|$ such that $\kappa^{|L|} = \kappa$. Then by (iii) we get that $\mathrm{ntp}_T(\kappa) \leq \kappa^{|L|} = \kappa < \operatorname{ded} \kappa$.
(iv)$\Rightarrow$(i): By Corollary 1.22. $\square$

We can finally give the main definition of this chapter.

Definition 1.37. We say that a formula $\phi(x, y) \in L$ is *stable* if it satisfies one of the equivalent conditions in Theorem 1.34. A theory T is stable if it satisfies one of the equivalent conditions in Corollary 1.36.

We chose to introduce the word "stable" only after having shown all properties to be equivalent because different authors attach the word to different properties among these, and so reading this text or browsing through the literature could be confusing. The word "stable" is thus not evocative of a single mathematical idea, it only conveys the vague meaning of tameness. Moreover, we wanted to make explicit in each result which properties were linked, without a possibility for misinterpretation.

1.6 Double limit property

Now we give a reformulation of the no-order property that is more suited for a generalization to the case of formulae taking truth values in an infinite linear order, as it is the case with the continuous logic we will be considering.

In such a context, we can properly speak of limits of sequences. To do so, we regard equality as the discrete distance

$$d(a,b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b. \end{cases}$$

Fix a model M. Every formula $\phi(x)$ induces a 'characteristic function' $\phi^M \colon M \to \{0,1\}$, defined by

$$\phi(a)^M = \begin{cases} 0 & \text{if } M \models \phi(a) \\ 1 & \text{if } M \not\models \phi(a). \end{cases}$$

Definition 1.38. We say that a formula $\phi(x,y)$ has the *double limit property in* M if for all sequences $(a_i)_{i\in\omega} \subseteq M_x$ and $(b_i)_{i\in\omega} \subseteq M_y$ we have

$$\lim_i \lim_j \phi(a_i,b_j)^M = \lim_j \lim_i \phi(a_i,b_j)^M \tag{DLP}$$

whenever all limits exist.

This is indeed just a rephrasing of the no-order property.

Lemma 1.39. *A formula* $\phi(x,y)$ *has* DLP *in* M *iff it has* NOP *in* M*.*

Proof. In both cases we prove the contrapositives.

$\underline{(\Rightarrow)}$: Suppose $\phi(a_i,b_j) = 0 \Leftrightarrow i < j$ for some $(a_i)_i \subseteq M_x$ and $(b_j)_j \subseteq M_y$. Then $\lim_j \phi(a_i,b_j) = 0$ and $\lim_i \phi(a_i,b_j) = 1$. Hence $\lim_i \lim_j \phi(a_i,b_j) = 0$ whereas $\lim_j \lim_i \phi(a_i,b_j) = 1$.
$\underline{(\Leftarrow)}$: Suppose ϕ does not have DLP. Without loss of generality there are sequences $(a_i)_i$ and $(b_j)_j$ such that

$$\lim_i \lim_j \phi(a_i,b_j) = 0, \qquad \lim_j \lim_i \phi(a_i,b_j) = 1,$$

which means that

$$\exists I \;\; \forall i > I \;\; \exists J_i \;\; \forall j > J_i \;\; (\phi(a_i,b_j) = 0) \tag{0.}$$

$$\exists J \;\; \forall j > J \;\; \exists I_j \;\; \forall i > I_j \;\; (\phi(a_i,b_j) = 1) \tag{1.}$$

Now we recursively construct a subsequence witnessing that $\phi(x,y)$ has OP. Use (1.) to find a'_0 and b'_0 such that $\phi(a'_0,b'_0) = 1$. So there is a sequence witnessing that ϕ has the 1-order property. Now suppose we have a'_i, b'_j for $i,j \leq n$ witnessing that ϕ has the n-order property, namely such that

$$\phi(a'_i,b'_j) = 0 \iff i < j.$$

We want to extend these sequences to sequences witnessing that ϕ has the $(n+1)$-order property. For each $k \leq n$, let i_k and j_k be the indexes such that $a'_k = a_{i_k}$ and $b'_k = b_{j_k}$. From (0.) it follows that for all $k \leq n$ there is J_{i_k} such that

$$\forall j > J_{i_k} \ \ (\phi(a_{i_k}, b_j) = 0).$$

Let $M \doteq \max\{J_{i_0}, \ldots, J_{i_n}\}$. Then

$$\forall j > M \ \ \big((\phi(a_{i_0}, b_j) = \cdots = (\phi(a_{i_n}, b_j) = 0\big).$$

Now we can repeat this argument with the roles of (0.) and (1.) switched to obtain a number N such that

$$\forall i > N \ \ \big((\phi(a_i, b_{j_0}) = \cdots = (\phi(a_i, b_{j_n}) = 1\big).$$

Now fix any $j' > M$. Condition (1.) yields $I_{j'}$ such that $\phi(a_i, b_{j'}) = 1$ for all $i > I_{j'}$. Fix any $i' > \max\{I_{j'}, N\}$. Finally, let $b'_{n+1} \doteq b_{j'}$ and $a'_{n+1} \doteq a_{i'}$. For every $n \in \omega$ we have constructed a sequence witnessing that ϕ has the n-order property, so by definition ϕ has OP. $\square$

Chapter 2

Compactness and continuous maps

In this chapter we recall some basic topology and functional analysis notions and then proceed towards Theorem 2.21, a characterization of compactness for sets of bounded continuous functions on a compact space. This result essentially says that weak compactness and pointwise compactness in this case agree; it subsumes [Gro52, Théorème 6] of Grothendieck and it is the key to prove the characterization of stability for continuous logic as done by Ben Yaacov in [Ben13].

We will see that what is actually needed to prove such a characterization of stability is only Corollary 2.19, which does not even involve weak compactness, unlike Grothendieck's theorem.

Anyway, the key to prove Theorem 2.21 is indeed the *double limit property* introduced by Grothendieck in [Gro52], the same paper where he proved his aforementioned theorem. As noted in [Ben13], this property is strikingly similar to the classical *no-order property* in model theory. In fact, we will see that the latter may be rephrased as a kind of double limit property.

2.1 Preliminary general topology and functional analysis

We recall some basic topology and functional analysis notions we need. To find more about these topics the reader may consult for instance respectively [Eng89] and [Bre11]. To start, recall the following basic terminology.

Definition 2.1. If X and Y are topological spaces and $F \subseteq Y^X$ is a family of functions, the *topology on X generated by F* is the smallest topology τ on X such that all maps in F are τ-continuous.

Now we introduce the weak and pointwise convergence topologies and state some remarks regarding their basic properties we will use and their mutual relationship.

2.1.1 Weak topology

Let E be a complex normed space and E^* its dual space of complex norm-bounded linear functionals on E. The *weak topology* on E is the topology generated by the family of maps E^*. We use the symbol

$$f_n \rightharpoonup f$$

to say that a sequence $(f_n)_n$ converges weakly to f.

Remark 2.2. A basis of open neighbourhoods for $x_0 \in E$ in the weak topology is given by sets $V_{x_0}(f_0, \dots, f_n; \epsilon)$ of the form

$$\{x \in E : |f_i(x) - f_i(x_0)| < \epsilon,\, i \leq n\} = \bigcap_{i \leq n} f_i^{-1}[B_\epsilon(f_i(x_0))]$$

for $f_0, \dots, f_n \in E^*$ and $\epsilon > 0$.

In the following we will need the following result.

Fact 2.3 (Riesz Representation Theorem, [Bre11, p. 4.14])**.** *Let X be a compact Hausdorff space. Then for every linear functional $T\colon C(X) \to \mathbb{R}$ there exists a unique Radon measure μ on X such that*

$$T(f) = \int_X f \, \mathrm{d}\mu$$

for all $f \in C(X)$.

2.1.2 Pointwise convergence topology

Let X and Y be any sets and $A \subseteq Y^X$. The *pointwise convergence* topology on A is the topology generated by the family of projections $\{\pi_x\}_{x \in X}$, where $\pi_x \colon A \to Y,\ f \mapsto f(x)$. Equivalently, the subset topology inherited from the product topology of Y^X. We use the symbol

$$f_n \to f$$

to say that the sequence $(f_n)_n$ converges pointwise to f.

Usually we will assume $Y = \mathbb{C}$ or $Y = [0, 1]$.

Remark 2.4. A basis of open neighbourhoods for $f_0 \in A \subseteq \mathbb{C}^X$ in the pointwise convergence topology on A is given by sets $V_{f_0}(x_0, \dots, x_n; \epsilon)$ of the form

$$\{f \in A : |f(x_i) - f_0(x_i)| < \epsilon,\, i \leq n\} = \bigcap_{i \leq n} \pi_{x_i}^{-1}[B_\epsilon(\pi_{x_i}(f_0))]$$

for $x_0, \dots, x_n \in X$ and $\epsilon > 0$.

Remark 2.5. This topology is Hausdorff whenever Y is. For the case $Y = \mathbb{C}$, if $f_0, f_1 \in A$ are distinct, then there is $x \in X$ where they do not agree; if $\epsilon \doteq |f_0(x) - f_1(x)| > 0$, it is easy to check that the sets $V_{f_0}(x; \epsilon/2)$ and $V_{f_1}(x; \epsilon/2)$ are disjoint open neighbourhoods respectively of f_0 and f_1.

Remark 2.6. If E is a normed space, the pointwise convergence topology on $A \subseteq \mathbb{C}^E$ is weaker than the weak topology on A, since $\{\pi_x\}_{x\in E} \subseteq A^*$. In particular, if $(f_n)_n \subseteq A$ then

$$f_n \rightharpoonup f \quad \Rightarrow \quad f_n \to f.$$

We state a very particular case of the Dominated Convergence theorem that we will use later. See for instance [Bar95, Theorem 5.6]) for reference.

Fact 2.7 (Dominated Convergence)**.** *Let X be a topological space and let $(f_n)_n \subseteq C_b(X)$ be a sequence such that $f_n \to f \in C_b(X)$. Then for all Radon measures on X we have $\int f_n \,\mathrm{d}\mu \to \int f \,\mathrm{d}\mu$.*

2.1.3 Compactness

First we recall the notion of net, which generalizes the concept of sequence and is useful to deal with compactness in more general settings.

Definition 2.8. A *net* in a topological space X is a pair (A, k) where A is a directed partially ordered set and $k\colon A \to X$. As with sequences, we use the notation $x_a = k(a)$ and denote k by $(x_a)_{a\in A}$.

The only result we will use is the (forward implication of the) following fact. We will not need to know what convergence of nets actually means.[1]

Fact 2.9 ([Ped89, Proposition 1.4.3])**.** *Let X and Y be topological spaces and $f\colon X \to Y$. The following are equivalent:*

(i) *The map f is continuous.*
(ii) *For each net $(x_a)_{a\in A} \subseteq X$, if $x_a \to x$ then $f(x_a) \to f(x)$.*

Recall the following characterization of compactness that holds in general for topological spaces.

Fact 2.10 ([Ped89, Theorem 1.6.2])**.** *For any topological space X, the following are equivalent:*

(i) *Every open covering of X has a finite subcovering.*
(ii) *Each family of closed sets of X with the finite intersection property has non-empty intersection.*
(iii) *Every net in X has a cluster point.*
(iv) *Every net in X has a convergent subnet.*

If X satisfies one of the following it is said compact*.*

[1] Anyway, the reader may refer to [Ped89, §1.3] for all details regarding nets: subnets, convergence, cluster points.

The next simple fact is needed in the following. It says that for Hausdorff topologies, a compact one is minimal with respect to inclusion.

Lemma 2.11. *If (X, τ) is a Hausdorff compact space, then τ is the minimal topology for which X is Hausdorff.*

Proof. Let $\sigma \subseteq \tau$ be an Hausdorff topology on X. We show that $\tau \subseteq \sigma$. Let $A \in \tau$. Then $A^{\complement}$ is τ-closed. Since X is τ-compact, $A^{\complement}$ is τ-compact and *a fortiori* σ-compact. Since σ is Hausdorff, $A^{\complement}$ is σ-closed and so $A \in \sigma$. □

Let us introduce some obvious terminology to describe various kinds of compactness. In the next sections we will proceed to analyse what these notions look like in the very specific case of certain sets of continuous functions.

Definition 2.12. Let E be a normed space, X any set and Y a topological space. We say that:

- $A \subseteq E$ is *weakly compact* if it is compact in the weak topology of E.
- $A \subseteq Y^X$ is *pointwise compact* if it is compact in the pointwise convergence topology of A.
- $A \subseteq Y$ is *precompact* if its closure $\mathrm{cl}(A) \subseteq Y$ is compact.
- $A \subseteq Y$ is *sequentially precompact* if every sequence $(a_n)_{n\in\omega} \subseteq A$ has a convergent subsequence.
- $A \subseteq Y$ is *sequentially compact* if every sequence $(a_n)_{n\in\omega} \subseteq A$ has a convergent subsequence whose limit is in A.

Remark 2.13. In general, compactness and sequential compactness are distinct. For example, the space $[0,1]^{[0,1]}$ with the pointwise convergence topology is compact by Tychonoff's Theorem but not sequentially compact. On the other hand, the ordinal space $[0, \omega_1)$ with the order topology is sequentially compact but not compact. See [SS78, Examples 105, 43] for details.

There are indeed spaces where the notions of compactness and sequential compactness agree, for example metric spaces; in particular, Banach spaces. As a matter of fact, for Banach spaces the same characterization remains true even for the weak topology, which need not be metrizable.

Fact 2.14 (Eberlein-Šmulian Theorem, [Die84, p. 18])**.** *Let E be any Banach space and $A \subseteq E$. The following are equivalent:*

(i) *A is weakly precompact.*
(ii) *A is weakly sequentially precompact.*
(iii) *Every sequence in A has a weak cluster point.*

Convention 2.15. If X is a topological space, we denote by $C(X)$ the set of continuous complex-valued functions on X.

A particular example of Banach space to which we will apply Fact 2.14 is the space $C(X)$ where X is Hausdorff compact, with the uniform norm.

Remark 2.16. Recall that sequences are nets but a subnet of a sequence need not be a subsequence. So if a topological space X is compact and $(x_n)_n \subseteq X$ is a sequence, we cannot say that it has a convergent subsequence; however, by Fact 2.10(iii) it surely has a cluster point.

2.2 Double limit property, weak compactness and pointwise compactness in $C(X)$

The proof of the main result of this section (Theorem 2.21) requires two preliminary propositions. The first to pass from DLP to pointwise compactness, the second to pass from pointwise compactness to sequential pointwise compactness. The other implications are trivial, modulo the Eberlein-Šmulian Theorem 2.14 and the Dominated Convergence Theorem 2.7.

