Measures on minimally generated Boolean algebras

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Abstract

We investigate properties of minimally generated Boolean algebras. It is shown that all measures defined on such algebras are separable but not necessarily weakly uniformly regular. On the other hand, there exist Boolean algebras small in terms of measures which are not minimally generated. We prove that under CH a measure on a retractive Boolean algebra can be nonseparable. Some relevant examples are indicated. Also, we give two examples of spaces satisfying some kind of Efimov property.

1. Introduction

In [18] Sabine Koppelberg introduced the notion of *minimally generated* Boolean algebra. Loosely speaking a Boolean algebra is minimally generated if it can be generated by small, indivisible steps (see the next sections for precise definitions and terminology used here). Among other results, Koppelberg showed that all such algebras are small in the sense they do not contain an uncountable independent sequence. On the other hand, almost all well–known subclasses of small Boolean algebras such as interval, tree or superatomic ones appeared to be minimally generated.

The studies originated in [18] were continued in [20], where some interesting counterexamples were indicated. In [21] one can find examples of forcing with minimally generated algebras. Several papers by Lutz Heindorf are closely related to the topic, see, e.g., [4]. This paper is a modest attempt to deepen the knowledge about this class of Boolean algebras.

In Section 2 we set up notation and terminology.

Section 3 is devoted to the study of the Stone spaces of minimally generated algebras. We try to find their place among well–known classes of topological spaces. We have not been able to give a topological characterization of the compact spaces whose algebras of clopen subsets are minimally generated. Nearly all results contained in the section are direct applications of Koppelberg's theorems (repeated without proofs at the beginning of the section), so we decided to call such spaces *Koppelberg compacta*. Quite unexpectedly, it appeared that all monotonically normal spaces are Koppelberg compact.

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The essential part of the paper presents several results on measures on minimally generated algebras. It is done in Section 4. We show that all measures admitted by such algebras are separable (in fact, they fulfil a certain stronger regularity condition). It sheds some new light on similar results obtained for interval algebras and monotonically normal spaces (see [27, 7] respectively).

Moreover, in Section 4 we prove that a Boolean algebra carries either a nonseparable measure or a measure which is uniformly regular. It is shown that all measures on a free product $\mathfrak{A} \oplus \mathfrak{B}$ of Boolean algebras are weakly uniformly regular if only all measures on $\mathfrak{A}$ and $\mathfrak{B}$ are weakly uniformly regular. We show that minimal generation cannot be characterized by measure theoretic conditions, at least not in any natural way. We point out that measures on retractive algebras can be nonseparable if CH is assumed. The retractive algebras are, thus, the only well–known subclass of small Boolean algebras which is not included in the class of minimally generated algebras. Using the above results we present some new examples of small (also, retractive) but not minimally generated Boolean algebras.

The last section deals with the connection between Koppelberg compacta and Efimov spaces, where by a Efimov space we mean a compact space that neither contains a nontrivial convergent sequence nor a copy of $\beta\omega$. It is not known if such spaces can be constructed in ZFC. However, many constructions of such spaces were carried out in several models of ZFC. Most of them (see [8, 10, 11]) use, explicitly or not, the notion of minimally generated Boolean algebra. Section 5 discusses this topic. We do not exhibit any new Efimov space, but we try to locate potential Efimov spaces within the class of Koppelberg compacta. We give here alternative and quite simple proof of Haydon's theorem stating that there is a compact but not sequentially compact space without a nonseparable measure. We finish with a construction of a *Efimov–like* space not involving minimally generated algebras.

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2. Preliminaries

We use the standard set theoretic notation. For any unexplained terminology the reader is referred to [19].

Throughout this paper all "algebras" are Boolean algebras, even if it is not stated explicitly. We denote the Boolean operations like in algebras of sets ($\cup$, ^{c}, and so on). Given a Boolean algebra $\mathfrak{A}$ we denote by Stone($\mathfrak{A}$) its Stone space, i.e. the space of ultrafilters on $\mathfrak{A}$. A topological space is said to be *Boolean* if it is compact and zero–dimensional.

By a measure on a Boolean algebra we mean a finitely additive function. We also occasionally mention Radon measures on topological spaces. If X is a topological space then μ is a Radon measure on X if it is a σ–additive measure defined on the σ–algebra of Borel sets on X. We treat here only finite measures.

Let $\mathfrak{A}$ be a Boolean algebra and let K be its Stone space. Recall that every (finitely additive) measure on $\mathfrak{A}$ can be transferred to the algebra of clopen subsets of K and then extended to the unique Radon measure.

A measure μ on a Boolean algebra $\mathfrak{A}$ is *atomless* if for every $\varepsilon > 0$ there is a finite partition of 1 into elements of measure at most ε. In [6] such a measure is called "strongly continuous". Notice that there are different notions of atomlessness of measure, not necessary equivalent to the above one. We say that a measure μ on a topological space (a Boolean algebra) is *strictly positive* if $\mu(A) > 0$ for every nonempty open set (nonempty element of algebra) A .

Let us fix some notation concerning Boolean algebras. If $\mathcal{A}$ is a family of subsets of X then $alg(\mathcal{A})$ is the subalgebra of $P(X)$ generated by $\mathcal{A}$. If $\mathfrak{A}$ is a Boolean algebra then $\mathfrak{A}(B) = alg(\mathfrak{A} \cup \{B\})$. Recall that in $\mathfrak{A}(B)$ all elements are of the form $(B \cap A_1) \cup (B^c \cap A_2)$, where A_1, A_2 belong to $\mathfrak{A}$. By $Fin(X)$ we denote the family of finite subsets of X (write Fin if $X = \omega$) and by $Fin\text{–}Cofin(X)$ the algebra $alg(Fin(X))$.

Recall that $\mathfrak{A} \oplus \mathfrak{B}$ ($\mathfrak{A} \times \mathfrak{B}$) is a *free product* (*product*) of Boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$ if it is the algebra of clopen sets of the product (disjoint union, respectively) of its Stone spaces.

For an algebra $\mathfrak{A}$ it is convenient to say that a sequence $(A_n)_{n\in\omega}$ in $\mathfrak{A}$ is *convergent to an ultrafilter* $p \in \mathrm{Stone}(\mathfrak{A})$ if for every $U \in p$ we have $A_n \subseteq U$ for almost all n. We say that a sequence $(p_n)_{n\in\omega}$ in $\mathrm{Stone}(\mathfrak{A})$ is convergent to p if for every $U \in p$ we have $p_n \in U$ for almost all n.

A family $\mathcal{A} \subseteq \mathfrak{A}$ is said to be *independent* if for arbitrary disjoint finite subsets $\{A_0, ..., A_n\}$ and $\{B_0, ..., B_m\}$ of $\mathcal{A}$ we have

$$A_0 \cap ... \cap A_n \cap B_0^c \cap ... \cap B_m^c \neq \emptyset.$$

We say that a Boolean algebra is *small* if it does not contain an uncountable independent sequence.

A family $\mathcal{P}$ of open sets is called a *π–base* for a topological space X provided every nonempty open set contains a nonempty member of $\mathcal{P}$. A Boolean algebra $\mathfrak{A}$ is *dense* in $\mathfrak{B}$ if $\mathfrak{A} \subseteq \mathfrak{B}$ and for every $B \in \mathfrak{B}$ we can find $A \subseteq B$ such that $A \in \mathfrak{A}$. Of course, then $\mathfrak{A}$ forms a π–base for $\mathrm{Stone}(\mathfrak{B})$.

For a Boolean algebra $\mathfrak{A}$ we say that $\mathcal{T} \subseteq \mathfrak{A}$ is a *pseudo–tree* if for every A, $B \in \mathcal{T}$ either $A \cap B = \emptyset$, $A \subseteq B$ or $B \subseteq A$. If, additionally, the family $\{S \in \mathcal{T} \colon T \subseteq S\}$ is well–ordered by "$\supseteq$" for every $T \in \mathcal{T}$, then $\mathcal{T}$ is a tree. The following simple fact is proved in [20].

Fact 2.1 (Koppelberg) *If a Boolean algebra $\mathfrak{A}$ admits a strictly positive measure then all trees in $\mathfrak{A}$ are countable.*

3. Minimally generated Boolean algebras and their Stone spaces

In this section we overview known results concerning minimal generation and translate them to the language of topology. We start by the definition of our main notion. It was introduced by Sabine Koppelberg in [18] although it was previously used implicitly by other authors.

Definition 3.1 *We say that $\mathfrak{B}$ is a minimal extension of $\mathfrak{A}$ if $\mathfrak{A} \subseteq \mathfrak{B}$ and there is no algebra $\mathfrak{C}$ such that $\mathfrak{A} \subsetneq \mathfrak{C} \subsetneq \mathfrak{B}$.*

An algebra $\mathfrak{B}$ is minimally generated over $\mathfrak{A}$ if there is a continuous sequence of algebras $(\mathfrak{A}_\alpha)_{\alpha\leq\kappa}$, such that $\mathfrak{A}_0 = \mathfrak{A}$, $\mathfrak{A}_{\alpha+1}$ is a minimal extension of $\mathfrak{A}_\alpha$ for every $\alpha < \kappa$ and $\mathfrak{A}_\kappa = \mathfrak{B}$.
Finally, a Boolean algebra is minimally generated if it is minimally generated over $\{0, 1\}$.

The notion of minimal extension corresponds to the idea of a *simple extension* in the inverse limits setting. Indeed, many authors considering problems similar to those presented in this paper prefer to use the language of inverse limits (see e.g. [8, 10]).