2.2.1 Pointwise compactness and the double limit property

We start by introducing the topological double limit property.

Definition 2.17. Let X and Y be topological spaces. We say that the *double limit property* holds for $A \subseteq Y^X$ and $X_0 \subseteq X$ if for all sequences $(f_n)_n \subseteq A$ and $(x_n)_n \subseteq X_0$ we have

$$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m) \tag{DLP(A, X_0)}$$

whenever all limits exist.

Now we turn to the first of the aforementioned propositions, which shows how the double limit property is connected with pointwise compactness for continuous functions.

Proposition 2.18. *Let X be a compact Hausdorff space, $X_0 \subseteq X$ dense and $A \subseteq C(X)$ bounded. If* DLP(A, X_0) *holds, then A is pointwise precompact in $C(X)$.*

Proof. Since A is bounded, there is $r \geq 0$ such that $A \subseteq \mathrm{cl}(B_0(r))^X$, which is compact by Tychonoff's theorem. So the closed set $\mathrm{cl}(A)$ is compact as well. We only have to prove that $\mathrm{cl}(A) \subseteq C(X)$.

Suppose there is a function $f \in \mathrm{cl}(A)$ which is not continuous in a point $x \in X$. By density of X_0, there is a neighbourhood U of $f(x)$ such that every neighbourhood of x contains a point $y \in X_0$ satisfying $f(y) \notin U$. We recursively construct sequences $(f_n)_n \subseteq A$ and $(x_n)_n \subseteq X_0$ for $n \geq 1$. Start from any $f_1 \in A$. If we have $f_1, \ldots, f_n$ and $x_1, \ldots, x_{n-1}$, by continuity of the f_i's and how we chose U, we can pick $x_n \in X_0$ such that

(i) $|f_m(x) - f_m(x_n)| < \frac{1}{n}$ for all $m = 1..n$
(ii) $f(x_n) \notin U$.

Now, since $f \in \mathrm{cl}(A)$ with respect to the pointwise topology, by Remark 2.4 we can pick $f_{n+1} \in A$ such that

(iii) $|f_{n+1}(x_m) - f(x_m)| < \frac{1}{n}$ for all $m = 1..n$
(iv) $|f_{n+1}(x) - f(x)| < \frac{1}{n}$.

Condition (i) ensures that $\lim_m f_n(x_m) = f_n(x)$ for all $n \in \omega$ and (iv) that $\lim_n f_n(x) = f(x)$. Thus

$$\lim_n \lim_m f_n(x_m) = f(x) \in U.$$

Now (iii) yields that $\lim_n f_n(x_m) = f(x_m) \notin U$ for all $m \in \omega$. By assumption $A \subseteq B_r(0)$ for some $r \geq 0$. Then $f \in \mathrm{cl}(A) \subseteq \mathrm{cl}(B_r(0))$ and so the image of f is contained in a compact set. Therefore, there is a convergent subsequence $(f(x_{m_j}))_j$. Since $f(x_{m_j}) \in U^\complement$ for all j, then $\lim_j f(x_{m_j}) \in \mathrm{cl}(U^\complement) = U^\complement$. We have obtained that

$$\lim_j \lim_n f_n(x_{m_j}) = \lim_j f(x_{m_j}) \notin U,$$

which contradicts our assumption that the double limits coincide. □

We state a simple corollary that we will crucially use for proving Proposition 4.16, i.e. how definability of types in continuous logic follows from DLP. Here we restrict to the case of functions with values in the unit interval.

Corollary 2.19. *Let X be a compact Hausdorff space, $X_0 \subseteq X$ dense and $A \subseteq C(X; [0,1])$ bounded. If $\mathrm{DLP}(A, X_0)$ holds, then every net $(f_i)_{i \in I} \subseteq A$ has a subnet that converges pointwise to some $f \in C(X)$.*

Proof. Observe that we can view $[0,1] \subseteq \mathbb{C}$. Suppose that $\mathrm{DLP}(A, X_0)$ holds. Then by Proposition 2.18 the set A is pointwise precompact in $C(X)$, i.e. $\mathrm{cl}(A)$ is pointwise compact in $C(X)$. By Fact 2.10(iv) and $A \subseteq \mathrm{cl}(A)$ it follows that each net in A has a subnet which converges in $C(X)$. □

2.2.2 Pointwise compactness and pointwise sequential compactness

Now we present the second proposition needed for proving the main theorem. It shows that for certain continuous functions pointwise compactness implies sequential pointwise compactness. This property plays a major part in the definition of *angelic* spaces, of which $C(X)$ with the pointwise convergence topology is one of the simplest examples.

Proposition 2.20. *Let X be a compact Hausdorff space and $A \subseteq C(X)$ be bounded. If A is pointwise precompact in $C(X)$, then it is pointwise sequentially precompact in $C(X)$.*

Proof. Let τ_X be the topology on X. Call τ_p the topology on X of pointwise convergence. Fix $(f_n)_n \subseteq A$. By assumption, $(f_n)_n$ has a τ_p-cluster point $f \in C(X)$. Let τ be the topology on X generated by the family $\{f_n\}_n$. Since the f_n's are τ_X-continuous, τ is Hausdorff and weaker than τ_X; by Lemma 2.11, the space (X, τ) is compact.

Claim. *The space (X, τ) is pseudo-metric.*[2]

Proof. Let $d(x, y) \doteq \sup_n 2^{-n}|f_n(x) - f_n(y)|$; it clearly is a pseudo metric. Let τ_d be the topology generated by d. Since both τ and τ_d are Hausdorff compact, by Lemma 2.11, it suffices to show that $\tau_d \subseteq \tau$. Let $x \in X$ and $\epsilon > 0$. Let $L \geq 0$ be a constant bounding A. Pick $N \in \omega$ such that $2L/2^N < \epsilon/2$. Consider the τ-open sets

$$U_n = f_n^{-1}[B_{\epsilon/2}(f_n(x))] = \{y \in X : |f_n(x) - f_n(y)| < \epsilon/2\}$$

for $n \in \omega$ and let $U = \bigcap_{n \leq N} U_n$. If $y \in U$, then

$$\begin{aligned} d(x, y) &\leq \sup_{n \leq N} 2^{-n}|f_n(x) - f_n(y)| + \sup_{n > N} 2^{-n}|f_n(x) - f_n(y)| \\ &< [\epsilon/2] + \Big[2^{-N}\big(\sup_{n>N}|f_n(x)| + \sup_{n>N}|f_n(y)|\big)\Big] \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

This means that $U \subseteq B_\epsilon(x)$. ◊

Compact (pseudo-)metric spaces are separable ([Eng89, Theorem 4.1.18]) so there is $X_0 = \{x_j\}_{j \in \omega}$ dense in (X, τ).

Since f is a τ_p-cluster point for $(f_n)_n$, the *bounded* sequence $(f_n(x_0))_n \subseteq \mathbb{C}$ has $f(x_0)$ as a cluster point and so by the Bolzano-Weierstrass Theorem there is a subsequence $(f_{0,n})_n$ such that $f_{0,n}(x_0) \to f(x_0)$. Now $(f_{0,n}(x_1))_n$ has $f(x_1)$ as a cluster point and so there is a subsequence $(f_{1,n})_n$ such that $f_{1,n}(x_1) \to f(x_1)$. Proceeding similarly, at each stage $j \in \omega$ we have a subsequence $(f_{j,n})_n$ such that $f_{j,n}(x_j) \to f(x_j)$. Let $f_{n_j} \doteq f_{j,j}$. Then $f_{n_j}(x_j) = f_{j,j}(x_j) \to f(x_j)$ for all $j \in \omega$; this means that $f_{n_j} \to f$ on X_0.

The topology τ_0 on A of pointwise convergence on X_0 is Hausdorff and weaker than τ_p, so it is also compact; it follows from Lemma 2.11 that $\tau_p = \tau_0$. Notice that f is the unique τ_p-cluster point of $(f_{n_j})_j$. Indeed, if g is a cluster point, then f and g have to agree on X_0 because $f_{n_j} \to f$ on X_0; but both are τ-continuous and X_0 is τ-dense, so $f = g$. Therefore $f_{n_j} \to f$. □

2.2.3 The compactness characterization theorem for $C(X)$

We are finally ready to prove the main result of this chapter, a way of characterizing compactness for bounded sets of continuous functions on a

[2] If one of the f_n's or f are injective, then (X, τ) is a metric space; anyway, pseudo-metric is enough for our needs.

compact Hausdorff space. This is done by combining the last two propositions with the Eberlein-Šmulian Theorem 2.14 and the Dominated Convergence Theorem 2.7.

Theorem 2.21. *Let X be a compact Hausdorff space and $A \subseteq C(X)$ be bounded. The following are equivalent:*

(i) *A is pointwise precompact in $C(X)$.*

(i*) *A is pointwise sequentially precompact in $C(X)$.*

(ii) *$\mathrm{DLP}(A, X_0)$ holds for some dense $X_0 \subseteq X$.*

(iii) *A is weakly precompact.*

(iii*) *A is weakly sequentially precompact.*

Proof. First we prove that (iii)⇔(iii*)⇔(i*).

(iii)⇔(iii*): By the Eberlein–Šmulian Theorem 2.14.
(iii*)⇒(i*): By Remark 2.6.
(i*)⇒(iii*): Let $(f_n)_n \subseteq A$. By (i*), passing to a subsequence if needed, there is $f \in C(X)$ such that $f_n \to f$. By the Dominated Convergence Theorem 2.7 we have $\int f_n \, \mathrm{d}\mu \to \int f \, \mathrm{d}\mu$ for all Radon measures μ on X, which are the same as functionals in $C(X)^*$ by the Riesz Representation Theorem 2.3; hence $f_n \rightharpoonup f$.

Now we prove that (i*)⇔(ii)⇔(i).

(i*)⇒(ii): Take $X_0 = X$. Let $(f_n)_n \subseteq A$ and $(x_n)_n \subseteq X$ and suppose the double limits exist. By (i*) there is $f \in C(X)$ such that $f_n \to f$. Therefore

$$\lim_n \lim_m f_n(x_m) = \lim_n f_n(x) = f(x) = \lim_m f(x_m) = \lim_m \lim_n f_n(x_m),$$

where the first, third and fourth equalities hold by continuity, while the second because $f_n \to f$.
(ii)⇒(i): By Proposition 2.18.
(i)⇒(i*): By Proposition 2.20.

This completes the proof.[3] □

Convention 2.22. We will sometimes use the notation

$$C_p(X)$$

for the space $C(X)$ with the pointwise convergence topology.

[3]Notice that with the above arguments it is not possible to prove (ii)⇒(iii) without using the starred conditions together with Proposition 2.20, i.e. passing back and forth to sequences. This is because the Dominated Convergence Theorem 2.7 does not hold in general for nets; if that were true, the proof of (i*)⇒(iii*) would 'lift up' to a direct proof of (i)⇒(iii), but this is not the case.

2.2.4 Grothendieck's theorem

The assumptions of Theorem 2.21 may be slightly relaxed. It is enough to demand X to be *countably compact* and Hausdorff.[4] Moreover, condition (ii) automatically forces A to be bounded; this is due to the following result.

Lemma 2.23. *Let E be a complex normed space. If a subset $A \subseteq E$ is weakly compact, then it is bounded and weakly closed.*

Proof. Let $\delta\colon E \to E^{**}$ be the canonical embedding $x \mapsto \delta_x$ defined by $\delta_x(f) \doteq f(x)$. For all $f \in E^*$ the set $f[A]$ is a continuous image of a (weakly) compact set. So it is a compact set in $\mathbb{C}$, hence bounded. Then for all $f \in E^*$ we have

$$\sup_{x\in A} |\delta_x(f)| = \sup_{x\in A} |f(x)| < \infty.$$

The space E^* is Banach so we can apply the Uniform Boundedness Principle ([Bre11, Theorem 2.2]) to $\{\delta_x : x \in A\} \subseteq E^{**}$ and get that $\sup_{x\in A}||\delta_x|| < \infty$. It follows from the Hahn-Banach theorem ([Bre11, Corollary 1.4]) that $||\delta_x|| = ||x||$ for all $x \in E$, and so A is bounded. Since the weak topology is Hausdorff, the weakly compact set A is also weakly closed. □

The equivalence (ii)⇔(iii) was originally stated by Grothendieck as follows, without any assumption on the space X. Recall that if X is just a topological space, the space $C_b(X)$ of bounded continuous complex-valued functions on X is still a Banach space (see [HS75, Theorem 7.9]).

Grothendieck's Theorem ([Gro52, Théorème 6])**.** *Let X be an* arbitrary *topological space, $X_0 \subseteq X$ a dense subset and $A \subseteq C_b(X)$. The following are equivalent:*

(i) *A is weakly precompact.*

(ii) *A is bounded and* $\mathrm{DLP}(A, X_0)$ *holds.*

The theorem is proved in [Gro52] by means of the "Čech compactification" βX of X. It is also quoted *verbatim* in [Ben13, Fact 2], without a proof. It seems reasonable to us doubting whether the result holds without any assumption on X. Aside from the fact that in all the works mentioned in the footnote some properties are assumed, the reasons for demanding them seem quite clear:

- The image of X into βX may not be dense if the space X is not at least Tychonoff (recall that a countably compact Hausdorff space is regular, hence Tychonoff).
- If X is not countably compact, the proof of Proposition 2.20 does not work for $C_b(X)$, hence we cannot pass down to sequences to use the Dominated Convergence Theorem 2.7, which does not hold for nets.

[4] This can can be seen in the huge monograph [Tka15, U.044] on $C_p(X)$ and [Fre03, Proposition 462F]. In [KL16, Lemma D.3] it is stated and proved only in the case of X Hausdorff compact.

Chapter 3

Continuous logic

In this chapter we introduce Continuous First Order Logic (CFO) from scratch, describing its syntax, semantics and fundamental theorems. Then we present types and some topologies on the type space, which will be needed to deal with stability. In the continuous setting, we will measure the size of type spaces in two ways: according to their size (as in classical logic) or to their density character (with respect to a uniform norm in the local case; with respect to the "logic topology", a generalization of the usual one in two-valued model theory, in the global case). Even if it is not strictly needed in following chapter, we introduce also another topology on global type spaces, the "metric topology", which, as the name suggests, is generated by a metric. We will also analyse some of the relationships between the logic and metric topologies. We base our exposition mainly on [Yaa+08], and to a less extent on [BU08].