Definition 3.2 *Let $(X_\alpha)_{\alpha\in\lambda}$ be an inverse limit and let $(f_{\alpha\beta})_{\alpha<\beta<\kappa}$ be the set of its bonding mappings. We say that $X_{\alpha+1}$ is a simple extension of X_α if there is exactly one point $x_\alpha \in X_\alpha$ such that $f^{-1}_{(\alpha)(\alpha+1)}(x)$ is a singleton for all $x \neq x_\alpha$ and consists of two points if $x = x_\alpha$.*

The connection can be explained by the following simple lemma. Indeed, if an algebra $\mathfrak{B}$ extends $\mathfrak{A}$ minimally then all ultrafilters in $\mathfrak{A}$ but (possibly) one has unique extensions in $\mathfrak{B}$. It is stated (in a slightly different language) in [18], but we prove it here for the reader's convenience.

Lemma 3.3 *Let $\mathfrak{A} \subseteq \mathfrak{B}$. Then $\mathfrak{B}$ extends $\mathfrak{A}$ minimally if and only if the set*

$$\mathcal{U} = \{A \in \mathfrak{A} \colon \exists B \in \mathfrak{B}\ A \cap B \notin \mathfrak{A}\}$$

is an ultrafilter on $\mathfrak{A}$ and only this ultrafilter is split by $\mathfrak{B}$, i.e. only this ultrafilter can be extended to two different ultrafilters on $\mathfrak{B}$.

Proof. Let $\mathfrak{A} \subseteq \mathfrak{B}$. It is easy to check that if $A_0 \in \mathcal{U}$ and $A_0 \subseteq A_1$ then $A_1 \in \mathcal{U}$. If $B \in \mathfrak{B} \setminus \mathfrak{A}$ then for every $A \in \mathfrak{A}$ either $A \cap B \notin \mathfrak{A}$ or $A^c \cap B \notin \mathfrak{A}$. Therefore, if $\mathcal{U}$ is closed under finite intersections then it is an ultrafilter.
Assume that $\mathfrak{B}$ extends $\mathfrak{A}$ minimally. Consider A_0, $A_1 \in \mathfrak{A}$ and B_0, $B_1 \in \mathfrak{B}$ such that $A_0 \cap B_0 \notin \mathfrak{A}$ and $A_1 \cap B_1 \notin \mathfrak{A}$. Suppose that $A_0 \cap A_1 \notin \mathcal{U}$. Then $C = A_0 \cap A_1 \cap B_0 \cap B_1 \in \mathfrak{A}$. Hence, $C_0 = A_0 \cap B_0 \setminus C \notin \mathfrak{A}$ and $C_1 = A_1 \cap B_1 \setminus C \notin \mathfrak{A}$, $C_0 \cap C_1 = \emptyset$ and $C_0 \cup C_1 \neq 1$. Therefore,

$$\mathfrak{A} \subsetneq \mathfrak{A}(C_0) \subsetneq \mathfrak{A}(C_0, C_1) \subseteq \mathfrak{B},$$

a contradiction. Thus, $\mathcal{U}$ is an ultrafilter.
Consider $p \in \mathrm{Stone}(\mathfrak{A})$ such that there is $A \in p \setminus \mathcal{U}$. Then $A \cap B \in \mathfrak{A}$ and $A \setminus B \in \mathfrak{A}$ for every $B \in \mathfrak{B} \setminus \mathfrak{A}$. Thus, either $A \cap B \in p$ and then we cannot extend p by B^c or $A \setminus B \in p$ but then we cannot extend p by B. Consequently, $\mathcal{U}$ is the only ultrafilter split by $\mathfrak{B}$.
It is easy to see that if $\mathfrak{B}$ is not a minimal extension of $\mathfrak{A}$, then there exist pairwise disjoint $B_0, B_1, B_2 \in \mathfrak{B} \setminus \mathfrak{A}$. Therefore, either $\mathcal{U}$ is not an ultrafilter on $\mathfrak{A}$ or it can be extended to at least three ultrafilters on $\mathfrak{B}$.

■

This gives some idea how minimal extensions look like. The following remark is a simple consequence of the definition and of Lemma 3.3 but it simplifies many considerations included in the next sections.

Proposition 3.4 *Let $\mathfrak{B}$ be a minimal extension of $\mathfrak{A}$. The following facts hold:*

- *if $B \in \mathfrak{B} \setminus \mathfrak{A}$ then $\mathfrak{B} = \mathfrak{A}(B)$;*
- *if we consider disjoint elements A_0, A_1 of $\mathfrak{A}$ and any element B of $\mathfrak{B}$ then $A_0 \cap B \in \mathfrak{A}$ or $A_1 \cap B \in \mathfrak{A}$.*

Now we review some basic facts concerning minimally generated Boolean algebras. The proofs of Proposition 3.5 and of Theorem 3.6 can be found in [18].

Proposition 3.5 *The class of minimally generated algebras is closed under the following operations:*

(a) taking subalgebras;

(b) homomorphic images;

(c) finite products.

A Boolean algebra is called an *interval algebra* if it is generated by a subset linearly ordered under the Boolean partial order. Similarly, an algebra generated by a tree is called a *tree algebra*. Every tree algebra is embeddable into some interval algebra. A Boolean algebra $\mathfrak{A}$ is said to be *superatomic* if every nontrivial homomorphic image of $\mathfrak{A}$ has at least one atom. Recall also that a topological space is said to be *ordered* if its topology is generated by open intervals of some linear order (for Boolean spaces, if it is a Stone space of some interval algebra, equivalently). A topological space X is called *scattered* if for every closed subspace Y of X the isolated points of Y are dense in Y (i.e. if it is a Stone space of some superatomic algebra in the case of Boolean spaces).

Theorem 3.6 (Koppelberg) [18] *The following classes are included in the class of minimally generated Boolean algebras:*

(a) subalgebras of interval algebras (and, thus, countable algebras, tree algebras);

(b) superatomic algebras.

If a Boolean algebra contains an uncountable independent set then it cannot be minimally generated (see [18] or Theorem 4.9 in the next section). The algebra $\mathfrak{C}$ of clopen subsets of $[0,1) \times ([0,1) \cap \mathbb{Q})$, where $[0,1)$ is endowed with the Sorgenfrey line topology, is an example of a small algebra which is not minimally generated (see [20]). It also shows that a free product of minimally generated Boolean algebras does not need to be minimally generated.

We translate now Koppelberg's results to the language of topology. Most of the following reformulations are trivial. Say that a topological space is *Koppelberg compact* if it is Boolean and the algebra of its clopen subsets is minimally generated.

Proposition 3.7 *The class of Koppelberg compacta is closed under the following operations:*

(a) continuous images;

(b) taking closed subspaces;

(c) finite disjoint unions.

Proof. Clearly, (a) and (c) are direct consequences of Proposition 3.5. For Boolean algebras $\mathfrak{A}, \mathfrak{B}$ let $f\colon \mathrm{Stone}(\mathfrak{A}) \to \mathrm{Stone}(\mathfrak{B})$ be a continuous mapping. The set $\{f^{-1}(B)\colon B \in \mathfrak{B}\}$ forms a subalgebra of $\mathfrak{A}$, on the other hand it is isomorphic to $\mathfrak{B}$. We conclude that the minimal generation of $\mathfrak{A}$ implies the minimal generation of $\mathfrak{B}$, by (a) of Proposition 3.5. The proof of (b) is complete.

■

We translate in the same way Theorem 3.6. We first recall the notion of monotonically normal spaces which has been intensively studied in a number of papers over last years.

Definition 3.8 *A topological space X is monotonically normal if it is T_1 and for every open $U \subseteq X$ and $x \in U$ we can find an open subset $h(U,x)$ such that $x \in h(U,x) \subseteq U$ and*

- *$U \subseteq V$ implies $h(U,x) \subseteq h(V,x)$ for every $x \in U$;*
- *$h(x, X \setminus \{y\}) \cap h(y, X \setminus \{x\}) = \emptyset$ for $x \neq y$.*

Theorem 3.9 *A Boolean space K is Koppelberg compact if one of the following conditions is fulfilled:*

(a) K is metrizable;

(b) K is ordered;

(c) K is scattered;

(d) K is monotonically normal.

Proof. Of these (a), (b) and (c) are trivial since the ordered Boolean spaces coincide with the Stone spaces of interval algebras and the class of scattered Boolean spaces is exactly the class of Stone spaces of superatomic algebras. To prove (d) recall Rudin's theorem (see [26]) stating that every compact monotonically normal space is a continuous image of compact ordered space. By (a) of Proposition 3.7 we are done.

■

The class of Koppelberg compact spaces is not included in any class mentioned in the above theorem, which is a trivial assertion in case of (a), (b) and (c). Also, monotone normality and minimal generation are not equivalent, even in the class of zero–dimensional spaces. Before exhibiting the example recall that by the result due to Heindorf (see [17]) every subalgebra of an interval algebra is generated by a pseudo–tree.

The example is following. Consider an algebra $\mathfrak{A} = alg(Fin \cup \{A_\alpha\colon \alpha \in \mathfrak{c}\})$, where $(A_\alpha)_{\alpha\in\mathfrak{c}}$ is an almost disjoint family of subsets of ω. It is clear that $\mathfrak{A}$ is minimally generated and that we cannot generate $\mathfrak{A}$ by a pseudo–tree. Therefore, $\mathfrak{A}$ is not embeddable in an interval algebra and, by Rudin's result, $\mathrm{Stone}(\mathfrak{A})$ is not monotonically normal.

Anyway, the connection between the class of interval algebras, tree algebras and minimal generation is stronger than just the inclusion. The proof of following theorem can be found in [20].

Theorem 3.10 (Koppelberg) *If a Boolean algebra $\mathfrak{A}$ is minimally generated then $\mathfrak{A}$ contains a dense tree subalgebra $\mathfrak{B}$ such that $\mathfrak{A}$ is minimally generated over $\mathfrak{B}$.*

The topological conclusion is as follows. Recall that two topological spaces are *co–absolute* if the algebras of their regular open sets are isomorphic.