3.1 Syntax

As in classical logic, in every language we include the same fixed *logical symbols*, which are:

- An infinite set of variables.
- A symbol d for the distance predicate, which is the continuous counterpart of equality.
- The symbols "inf" and "sup" for continuous quantifiers, corresponding respectively to $\exists$ and $\forall$.
- A symbol for each *continuous* function $u\colon [0,1]^n \to [0,1]$ for any $n \in \omega$. These play the role of connectives.

The interpretation of the distance predicate in a metric structure requires to have a fixed metric space (M, d^M), which we always assume to be *complete* and *bounded*, without loss of generality with $d^M \leq 1$. So here we have a different class of structures for every such metric space. In each class – as it is the case with FOL – 'equality' has always the same meaning, namely the concrete metric in the underlying space.

Definition 3.1. A *metric language* consists of the set of logical symbols, together with a set of *non-logical symbols* of the following form:

$$\langle R_i, n_{R_i}, \Delta_{R_i} \mid i \in I\rangle \cup \langle f_j, n_{f_j}, \Delta_{f_j} \mid j \in J\rangle \cup \langle c_k \mid k \in K\rangle.$$

The cardinality of a language L is the cardinal $|L| \doteq |I| + |J| + |K| + \aleph_0$.

As in classical logic, these are respectively symbols for relations, functions and constants, with their arities. In addition, we have symbols for moduli of uniform continuity, since we will require interpretations of relation and function symbols to always be *uniformly continuous* maps, in order to make the theory work smoothly. We recall what this means.

Definition 3.2. Let (M_1, d_1) and (M_2, d_2) be metric spaces and take any $f\colon M_1 \to M_2$. A map $\Delta_f\colon (0,1] \to (0,1]$ is called a *modulus of uniform continuity for f* if for every $\epsilon \in (0,1]$ and every $x, y \in M_1$ we have

$$d_1(x,y) < \Delta_f(\epsilon) \quad \Rightarrow \quad d_2(f(x), f(y)) < \epsilon.$$

We say that the function f is *uniformly continuous* if it has a modulus of uniform continuity.

Remark 3.3. Let $f\colon M_1 \to M_2$ be uniformly continuous. Then for each $\lambda > 0$ and $\epsilon > 0$ such that $\lambda\epsilon \in (0,1]$ we have $\Delta_f(\lambda\epsilon) < (1+\lambda)\Delta_f(\epsilon)$.

Structures are defined in the following natural way, by specifying concrete meanings for all symbols.

Definition 3.4. Fix L a metric language as above. A *metric L-structure* based on a (complete and bounded) metric space (M, d^M) consists of

$$\langle R_i^M, n_{R_i}^M, \Delta_{R_i}^M \mid i \in I\rangle \cup \langle f_j^M, n_{f_j}^M, \Delta_{f_j}^M \mid j \in J\rangle \cup \langle c_k^M \mid k \in K\rangle,$$

where $c_k^M \in M$, and the natural numbers $n_{R_i}^M$ and $n_{f_j}^M$ are the arities of the *uniformly* continuous maps (with respect to d^M)

$$R_i^M\colon M^{n_{R_i}^M} \to [0,1], \qquad f_j^M\colon M^{n_{f_j}^M} \to M,$$

with $\Delta_{R_i}^M$ and $\Delta_{f_j}^M$ their respective moduli of uniform continuity.

Convention 3.5. To keep the notation simple, we usually use the same symbol M for the domain of the metric space, the metric space itself and the structure based on it.

Convention 3.6. Careful readers may have noticed that we have not specified yet which metric we endow M^n with if $n > 1$. We can take any *product metric* of the form

$$d_p((a_1, \dots, a_n), (b_1, \dots, b_n)) = ||(d(a_1, b_1), \dots, d(a_n, b_n))||_p$$

for $p \in [1, \infty]$, since in finite dimensions they are all equivalent and thus induce the same topology. For simplicity, we take $p = \infty$ and use the same symbol d instead of d_∞. So for $a, b \in M^n$ we have

$$d(a, b) \doteq \max_{1 \le i \le n} d(a_i, b_i)$$

We can define embeddings, isomorphisms and substructures almost as in the classical case: here we should also demand distances to be preserved.

Definition 3.7. Let M and N be L-structures. An *embedding* from M to N is an isometry $T \colon (M, d^M) \to (N, d^N)$ such that for all $\vec{a} \subseteq M$ we have:

- $R^M(\vec{a}) = R^N(T(\vec{a}))$
- $T(f^M(\vec{a})) = f^N(T(\vec{a}))$
- $T(c^M) = c^N$

An *isomorphism* is a surjective (thus bijective) embedding. We say that M is a *substructure* of N if the inclusion map $M \hookrightarrow N$ is an embedding, and with a little abuse of notation we write $M \subseteq N$.

Terms are defined inductively, exactly as in classical logic.

Definition 3.8. *Terms* of a language L consist of variables and constant symbols, and if $t_1, \dots, t_n$ are L-terms and f is a n-ary function symbol, also $f(t_1, \dots, t_n)$ is an L-term. All terms are built this way. The *interpretation* in M of an L_M-term $t(x_1, \dots, x_n)$ is the function

$$t^M \colon M^n \to M, \ \vec{a} \mapsto t[\vec{a}/\vec{x}]$$

obtained by replacing every free occurrence of every x_i with a_i.

Now we define formulae, which are built essentially as in classical logic.

Definition 3.9. Let L be a fixed language. The set of *L-formulae* consists of the following expressions:

- Atomic L-formulae, that are expressions of the form $R(t_1, \dots, t_n)$ or $d(t_1, t_2)$, where R is an n-ary relation symbol and $t_1, \dots, t_n$ terms.
- $u(\phi_1, \dots, \phi_n)$, where u is a connective and $\phi_1, \dots, \phi_n$ are L-formulae.
- $\sup_x \phi$ and $\inf_x \phi$, where ϕ is an L-formula and x a variable.

All formulae are built this way.

We define exactly as in classical logic concepts like quantifier-free formula, subformula, sentence, syntactic substitution, free and bound occurrences of variables, expansion of a language and so on.

Example 3.10. Some basic examples of metric structures:

- A complete metric space (M, d) bounded by 1 with no relations, no functions and no constants is the minimal example of a metric structure. The language here is empty.
- Each first order structure can be viewed as a metric structure by taking the discrete metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

  and identifying each predicate R with its characteristic function χ_R.
- Take M to be the unit ball of a real Banach space $(E, ||\cdot||)$, the distance $d(x,y) = ||x - y||$ and a function symbol $f_{a,b}$ for each $a, b \in \mathbb{R}$ such that $|a| + |b| \leq 1$, which is interpreted as $f_{a,b}(x,y) = ax + by$.
- More examples (probability spaces, Hilbert spaces, L^p spaces) can be found in [Yaa+08].

3.2 Semantics

It is in its semantic that continuous logic is particularly more refined than the classical one: in a model M, every sentence σ has a truth value

$$\sigma^M \in [0, 1]$$

which is defined inductively as follows. As usual, we will just write $\phi(a)$ instead of $\phi[x/a]$ to denote the sentence obtained by substituting every free occurrence of x in ϕ with a.

Definition 3.11. Let M be a structure. For all closed L_M-terms $t_1, \ldots, t_n$, all L_M-sentences $\sigma_1, \ldots, \sigma_n$ and all L_M-formulae $\phi(x)$ we define:

- $d(t_1, t_2)^M \doteq d^M(t_1^M, t_2^M)$
- $R(t_1, \ldots, t_n)^M \doteq R^M(t_1^M, \ldots, t_n^M)$
- $u(\sigma_1, \ldots, \sigma_n)^M \doteq u(\sigma_1^M, \ldots, \sigma_n^M)$
- $(\sup_x \phi(x))^M \doteq \sup_{a \in M} \phi(a)^M$
- $(\inf_x \phi(x))^M \doteq \inf_{a \in M} \phi(a)^M$.

for any n-ary relation symbol R and any n-ary connective u.

Remark 3.12. Every L_M-formula $\phi(x_1, \dots, x_n)$ yields a *uniformly continuous* function

$$\phi^M : M^n \to [0,1], \ \vec{a} \mapsto \phi(\vec{a})^M.$$

This is due to the fact that all interpretations of relations and functions are taken to be uniformly continuous.[1]

Convention 3.13. Let M be an L_A-structure. We use the notation

$$\mathcal{L}(A) \doteq \{\phi^M \mid \phi \in L_A\}$$

for the family of maps induced by L_A-formulae.

Just as in classical logic, two formulae are indeed called logically equivalent if they always have the same truth values.

Definition 3.14. We say that two formulae ϕ and ψ are *logically equivalent* if for every structure M the maps ϕ^M and ψ^M coincide.

We can define a distance between formulae. Two formulae are logically equivalent iff the logical distance between them is zero.

Definition 3.15. The *logical distance* between two L-formulae ϕ, ψ is

$$d_0(\phi, \psi) \doteq \sup\{|\phi(a)^M - \psi(a)^M| : M \text{ is an } L\text{-structure, } a \in M\}.$$

It is clearly a pseudo-metric; it is a metric if formulae are considered up to logical equivalence.

In classical logic, having only two truth values, we can identify without harm every sentence σ with the statement "σ is true". In continuous logic we can do the same but the identification is much 'stronger': we pick the somehow natural value 0 among the infinitely many possible ones, and we say that a sentence σ is *true* in a model M when $\sigma^M = 0$. More formally:

Definition 3.16. Let M be an L-structure and σ an L-sentence. We define

$$M \models \sigma \quad \iff \quad \sigma^M = 0$$

and say that σ *is true in* M. If $\phi(x)$ is an L-formula, we say that $a \in M$ *realizes* $\phi(x)$ if $\phi(a)^M = 0$, and we write $M \models \phi(a)$.

Remark 3.17. Note that the map

$$[0,1]^2 \to [0,1], \ (x,y) \mapsto |x - y|$$

is a connective. Since we can regard every $r \in [0,1]$ as a constant connective, expressions of the form $|\phi(x) - r|$ are formulae, which we abbreviate by $\phi(x) - r$. Therefore, we have

$$M \models \sigma - r \quad \iff \quad \sigma^M = r$$

for all sentences σ.

[1] It can be checked by induction on the complexity of formulae, using basic analysis tools about uniform continuity. For details see [Yaa+08, §2].

A different approach to define the satisfaction relation could be to introduce the concept of *conditions*, i.e. formal expressions of the form $\sigma = r$ where σ is a sentence and $r \in [0, 1]$, and define theories as sets of conditions. Satisfaction of conditions is defined in the obvious way: a model M satisfies the condition $\sigma = r$ if $\sigma^M = r$.

From a certain perspective, this would indeed be a more intuitive approach, because it avoids the identification between sentences and statements asserting they are 'true'. Moreover, allowing a generic value $r \in [0, 1]$ makes the 'translation' trick mentioned in Remark 3.17 unnecessary.

On the other hand, there are two important reasons that make us lean towards discarding the approach with conditions. Firstly, most theories are naturally axiomatized by conditions of the form $\sigma = 0$, in which case there is no need for other values. Secondly and most importantly, defining theories as just sets of *sentences* is precisely what we do in classical logic and in the proofs it is simpler to handle a sentence σ than a condition $\sigma = 0$, which is anyway completely determined by σ itself.

Definition 3.18. An L-*theory* is a set of L-sentences. A structure M *satisfies* a theory T, in symbols $M \models T$, if $M \models \sigma$ for every $\sigma \in T$. The *theory* of M is the set $\mathrm{Th}(M)$ of L-sentences realized by M. Theories of this form are called *complete*.

The concept of elementariness can be readily adapted to continuous logic in a natural way. Indeed, two structures are elementary equivalent if they are indistinguishable by considering their continuous first order properties.

Definition 3.19. Let M and N be L-structures. We say that:

- M and N are *elementarily equivalent*, in symbols $M \equiv N$, if for every sentence $\sigma \in L$ we have $\sigma^M = \sigma^N$.
- A substructure M of N is *elementary*, in symbols
$$M \preceq N$$
if for every $\phi(\vec{x}) \in L$ and $\vec{a} \subseteq M$ we have $\phi(\vec{a})^M = \phi(\vec{a})^N$.
- A partial function $F\colon M \to N$ is *elementary* if for every $\phi(\vec{x}) \in L$ and every $\vec{a} \subseteq \mathrm{dom}(F)$ we have $\phi(\vec{a})^M = \phi(F(\vec{a}))^N$. We say that F is an *elementary embedding* if it is also total.

As in classical logic, isomorphisms are elementary embeddings but the converse need not be true. Notice also that elementary maps are distance preserving.

3.3 Fundamental theorems

Ultraproducts of metric structures can be defined as in classical logic and are well-defined due to the fact that we work only with complete and bounded metric spaces. They can indeed be used to prove a continuous version of Łos' theorem, which in turn easily implies the following continuous version of the Compactness Theorem.[2]

Fact 3.20 (Continuous Compactness Theorem, [BU08, Theorem 2.16])**.** *Let L be any metric language. If an L-theory is finitely satisfiable, then it has a model.*

Continuous versions of the Löwenheim-Skolem theorems also hold, but in the statements the concept of cardinality has to be replaced with that of *density character*. Recall the density character of a topological space X is the smallest cardinality of a dense subset of X. We will use the symbol

$$||X||$$

to denote it. Note that $||X|| \leq |X|$ and if $A \subseteq X$ then $||A|| \leq ||X||$.

Fact 3.21 (Continuous Downward Löwenheim-Skolem Theorem, [Yaa+08, Theorem 7.3])**.** *If M is an L-structure and A a subset of M, then there exists $A \subseteq N \preceq M$ with $||N|| \leq ||A|| + |L|$. In particular, if $||A|| \geq |L|$ then $||N|| = ||A||$.*

For the upward version, the assumption that the structure is infinite is replaced with demanding it to be non-compact as a metric space.

Fact 3.22 (Continuous Upward Löwenheim-Skolem Theorem, [BU08, Theorem 2.18])**.** *Let M be an L-structure whose underlying metric space is not compact. Then for every cardinal $\kappa \geq ||M|| + |L|$ there exists $N \succeq M$ with $||N|| = \kappa$.*

The concepts of saturation and homogeneity are defined exactly as in the classical case.

Definition 3.23. Let M be an L-structure. We say that:

- M is *κ-saturated* if for every $A \in [M]^{<\kappa}$, any set Γ of L_A-formulae is finitely satisfiable in M_A.
- M is *κ-homogeneous* if every partial elementary map $F\colon M \to M$ with $|F| < \kappa$ can be extended to an automorphism of M.

Elementary superstructures with these properties always exist also in continuous logic.

Fact 3.24 ([Yaa+08, Theorem 7.12])**.** *Let M be an L-structure. For every infinite cardinal κ there exists a κ-saturated and κ-homogeneous elementary L-superstructure $N \succeq M$.*

So also in continuous logic we can always work in some big monster model. This will be useful to deal with stability.

[2] The proofs are essentially the same and can be found in [Yaa+08, §5].

3.4 Types in continuous logic

Global types in continuous logic are defined exactly as in the classical case. This is made possible by how we defined the satisfaction relation and we hope it may result pleasant to the reader.

Definition 3.25. Fix an L-structure M and $A \subseteq M$.

- A *partial type over* A (in M) is a set Φ of L_A-formulae which is finitely satisfiable in M_A.
- A partial type p over A is *complete* if it is maximal finitely satisfiable in M_A. For $b \in M$, the complete type of b over A is

  $$\mathrm{tp}^M(b/A) \doteq \{\phi(x) \in L_A : M \models \phi(b)\}.$$

  The set of complete types over A (in M) is denoted by $S^M(A)$.
- We define n-types and x-types as in the classical case and use the same notation.

We start with a simple observation which is nonetheless important and allows a simple characterization of types.

Lemma 3.26. *Let $p(x) \in S(A)$. If p is realized in the model M by $b \in M$, then for all $\phi(x) \in L_A$ the value $r = \phi(b)^M$ is the unique $r \in [0,1]$ such that $\phi(x) - r \in p$.*

Proof. Uniqueness: if $\phi(x) - r_1$ and $\phi(x) - r_2$ are in p, then $r_1 = \phi(b)^M = r_2$. Existence: the model M witnesses that $p \cup \{\phi(x) - r\}$ is consistent, and so by maximality $\phi(x) - r \in p$. $\square$

In other words, in continuous logic a complete type p assigns to every formula ϕ exactly one truth value $r \in [0,1]$.

Convention 3.27. Since the number r depends only on ϕ and p, we will use the symbol

$$\phi^p$$

to denote it. We can think of it as the value of ϕ according to p.

Remark 3.28. Note that each $p \in S(A)$ is completely determined by the values ϕ^p for $\phi \in L_A$. In other words, $p(x) = \{\phi(x) - \phi^p \mid \phi \in L_A\}$.

Remark 3.29. For realized types, we have $\phi^{\mathrm{tp}^M(b/A)} = \phi(b)^M$ and when the context is clear we may denote this number just by ϕ^b.

In the two valued-case, each type induces a $\{0,1\}$-valued *Keisler measure*, i.e. a probability measure on the Boolean algebra of definable sets. Here we cannot do the same since sets of the form $\phi(M) \doteq \{b \in M : M \models \phi(b)\}$ do not behave well with respect to set-theoretic difference. Instead, in CFO each type $p \in S(A)$ induces a functional

$$T_p \colon (L_A, d_0) \to [0,1],\ \phi \mapsto \phi^p.$$

3.5 The logic topology

Now we define a topology on $S_n(A)$ in a natural way, similar to what is normally done in classical logic.

Definition 3.30. The *logic topology* on $S_n(A)$ has basic open neighbourhoods for $p \in S_n(A)$ of the form

$$[\phi < r] \doteq \{p \in S_n(A) : \phi^p < r\}$$

for $\phi \in p$ and $r \in [0,1]$. We denote it by τ_0.

It is useful to introduce also the notation $[\phi \leq r]$ defined in the obvious way. This set is closed since its complement can be written as $[1-\phi < 1-r]$. Similarly, $[\phi > r]$ is open and $[\phi \geq r]$ is closed.

Remark 3.31. Note that the map

$$x \dot{-} y \doteq \max\{x-y, 0\}$$

is a binary connective.

Lemma 3.32. *Closed subsets of $S_n(A)$ in the logic topology are exactly the sets of the form $C_\Gamma = \{p \in S_n(A) : \Gamma \subseteq p\}$, where Γ is a set of L_A-formulae.*

Proof. Since $C_\Gamma = \bigcap_{\phi \in \Gamma}[\phi \leq 0]$, these sets are closed. Take now a closed set C and let $p \notin C$. Since $p \in S_n(A) \setminus C$ which is open, by definition of logic topology there is $\phi \in p$ and $r > 0$ such that $p \in [\phi < r] \subseteq S_n(A) \setminus C$. Then $C \subseteq [\phi \geq r] = [r \dot{-} \phi \leq 0]$. Let Γ be the set of all formulae of the form $r \dot{-} \phi$ obtained this way. It is clear that $C = C_\Gamma$. □

This topology has some nice properties that spaces of types in classical logic have. They indeed follow from the Compactness Theorem 3.20.

Proposition 3.33. *The space $(S_n(A), \tau_0)$ is Hausdorff compact.*

Proof. Hausdorff: Let $p \neq q$ and pick without loss of generality $\phi \in p \setminus q$. For $r = \phi^q > 0$, we have the disjoint open sets $p \in [\phi < r/2]$ and $q \in [\phi > r/2]$. Compact: Let $\{C_{\Gamma_i}\}_{i \in I}$ be a family of closed sets with the finite intersection property. Then for all $i_0, \ldots, i_n \in I$

$$\emptyset \neq \bigcap_{j=0}^{n} C_{\Gamma_{i_j}} = \Big\{p \in S_n(A) : \bigcup_{j=0}^{n} \Gamma_{i_j} \subseteq p\Big\}.$$

This means that the set of L_A-formulae $\bigcup_{i \in I} \Gamma_i$ is finitely satisfiable, since its finite portions are contained in a type. By the Compactness Theorem, it is satisfiable and so there is a type p extending it. Then $p \in \bigcap_{i \in I} C_{\Gamma_i}$. □

The next fact is a useful topological criterion to show that a substructure is elementary.

Lemma 3.34 (Continuous Tarski-Vaught Test)**.** *Let M be an L-structure and $A \subseteq M$ a* closed *subset. The following are equivalent:*

(i) *A is (the domain of) an elementary substructure of M.*

(ii) *For every L-formula $\phi(x, \vec{y})$ and every $\vec{a} \subseteq A$ we have*

$$\inf_{b \in A} \phi(b, \vec{a})^M = \inf_{b \in M} \phi(b, \vec{a})^M.$$

(iii) *The set of realized types $\{\mathrm{tp}^M(a/A) \mid a \in A\}$ is dense in $(S_1(A), \tau_0)$.*

Proof. (i)⇒(ii): By definition of elementary substructure applied to the formula $\inf_x \phi(x, \vec{y})$.
(ii)⇒(i): First we show that A is (the universe of) a substructure; it suffices that it is closed under function symbols, i.e. $f(\vec{a}) \in A$ for each $\vec{a} \subseteq A$. For any $\vec{a} \subseteq A$, applying (ii) to the formula $d(y, f(\vec{x}))$ yields

$$d(A, f(\vec{a})) \doteq \inf_{b \in A} d(b, f(\vec{a}))^M = \inf_{b \in M} d(b, f(\vec{a}))^M = 0$$

which means that $f(\vec{a}) \in \mathrm{cl}(A) = A$, since A is closed. Now (ii) can be used to show by induction on the complexity of formulae that (i) holds, exactly as in classical logic.
(iii)⇒(ii): Fix an L-formula $\phi(x, \vec{y})$ and $\vec{a} \subseteq A$ and $r \doteq (\inf_x \phi(x, \vec{a}))^M$. Fix $\epsilon \in (0, 1-r]$. By density, we can pick $\mathrm{tp}^M(c/A)$ in the non-empty open set $[\phi(x, \vec{a}) < r + \epsilon]$ for some $c \in A$. Then

$$\inf_{b \in A} \phi(b, \vec{a})^M \leq \phi(c, \vec{a})^M = \phi(x, \vec{a})^{\mathrm{tp}^M(c/A)} < r + \epsilon = \inf_{b \in M} \phi(b, \vec{a})^M + \epsilon.$$

Since ϵ was arbitrary, this means that $\inf_{b \in A} \phi(b, \vec{a})^M \leq \inf_{b \in M} \phi(b, \vec{a})^M$. The other inequality is trivial.
(ii)⇒(iii): It suffices to show that each non-empty basic open set $[\phi(x, \vec{a}) < r]$ contains some $\mathrm{tp}^M(b/A)$ for $b \in A$. By (ii) we can rewrite this as follows:

$$\begin{aligned} \exists b \in A\ (\mathrm{tp}^M(b/A) \in [\phi(x, \vec{a}) < r]) \quad &\iff \quad \exists b \in A\ (\phi(b, \vec{a})^M < r) \\ &\iff \quad \inf\nolimits_{b \in A} \phi(b, \vec{a})^M < r \\ &\iff \quad \inf\nolimits_{b \in M} \phi(b, \vec{a})^M < r. \end{aligned}$$

In order to show that the last inequality holds, pick any $p \in [\phi(x, \vec{a}) < r]$. Let $t \doteq \phi(x, \vec{a})^p < r$ so that $\phi(x, \vec{a})^p - t \in p$. By definition, p is finitely satisfiable in M_A, so there is $b \in A$ such that $\phi(b, \vec{a})^M = t$. It follows that $\inf_{b \in M} \phi(b, \vec{a})^M \leq t < r$, hence our thesis. □

It follows that M is dense in $S_n(M)$ for the logic topology, just as in classical logic.

Corollary 3.35. *For every L-structure M, the set $\{\mathrm{tp}^M(a/M) \mid a \in M\}$ of realized types is dense in $(S_n(M), \tau_0)$.*

Proof. The set $A = M$ is obviously closed and an elementary substructure of itself. From Lemma 3.34(iii) the thesis for $n = 1$ directly follows. For $n > 1$ it can be obtained by a simple induction, by seeing formulae of the form $\phi((x_1, \dots, x_n), (y_1, \dots, y_m))$ as $\phi((x_1, \dots, x_{n-1}), (x_n, y_1, \dots, y_m))$. $\square$

Combining the last result with the fact that spaces of types are compact we get that $S_n(M)$ is a compactification of M.

Proposition 3.36. *Let $i\colon (M, d) \to (S_n(M), \tau_0)$ be the map $a \mapsto \mathrm{tp}^M(a/M)$. Then $(S_n(M), \tau_0, i)$ is a compactification of (M, d).*

Proof. The space $(S_n(M), \tau_0)$ is compact by Proposition 3.33 and (M, d) is mapped densely in it by Corollary 3.35. The only things left to check are the properties of the map i.

Injective: if $a, b \in M$ are distinct, then $d(a, b) > 0$ and so

$$d(x, a) \in \mathrm{tp}(a/M) \setminus \mathrm{tp}(b/M).$$

Continuous: for basic τ_0-open sets we have

$$\begin{aligned} i^{-1}[[\phi < r]] &= \{a \in M \mid \mathrm{tp}(a/M) \in [\phi < r]\} = \{a \in M \mid \phi^{\mathrm{tp}(a/M)} < r\} \\ &= \{a \in M \mid \phi(a)^M < r\} = \phi^{-1}[[0, r)]. \end{aligned}$$

The last set is open because formulae are (uniformly) continuous.

Open on its image: on basic balls we have

$$i[B_r(a)] = \{\mathrm{tp}(b/M) : d(a, b) < r\} = [d(x, a) < r] \cap i[M]$$

which is by definition open for the logic topology restricted to $i[M]$. $\square$

3.6 The metric topology

Now we define another natural topology on $S_n(A)$, which is induced by the metric d on M. It refines the logic topology and agrees with it on realized types. We will use the same symbol d to denote it. It is particularly useful since this way $S_n(A)$ becomes a metric space, allowing us to import techniques from functional analysis.

Convention 3.37. Throughout this section we fix a model M and $A \subseteq M$ such that M realizes every type in $S_n(A)$ for all $n \geq 1$. When the model is omitted as a superscript, we implicitly assume we are working in M.

Definition 3.38 (The metric on types)**.** For $p, q \in S_n(A)$ define

$$d(p, q) \doteq \inf\{d(b, c) : M \models p(b),\ M \models q(c)\}$$

where $d(b, c)$ is in the sense of Convention 3.6.

Notice that the definition of the metric does not depend on the choice of the model M since it realizes all n-types over A.

Remark 3.39. For all $a, b \in M$ we have

$$d(\mathrm{tp}(a/A), \mathrm{tp}(b/A)) \leq d(a, b).$$

In general, we cannot hope for the equality; for instance, if $a, b \in M$ satisfy $a \neq b$ but $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$, then $d(\mathrm{tp}(a/A), \mathrm{tp}(b/A)) = 0 < d(a, b)$.[3]

The next result is crucial to show that this map actually defines a metric on the type space and gives also us a way to easily handle computations.

Lemma 3.40. *For any types $p, q \in S_n(A)$ there are $b \models p$ and $c \models q$ in M such that $d(p, q) = d(b, c)$.*

Proof. Consider the set of L-formulae

$$\Phi(x, y) \doteq p(x) \cup q(y) \cup \{d(x, y) \dot{-} (D + 1/n) : n \in \omega\}$$

where $D \doteq d(p, q)$.

<u>Claim.</u> *The set Φ is finitely satisfiable.*

<u>Proof.</u> Take a finite subset, i.e.

$$\Phi_0(x, y) = p_0(x) \cup q_0(y) \cup \{d(x, y) \dot{-} (D + 1/n) : n \in S\}$$

where $p_0 \subset p$ and $q_0 \subseteq q$ and $S \subseteq \omega$ are all finite. For each $n \in S$ we have

$$D + 1/n > D = \inf\{d(b, c)^M : M \models p(b),\ M \models q(c)\},$$

and so there exist $b_n, c_n \in M$ with $b_n \models p(x)$ and $c_n \models q(y)$ such that $d(b_n, c_n) \leq D + 1/n$, which means that $M \models d(b_n, c_n) \dot{-} (D + 1/n)$. Therefore we have $(b', c') \doteq \arg\min_{n \in S} d(b_n, c_n) \models \Phi_0(x, y)$. $\Diamond$

By compactness, Φ is a partial type, i.e. $\Phi(x, y) \in S_{2n}(\emptyset) \subset S_{2n}(A)$. By our assumptions on M, it is realized in M, so there are $b, c \in M$ with $b \models p$ and $c \models q$ such that $d(b, c) \leq D = d(p, q)$. It follows from the definition of the metric on types that $d(p, q) = d(b, c)$. $\square$

Corollary 3.41. *The function $d(p, q)$ is a metric on $S_n(A)$.*

Proof. Clearly it is a pseudo-metric. Now suppose that $d(p, q) = 0$. Then by Lemma 3.40 $d(b, c) = 0$ for some $b \models p$ and $c \models q$. Since d is a metric on M, it follows that $b = c$. So $p = \mathrm{tp}(b) = \mathrm{tp}(c) = q$. $\square$

So the metric $d(p, q)$ induces a topology on $S_n(A)$, which we will call *the metric topology*. We can compare it the our logic topology.