Theorem 3.11 *Let K be Koppelberg compact. Then the following conditions are fulfilled for every closed subspace F of K:*

(a) F is co–absolute with an ordered space (i.e. its algebra of regular open sets is isomorphic to the algebra of regular open sets of some ordered space);

(b) F has a tree π–base.

Proof. First, we sketch the proof that every tree algebra has a dense tree. Let $\mathfrak{A}$ be an algebra generated by a tree $\mathcal{T}$. Then $\mathcal{T}$ can be extended to a tree $\mathcal{T}' \subseteq \mathfrak{A}$ being dense in $\mathfrak{A}$. Indeed, if $A \in \mathfrak{A}$ and $\mathcal{S} \subseteq \mathfrak{A}$ is a tree generating $\mathfrak{A}$, such that no element $S \in \mathcal{S}$ fulfils $S \subseteq A$, then without loss of generality we can assume that A^c is of the form

$$A^c = \bigcup_{i \leq n} T_i,$$

where $T_i \in \mathcal{S}$ for $i \leq n$. Therefore, there is no $T \in \mathcal{S}$ disjoint with every T_i. It follows that there is a level of $\mathcal{S}$ which is not a partition of 1 and we can extend this level by an element below A. For trees $\mathcal{T}_0$, $\mathcal{T}_1$ we say that $\mathcal{T}_0 \leq \mathcal{T}_1$ if $\mathcal{T}_0 \subseteq \mathcal{T}_1$ and no element of $\mathcal{T}_1 \setminus \mathcal{T}_0$ has an element of $\mathcal{T}_0$ below. It is easy to see that there is a $\leq$–maximal tree in $\mathfrak{A}$. Such a tree is dense in $\mathfrak{A}$ (otherwise, it would not be maximal, by the above remark).

By Theorem 3.10 every minimally generated Boolean algebra has a dense tree algebra, so it has a dense tree. Therefore K has a tree π–base. Both implications for $F = K$ are proved in [18]. By (b) of Proposition 3.7 we are done.

■

The class of spaces with tree π–bases is surprisingly wide. By the result due to Balcar, Pelant and Simon (see [3]) even $\beta\omega \setminus \omega$ has a tree π–base. This property is usually not inherited by all closed subspaces, though. It is the reason why we have formulated Theorem 3.11 in the above way. Nevertheless, it would be desirable to find some stronger conditions implied by minimal generation, in particular to have a topological characterization of the Koppelberg compacta. It could allow us to get rid of (artificial, in principle) assumption of zero–dimensionality in the definition without referring to the idea of inverse limits. We have not been able to exhibit any example of a space which is not Koppelberg compact such that every closed subspace and every continuous image of it has a tree π–base, but we believe the properties listed in Theorem 3.11 do not characterize the Koppelberg compacta.

It is worth here to recall the idea of *discretely generated* topological spaces (formulated by Dow, Tkachuk, Tkachenko and Wilson in [9]).

Definition 3.12 *A topological space X is called discretely generated if for every subset $A \subseteq X$ we have*

$$\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(D) \colon D \subseteq A \text{ and } D \text{ is a discrete subspace of } X\}.$$

Problem 3.13 *Is every Koppelberg compactum discretely generated?*

One may ask when a given Boolean algebra $\mathfrak{A}$ has a proper minimal extension in a given algebra $\mathfrak{B} \supseteq \mathfrak{A}$. If $\mathfrak{B} = P(\mathrm{Stone}(\mathfrak{A}))$ then $\mathfrak{A}$ can be extended minimally by a point of its Stone space. On the other hand, in Section 5 we will consider only subalgebras of $P(\omega)$. In this case there do exist *maximal* minimally generated algebras, i.e. such subalgebras of $P(\omega)$ that no new subset of ω can extend them minimally. We present here a condition under which we can extend a Boolean algebra $\mathfrak{A}$ in $P(\mathrm{Stone}(\mathfrak{A}))$ in quite a natural way.

Lemma 3.14 *Let $(A_n)_{n\in\omega}$ be a disjoint sequence of clopen subsets of a Boolean space K converging to $p \in K$. Then we can extend $\mathfrak{A} = \mathrm{Clopen}(K)$ minimally by a set A of the form $A = \bigcup\{A_n \colon n \in T\}$, where T is an infinite co–infinite subset of ω. In particular, if $\mathfrak{B} \supseteq \mathfrak{A}$ is a σ–complete Boolean algebra, then we can extend $\mathfrak{A}$ minimally by an element of $\mathfrak{B}$.*

Proof. Let $Z = \bigcup_{n\in\omega} A_{2n}$. Of course, Z does not belong to $\mathfrak{A}$ as then either Z or Z^c would belong to p. $\mathfrak{A}(Z)$ splits the ultrafilter p but this is the only ultrafilter split by $\mathfrak{A}(Z)$.
Indeed, if $q \neq p$ then we have $B \in q$ such that $A_n \cap B = \emptyset$ for almost all n. Let then

$$A = \bigcup\{A_n \colon A_n \cap B \neq \emptyset\}.$$

Since $A \cap Z \in \mathfrak{A}$ either

- $A \cap Z \in q$ but then $(A \cap Z) \cap Z^c = \emptyset$ so q can be extended only by Z or
- $(A \cap Z)^c \in q$. Thus, $B \cap (A \cap Z)^c \in q$ and $B \cap (A \cap Z)^c \cap Z = \emptyset$ so we cannot extend q by Z.

■

Proposition 3.15 *If K is a Boolean space without isolated points and there is a G_δ point in K then $\mathfrak{A} = \mathrm{Clopen}(K)$ can be extended minimally by an open F_σ subset of K.*

Proof. Assume p is a G_δ point in K. Enumerate by $(U_n)_{n\in\omega}$ a countable base of p. Let $A_0 = U_0 \setminus U_1$. For $n \in \omega$ let $A_{n+1} = \bigcup_{m\leq n} U_m \setminus U_{n+1}$. It is easy to check that $(A_n)_{n\in\omega}$ is a disjoint sequence converging to p. By Lemma 3.14 we are done.

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It is easy to see that usually we can find many sequences witnessing that a Boolean algebra is minimally generated and these sequences can have different sizes. By the *length* of a minimally generated Boolean algebra $\mathfrak{A}$ we mean the least ordinal demonstrating the minimal generation of $\mathfrak{A}$.

4. Measures on minimally generated Boolean algebras

We recall several measure theoretic definitions. For a wider background the reader is referred to Fremlin's monograph [13].

Definition 4.1 *A measure μ on a Boolean algebra $\mathfrak{A}$ is said to be separable if there exists a countable $\mathcal{B} \subseteq \mathfrak{A}$ such that for every $A \in \mathfrak{A}$ and $\varepsilon > 0$ we have $B \in \mathcal{B}$ such that $\mu(A \triangle B) < \varepsilon$.*

A Radon measure satisfying the analogous condition is called a *measure of (Maharam) type* ω. The following two definitions are not so well–known as the above one.

Definition 4.2 *A measure μ on a compact space K is uniformly regular if the family μ is inner regular on the family of open subsets of K with respect to zero subsets of K (i.e. for every open $U \subseteq K$ and $\varepsilon > 0$ there is a zero subset $F \subseteq K$ such that $F \subseteq U$ and $\mu(U \setminus F) < \varepsilon$).*

Note that a measure μ on a Boolean algebra $\mathfrak{A}$ is uniformly regular if there is a countable set $\mathcal{A} \subseteq \mathfrak{A}$ such that μ is inner regular with respect to $\mathcal{A}$. We say that $\mathcal{A}$ approximates μ from below.
Sometimes uniformly regular measures are called "strongly countably determined", see [2] or [24] for further reading. The following simple modification of the above definition will be particularly useful.

Definition 4.3 *A measure μ on a Boolean algebra $\mathfrak{A}$ is weakly uniformly regular (w.u.r., for brevity) if there is a countable set $\mathcal{A} \subseteq \mathfrak{A}$ such that μ is inner regular with respect to the class $\{A \setminus I \colon A \in \mathcal{A},\ \mu(I) = 0\}$. We say that $\mathcal{A}$ weakly approximates μ from below.*

We can make this definition a little bit more understandable by switching to the topological point of view. A measure is weakly uniformly regular on Clopen(K), where K is a Boolean space, if the corresponding measure on K is uniformly regular on its support.
It is clear that the following implications hold:

$$\text{uniformly regular} \implies \text{weakly uniformly regular} \begin{cases} \implies \text{of Maharam type } \omega \\ \implies \text{has a separable support} \end{cases}$$

None of the above implications can be reversed. Consider the following examples:

(a) the usual 0–1 measure on the algebra $Fin\text{–}Cofin(\omega_1)$ is weakly uniformly regular but not uniformly regular;

(b) if $\mathfrak{A}$ is the algebra of Lebesgue measure on $[0,1]$ then the standard measure on Stone$(\mathfrak{A})$ is of Maharam type ω, its support is not separable, though, and thus it is not w.u.r.;

(c) the usual product measure on 2^{ω_1} has a separable support but is not of Maharam type ω (hence, is not w.u.r.).

We ought to remark here that example (b) exhibits one more property of uniform regularity. Notice that the Lebesgue measure on $[0,1]$ is uniformly regular but the measure from example (b) is not, although these measures has the same measure algebra. Hence, the uniform regularity of measure depends on its domain. This property plays no role in our considerations as we discuss here only measures on Boolean algebras and their Stone spaces.

Before we start an examination of measures on Koppelberg compacta, we prove a general theorem concerning the connections between uniformly regular measures and separable measures. Recall that if $\mathfrak{A}$ is contained in some larger algebra $\mathfrak{B}$ then every measure μ defined on $\mathfrak{A}$ can be extended to some measure ν defined on $\mathfrak{B}$. We say that $\mathfrak{A}$ is *ν–dense* in $\mathfrak{B}$ if

$$inf\{\nu(B \triangle A)\colon A \in \mathfrak{A}\} = 0$$

for every $B \in \mathfrak{B}$. We will need the following theorem due to Plachky (see [22]).

Theorem 4.4 (Plachky) *Let μ be a measure on a Boolean algebra $\mathfrak{B}$ containing an algebra $\mathfrak{A}$. The algebra $\mathfrak{A}$ is μ–dense in $\mathfrak{B}$ if and only if μ is an extreme point of the set*

$$\{\lambda\colon \lambda \textit{ is defined on } \mathfrak{B} \textit{ and } \lambda|\mathfrak{A} = \mu|\mathfrak{A}\}.$$

We use Plachky's criterion to prove the following result.

Lemma 4.5 *Let $\mathfrak{A}$ be a Boolean algebra carrying a measure μ. If $\mathfrak{A} \subseteq \mathfrak{B}$ then there is an extension of μ to a measure ν defined on $\mathfrak{B}$ such that $\mathfrak{A}$ is not ν–dense in $\mathfrak{B}$ if and only if there is $B \in \mathfrak{B}$ with the property $\mu_*(B) < \mu^*(B)$.*

Proof. Assume that $\mu_*(B_0) < \mu^*(B_0)$ for some $B_0 \in \mathfrak{B}$. It can be easily shown that the formulas

$$\mu'(B) = \mu^*(B \cap B_0) + \mu_*(B \setminus B_0),$$

$$\mu''(B) = \mu_*(B \cap B_0) + \mu^*(B \setminus B_0)$$

define extensions of μ to measures on the algebra $\mathfrak{A}(B_0)$. In turn, μ', μ'' can be extended to ν', ν'' on $\mathfrak{B}$. As $\nu' \neq \nu''$ it follows that $\nu = 1/2(\nu' + \nu'')$ is not an extreme extension, so by Plachky's criterion $\mathfrak{A}$ is not ν–dense in $\mathfrak{B}$.

The converse is obvious.

■

Theorem 4.6 *Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ carries either a uniformly regular measure or a measure which is not separable.*

Proof. Suppose that there is no uniformly regular measure on $\mathfrak{A}$. We construct a nonseparable measure ν defined on $\mathfrak{A}$. Namely, we construct a sequence of countable Boolean algebras $\{\mathfrak{B}_\alpha\colon \alpha < \omega_1\}$ and a sequence of measures $\{\mu_\alpha\colon \alpha < \omega_1\}$ such that for every $\alpha < \beta < \omega_1$ the following conditions are fulfilled:

- $\mathfrak{B}_\alpha$ carries μ_α;
- $\mathfrak{B}_\alpha \subseteq \mathfrak{B}_\beta \subseteq \mathfrak{A}$;
- μ_β extends μ_α;
- $\mathfrak{B}_\alpha$ is not μ_β–dense in $\mathfrak{B}_\beta$.

Assume that we have already constructed $\mathfrak{A}_\alpha$ and μ_α. We can extend μ_α to a measure τ on $\mathfrak{A}$. By our assumption, the measure τ is not uniformly regular so we can find an element A such that

$$\inf\{\tau(A \setminus U) \colon U \in \mathfrak{B}_\alpha,\, U \subseteq A\} > 0.$$

Set $\mathfrak{B}_{\alpha+1} = \mathfrak{B}_\alpha(A)$ and use Lemma 4.5 to find a measure $\mu_{\alpha+1}$ extending μ_α and such that $\mathfrak{B}_\alpha$ is not $\mu_{\alpha+1}$–dense in $\mathfrak{B}_{\alpha+1}$. At a limit step γ set $\mathfrak{B}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha$ and μ_γ to be the unique extension of all members of $\{\mu_\alpha \colon \alpha < \gamma\}$. Finally, set $\mathfrak{B} = \bigcup_{\alpha<\omega_1} \mathfrak{B}_\alpha$ and take the unique extension of all constructed μ_α's for μ. Every extension of μ to a measure ν on $\mathfrak{A}$ is not separable.

■

We turn now to the proper topic of this section. First, we will see how a measure behaves when considered on a minimal extension of its domain.

Lemma 4.7 *Let μ be an atomless measure on a Boolean algebra $\mathfrak{A}$ and let $\mathfrak{B}$ be a minimal extension of $\mathfrak{A}$. Then for every $B \in \mathfrak{B}$ we have $\mu_*(B) = \mu^*(B)$.*

Proof. Consider $B \in \mathfrak{B}$ and $\varepsilon > 0$. We will show that $\mu^*(B) - \mu_*(B) < \varepsilon$. Assume that $(A_n)_{n<N}$ is a partition of $1_{\mathfrak{A}}$ witnessing that μ is atomless (for our ε). From Lemma 3.3 we deduce that there is only one $k < N$ such that $A_k \cap B \notin \mathfrak{A}$ (we exclude the trivial case of $B \in \mathfrak{A}$). Since

$$\sum_{k\neq n<N} \mu(A_n \cap B) = \mu(B \setminus A_k) \leq \mu_*(B) \leq \mu^*(B) \leq \mu(B \setminus A_k) + \varepsilon,$$

we conclude that the demanded inequality holds. As ε was arbitrary, $\mu_* = \mu^*$ on $\mathfrak{B}$.

■

The above lemma expresses the fact that minimal extensions do not enrich atomless measures. This observation lies in the heart of the following facts.

Proposition 4.8 *If $\mathfrak{B}$ is minimally generated over $\mathfrak{A}$ and μ is a measure on $\mathfrak{B}$ such that $\mu|\mathfrak{A}$ is atomless and uniformly regular then μ is uniformly regular.*

Proof. It is a direct consequence of Lemma 4.7.

■

Theorem 3.9 allows us to see the following theorem as a generalization (of course only for the zero–dimensional case) of Theorem 9 of [7] (stating that every atomless measure on a monotonically normal space is of countable Maharam type) and of Theorem 3.2(i) of [27] (stating that every atomless measure on an ordered space is uniformly regular on its support).

Theorem 4.9 *Every measure μ on a minimally generated Boolean algebra $\mathfrak{A}$ is separable.*

Proof. Assume *a contrario* that there is a measure μ on $\mathfrak{A}$ which is not separable. Assume that the sequence $(\mathfrak{A}_\alpha)_{\alpha\leq\beta}$ witnesses that $\mathfrak{A}$ is minimally generated (where $\mathfrak{A}_\beta = \mathfrak{A}$) and let $\mu_\alpha = \mu|\mathfrak{A}_\alpha$ for every α. Denote

$$\kappa = \min\{\alpha\colon \mu_\alpha \text{ is not separable on } \mathfrak{A}_\alpha\}$$

and notice that $\mathrm{cf}(\kappa)$ is uncountable. Without loss of generality we can assume that μ_κ is atomless. If it is not then we can apply the Sobczyk–Hammer Decomposition Theorem (see Theorem 5.2.7 in [6]), i.e. split μ_κ into

$$\mu_\kappa = \nu_0 + \sum_{n\in\omega} a_n\nu_n,$$

where ν_0 is atomless and for $n \geq 1$ the measure ν_n is 0–1 valued. Of course $\sum_{n\in\omega} a_n\nu_n$ is separable so we can assume that $\mu_\kappa = \nu_0$. Denote now

$$\lambda = \min\{\alpha\colon \mu_\alpha \ \text{ is atomless}\}.$$

Of course $\lambda \leq \kappa$. Notice that $\mathrm{cf}(\lambda) = \aleph_0$. Indeed, if $\alpha(n)$ is the least ordinal such that there is a partition of 1 into sets from $\mathfrak{A}_{\alpha(n)}$ of μ–measure $< 1/n$, then μ on $\bigcup_{n\in\omega}\mathfrak{A}_{\alpha(n)}$ is atomless. Hence, $\lambda < \kappa$. But the measure μ_λ on $\mathfrak{A}_\lambda$ fulfils the conditions of Lemma 4.7 so for every $\alpha > \lambda$ the measure μ_α on $\mathfrak{A}_\alpha$ is a separable, in particular so is μ on $\mathfrak{A}$, a contradiction.

■

In fact, using this method one can prove that every measure on a minimally generated Boolean algebra is a countable sum of weakly uniformly regular measures.

The following corollary is proved directly in [18]. Recall that if we can map continuously a topological space K onto $\{0,1\}^{\omega_1}$ then there exists a measure of uncountable type on K (by Fremlin's theorem, under MA_{ω_1} the above conditions are in fact equivalent, see [12]). We should also remind here that a compact space K contains a copy of $\beta\omega$ if and only if it can be mapped continuously onto $\{0,1\}^{\mathfrak{c}}$. Now we can finally formulate the corollary.

Corollary 4.10 *If $\mathfrak{A}$ is a minimally generated Boolean algebra then $\mathfrak{A}$ does not contain an uncountable independent sequence. Therefore,* $\mathrm{Stone}(\mathfrak{A})$ *cannot be mapped continuously onto $\{0,1\}^{\omega_1}$ and there is no copy of $\beta\omega$ in* $\mathrm{Stone}(\mathfrak{A})$.

It is worth to point out here one more remark. Some axioms (such as CH) imply the existence of examples of small Boolean algebras carrying nonseparable measures. By Theorem 4.9 these examples turn out to be also examples of small but not minimally generated Boolean algebras.

The following fact can be easily deduced from the proof of Theorem 4.9.