[3] Notice that such elements cannot be in A, because if $a, b \in A$ and $a \neq b$, then $\mathrm{tp}(a/A) \neq \mathrm{tp}(b/A)$, since $d(x, a) \in \mathrm{tp}(a/A) \setminus \mathrm{tp}(b/A)$.

Proposition 3.42. *The logic topology on $S_n(A)$ is weaker than the metric topology. Moreover, they coincide on the subset of realized types.*

Proof. Let $[\phi < r]$ be a neighbourhood of $p \in S_n(A)$; in particular we have $\phi \in p$, i.e. $\phi^p = 0$. It is enough to check that it contains a d-ball. Take any $q \in S_n(A)$ with $d(p,q) < \Delta_\phi(r) \doteq R$. By Lemma 3.40, there are $b, c \in M^n$ such that $d(p,q) = d(b,c)$ with $b \models p$ and $c \models q$. Since $d(b,c) < \Delta_\phi(r)$, we get

$$\phi^q = |\phi^p - \phi^q| = |\phi(b)^M - \phi(c)^M| < r.$$

We obtained that $B_R(p) \subseteq [\phi < r]$.

Now the moreover part. Let $j\colon M \to S_n(A)$ be the map $b \mapsto \mathrm{tp}(b/A)$. It follows directly from the definition of the metric on types and Remark 3.29 that

$$B_r(\mathrm{tp}(b/A)) \cap j[M] \supseteq [d(x,b) < r] \cap j[M]$$

which implies that in $j[M]$ the metric topology is weaker than the logic topology, hence they coincide on realized types. $\square$

Remark 3.43. One can wonder whether the logic topology is *strictly* weaker than the metric topology. On this topic, in [BU08, p. 24] it is said that

> By a theorem of Henson (for Banach space structures in positive bounded logic, but it boils down to the same thing), for a complete countable theory T, the metric on $S_n(T)$ coincides with the logic topology for all n if and only if T is separably categorical, i.e., if and only if it has a unique separable model up to isomorphism.

So in general the two topologies are distinct.

We assumed at the beginning that a metric structure (M,d) is always complete. The set M can be mapped to the space $S_n(A)$ via the function $j\colon b \mapsto \mathrm{tp}(b/A)$. We defined a metric on $S_n(A)$, thus showing that it is metrizable; Proposition 3.42 implies that this metric actually coincides with the logic topology on realized types. Assume that $A \preceq M$ is closed. Then $j[M]$ is dense in $(S_n(A), \tau_0)$ by Lemma 3.34. We can summarize all this information in the following picture.

$$\begin{tikzcd}
 & (j[M],d) \arrow[r, hook] \arrow[d, equal, "3.42"] & (S_n(A),d) \\
\underset{\text{complete}}{(M,d)} \arrow[ur, "j"] \arrow[dr, "j"'] & & \\
 & (j[M],\tau_0) \arrow[r, hook, "\text{dense}"] & (S_n(A),\tau_0)
\end{tikzcd}$$

It is then somewhat natural to think that the metric space $(S_n(A), d)$ may also be complete. It is actually the case, as the next result shows. This is true in general for spaces that are called in [BU08, p. 25] *topometric spaces*, i.e. spaces (X, τ, d) where τ is a compact Hausdorff topology on X, d a metric refining τ and for each closed set $F \subseteq X$ and each $\epsilon > 0$ the closed ϵ-neighbourhood of F is also closed in X (see [BU08, Lemma 4.13]). Anyway, we provide a direct proof in our specific case.

Proposition 3.44. *The metric space $(S_n(A), d)$ is complete.*

Proof. Let $(p_i)_{i<\omega}$ be a Cauchy sequence in $(S_n(A), d)$.

Claim. *For every $a \models p_i$ there exists $b \models p_{i+1}$ such that $d(a, b) = d(p_i, p_{i+1})$.*

Proof. By Lemma 3.40, $d(p_i, p_{i+1}) = d(c, e)$ for some $c \models p_i$ and $e \models p_{i+1}$. So $\mathrm{tp}^M(a/A) = \mathrm{tp}^M(c/A)$, i.e. the map $c \mapsto a$ is elementary in M_A. By ω-homogeneity there is $f \in \mathrm{Aut}(M_A)$ such that $f(c) = a$. Thus

$$d(p_i, p_{i+1}) = d(c, e) = d(f(c), f(e)) = d(a, f(e)),$$

hence we may take $b = f(e) \models p_{i+1}$. $\Diamond$

So we can recursively construct a sequence $(b_i)_{i<\omega} \subseteq M$ such that for each $i < \omega$ we have $b_i \models p_i$ and $d(b_i, b_{i+1}) = d(p_i, p_{i+1})$. Since $(p_i)_i$ is Cauchy sequence, by Corollary 3.41 we get that $(b_i)_i$ is Cauchy in M, which is complete. Then $b_i \to b$ for some $b \in M$. Finally,

$$d(\mathrm{tp}(b/A), p_i) = d(\mathrm{tp}(b/A), \mathrm{tp}(b_i/A)) \leq d(b, b_i) \to 0.$$

This means that $p_i \to \mathrm{tp}(b/A) \in S_n(A)$, which concludes our proof. $\square$

Remark 3.45. For $p \in S_n(A)$, define the set of realizations of p as

$$S_p \doteq \{b \in M : b \models p(x)\} = \{b \in M : \forall \phi \in p \ (M \models \phi(b))\} = \bigcap_{\phi \in p} \phi^{-1}[\{0\}].$$

It is closed because formulae are (uniformly) continuous. It can be thought as a set determined by an infinite number of 'equations'; such sets are usually indeed closed. The distance between types can be understood as the distance between two sets:

$$d(p, q) = d(S_p, S_q) \doteq \inf\{d(a, b) : a \in S_p,\ b \in S_q\}.$$

In a complete metric space, when two sets are compact, their distance is always attained for some points, but when they are only closed this need not be true, even if they are bounded[4] (though not in $\mathbb{R}^n$, where the Heine-Borel theorem ensures it). Combining Lemma 3.40 and Proposition 3.44 provides an interesting example of a distance between bounded closed (in general non-compact[5]) sets in a complete metric space which is always attained.

[4] For instance, consider the two sets $S_1 = \{y = 0\}$ and $S_2 = \{1/x = 0\}$ in $\mathbb{R}^2$ and the metric $d(x, y) = \min\{||x - y||, 1\}$. They are still closed since d is equivalent to the usual metric and clearly now they are bounded. However, the distance $d(S_1, S_2) = 0$ is not attained.

[5] For instance, take a non-compact metric structure (M, d) in the empty language; then $S^M(\emptyset) = \{p\}$, hence $S_p = M$ is not compact.

3.7 Predicates and functions induced by formulae

Consider the usual topology τ_0 on $S_n(A)$ in classical logic. Each L_A-formula $\phi(x_1, \ldots, x_n)$ induces the clopen set $[\phi] = \{p \in S_n^M(A) : \phi \in p\}$ and also the map

$$S_n^M(A) \to \{0,1\},\ p \mapsto \begin{cases} 0 & \text{if } \phi \in p \\ 1 & \text{if } \phi \notin p \end{cases}$$

which is continuous, since its preimages are precisely the basic open sets $[\phi]$ and $[\neg\phi]$. With some abuse of notation we denote this map also by ϕ. This is legitimate since obviously two formulae are logically equivalent if and only if their associated maps are equal. It is not hard to see that every continuous map in $2^{S_n^M(A)}$ is the map induced by (an equivalence class of) a formula. So the correspondence $[\phi] \mapsto \phi \in 2^{S_n^M(A)}$ between clopen sets and continuous maps is bijective. Moreover, the topology τ generated by these maps is exactly τ_0. Since they are continuous for τ_0, by minimality $\tau \subseteq \tau_0$. For the converse, just observe that for every basic τ_0-open set we have $[\phi] = \phi^{-1}[\{0\}]$, which is τ-open by definition.

Remark 3.46. The situation in continuous logic is similar to some extent: every L_A-formula $\phi(x_1, \ldots, x_n)$ induces a map

$$S_n^M(A) \to [0,1],\ p \mapsto \phi^p$$

also denoted by ϕ for the same reason as before. Often we omit the superscript M when there are no ambiguities. The map ϕ is continuous for the logic topology, since $\phi^{-1}[(r,t)] = [\phi > r] \cap [\phi < t]$ for any $r, t \in [0,1]$.

Again, maps induced by formulae actually generate the logic topology by a similar argument, namely because they are continuous and for every basic open neighbourhood of p we can write $[\phi < r] = \phi^{-1}[[0,r)]$.

This indeed gives motivation to how we defined the logic topology in the first place: if we think (as we may) of the usual topology on types in classical logic as being the topology generated by (maps induced by) formulae, then what we did is just using the exact same definition in continuous logic.

Convention 3.47. Let M be an L-structure and $A \subseteq M$. We denote by $\mathfrak{L}^M(A)$ the family of maps on spaces $S_n^M(A)$ induced by L_A-formulae. So now we have the three families:

$$\begin{aligned} L_A &\doteq \{\phi : \phi \text{ is an } L_A\text{-formula}\} \\ \mathcal{L}^M(A) &\doteq \{\phi\colon M^n \to [0,1] \mid \phi \in L_A\} \\ \mathfrak{L}^M(A) &\doteq \{\phi\colon S_n^M(A) \to [0,1] \mid \phi \in L_A\}. \end{aligned}$$

Remark 3.48. By Remark 3.29, the map $\phi\colon S_n^M(A) \to [0,1]$ is an extension of the (uniformly) continuous function $\phi\colon M^n \to [0,1]$.

We already saw that this extension is continuous for the logic topology. Now we shall see that it is also uniformly continuous for the metric topology.

Lemma 3.49. *Maps $\phi\colon S_n^M(A) \to [0,1]$ induced by formulae are uniformly continuous for the metric topology.*

Proof. As noticed in Remark 3.12, there is a modulus Δ_ϕ of uniform continuity for ϕ^M. Now we show that it is a modulus of uniform continuity also for $\phi\colon S_n^M(A) \to [0,1]$. Let $\epsilon > 0$ and suppose that $p, q \in S_n^M(A)$ satisfy $d(p,q) < \Delta_\phi(\epsilon)$. By Lemma 3.40, there here are $a, b \in M^n$ realizing respectively p, q such that $d(p,q) = d(a,b)$. Then by Remark 3.48 we get

$$|\phi(p) - \phi(q)| = |\phi(\mathrm{tp}(a/A)) - \phi(\mathrm{tp}(b/A))| = |\phi(a)^M - \phi(b)^M| < \epsilon$$

where the inequality holds because $d(a,b) < \Delta_\phi(\epsilon)$. □

We summarize what we obtain so far in the following result.

Corollary 3.50. *Let ϕ be an L-formula. Then the map $\phi\colon S_n^M(A) \to [0,1]$ extends $\phi\colon M^n \to [0,1]$ and is continuous for the logic topology and uniformly continuous for the metric topology.*

Proof. Directly by Remark 3.46, Remark 3.48 and Lemma 3.49. □

This yields a nice characterization for continuity. To prove it we need the following result.

Fact 3.51 (Corollary of the lattice version of the Stone-Weierstrass Theorem, [BU08, Proposition 1.4])**.** *Let X be a compact Hausdorff space containing at least two points and $\mathcal{F} \subseteq C(X;[0,1])$ such that:*

- *For all $f, g \in \mathcal{F}$ also the following maps are in $\mathcal{F}$:*

$$x \mapsto 1 - f(x), \qquad x \mapsto f(x) \dot{-} g(x), \qquad x \mapsto f(x)/2.$$

- *For all distinct $x, y \in X$ there is $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.*

Then $\mathcal{F}$ is dense in $C(X;[0,1])$ with the uniform convergence topology.

Let us introduce some notation for this topology.

Definition 3.52. The *uniform distance* of $\Phi, \Psi\colon S_n^M(A) \to [0,1]$ is

$$d_\infty^M(\Phi,\Psi) \doteq ||\Phi - \Psi||_\infty^M = \sup_{p \in S_n^M(A)} |\Phi(p) - \Psi(p)|.$$

Moreover, for formulae $\phi, \psi \in L_A$ we let

$$d_\infty(\phi,\psi) = \sup\{d_\infty^M(\phi,\psi) \mid M \text{ an } L\text{-structure}\}.$$

The metrics d_0 and d_∞ coincide.

Lemma 3.53. *For each $\phi, \psi \in L_A$ we have $d_0(\phi,\psi) = d_\infty(\phi,\psi)$. In particular, the spaces $(\mathcal{L}(A), d_0)$ and $(\mathfrak{L}(A), d_\infty)$ are isometric.*

Proof. The $\leq$ part is due to Remark 3.29 and the fact that we can see elements of any structure M as realized types in $S_n^M(A)$. Now the $\geq$ part. Let $p \in S_n^M(A)$. By τ_0-density of realized types, we can fix a net $(b_i)_{i\in I} \subset M$ which is τ_0-convergent to p. By Remark 3.46 the map $p \mapsto |\phi^p - \psi^p|$ is τ_0-continuous, hence we have

$$|\phi^p - \psi^p| = \lim_i |\phi(b_i)^M - \psi(b_i)^M| \leq d_0(\phi, \psi).$$

This concludes the proof. □

We obtain the following characterization, which proves that in the continuous logic setting it is much more interesting to study continuous maps $S_n(A) \to [0,1]$, since they usually do not precisely correspond to (equivalence classes of) formulae: this holds only up to uniform approximation.

Proposition 3.54. *For any function $\Phi\colon S_n^M(A) \to [0,1]$ the following are equivalent:*

(i) *Φ is continuous for the logic topology.*
(ii) *Φ is the d_∞^M-limit of a sequence $(\phi_k)_k \subset \mathfrak{L}^M(A)$.*
(iii) *Φ is continuous for the logic topology and uniformly continuous for the metric topology.*

Proof. (i)⇒(ii): Suppose that Φ is continuous for the logic topology. Maps induced by formulae separate points: if $p \neq q$, then there is ϕ which belongs to one of them and not to the other, hence by Lemma 3.26 $\phi^p \neq \phi^q$. The maps $x \mapsto 1 - x$ and $(x, y) \mapsto x \dot{-} y$ and $x \mapsto x/2$ are all continuous, hence connectives. So the set of formulae is closed with respect to them. Therefore, we can apply Fact 3.51 to the set of maps induced by formulae and get a sequence which uniformly converges to Φ. (ii)⇒(iii): Maps induced by formulae are continuous for the logic topology and uniformly continuous for the metric topology. These properties are preserved under uniform convergence. (iii)⇒(i): Trivial. □

Uniform limits of (maps induced by) formulae, equivalently continuous maps for the logic topology, are important for the following and thus deserve a name.