Corollary 4.11 *Every atomless measure μ on a minimally generated Boolean algebra of length at most ω_1 is uniformly regular.*

We show that the above corollary cannot be strengthened in the obvious way.

Example 4.12 *There is a Boolean algebra of length at most $\omega_1+\omega$ carrying an atomless measure which is not uniformly regular.*

Proof. Let $A(\omega_1)$ denote the Alexandrov compactification of ω_1 endowed with the discrete topology, i.e. the space $\omega_1 \cup \{\infty\}$ with the topology generated by $\{\alpha\}$ for $\alpha \in \omega_1$ and $\{\infty\} \cup (\omega_1 \setminus I)$ for finite sets I. Consider the algebra $\mathfrak{A} = \mathrm{Clopen}(A(\omega_1) \times C)$, where C is the Cantor set.

Claim 1. The algebra $\mathfrak{A}$ is minimally generated.

We can construct in a minimal way the algebra $\{0\} \times \mathrm{Clopen}(C)$ in the first ω steps. There are no obstacles (for the minimality of extensions) to repeat this construction for $\{1\} \times \mathrm{Clopen}(C)$ and proceed in this manner obtaining finally (in ω_1 steps) the algebra generated by sets of the form $\{\alpha\} \times K$, where $\alpha \in \omega_1$ and K is a clopen subset of C. Then we can add by minimal extensions all sets of the form $(\{\infty\} \cup \omega_1) \times K$, where K is a clopen subset of C. As a result, we obtain $\mathfrak{A}$.

Consider now the following measure μ on $\mathfrak{A}$:

$$\mu(A) = \lambda(A \cap (\{\infty\} \times C)),$$

where λ is the standard measure on C.

Claim 2. The measure μ is atomless but not uniformly regular.

Indeed, suppose that there is a countable family $\mathcal{A} \subseteq \mathfrak{A}$ approximating μ from below. For every $A \in \mathcal{A}$ of positive measure $\infty \in \pi(A)$, where $\pi\colon A(\omega_1) \times C \to A(\omega_1)$ is the projection to the first coordinate, so $\pi(A) = \omega_1 \setminus I_A$, where I_A is finite. Let

$$\alpha = sup \bigcup \{I_A \colon A \in \mathcal{A}\} + 1.$$

Let $B = (\{\infty\} \cup (\omega_1 \setminus \{\alpha\})) \times C$. It is easily seen that

- $B \in \mathfrak{A}$;
- $\mu(B) = 1$;
- there is no $A \in \mathcal{A}$ such that $\mu(A) > 0$ and $A \subseteq B$ (if $\mu(A) > 0$ and $A \in \mathcal{A}$ then by the definition of α we see that $\{\alpha\} \times C \subseteq A$).

■

From the above example we deduce that the length of a minimally generated algebra is not necessarily a cardinal number. The above algebra $\mathfrak{A}$ cannot be generated in ω_1 steps as then every atomless measure admitted by $\mathfrak{A}$ should be uniformly regular. Anyway, the following fact implies that the lengths of minimally generated algebras are limit ordinal numbers.

Proposition 4.13 *Let $\mathfrak{A}$ be a minimally generated subalgebra of a Boolean algebra $\mathfrak{C}$. Then the algebra $\mathfrak{A}(B)$ is minimally generated for every $B \in \mathfrak{C}$.*

Proof. Let $(A_\alpha)_{\alpha\in\kappa}$ be such that $\mathfrak{A}_{\alpha+1} = \mathfrak{A}_\alpha(A_\alpha)$ for every $\alpha < \lambda$, where $(\mathfrak{A}_\alpha)_{\alpha\in\lambda}$ is a sequence witnessing the minimal generation of $\mathfrak{A}$. We will construct a sequence of minimal extensions generating $\mathfrak{B} = \mathfrak{A}(B)$. Recall that an ordinal number λ is called *even* if it can be represented as $\lambda = \gamma + 2n$, where γ is a limit ordinal or 0 and $n \in \omega$. For such ordinals let $h(\gamma + 2n) = \gamma + n$.

Let $\mathfrak{B}_0 = \{0, 1, B, B^c\}$. Define

$$\mathfrak{B}_{\alpha+1} = \begin{cases} \mathfrak{B}_\alpha(B \cap A_{h(\alpha+2)}) & \text{if } \alpha \text{ is even;} \\ \mathfrak{B}_\alpha(A_{h(\alpha+1)}) & \text{else.} \end{cases}$$

At a limit step γ we set $\mathfrak{B}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{B}_\alpha$.

Our new sequence generates the demanded algebra in a minimal way. Let ξ be even. Then $\mathfrak{B}_\xi$ is extended to $\mathfrak{B}_{\xi+1}$ by an element of the form $B \cap A$, where $A \in \mathfrak{A}$. The following equality holds:

$$\{C \in \mathfrak{B}_\xi \colon C \cap (A \cap B) \notin \mathfrak{B}_\xi\} = \{C \in \mathfrak{A} \cap \mathfrak{B}_\xi \colon C \cap A \notin \mathfrak{A} \cap \mathfrak{B}_\xi\}.$$

Since the latter is an ultrafilter in $\mathfrak{A} \cap \mathfrak{B}_\xi$ and this ultrafilter is the only one split by A, using Lemma 3.4 we obtain that our extension is minimal.

Similar arguments work for the case of odd ξ.

■

We will show now that the property of admitting only w.u.r. measures is closed under free products. By the result due to Sapounakis (see [27]) interval Boolean algebras admit only w.u.r. measures. It follows that Koppelberg's example $\mathfrak{C}$ mentioned on page 5 carries only w.u.r. measures (since it is a free product of interval algebras) but it is not minimally generated. Therefore, every measure on a minimally generated algebra is separable but there is a Boolean algebra admitting only w.u.r. measures which is not minimally generated. Consequently, minimal generation cannot be characterized by any measure theoretic property mentioned in this section.

Theorem 4.14 *If every measure on a Boolean algebra $\mathfrak{A}$ is w.u.r. and every measure on $\mathfrak{B}$ is w.u.r., then every measure on $\mathfrak{A} \oplus \mathfrak{B}$ is w.u.r.*

Proof. For simplicity assume that the considered algebras are contained in $P(X)$ for some set X.

It is enough to show that we can weakly approximate from below all the rectangles since every member of $\mathfrak{A} \oplus \mathfrak{B}$ is a finite union of rectangles. Let μ be a measure on $\mathfrak{A} \oplus \mathfrak{B}$. Define

$$\mu_1(A) = \mu(A \times X)$$

and for $A \in \mathfrak{A}$

$$\mu_A(B) = \mu(A \times B).$$

By the assumption the measure μ_1 is weakly uniformly regular so there is a countable set $\mathcal{A}$ weakly approximating μ_1 from below. For every $A \in \mathcal{A}$ the measure μ_A is also w.u.r. and has an approximating set $\mathcal{B}\ (A)$.

We will show that $\{A_0 \times B_0 \colon A_0 \in \mathcal{A}, B_0 \in \mathcal{B}\ (A_0)\}$ weakly approximates μ from below. Indeed, consider $A \in \mathfrak{A}, B \in \mathfrak{B}$ and $\varepsilon > 0$. Then, by the definition we can find:

- $A_0 \in \mathcal{A}$ such that $\mu_1(A \setminus A_0) < \frac{\varepsilon}{2}$ and $\exists F\ \mu_1(F) = 0, A_0 \setminus F \subseteq A$;
- $B_0 \in \mathcal{B}(A_0)$ such that $\mu_{A_0}(B \setminus B_0) < \varepsilon/2$ and $\exists G\ \mu_{A_0}(G) = 0, B_0 \setminus G \subseteq B$.

Now $\mu((A \times B) \setminus (A_0 \times B_0)) < \varepsilon$ since

$$(A \times B) \setminus (A_0 \times B_0) = A_0 \times (B \setminus B_0) \cup (A \setminus A_0) \times B$$

but

$$\mu(A_0 \times (B \setminus B_0)) = \mu_{A_0}(B \setminus B_0) < \varepsilon/2$$

and

$$\mu((A \setminus A_0) \times B) \leq \mu((A \setminus A_0) \times X) = \mu_1(A \setminus A_0) < \varepsilon/2.$$

It suffices to show that there exists an element H such that $\mu(H) = 0$ and $(A_0 \times B_0) \setminus H \subseteq (A \times B)$. Clearly, $H = (F \times X) \cup (A_0 \times G)$ is such an element.

■

We continue the measure theoretic examination of minimally generated Boolean algebras. The existence of uniformly regular measures on such algebras follows from Theorem 4.6 and Theorem 4.9. Anyway, such measures can be easily constructed directly using Theorem 3.10. Under certain conditions we can force these measures to have additional properties.

Theorem 4.15 *Let $\mathfrak{A}$ be an atomless minimally generated Boolean algebra. Then $\mathfrak{A}$ carries an atomless uniformly regular measure μ. Moreover, if any of the following conditions is fulfilled then we can demand that μ is strictly positive as well:*

- *if $\mathfrak{A}$ carries a strictly positive measure;*
- *if $\mathfrak{A}$ is c.c.c. and the Suslin Conjecture is assumed;*
- *if $\mathfrak{A}$ is strongly c.c.c., i.e. it does not contain any uncountable set of pairwise incomparable elements.*

Proof. Let $T \subseteq \mathfrak{A}$ be a tree as in Theorem 3.10.

We can easily find a countable dyadic tree $T_0 \subseteq T$. For an element $A \in T_0$ put $\mu(A) = 1/2^n$ if A belongs to the n–th level of T_0. In this way we obtain a measure defined on the algebra generated by T_0. It is atomless and uniformly regular, so by Lemma 4.7 its extension to ν defined on $\mathfrak{A}$ will be uniformly regular as well.

Claim. If T can be assumed to be countable then $\mathfrak{A}$ carries a strictly positive uniformly regular measure.

Indeed, we can easily find a tree $T_0 \subseteq T$ isomorphic to $\omega^{<\omega}$ such that every level of T_0 forms a maximal antichain in $\mathfrak{A}$ and T_0 is dense in $\mathfrak{A}$. Define a strictly positive measure μ on T_0. By a similar argument as before the extension of μ to the measure ν on $\mathfrak{A}$ will be uniformly regular. Clearly, ν is strictly positive and the claim is proved.

To complete the proof we show that the assumptions listed above imply that T can be conceived as countable.

If $\mathfrak{A}$ carries a strictly positive measure then, according to Fact 2.1, every tree contained in $\mathfrak{A}$ is countable, and so is T.

If $\mathfrak{A}$ is c.c.c. then it does not contain neither an uncountable chain nor an uncountable antichain so every uncountable tree contained in $\mathfrak{A}$ is Suslin. Hence, the Suslin Conjecture implies that T is countable.

Finally, by the theorem of Baumgartner and Komjáth, if $\mathfrak{A}$ is strongly c.c.c. then it contains a countable dense subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$ (see [5] for the details). Therefore, the Stone space of $\mathfrak{A}$ is separable and thus it supports a strictly positive measure (for the proofs of the last implications we refer the reader to [28]).

■

It follows that in the class of Koppelberg compacta the property of having a strictly positive measure is equivalent to separability. If the Suslin Conjecture is assumed these properties are equivalent also to c.c.c. We can use these remarks to answer the question which seems to be natural in the context of Theorem 4.9.

Theorem 4.16 *There is a minimally generated Boolean algebra supporting a measure which is not w.u.r.*

Proof. Denote by $\mathfrak{B}$ the algebra of Lebesgue measure on $[0,1]$. Let $\mathfrak{A} \subseteq \mathfrak{B}$ be a minimally generated Boolean algebra such that for no $B \in \mathfrak{B} \setminus \mathfrak{A}$ the extension $\mathfrak{A}(B)$ is minimal over $\mathfrak{A}$. Notice that according to the proof of Theorem 3.15 and the completeness of $\mathfrak{B}$ no $p \in K = \mathrm{Stone}(\mathfrak{A})$ is a G_δ point.

Since $\mathfrak{A}$ carries a strictly positive measure the space K is separable (by Theorem 4.15). Let $\{x_n : n \geq 1\}$ be dense in K. Consider the following measure:

$$\mu = \sum_{n \geq 1} \delta_{x_n}/2^n.$$

It is not w.u.r. Otherwise, it would be uniformly regular because μ is strictly positive. But δ_x is uniformly regular only if x is G_δ and there are no such points in K. Therefore, the measure δ_{x_1} is not uniformly regular and, accordingly, μ is not w.u.r.

■

We finish this section with a short analysis of the behavior of measures on other well–known subclass of small Boolean algebras.

Definition 4.17 *A Boolean algebra $\mathfrak{A}$ is retractive if for every epimorphism $e\colon \mathfrak{A} \to \mathfrak{B}$ there is a monomorphism (lifting) $m\colon \mathfrak{B} \to \mathfrak{A}$ such that $e \circ m = id_{\mathfrak{B}}$.*

Notice that a Boolean algebra is retractive if and only if its Stone space K is co–retractive, i.e. every closed subspace of K is a retract of K. J. Donald Monk showed that no retractive Boolean algebra contains an uncountable independent sequence. It is also known that not every minimally generated algebra is retractive. In [20] Koppelberg gave an example of a retractive but not minimally generated Boolean algebra. However, the construction was carried out under CH. We present here an example of a retractive algebra which is not minimally generated and additionally carries a nonseparable measure. It requires the following assumption:

$$\mathrm{cof}(\mathcal{N}) = min\{|\mathcal{A}|\colon \mathcal{A} \subseteq \mathcal{N}\ \forall N \in \mathcal{N}\ \exists A \in \mathcal{A}\ N \subseteq A\} = \omega_1,$$

where $\mathcal{N}$ denotes the ideal of Lebesgue measure zero sets. Of course CH implies $\mathrm{cof}(\mathcal{N}) = \aleph_1$, on the other hand e.g. in the Sacks model $\mathfrak{c} = \aleph_2$ and, nevertheless, $\mathrm{cof}(\mathcal{N}) = \aleph_1$.

In the following theorem we simply take advantage of the construction carried out by Plebanek in [23]. Recall that a Boolean space K is *Corson compact* if there exists a point–countable family $\mathcal{D}$ of clopen subsets of K such that $\mathcal{D}$ separates points of K. For our purposes it is important that the separable Corson compact spaces are metrizable (see [1]).

Theorem 4.18 *Assume* $\mathrm{cof}(\mathcal{N}) = \aleph_1$. *Then there is a retractive Boolean algebra* $\mathfrak{A}$ *carrying a nonseparable measure and without a tree* π*–base.*

Proof. The equality $\mathrm{cof}(\mathcal{N}) = \aleph_1$ implies the existence of a Corson compact space K carrying a strictly positive nonseparable measure μ such that for every nowhere dense $F \subseteq K$ the set F is metrizable (see [23]).

To verify the retractiveness of Boolean algebra $\mathfrak{A} = \mathrm{Clopen}(K)$ one needs only to check if for every dense ideal $I \subseteq \mathfrak{A}$ the algebra $\mathfrak{A}/I$ is countable (see Theorem 4.3 (c) in [25]). If an ideal I is dense then $F = \mathrm{Stone}(\mathfrak{A}/I)$ is a closed nowhere dense subspace of K. Thus, it is metrizable. So $\mathfrak{A}/I$ is countable.

Assume now for a contradiction that $\mathfrak{A}$ has a tree π–base T. Since μ is strictly positive, by Fact 2.1, T has to be countable. Thus, K is separable and, since it is Corson compact, K is metrizable. It follows that every measure on K is of countable Maharam type, a contradiction.

■

On the other hand, as we have already mentioned, it is consistent to assume that small Boolean algebras carry only separable measures. Combining Fremlin's theorem mentioned on page 12 and the fact that retractive algebras are small we obtain the following.

Theorem 4.19 *If* MA_{ω_1} *holds then retractive algebras admit only separable measures.*

It is not known if it is consistent to assume that every retractive Boolean algebra is minimally generated (or, at least, has a tree π–base).

5. Connection to Efimov Problem

We recall the longstanding Efimov problem.

Problem 5.1 *Is there an infinite compact space which neither contains a nontrivial convergent sequence nor a copy of* $\beta\omega$*?*

Such spaces (we call them *Efimov spaces*) can be constructed if certain set theoretic axioms are assumed. The question if one can construct a Efimov space in ZFC is still unanswered. For example, it is not known if Martin's Axiom implies the existence of Efimov spaces.

Consider a sequence $(r_n)_{n\in\omega}$ and a subsequence $(l_n)_{n\in\omega}$ in a topological space X. We say that $K \subseteq X$ separates $L = \{l_n \colon n \in \omega\}$ in $R = \{r_n \colon n \in \omega\}$ if $R \cap K = L$. To make a Boolean space Efimov we have to add many clopen sets to ensure that every sequence of distinct points has a subsequence separated by a clopen set. On the other hand, if our space is too rich, then it contains a sequence all of whose subsequences are separated and, thus, it would contain a copy of $\beta\omega$.

By Corollary 4.10 minimal generation gives us a tool for constructing compact zero–dimensional spaces without copies of $\beta\omega$. Fedorčuk's Efimov space (see [11]) has been constructed using simple extensions as well as the example presented by Dow in [8]. The first one requires CH, the latter a certain axiom connected to the notion of *splitting number*. For another construction (using $\diamondsuit$) see also [20].

We consider compactifications of ω. Notice at once that if there exists a Efimov space then by taking the closure of countable discrete subspace we can obtain a compactification of ω which is Efimov.

We will employ the idea of pseudo–intersection number. Write $A \subseteq^* B$ if $A \setminus B$ is finite. We say that $P \subseteq X$ is a *pseudo–intersection* for a family $\mathcal{P} \subseteq P(X)$ provided for every $A \in \mathcal{P}$ we have $P \subseteq^* A$. A family $\mathcal{P}$ is said to have strong finite intersection property (*sfip* for brevity) if every finite subfamily has an infinite intersection. The definition of the *pseudo–intersection number* is as follows

$$\mathfrak{p} = min\{|\mathcal{P}| \colon \mathcal{P} \subseteq [\omega]^\omega \text{ has } \textit{sfip} \text{ but no } X \in [\omega]^\omega \text{ is a pseudo–intersection for } \mathcal{P}\}.$$

The assumption $\mathfrak{p} = \mathfrak{c}$ is equivalent to Martin's Axiom for σ–centered families (see, e.g., [14]).

For a topological space X and a cardinal α we say that $S \subseteq X$ is G_α if there is a family of open sets $\{U_\xi \colon \xi \in \alpha\}$ such that $S = \bigcap_{\xi\in\alpha} U_\xi$. It is convenient to say that S is $G_{<\alpha}$ if there is a $\beta < \alpha$ such that S is G_β.

Theorem 5.2 *There is a Koppelberg compactification K of ω without a convergent sequence of distinct $G_{<\mathfrak{p}}$ points. In particular, if MA is assumed then K does not contain a convergent sequence of distinct $G_{<\mathfrak{c}}$ points.*

Proof. We will indicate a Koppelberg compactification of ω without a convergent subsequence of ω such that no point of its remainder is $G_{<\mathfrak{p}}$. We first show two claims.