Definition 3.55. A function $\Phi\colon S_n^M(A) \to [0,1]$ satisfying one of the equivalent conditions in Proposition 3.54 is called an *n-predicate over A* (in M). We denote by

$$\mathfrak{P}_n^M(A)$$

the family of predicates over A and we set $\mathfrak{P}^M(A) \doteq \bigcup_n \mathfrak{P}_n^M(A)$. Sometimes we also use the notation $\mathfrak{P}_x^M(A)$ for the predicates $S_x^M(A) \to [0,1]$.

Using Corollary 3.50, i.e. the fact that maps induced by formulae are continuous for the logic topology, we can rephrase Proposition 3.54(ii) as follows.

Corollary 3.56. *The set $\mathfrak{L}^M(A)$ is d_∞^M-dense in $\mathfrak{P}^M(A)$.*

3.8 Restricted formulae

The problem with taking as connectives *all* uniformly continuous maps $[0,1] \to [0,1]$ is that there are continuum-many of them. This number could be bigger than the cardinality of the language L, for instance when L is countable; this prevents many arguments from working smoothly. However, we can overcome this issue by restricting the class of connectives to families that 'approximate well' all formulae. We will see that there is a simple case of such a smaller class which is sufficient for our needs.

Definition 3.57 (Systems of connectives)**.**

- A *system of connectives* is a family $\mathcal{C} = \{C_n : n \in \omega\}$ where each C_n is a set of n-ary connectives $u\colon [0,1]^n \to [0,1]$.
- The *closure* $\overline{\mathcal{C}}$ of $\mathcal{C}$ is the smallest system of connectives containing $\mathcal{C}$, all projections and closed under composition. We say that $\mathcal{C}$ is *closed* if $\overline{\mathcal{C}} = \mathcal{C}$.
- We say that $\mathcal{C}$ is *full* if $\overline{\mathcal{C}}$ is uniformly dense in the set of all connectives, i.e. for any $\epsilon > 0$ and any connective $u(t_1, \dots, t_n)$ there is an n-ary connective $v \in \overline{\mathcal{C}}$ such that for all $t_1, \dots, t_n \in [0,1]$ we have
$$|u(t_1, \dots, t_n) - v(t_1, \dots, t_n)| \leq \epsilon.$$
- The set of $\mathcal{C}$*-restricted* formulae consists of formulae built using only connectives in $\mathcal{C}$.

The importance of a full system of connectives is that $\mathcal{C}$-restricted formulae are dense in the set of all formulae with respect to the logical distance.

Lemma 3.58. *If $\mathcal{C}$ is a full system of connectives, then for every $\epsilon > 0$ and every L-formula $\phi(x)$ there is a $\mathcal{C}$-restricted L-formula $\psi(x)$ such that $d_0(\phi, \psi) \leq \epsilon$, where d_0 is the logical distance defined in* Definition 3.15*. In other words, the set of $\mathcal{C}$-restricted formulae is d_0-dense in $\mathcal{L}$.*

Proof. By induction on the complexity of ϕ. Fix $\epsilon > 0$.

$\underline{\phi \text{ is atomic:}}$ There is nothing to prove as we may take $\psi = \phi$.
$\underline{\phi = u(\theta_1, \dots, \theta_n):}$ By inductive hypothesis, there are $\delta_1, \dots, \delta_n$ $\mathcal{C}$-restricted formulae such that $d_0(\theta_i, \delta_i) \leq \Delta_u(\epsilon/2)/n$ for all $1 \leq i \leq n$. Let us abbreviate $\theta = (\theta_1, \dots, \theta_n)$ and $\delta = (\delta_1, \dots, \delta_n)$. It follows by Convention 3.6 that

$$d_0(\theta, \delta) \leq \frac{\Delta_u(\epsilon/2)}{n} n = \Delta_u(\epsilon/2). \tag{3.1}$$

Since $\mathcal{C}$ is full, there is some $v \in C_n \in \mathcal{C}$ such that

$$d_0(u(\delta), v(\delta)) \leq \epsilon/2. \tag{3.2}$$

Let $\psi = v(\delta_0, \dots, \delta_n)$, a $\mathcal{C}$-restricted formula. By (3.1) and (3.2) respectively, we get that

$$d_0(\phi, \psi) = d_0(u(\theta), v(\delta)) \leq d_0(u(\theta), u(\delta)) + d_0(u(\delta), v(\delta)) \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

$\phi(x) = \inf_y \theta(x, y)$: By inductive hypothesis, there is a $\mathcal{C}$-restricted formula $\delta(x, y)$ such that $d_0(\theta, \delta) \leq \epsilon$. Let $\psi(x) = \inf_y \delta(x, y)$, a $\mathcal{C}$-restricted formula. We show that $d_0(\phi, \psi) \leq \epsilon$. For each structure M and each $a \in M$ we have:

$$\begin{aligned} |\phi(a)^M - \psi(a)^M| &= |\inf_{b \in M} \theta(a, b)^M - \inf_{b \in M} \delta(a, b)^M| \\ &\leq \sup_{b \in M} |\theta(a, b)^M - \delta(a, b)^M| \\ &\leq d_0(\theta, \delta) \leq \epsilon, \end{aligned}$$

where the first inequality holds for the general fact that for any functions $f, g \colon Y \to [0, 1]$ we have

$$|\inf_Y f - \inf_Y g| \leq \sup_Y |f - g|;$$

in this case $f(y) = \theta(a, y)$ and $g(y) = \delta(a, y)$. It follows that $d_0(\phi, \psi) \leq \epsilon$.
$\phi(x) = \sup_y \theta(x, y)$: Observe that

$$\phi(x) = \sup_y \theta(x, y) = 1 \dot{-} \inf_y [1 \dot{-} \theta(x, y)]$$

and use the inductive hypothesis. All cases are thus covered. □

It turns out that there is a very simple full system of connectives, in particular a finite one.

Definition 3.59. Let $\mathcal{C}_0 = \{C_n : n \in \omega\}$ be the system of connectives with $C_0 = \{0, 1\}$, $C_1 = \{x/2\}$, $C_2 = \{\dot{-}\}$ and all other C_n empty. We say that a formula is *restricted* if it is $\mathcal{C}_0$-restricted. Finally, we write

$$\phi \in L_0(A)$$

to say that ϕ is a restricted L_A-formula and we use the notations

$$\begin{aligned} \mathcal{L}_0^M(A) &\doteq \{\phi^M \colon M^n \to [0, 1] \mid \phi \in L_0(A)\} \\ \mathfrak{L}_0^M(A) &\doteq \{\phi \colon S_n^M(A) \to [0, 1] \mid \phi \in L_0(A)\} \end{aligned}$$

for the families of induces maps.

Fact 3.60 ([Yaa+08, Proposition 6.6])**.** *The system of connectives $\mathcal{C}_0$ is full.*

Proof. Based on the fact that $\mathcal{C}_0$ contains many simple fundamental connectives and using Stone-Weierstrass Theorem. We omit it. It can be found in the textbook from which this result is taken. □

Remark 3.61. The importance of $\mathcal{C}_0$ lies in the fact that it is countable; hence $|\mathcal{L}_0| = |L|$. Moreover, $|\mathcal{L}_0(A)| \leq |A| + |L|$. By Fact 3.60 and Lemma 3.58 it follows that

$$||(\mathcal{L}(A), d_0)|| \leq |\mathcal{L}_0(A)| \leq |A| + |L|.$$

The bottom line is that there are as many restricted formulae as the cardinality of the language and approximating with them is often sufficient.

We have two notions of 'approximation': one regarding elements in a structure M with respect to the metric d^M, the other given by restricted formulae. Since density is transitive, we can combine them to obtain formulae which still approximate nicely.

Lemma 3.62. *Let M be an L-structure and $D \subseteq M$ dense. Then we have the d_0-dense inclusion $\mathcal{L}_0(D) \subseteq \mathcal{L}(M)$.*

Proof. We show that each d_0-ball in $\mathcal{L}(M)$ contains some map in $\mathcal{L}_0(D)$. Fix $\phi(x, a)$ an L_M-formula with $a \in M$ the tuple of its parameters and $\epsilon > 0$. We have to find $\psi(x, c) \in L_0(D)$ with $d_0(\phi(x, a), \psi(x, c)) < \epsilon$.

Let $\phi(x, y)$ be the L-formula obtained by substituting to a some fixed tuple y of new variables. By Remark 3.3 we can find $\epsilon' < \epsilon$ such that $\inf_{b \in M} \Delta_{\phi(b,y)}(\epsilon') > 0$. Since D is dense in M, there is $c \in D$ such that

$$d(a, c) < \inf_{b \in M} \Delta_{\phi(b,y)}(\epsilon')$$

so that for all $b \in M$ we have $|\phi^M(b, a) - \phi^M(b, c)| < \epsilon'$, i.e.

$$d_0(\phi(x, a), \phi(x, c)) < \epsilon'.$$

By Lemma 3.58, there is $\psi(x, y) \in L_0$ with $d_0(\phi(x, y), \psi(x, y)) \leq \epsilon - \epsilon'$; in particular

$$d_0(\phi(x, c), \psi(x, c)) \leq \epsilon - \epsilon'.$$

Therefore the $L_0(D)$-formula $\psi(x, c)$ is such that $d_0(\phi(x, a), \psi(x, c)) < \epsilon$, which concludes our proof. □

Combining Lemma 3.62 with Lemma 3.53 and Corollary 3.56 yields:

Corollary 3.63. *Let M be an L-structure and $D \subseteq M$ be dense. Then we have the d_∞^M-dense inclusions*

$$\mathfrak{L}_0^M(D) \subseteq \mathfrak{L}^M(M) \subseteq \mathfrak{P}^M(M).$$

Chapter 4

Stability in continuous logic

In this chapter we work in the continuous logic setting we have just presented to eventually prove also here the 'fundamental theorem of stability', following the main idea of [Ben13], but using Corollary 2.19 instead of Grothendieck's theorem.

Convention 4.1. Throughout this chapter we fix a metric language L, a complete L-theory T and a monster model $\mathbb{M} \models T$ of size $\kappa(\mathbb{M})$ bigger than any other cardinality we will encounter. So "model" means "model of T".

4.1 No-order property and double limit property

First we define the double limit property and the order property essentially as we did in the classical case. The DLP is defined exactly in the same way.

Definition 4.2. A formula $\phi(x, y)$ has the *double limit property in* $M \models T$ if for all sequences $(a_i)_i \subseteq M$ and $(b_i)_i \subseteq M$ the following holds

$$\lim_i \lim_j \phi(a_i, b_j)^M = \lim_j \lim_i \phi(a_i, b_j)^M \tag{DLP}$$

whenever all limits exist. A theory T has DLP if all L-formulae do.

The no-order property needs to be slightly adjusted.

Definition 4.3. Let M be a model. We say that an L-formula $\phi(x, y)$ has the *order property in* $M \models T$ if there are *distinct* $r, s \in [0, 1]$ and two sequences $(a_i)_{i<\omega} \subseteq M_x$ and $(b_i)_{i<\omega} \subseteq M_y$ such that for all $i, j < \omega$ we have

$$\phi(a_i, b_j)^M = \begin{cases} r & \text{if } i < j \\ s & \text{if } i \geq j. \end{cases} \tag{OP}$$

Otherwise, we say that $\phi(x, y)$ has NOP in M. As usual, a theory T has NOP if *no* L-formula has the order property in $\mathbb{M}$, while it has OP if there is *an* L-formula with the order property in $\mathbb{M}$.

By taking the proof of Lemma 1.39 and just exchanging r and s in the place of 0 and 1 we obtain the following.

Lemma 4.4. *Let $\phi(x, y) \in L$ and $M \models T$. Then ϕ has* DLP *in M iff ϕ has* NOP *in M.*

It follows that the same observations made in Remark 1.11 apply here.

4.2 Local types in CFO

In this section we introduce local types and the logic topology on them in the obvious way. We will then import results already obtained to get the usual properties on their type space.

Definition 4.5. Let $\phi(x, y)$ be a fixed formula and M a model. A (complete) *ϕ-type over $B \subseteq M_y$* (in M) is a set of formulae of the form

$$\{\phi(x, b) - r_b \mid b \in B,\ r_b \in [0, 1]\}$$

which is maximal finitely satisfiable in M. For $a \in M$, the ϕ-type over B realized by a is

$$\mathrm{tp}^M_\phi(a/B) \doteq \{\phi(x, b) - \phi(a, b)^M : b \in B\}.$$

We denote by $S^M_\phi(B)$ the space of ϕ-types over B (in M). Often we omit the superscripts M.

What are the numbers r_b in the definition? Suppose we have a maximal finitely satisfiable set $\Phi(x) = \{\phi(x, b) - r_b : b \in B\}$ and pick any $p \in S_x(B)$ extending $\Phi(x)$. Then for each $b \in B$ we have $r_b = \phi(x, b)^p$. So we can think of local types as restriction of global types to their subset of formulae of the specific form $\phi(x, b) - r_b$.

Remark 4.6. Another way to define local types in continuous logic is as quotients of global types in $S_x(B)$ by the family of functions $p \mapsto \phi(x, b)^p$ induced by the formulae $\phi(x, b)$ for $b \in B$. This means that we define the equivalence relation on $S_x(B)$

$$p \sim q \qquad \iff \qquad \phi(x, b)^p = \phi(x, b)^q \text{ for all } b \in B.$$

The two approaches are equivalent, since it easily checked that the map

$$\{\phi(x, b) - r_b : b \in B\} \longmapsto [p]_\sim$$

where $p \in S_x(B)$ is any type extending $\{\phi(x, b) - r_b : b \in B\}$, is well-defined and bijective. We have chosen our approach because it avoids having to deal with quotients and thus every local type is set-theoretically included in a global one, just as in classical logic.

4.2.1 Size of type spaces in CFO

We can easily mimic the proof of Proposition 1.20 to obtain an analogous result for continuous logic.