Claim 1. Let $\mathfrak{A}$ be a subalgebra of $P(\omega)$ cointaining the algebra $Fin\text{–}Cofin$. Then there is a nontrivial convergent subsequence of ω in $K = \text{Stone}(\mathfrak{A})$ if and only if there is $p \in K$ with an infinite pseudo–intersection.

Indeed, assume that a sequence $(n_k)_{k\in\omega}$ converges to p. Thus, for every $A \in p$ we have $N = \{n_0, n_1, ...\} \subseteq^* A$ and, consequently, N is a pseudo–intersection of p. Conversely, an enumerated pseudo–intersection of p forms a subsequence of ω convergent to p.

Claim 2. Let $\mathfrak{A}$ be an algebra minimally generated over $Fin\text{–}Cofin$ with an ultrafilter p with infinite pseudo–intersection P. Then $\mathfrak{A}(P)$ is a minimal extension of $\mathfrak{A}$.

It is so because for every $A \in \mathfrak{A}$ either $A \cap P \in Fin$ or $P \subseteq^* A$ and, therefore, either $A \cap P \in \mathfrak{A}$ or $A^c \cap P \in \mathfrak{A}$. By Lemma 3.3 we are done.

Let $\mathfrak{A} \subseteq P(\omega)$ be a Boolean algebra minimally generated over Fin–$Cofin$ such that $\mathfrak{A}(A)$ is not a minimal extension of $\mathfrak{A}$ for any $A \in P(\omega) \setminus \mathfrak{A}$. By Claim 2 no $p \in K = \mathrm{Stone}(\mathfrak{A})$ has an infinite pseudo–intersection and by Claim 1 there is no convergent subsequence of ω in K. Since no $p \in K \setminus \omega$ is a $G_{<\mathfrak{p}}$ point and K is Koppelberg compact, we are done.

■

As a corollary we get the following theorem proved by Haydon in [15].

Corollary 5.3 (Haydon) *There is a compact space which is not sequentially compact but which carries no measure of uncountable type.*

Proof. Let K be as in Theorem 5.2. Then the natural numbers form a sequence witnessing that K is not sequentially compact. By Theorem 4.9 every measure on K has a countable Maharam type.

■

In fact, as can easily be seen in the proof of Theorem 5.2, every Boolean algebra $\mathfrak{A}$ minimally generated over Fin–$Cofin$ can be extended to $\mathfrak{B} \subseteq P(\omega)$ such that $\mathrm{Stone}(\mathfrak{B})$ fulfills the conditions of Theorem 5.2 and Corollary 5.3. Thus, we can produce a lot of examples of such spaces.

Moreover, using Theorem 3.10 we can easily indicate tree algebras with the same property as in the above theorems. In fact, tree algebras can be unexpectedly rich. By the theorem already mentioned in Section 2 there is a tree algebra $\mathfrak{A}$ dense in $P(\omega)/Fin$, i.e. such that for every infinite $N \subseteq \omega$ there is an infinite set $M \subseteq^* N$ such that $M \in \mathfrak{A}$.

Theorem 5.2 can be counterpointed by the following theorem. Let us say that a compact space K is *Grothendieck* if $C(K)$ is Grothendieck, i.e. if every weak* convergent sequence in the space $C^*(K)$ weakly converges, which means that in a sense $C^*(K)$ does not contain nontrivial convergent sequences of measures and, thus, there is no nontrivial convergent sequences of points in K (as the convergence of $(x_n)_{n\in\omega}$ is equivalent to the convergence of $(\delta_{x_n})_{n\in\omega}$). So, the notion of a Grothendieck space is a strengthening of the property of not containing nontrivial convergent sequences.

Definition 5.4 *Let $\mathcal{F}$ be a family of subsets of a compact space K. We say that K contains a copy of $\beta\omega$ consisting of $\mathcal{F}$ sets if there is a disjoint sequence $(F_n)_{n\in\omega}$ of elements of $\mathcal{F}$ such that for every $T \subseteq \omega$ there is $A \in \mathrm{Clopen}(K)$ such that*

$$A \cap \bigcup_{n\in\omega} F_n = \bigcup_{n\in T} F_n.$$

Denote by (*) the following assumption:

$$2^\kappa \leq \mathfrak{c} \text{ if } \kappa < \mathfrak{c}.$$

Recall that (*) implies that $\mathfrak{c}$ is regular and that MA implies (*). We prove the following theorem.

Theorem 5.5 *There is a Grothendieck space not containing copies of $\beta\omega$ consisting of G_δ sets. Moreover, if (*) is assumed then there is a Grothendieck space without copies of $\beta\omega$ consisting of $G_{<\mathfrak{c}}$ sets.*

Thus, although it is not known if one can construct a Efimov space under MA, some sorts of Efimov spaces can be, nevertheless, indicated: either if we admit the existence of a convergent sequence of $G_\mathfrak{c}$ points or if we admit $\beta\omega$ to be embeddable but only in such a way that natural numbers are mapped on $G_\mathfrak{c}$ sets.

In fact, our construction has a slightly stronger property. We say that a Boolean algebra $\mathfrak{A}$ has the *Subsequential Completeness Property* (*SCP*, for brevity) if for every disjoint sequence in $\mathfrak{A}$ there is an infinite co–infinite subset $T \subseteq \omega$ such that $(A_n)_{n\in T}$ has a least upper bound in $\mathfrak{A}$. A compact space K has SCP if Clopen(K) has SCP. Haydon showed that the spaces with *SCP* are Grothendieck (see [15]).

Definition 5.6 *Let $\mathfrak{A}$ be a Boolean algebra. Let $R = \{F_n \colon n \in \omega\}$ be a set of filters on $\mathfrak{A}$ and let $L = \{F_n \colon n \in T\}$ for some $T \subseteq \omega$. We say that $A \in \mathfrak{A}$ separates L in R if*

- *$A \in F_n$ for $n \in T$;*
- *$A^c \in F_n$ for $n \notin T$.*

The algebra $\mathfrak{A}$ separates L in R if there is $A \in \mathfrak{A}$ separating L in R.

Notice that a sequence $(F_n)_{n\in\omega}$ of closed sets in Stone$(\mathfrak{A})$ is a copy of $\beta\omega$ if and only if every subsequence of $(F_n)_{n\in\omega}$ is separated in $(F_n)_{n\in\omega}$ by $\mathfrak{A}$.

Thus, the assertion that K does not contain copies of $\beta\omega$ consisting of clopen sets has a simple algebraic interpretation. It means that for every pairwise disjoint sequence $(A_n)_{n\in\omega}$ from $\mathfrak{A} =$ Clopen(K) the algebra $\mathfrak{A}$ contains a least upper bound of $(A_n)_{n\in T}$ for some infinite co–infinite $T \subseteq \omega$ but there is also $N \subseteq \omega$ such that $(A_n)_{n\in N}$ is non–separated in $(A_n)_{n\in\omega}$ by $\mathfrak{A}$.

The construction proceeds as follows, in the spirit of Haydon's construction from [15].

Consider a Boolean algebra $\mathfrak{A}$ and a sequence $(F_n)_{n\in\omega}$ of filters on $\mathfrak{A}$. We will say that a sequence $(p_n)_{n\in\omega}$ is *an extension of* $(F_n)_{n\in\omega}$ *in* $\mathfrak{A}$ if p_n is an extension of F_n to an ultrafilter in $\mathfrak{A}$ for every $n \in \omega$. We will use the following trivial observation.

Fact 5.7 *Let R be a sequence of filters on a Boolean algebra $\mathfrak{A}$ with a subsequence L separated in R by $\mathfrak{A}$. If R' and L' are extensions of R and L in $\mathfrak{A}$ then L' is still separated in R' by $\mathfrak{A}$.*

Before we prove Theorem 5.5 we have to show the following lemma.

Lemma 5.8 *Let $\mathfrak{A} \subseteq P(X)$ be a Boolean algebra. Assume that $\{(L_\alpha, R_\alpha) \colon \alpha < \kappa < \mathfrak{c}\}$ is such that R_α is a nontrivial sequence in* Stone$(\mathfrak{A})$ *and L_α is its subsequence no separated in R_α by $\mathfrak{A}$ for every $\alpha < \kappa$. Let $(A_n)_{n\in\omega}$ be a disjoint sequence in $\mathfrak{A}$. Then there is an infinite, co–infinite $\sigma \subseteq \omega$ and a collection $\{(L'_\alpha, R'_\alpha) \colon \alpha < \kappa < \mathfrak{c}\}$ such that for every $\alpha < \kappa$ and $n \in \omega$ we have: $R'_\alpha(n)$ is an extension of $R_\alpha(n)$ to an ultrafilter in $\mathfrak{A}(\bigcup_{n\in\sigma} A_n)$, L'_α is the corresponding subsequence of R'_α and $\mathfrak{A}(\bigcup_{n\in\sigma} A_n)$ does not separate L'_α in R'_α.*

Proof. For $\sigma \subseteq \omega$ denote

$$A_\sigma = \bigcup_{n\in\sigma} A_n.$$

Consider the algebras $\mathfrak{A}(A_\sigma)$ for $\sigma \subseteq \omega$. Fix $\alpha < \kappa$ and $n \in \omega$. We define $R^\sigma_\alpha(n)$ in the following way. If A_σ does not split the ultrafilter $\mathcal{F} = R_\alpha(n)$ then $R^\sigma_\alpha(n)$ is the unique extension of $\mathcal{F}$ in $\mathfrak{A}(A_\sigma)$. If A_σ splits $\mathcal{F}$ then let $R^\sigma_\alpha(n)$ be defined as the extension of $\mathcal{F}$ by A^c_σ. $L^\sigma_\alpha(n) = R^\sigma_\alpha(m)$ if $L_\alpha(n) = R_\alpha(m)$.

Consider an almost disjoint family Σ of infinite subsets of ω of cardinality $\mathfrak{c}$. We show that there is $\sigma \in \Sigma$ such that no L^σ_α is separated in R^σ_α by $\mathfrak{A}(A_\sigma)$. Suppose otherwise; then, by a cardinality argument, there are $\alpha < \kappa$, $\sigma, \tau \in \Sigma$ and $U_1, U_2 \in \mathfrak{A}$ such that $\sigma \neq \tau$ and

$$Z_\sigma = (A_\sigma \cap U_1) \cup (A^c_\sigma \cap U_2) \text{ separates } L^\sigma_\alpha \text{ in } R^\sigma_\alpha$$

and

$$Z_\tau = (A_\tau \cap U_1) \cup (A^c_\tau \cap U_2) \text{ separates } L^\tau_\alpha \text{ in } R^\tau_\alpha.$$

Set

$$A = (A_{\sigma\cap\tau} \cap U_1) \cup (A^c_{\sigma\cap\tau} \cap U_2),$$

and notice that $A \in \mathfrak{A}$ (as $\sigma \cap \tau$ is finite). It suffices to show that the set A separates L_α in R_α.

Consider $\mathcal{F} = L_\alpha(n)$ for some $\alpha < \kappa$ and $n \in \omega$. We show that $A \in \mathcal{F}$. Denote $\mathcal{F}^\sigma = L^\sigma_\alpha(n)$ and $\mathcal{F}^\tau = L^\tau_\alpha(n)$. Obviously, $Z_\sigma \in \mathcal{F}^\sigma$ and $Z_\tau \in \mathcal{F}^\tau$. It means that either $Z^1_\sigma = (A_\sigma \cap U_1) \in \mathcal{F}^\sigma$ or $Z^2_\sigma = (A^c_\sigma \cap U_2) \in \mathcal{F}^\sigma$ and either $Z^1_\tau = (A_\tau \cap U_1) \in \mathcal{F}^\tau$ or $Z^2_\tau = (A^c_\tau \cap U_2) \in \mathcal{F}^\tau$. To show that $A \in \mathcal{F}$ we have to consider three cases. We will repeatedly use basic properties of ultrafilters.

1. If $Z^1_\sigma \in \mathcal{F}^\sigma$ and $Z^2_\tau \in \mathcal{F}^\tau$ or $Z^2_\sigma \in \mathcal{F}^\sigma$ and $Z^1_\tau \in \mathcal{F}^\tau$ then both U_1, U_2 belong to $\mathcal{F}$ and, since either $A_{\sigma\cap\tau} \in \mathcal{F}$ or $A^c_{\sigma\cap\tau} \in \mathcal{F}$, $A \in \mathcal{F}$.
2. If $Z^2_\sigma \in \mathcal{F}^\sigma$ and $Z^2_\tau \in \mathcal{F}^\tau$ then the set $A_{\sigma\cap\tau}$ cannot belong to $\mathcal{F}$ (because then $\emptyset \in \mathcal{F}^\sigma$), so $A^c_{\sigma\cap\tau} \in \mathcal{F}$ but $U_2 \in \mathcal{F}$ and, therefore, $A \in \mathcal{F}$.
3. Assume that $Z^1_\sigma \in \mathcal{F}^\sigma$ and $Z^1_\tau \in \mathcal{F}^\tau$. Notice first that in this case $\mathcal{F}^\sigma, \mathcal{F}^\tau$ have to be unique extensions of $\mathcal{F}$ (by A_σ, A_τ respectively). Therefore, $\mathcal{F}$ has a unique extension in $\mathfrak{A}(A_\sigma, A_\tau)$ and $Z^1_\sigma \cap Z^1_\tau = A_{\sigma\cap\tau} \cap U_1$ belongs to this extension. But $A_{\sigma\cap\tau} \cap U_1 \in \mathfrak{A}$ and, again, $A \in \mathcal{F}$.

Similar methods are used to prove that for every element $\mathcal{F}$ of R_α not belonging to L_α we have $A \notin \mathcal{F}$. Hence, A separates L_α in R_α, a contradiction.

It follows that there is $\sigma \in \Sigma$ such that $\mathfrak{A}(A_\sigma)$ does not separate $L'_\alpha = L^\sigma_\alpha$ in $R'_\alpha = R^\sigma_\alpha$ for $\alpha < \kappa$; σ is infinite and co–infinite.

■

Proof. (of Theorem 5.5) Let $\rho\colon \mathfrak{c} \to \mathfrak{c} \times \mathfrak{c}$ be a surjection such that if $\rho(\alpha) = (\gamma, \beta)$ then $\gamma \leq \alpha$ and $\rho(0) = (0, 0)$. We construct an increasing sequence of Boolean algebras $(\mathfrak{A}_\alpha)_{\alpha\in\mathfrak{c}}$ each of size less than $\mathfrak{c}$. For every $\xi < \mathfrak{c}$ fix

- an enumeration $\{\mathcal{A}^\xi_\beta\colon \beta < \mathfrak{c}\}$ of disjoint sequences in $\mathfrak{A}_\xi$;
- an enumeration $\{S^\xi_\beta\colon \beta < \mathfrak{c}\}$ of disjoint sequences of G_δ sets in $\mathrm{Stone}(\mathfrak{A}_\xi)$.

Let $\mathfrak{A}_0 = \mathrm{Clopen}(2^\omega)$. Fix $R^0_0(0)$ to be an extension of S^0_0 in $\mathfrak{A}_0$ and $L^0_0(0)$ to be some non–separated subsequence.

Assume that $\mathfrak{A}_\alpha$ is constructed and we have a family $\{(L^\xi_\delta(\alpha), R^\xi_\delta(\alpha))\colon (\xi,\delta)=\rho(\eta), \eta < \alpha\}$ of sequences of ultrafilters and their non–separated subsequences. Let $\rho(\alpha) = (\gamma,\beta)$. Define $R^\gamma_\beta(\alpha)$ to be an extension of S^γ_β in $\mathfrak{A}_\alpha$. Fix a subsequence $L^\gamma_\beta(\alpha)$ non–separated by $\mathfrak{A}_\alpha$ (such a subsequence exists since $|\mathfrak{A}_\alpha| < \mathfrak{c}$). Apply Lemma 5.8 to the sequence $\mathcal{A}^\gamma_\beta$ and to $\{(L^\xi_\delta(\alpha), R^\xi_\delta(\alpha))\colon (\xi,\delta)=\rho(\eta), \eta \le \alpha\}$ to produce $\mathfrak{A}_{\alpha+1}$. Let $L^\xi_\delta(\alpha+1) = L'_{\rho^{-1}(\xi,\delta)}$ and $R^\xi_\delta(\alpha+1) = R'_{\rho^{-1}(\xi,\delta)}$ for every pair (ξ,δ) such that there is $\eta \le \alpha$ and $\rho(\eta) = (\xi,\delta)$.

On a limit step α take $\mathfrak{A}_\alpha = \bigcup_{\beta<\alpha}\mathfrak{A}_\beta$. Set $R^\xi_\delta(\alpha)$ to be the unique extension of all $R^\xi_\delta(\beta)$ and $L^\xi_\delta(\alpha)$ to be the unique extension of all $L^\xi_\delta(\beta)$ for $\beta < \alpha$ and pair (ξ,δ) such that there is $\eta < \alpha$ and $\rho(\eta) = (\xi,\delta)$. It is easy to see that in this way the limit steps preserve the property that L^ξ_δ is non–separated in R^ξ_δ.

Finally, let $\mathfrak{A} = \bigcup_{\alpha<\mathfrak{c}}\mathfrak{A}_\alpha$ and $K = \mathrm{Stone}(\mathfrak{A})$. We demonstrate that K satisfies all the required conditions.

Indeed, it is easy to see that K has SCP (and, therefore, is Grothendieck). If $\mathcal{A} = \{A_n\colon n\in\omega\}$ is a disjoint sequence in $\mathfrak{A}$ then there is $\alpha < \mathfrak{c}$ such that $A_n \in \mathfrak{A}_\alpha$ for every n. It is then enumerated as $\mathcal{A}^\alpha_\beta$ for some β and, thus, $\bigcup_{n\in N} A_n$ is added at step $\rho^{-1}(\alpha,\beta)$, for some infinite N.

Similarly, consider a disjoint sequence $(F_n)_{n\in\omega}$ of closed G_δ sets together with fixed countable bases. Since the cofinality of $\mathfrak{c}$ is uncountable all elements of these bases appear in $\mathfrak{A}_\alpha$ for some $\alpha < \mathfrak{c}$. The sequence $(F_n|\ \mathfrak{A}_\alpha)_{n\in\omega}$ is labeled as R^α_β for some β. From that point using Fact 5.7 we bother to keep L^α_β not separated in R^α_β. Therefore, $(F_n)_{n\in\omega}$ is not a copy of $\beta\omega$.

If we assume (*) then for every $\xi < \mathfrak{c}$ the set of disjoint sequences of closed sets in $\mathfrak{A}_\xi$ is of cardinality $\mathfrak{c}$. Therefore, for every $\xi < \mathfrak{c}$ we can think about $\{S^\xi_\beta\colon \beta<\mathfrak{c}\}$ as being an enumeration of disjoint sequences of closed sets in $\mathfrak{A}_\xi$. Since (*) implies also that $\mathfrak{c}$ is regular, the above proof shows that K does not contain copies of $\beta\omega$ consisting of $G_{<\mathfrak{c}}$ sets.

■

By Argyros's theorem (see [16]), every Boolean algebra with SCP contains an independent sequence of size ω_1, so K from the above theorem is not Koppelberg compact and, what is more important, under CH $\beta\omega$ is embeddable in K. Therefore, one cannot hope that the above example will turn out to be a Efimov space in ZFC.

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