Proposition 4.7. *If an L-formula $\phi(x, y)$ has* OP*, then for all $\kappa \geq |L|$ there is a model $M \models T$ of size κ with $|S_\phi(M)| \geq \operatorname{ded} \kappa$.*

Proof. Let $\phi(x, y) \in L$ have OP and fix $M \models T$ of size $\kappa \geq |L|$, as witnessed by two distinct $r, s \in [0, 1]$. As in Lemma 1.12, let I be a dense linear order of size κ and $(a_i)_{i\in I} \subseteq M_x$ and $(b_i)_{i\in I} \subseteq M_y$ such that for all $i, j \in I$ we have

$$\phi(a_i, b_j)^M = \begin{cases} r & \text{if } i < j \\ s & \text{if } i \geq j. \end{cases}$$

For every cut $C \subseteq I$, the set

$$\Phi_C \doteq \{\phi(x, b_j) - r : j \in C^{\complement}\} \cup \{\phi(x, b_j) - s : j \in C\}$$

is by compactness and density a partial ϕ-type over M. Let p_C be a complete ϕ-type over M extending Φ_C. Clearly if $C_1 \neq C_2$ then $\Phi_{C_1} \neq \Phi_{C_2}$ and *a fortiori* $p_{C_1} \neq p_{C_2}$. So the map $C \mapsto p_C$ is injective, hence our thesis. □

4.2.2 The logic topology on local types

We can endow the space $S_\phi(B)$ with a logic topology in the same vein of the case of global types, using the characterization in Remark 3.46 or a restriction of the open basic open sets in the global case.

Definition 4.8. Let $\phi(x, y)$ be an L-formula, M an L-structure and $B \subseteq M_y$. The *logic topology* on $S_\phi(B)$ is the topology τ_0 generated by the family of (maps induced by) L_B-formulae of the form $\phi(x, b)$ for $b \in B$, i.e. the maps

$$\phi(x, b) \colon S_\phi(B) \to [0, 1], \; p \mapsto \phi(x, b)^p,$$

or equivalently, the topology on $S_\phi(B)$ generated by sets of the form

$$[\![\phi(x, b) < r]\!] \doteq \{p \in S_\phi(M) : \phi(x, b)^p < r\}$$

for $r \in [0, 1]$ and $b \in B$.

With the same type of arguments we used in Section 3.5 it is easy to show that also $S_\phi(B)$ is topologically well-behaved, as it shares the same properties of $S_n(B)$.

Proposition 4.9. *The space $(S_\phi(B), \tau_0)$ is Hausdorff compact.*

We can use what we have already proved for global types to prove that also realized local types are dense for the logic topology.

Corollary 4.10. *The set* $\{\mathrm{tp}_\phi(a/M) : a \in M\} \subset S_\phi(M)$ *of realized types is dense for the logic topology.*

Proof. Let $[\![\phi(x,b) < r]\!]$ be a basic open set of $S_\phi(M)$. By Corollary 3.35, we can pick $\mathrm{tp}(a/M) \in [\phi(x,b) < r]$. This means that $\phi(a,b)^M < r$, which implies that $\mathrm{tp}_\phi(a/M) \in [\![\phi(x,b) < r]\!]$. □

4.2.3 Local predicates

'Local predicates' are defined exactly as in the global case, as continuous maps on the local type space.

Definition 4.11. A *ϕ-predicate over B* is a map $\Phi\colon S^M_\phi(B) \to [0,1]$ which is continuous for the logic topology. We denote by

$$\mathfrak{P}^M_\phi(B)$$

the family of ϕ-predicates over B (in M).

Remark 4.12*.* Using this definition in classical logic, a ϕ-predicate over A would be a continuous map $\Phi\colon S_\phi(B) \to \{0,1\}$, equivalently a clopen set in $S_\phi(B)$, i.e. a Boolean combination of sets of the form $[\phi(x,b)]$ for $b \in B$. So Φ is (the map induced by) a ϕ-predicate over B, in the sense of Definition 1.32, hence the two definitions agree.

In the continuous case, by Proposition 3.54, the above definition is almost always more general, i.e. except when the set of L-formulae is closed under uniform limits.

4.3 Definability of types from DLP

Introducing definable types in continuous logic requires a careful definition, which is semantic in nature since predicates need not be formulae and so they could have no syntactic meaning.[1]

Definition 4.13 (Definability of types)**.**

- Fix $\phi(x,y) \in L$. We say that a type $p(x) \in S^M_\phi(A)$ is *definable* if there is a $\Phi_p \in \mathfrak{P}^M_{\phi^*}(A)$ such that for all $a \in A$ we have

$$\phi(x,a)^p = \Phi_p(\mathrm{tp}^M_{\phi^*}(a/A)) \doteq \Phi_p(a). \tag{4.1}$$

[1] Actually it is possible to give predicates syntactic meaning by introducing a device called *forced limit* of a sequence (see [BU08, §3.2]) or to represent them as 'formulae' by extending the notion of formula by allowing also infinitary maps $[0,1]^{\mathbb{N}} \to [0,1]$ as connectives (see [Yaa+08, Proposition 9.3]). Both approaches go beyond the scope of this work.

- A type $p \in S_x(A)$ is *definable over* B if for every $\phi(x, y) \in L$ the ϕ-type $p|_\phi$ is definable over B, where

$$p|_\phi \doteq \{\phi(x, a) - \phi(x, a)^p : a \in A\} \subset p.$$

- A type is *definable* if it is definable in its domain.

Remark 4.14. This definition is indeed (almost) a generalization of the classical one. In fact, by Remark 4.12, the ϕ^*-predicate Φ_p over A in the above definition is (the map induced by) a ϕ^*-predicate ψ over A in the sense of Definition 1.32, which is in particular an L_A-formula. Hence, by Remark 3.48 we have $\psi(\mathrm{tp}_{\phi^*}(a/A)) = \psi(a)^{\mathbb{M}}$. So equation (4.1) becomes the equivalence

$$\phi(x, a) \in p \quad \iff \quad \models \psi(a),$$

which is exactly how we defined definable types in Definition 1.28, except for the fact that the L_A-formula $\psi(y)$ was there not required to be a ϕ^*-predicate. However, this is the form we actually obtained in Corollary 1.33.

The next simple fact will be needed in the following proof.

Lemma 4.15. *Let M be a model, $A \subseteq M$ and $\phi(x, y)$ an L_A-formula. Then $\phi(x, b)^{\mathrm{tp}_\phi(a/A)} = \phi^*(y, a)^{\mathrm{tp}_{\phi^*}(b/A)}$ for all $a \in M_x$ and $b \in M_y$.*

Proof. $\phi^*(y, a)^{\mathrm{tp}_{\phi^*}(b/A)} = \phi^*(b, a)^M = \phi(a, b)^M = \phi(x, b)^{\mathrm{tp}_\phi(a/A)}$. $\square$

We are ready to prove definability of types from DLP, using Corollary 2.19. The next result and its proof are the culmination of the results in this thesis.

Proposition 4.16 ([Ben13, Theorem 3])**.** *If a formula $\phi(x, y)$ has* DLP *in the model M, then every type $p \in S_\phi^M(M)$ is definable by a (unique) predicate $\Phi_p \in \mathfrak{P}_{\phi^*}^M(M)$.*

Proof. Fix the model M. Let $p(x) \in S_\phi(M)$. Since by Corollary 4.10 realized local types are τ_0-dense, there is a net $(a_i)_{i\in I} \subseteq M$ such that

$$\mathrm{tp}_\phi(a_i/M) \to p.$$

Let $X = S_{\phi^*}(M)$ and let $X_0 \subseteq X$ be the set of ϕ^*-types realized in M, which is τ_0-dense. For $a \in M$, let $\phi_a : X \to [0, 1]$ be the τ_0-continuous map induced by the L_A-formula $\phi^*(y, a)$, which is defined by

$$\phi_a(q) = \phi^*(y, a)^q.$$

Let $A \doteq \{\phi_a \mid a \in M\} \subseteq C(X)$. It follows respectively from the definition of ϕ_a, Lemma 4.15 and Remark 3.48 that for all sequences $(c_n)_n \subseteq M_x$ and all $(b_n)_n \subseteq M_y$ we have

$$\phi_{c_n}(\mathrm{tp}_{\phi^*}(b_m/M)) = \phi^*(y, c_n)^{\mathrm{tp}_{\phi^*}(b_m/M)} = \phi(x, b_m)^{\mathrm{tp}_\phi(c_n/M)} = \phi(c_n, b_m)^M.$$

Since ϕ has DLP, this implies that

$$\lim_n \lim_m \phi_{c_n}(\mathrm{tp}_{\phi^*}(b_m/M)) = \lim_m \lim_n \phi_{c_n}(\mathrm{tp}_{\phi^*}(b_m/M))$$

whenever all limits exist. Therefore, all the assumptions for applying Corollary 2.19 to A, X_0, X are satisfied, so we can assume without loss of generality that the net $(\phi_{a_i})_i \subseteq A$ converges pointwise to some $\Phi_p \in C(X) = \mathfrak{P}^M_{\phi^*}(M)$. We obtain that

$$\begin{aligned}\phi(x,b)^p &= \lim_i \phi(x,b)^{\mathrm{tp}_\phi(a_i/M)} = \lim_i \phi^*(y,a_i)^{\mathrm{tp}_{\phi^*}(b/M)} \\ &= \lim_i \phi_{a_i}(\mathrm{tp}_{\phi^*}(b/M)) = \Phi_p(\mathrm{tp}_{\phi^*}(b/M)) \doteq \Phi_p(b)\end{aligned}$$

where the equalities hold respectively for the following reasons: by τ_0-continuity of $p \mapsto \phi^p$ and Fact 2.9, by Lemma 4.15, by definition of ϕ_a and by the fact that $\phi_{a_i} \to \Phi_p$. So p is defined by the ϕ^*-predicate Φ_p over M.

The uniqueness of Φ_p is by τ_0-density of realised types: two ϕ^*-predicates over M defining p are τ_0-continuous maps on X agreeing on the τ_0-dense set X_0. Therefore they must coincide. □

4.4 The fundamental theorem of stability for CFO

In this section we prove the 'fundamental theorem of stability' in continuous logic. In this setting, there are two ways of measuring the 'size' of a type space: via cardinality or via density character with respect to the following uniform metric.

4.4.1 The metric topology on local types

Definition 4.17. Fix an L-formula $\phi(x,y)$. The *ϕ-distance* of $p, q \in S_\phi(B)$ in M is

$$d_\phi(p,q) \doteq ||p-q||_\phi = \sup_{b\in M} |\phi(x,b)^p - \phi(x,b)^q|.$$

We endow $S_\phi(B)$ with the topology generated by this metric.

As in the global case we define the uniform distance. For simplicity we denote it with the same symbol.

Definition 4.18. The *uniform distance* of $\Phi, \Psi \colon S^M_\phi(B) \to [0,1]$ in M is

$$d^M_\infty(\Phi,\Psi) \doteq ||\Phi - \Psi||^M_\infty = \sup_{p\in S^M_\phi(B)} |\Phi(p) - \Psi(p)|.$$

Often we omit the superscript M.

The metric d_ϕ on the subset of *definable* types essentially coincides with the metric d_∞ on ϕ-predicates.

Lemma 4.19. *If $p, q \in S_\phi(B)$ are defined respectively by $\Phi_p, \Phi_q \in \mathfrak{P}_\phi(B)$, then*

$$d_\phi(p, q) = d_\infty(\Phi_p, \Phi_q).$$

Proof. We have

$$d_\phi(p, q) = \sup_{b \in M} |\phi(x, b)^p - \phi(x, b)^q| = \sup_{b \in M} |\Phi_p(b) - \Phi_q(b)|$$
$$d_\infty(\Phi_p, \Phi_q) = \sup_{r \in S_\phi(B)} |\Phi_p(r) - \Phi_q(r)|.$$

Now proceed as in Lemma 3.53, by using the fact that predicates are τ_0-continuous and realized types are τ_0-dense. □

4.4.2 Local fundamental theorem in CFO

Measuring the 'size' of $S_\phi(M)$ via cardinality or density character is essentially the same, as the next theorem shows. It is the continuous analogue of Theorem 1.34. First, let us state two simple observations that we will use in the next proofs.

Remark 4.20. We may embed

$$S_\phi(B) = \frac{S_x(B)}{\sim} \hookrightarrow S_x(B)$$

via the map $p \mapsto \overline{p}$, where $\overline{p} \in S_x(B)$ is any x-type such that $\overline{p}|_\phi = p$. This map is clearly injective. Similarly, by the universal property of quotient spaces, any continuous map $\Phi \colon \frac{S_x(B)}{\sim} \to [0, 1]$ has a continuous extension to $\overline{\Phi} \colon S_x(B) \to [0, 1]$. So we may think of $S_\phi(B) \subseteq S_x(B)$ and $\mathfrak{P}_\phi(B) \subseteq \mathfrak{P}_x(B)$.

Remark 4.21. A metric space (M, d) is first-countable: every $a \in M$ has the system of neighbourhoods $\{B_a(1/n) : n \in \omega\}$ given by the open balls. If $D \subseteq M$ is dense, then for each $a \in M$ we can pick a sequence $(a_n)_n \subset D$ converging to a; the map $a \mapsto (a_n)_n$ is injective by the uniqueness of limits and so $|M| \leq |D|^{\aleph_0}$. It follows that $|M| \leq ||M||^{\aleph_0}$.

Theorem 4.22. *For an L-formula $\phi(x, y)$, the following are equivalent:*

(i) *The formula ϕ has* NOP *(in T).*
(ii) *All ϕ-types over any $M \models T$ are definable (by ϕ^*-predicates over M).*
(iii) *For all $\kappa \geq |L|$ and $M \models T$, if $||M|| \leq \kappa$ then $||S_\phi(M)|| \leq \kappa$.*
(iv) *There exists some $\kappa \geq |L|$ such that for all $M \models T$, if $|M| \leq \kappa$ then $|S_\phi(M)| < \operatorname{ded} \kappa$.*

Proof. (i)⇒(ii): By Lemma 4.4 and Proposition 4.16.
(ii)⇒(iii): Let $\kappa \geq |L|$ and fix $D \subseteq M$ dense of size $|D| \leq \kappa$. Respectively by (ii), Remark 4.20, Corollary 3.63 and Remark 3.61 we get that

$$||(S_{\phi^*}(M), d_{\phi^*})|| = ||(\mathfrak{P}_{\phi^*}(M), d_\infty)|| \leq ||(\mathfrak{P}_y(M), d_\infty)||$$
$$\leq ||(\mathfrak{P}(M), d_\infty)|| \leq |\mathfrak{L}_0(D)| \leq |D| + |L| \leq \kappa.$$

It follows easily that $||(S_\phi(M), d_\phi)|| \leq \kappa$.

(iii)⇒(iv): Let $\kappa \doteq |L|^{\aleph_0}$. Pick $M \models T$ with $|M| = \kappa$. Then by (iii) we get that $||S_\phi(M)|| \leq \kappa$. By Remark 4.21 we get that $|S_\phi(M)| \leq \kappa^{\aleph_0} = \kappa < \operatorname{ded}\kappa$.
(iv)⇒(i): By Proposition 4.7. □

4.4.3 Global fundamental theorem in CFO

We can translate what we have obtained to the case of a complete theory, in the very same vein of what we did for the classical case. This time we have two 'counting' cardinal functions associated to a theory, counting respectively the *number* and the *density* of types.

Definition 4.23. For a theory T define the cardinal functions:

$$\begin{aligned} \operatorname{ntp}_T(\kappa) &\doteq \sup\{|S_n(M)| : M \models T,\ |M| = \kappa,\ n \in \omega\} \\ \operatorname{dtp}_T(\kappa) &\doteq \sup\{||S_n(M)|| : M \models T,\ ||M|| = \kappa,\ n \in \omega\}, \end{aligned}$$

where $S_n(M)$ is endowed with the logic topology.

The global fundamental theorem follows.

Corollary 4.24. *Let T be a complete theory. The following are equivalent:*

(i) *The theory T has* NOP*.*
(ii) *All types over any model of T are definable.*
(iii) *For all $\kappa \geq |L|$, we have $\operatorname{dtp}_T(\kappa) \leq \kappa^{|L|}$.*
(iv) *There is $\kappa \geq |L|$ such that $\operatorname{ntp}_T(\kappa) < \operatorname{ded}\kappa$.*

Proof. As in the proof of Corollary 1.36, we have that (ii) holds if and only if for each $\phi \in L$ all ϕ-types are definable over any model.

(i)⇒(ii): By what we just observed and Theorem 4.22.
(ii)⇒(iii): Suppose all ϕ-types are definable for every $\phi \in L$. Fix $\kappa \geq |L|$ and $M \models T$ with $||M|| = \kappa$. By Theorem 4.22, for all $\phi(x, y) \in L$ we have $||S_\phi(M)|| \leq \kappa$; by Remark 4.21 we get that $|S_\phi(M)| \leq \kappa^{\aleph_0} \leq \kappa^{|L|}$.

Claim. *For all finite variables x, let $\mathcal{M}_x \doteq \{\operatorname{tp}(a/M) : a \in M_x\}$ be the set of realized x-types over M. Then the function*

$$\mathcal{M}_x \ni p \mapsto f_p \qquad \textit{where} \qquad f_p \colon L_0 \to \textstyle\bigcup_{\phi \in L_0} S_\phi(M),\ \phi(x, y) \mapsto p|_\phi$$

is injective.

Proof. Let $p, q \in \mathcal{M}$ be respectively realized by a, b and suppose that $f_p = f_q$, i.e. for every $\phi(x, y) \in L_0$ we have $p|_\phi = q|_\phi$. This means that for every $\phi(x, y) \in L_0$ and every $c \in M_x$ we have $\phi(a, c)^M = \phi(b, c)^M$; equivalently $\phi(a)^M = \phi(b)^M$ for every $\phi(x) \in L_0(M)$.

The function $P\colon (\mathcal{L}(M), d_0) \to [0,1],\ \phi \mapsto \phi^p = \phi(a)^M$ is continuous:

$$\begin{aligned} P^{-1}[[0,r)] &= \{\phi \in \mathcal{L}(M) : \phi(a)^M \in [0,r)\} \\ &\supseteq \{\phi \in \mathcal{L}(M) : d_0(\phi,(x=x)) < r\} \end{aligned}$$

is clearly open; the same holds for the map Q induced by q. These continuous maps coincide on the set $\mathcal{L}_0(M)$ by our assumption, which by Lemma 3.58 is dense, hence so they do on $\mathcal{L}(M)$. By Remark 3.28 we have $p = q$. $\diamondsuit$

By Corollary 3.35, $\mathcal{M}_x$ is τ_0-dense in $S_x(M)$. By Remark 3.61, the family of L_0-formulae has size at most $|L| \leq \kappa$. Hence from the Claim it follows that $||S_x(M)|| \leq |\mathcal{M}_x| \leq \kappa^{|L|}$ for all finite variables x. Therefore we obtain that $\mathrm{dtp}_T(\kappa) \leq \aleph_0 \cdot \kappa^{|L|} = \kappa^{|L|}$.
<u>(iii)⇒(iv):</u> Let $\kappa = 2^{|L|}$. Suppose that $|M| \leq \kappa$. Then by (iii) we get that $\mathrm{dtp}_T(\kappa) \leq \kappa^{|L|} = \kappa$. By Remark 4.21, $\mathrm{ntp}_T(\kappa) \leq \kappa^{\aleph_0} \leq \kappa^{|L|} = \kappa < \mathrm{ded}\,\kappa$.
<u>(iv)⇒(i):</u> Let $\phi(x,y) \in L$. By Remark 4.20 $|S_\phi(M)| \leq |S_x(M)|$, hence by Theorem 4.22 the formula ϕ has NOP in M. Since ϕ was arbitrary, also T has NOP. $\square$

4.5 Comparison of different proofs of definability of types

In this section, we try to compare the different proofs of Proposition 4.16 (i.e. how definability of types follows from the double limit property) with particular emphasis on the reasons for the amazing simplicity of Ben Yaacov's proof in [Ben13, Theorem 3]. We make some remarks regarding the mathematical, heuristic and historical meaning of these different approaches.

Consider the following three proofs of Proposition 4.16:

(i) Via Corollary 2.19 (i.e. `DLP` ⇒ `pointwise compactness`).

This is the way we have proved it in this thesis.

(i*) Via Grothendieck's Theorem (i.e. `DLP` ⇒ `weak compactness`).

This is the way Ben Yaacov proves it in [Ben13].

(ii) `'Mimicking' the FOL` proof of Lemma 1.31.

This is the method employed in [BU08, §7].

We proceed to compare and analyse these proofs.

4.5.1 Weak compactness is overkill

It is clear from the proof of Proposition 4.16 (which is based on [Ben13, Proposition 3]) that weak compactness is not needed at all to get definability of types. As a matter of fact, what is really needed is pointwise compactness.

The point is that weak compactness trivially implies pointwise compactness by Remark 2.6.

So the reason why Ben Yaacov quoted and then used Grothendieck's theorem instead of Corollary 2.19 must simply be historical: what is crucial to link stability and compactness is the double limit property, and most importantly the easy yet smart observation that it is just a rephrasing of the no-order property, which is a familiar tool in model theory. The double limit property had been introduced by Grothendieck himself in [Gro52], where he proved his aforementioned theorem. This makes it somewhat natural to use that theorem in its full force, not bothering to observe that actually a weaker version would be sufficient.

But how much weaker? Even not considering the fact that – as we have already discussed – the original Grothendieck's theorem had weaker assumptions, its proof (of the backward direction, i.e. direction (ii)$\Rightarrow$(iii) of Theorem 2.21) is considerably harder than that of Corollary 2.19: it requires Proposition 2.20 (i.e. the fact that in this case pointwise compactness implies pointwise sequential compactness), the Dominated Convergence Theorem 2.7 and the Eberlein-Šmulian Theorem 2.14.

So if we try to read Proposition 4.16 as a 'corollary of functional analysis' as Ben Yaacov implicitly does in [Ben13], then this claim of his is made even stronger, since Corollary 2.19 is weaker (and easier to prove!) than Grothendieck's theorem. Moreover, the notions involved are also conceptually simpler (pointwise topology instead of weak topology); one might also say that Corollary 2.19 is not even functional analysis but just general topology.

4.5.2 Working in CFO is not a crucial factor

Ben Yaacov's discussion in [Ben13] of his discovery takes into account its purely historical importance: he argues that the concept of stability essentially originated in Grothendieck's work, way earlier than Shelah's work in classification theory in the 70s. This fact in itself is already striking.

But there is another implication which might possibly amaze even more: the hardest direction of the 'fundamental theorem of stability' – whose 'classical' proofs, as our proof of Lemma 1.31 shows, are all quite clunky, even in FOL – is made simpler (in CFO!) by importing techniques from functional analysis, and most importantly by realizing that a suitable 'translation' can be naturally carried out.

Continuous logic generalizes considerably classical logic; one could naturally expect the proofs of those result which (appropriately adjusted) still hold in CFO to be more complicated, or at least not less, as it seems to be the case here. Truth be told, it is not rare in mathematics to find simplicity and generality walking hand in hand.[2] After all, it is much more common

[2]This is quite interestingly a feature that is often attributed to Grothendieck's work itself. In an obituary David Mumford and John Tate wrote: "Although mathematics became more and more abstract and general throughout the 20th century, it was Alexander

in functional analysis to consider functions with values in $[0,1]$ instead of $\{0,1\}$ so it is not so strange that Ben Yaacov's paper [Ben13] blossomed from the CFO setting.

In [Pil18], Pillay made a further attempt to interpret Grothendieck's theorem in a model-theoretic fashion (in FOL). The double limit property obviously corresponds to the no-order property. More interestingly, weak compactness (equivalently, pointwise compactness) corresponds to "generic stability" of types; a type p over M is *generically stable* if it has an extension to a type p' over some N which is satisfiable in N and definable in N.

This is interesting in its own right. Anyway, the 'downgrade' in [Pil18] of the setting from continuous to two-valued also allows to see clearer if – once the heuristic component of working with CFO has given its fruits – translating Ben Yaacov's proof back to FOL sheds some light on the alleged higher simplicity of Ben Yaacov's proof compared to the traditional one. In particular, the direction (b)⇒(a) of [Pil18, Proposition 2.2] is essentially the two-valued model-theoretic version of our Proposition 2.18. It seems clear for us that the proof is not simpler; actually it is essentially the same.[3]

The bottom line is that it appear to us that the translation to FOL may improve only the understanding of the model-theoretic content of Grothendieck's theorem and not of the reason why the proof of definability of types from NOP is made so smoother by its use.

4.5.3 Simplicity comes at a price

While the proof of Lemma 1.31 (definability of types in FOL) and (the translation to FOL of) Proposition 4.16 could perhaps be seen as not much different in terms of difficulty, we believe the same cannot be said in the continuous setting, as we try to sketch below.

There is a quite natural way to translate our classical proof of Lemma 1.31 to CFO.[4] The ingredients to 'lift up' the proof essentially are:

- Introducing an "ϵ-order property", which weakens the usual order property and is more suitable for the continuous setting.
- Singling out a new "median value" connective, which morally plays the role of the stacked conjunctions and disjunctions in the proof of Lemma 1.31.

Grothendieck who was the greatest master of this trend. His unique skill was to eliminate all unnecessary hypotheses and burrow into an area so deeply that its inner patterns on the most abstract level revealed themselves – and then, like a magician, show how the solution of old problems fell out in straightforward ways now that their real nature had been revealed." [MT14]

[3] On a side note, in the proof of (a)⇒(b) – which is deemed as "easy" – there is an implicit (unnoticed?) use of the fact that for those functions pointwise compactness implies sequential pointwise compactness (our Proposition 2.20), a fact which does not seem to be trivialized by the simplified context.

[4] This is what is done in [BU08, §7] by Ben Yaacov and Usvyatsov (years before the publishing of [Ben13]).

- Developing the theory of imaginaries and canonical parameters: this is because in CFO a continuous map may be the uniform limit of countably-many formulae, thus it may depend on an infinite number of parameters; it is then necessary to 'shrink' those relations to imaginary elements.
- Introducing the concept of "forced limit" of a sequence, a way to be able to treat predicates as syntactic objects.

It should now seem quite clear that the proof of Proposition 4.16 is easier than that of [BU08, Lemma 7.4] (the translation to CFO of Lemma 1.31).

Does the latter proof really have no advantage at all? Actually, there is one, namely the fact that it proves *uniform* definability, i.e. that if ϕ has ϵ-NOP for all $\epsilon > 0$ then all types are definable *by the same predicate* (modulo the parameters). Our proof of Lemma 1.31 can actually be adapted to get the same in FOL.[5] So perhaps the increased difficulty reflects this gain obtained by using what is *de facto* a syntactic approach.

4.5.4 Continuous logic and functional analysis

In this final part, we look at the relationship between Proposition 2.18 and Proposition 4.16 from a higher perspective.

One of the greatest qualities of logic is its presence at all levels of language. In this case we have:

- The (formal) *language* which CFO deals with.
- The (semi-formal) *meta-language* of mathematics. Here both CFO and FA belong and this is where Proposition 2.18 lives.
- The (informal) *meta-meta-language* of ordinary speech.[6]

Consider the following scheme summarizing the aforementioned results.

Functional analysis (FA)	Continuous logic (CFO)
Proposition 2.18:	Proposition 4.16:
DLP $\Rightarrow$ pointwise compactness (PC)	NOP $\Rightarrow$ definability of types (DT)

What we did (after Ben Yaacov's [Ben13]) to prove Proposition 4.16 can be represented by:

$$\text{NOP} \Rightarrow \text{DLP} \overset{2.18}{\Longrightarrow} \text{PC} \Rightarrow \text{DT}. \tag{*}$$

The argument (*) is comprised of two ingredients:

[5] It is done for instance in [Che17, Proposition 2.22].

[6] To our knowledge, no philosopher has ever doubted that *some* amount of 'intuitive logic' (or rationality, which is embodied in the very term $\lambda o \gamma o \varsigma$ from which "logic" comes from) is found in ordinary language.

- The first and the third implication are made possible by the fact that formulae – *formal* syntactic objects in CFO – come with an attached semantic object (a function) which is sufficiently well-behaved (continuous). This is because of how CFO was built in the first place.
- The second implication – the core of (*) – is given by Proposition 2.18, a result which takes place in FA.

As discussed in Subsection 4.5.1, the argument (*) may be interpreted as showing an instance where CFO is a 'corollary' of FA. Seeing a result in CFO, a sub-branch of model theory (itself a branch of mathematical logic), as a corollary of a result in 'ordinary mathematics' (FA, or even general topology) may seem curious; one may then be tempted to try to see clearer why this is the case. One possible way of investigating this issue could be trying to use continuous model-theoretic techniques to dig deeper into the (model-theoretic formalization of the) space $C_p(X)$.[7] The problem is that in general[8] it is not metrizable, hence not treatable as a metric CFO structure. Is there a logic which is able to formalize $C_p(X)$?

[7]The subscript indicates the pointwise convergence (equivalently, weak) topology.
[8]For instance when X is uncountable.

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