Note on orderings on ideals and some cardinal coefficients

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Abstract

This is *a leftover note* containing some facts and proofs which were excluded from [2]. We present here several facts concerning some orderings on ideals on ω and cardinal invariants connected to these orderings.

1 Orderings on ideals

All unexplained terminology is in [2] and [1]. Recall some classical partial orderings on the family of ideals (on ω).

(RB) *Rudin-Blass order*: $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$ if there is a finite–to–one $f\colon \omega \to \omega$ such that $A \in \mathcal{I} \Leftrightarrow f^{-1}[A] \in \mathcal{J}$,

(RK) *Rudin-Keisler order*: $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$ if there is an $f\colon \omega \to \omega$ such that $A \in \mathcal{I} \Leftrightarrow f^{-1}[A] \in \mathcal{J}$,

(KB) *Katětov-Blass order*: $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$ if there is a finite–to–one $f\colon \omega \to \omega$ such that $A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}$,

(K) *Katětov order*: $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ if there is an $f\colon \omega \to \omega$ such that $A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}$.

Clearly we have the following diagram of implications between these orders:

$$\begin{tikzcd}
\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J} \arrow[r] & \mathcal{I} \leq_{\mathrm{K}} \mathcal{J} \\
\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J} \arrow[r] \arrow[u] & \mathcal{I} \leq_{\mathrm{RK}} \mathcal{J} \arrow[u]
\end{tikzcd}$$

Of course, we can use these orders for filters as well, for example $\mathcal{F} \leq_{\mathrm{RB}} \mathcal{G}$ iff $\mathcal{F}^* \leq_{\mathrm{RB}} \mathcal{G}^*$. Several deep results were proved about these partial orders (see e.g. [5]).

Definition 1.1 *A filter $\mathcal{F}$ on ω is feeble if there is a finite-to-one function $f\colon \omega \to \omega$ such that $f''[\mathcal{F}]$ is the Frechet filter* Fin^**, i.e.* $\mathrm{Fin} \leq_{\mathrm{RB}} \mathcal{F}^*$.

The theorem due to Jalali–Naini and Talagrand (see e.g. [9]) implies that every analytic filter is feeble, so there is a lot of non–trivial feeble filters.

Fact 1.2 *If $\mathcal{I}$ is an analytic P–ideal, then $\mathcal{I}^*$ is feeble.*

We investigate three orders stronger than the Katětov order:

(pm) $\mathcal{I} \leq_{\mathrm{pm}} \mathcal{J}$ if there is a permutation $f\colon \omega \to \omega$ such that $A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}$,

(1-1) $\mathcal{I} \leq_{1-1} \mathcal{J}$ if there is a one–to–one $f\colon \omega \to \omega$ such that $A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}$,

(s.i.) $\mathcal{I} \leq_{\mathrm{s.i.}} \mathcal{J}$ if there is a strictly increasing $f\colon \omega \to \omega$ such that $A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}$.

Clearly, $\mathcal{I} \leq_{\mathrm{pm}} \mathcal{J}$ means that $\mathcal{I}$ can be permuted into $\mathcal{J}$ (by f^{-1}). We can give alternative definitions of $\leq_{1-1}$ and $\leq_{\mathrm{s.i.}}$. Assume $\mathcal{I}$ is an ideal on ω. Then we will use the following notations:

- For a set $X \in \mathcal{I}^+$ let $\mathcal{I} \restriction X = \{A \subseteq \omega : A \cap X \in \mathcal{I}\}$.
- For a set $X = \{x_0 < x_1 < \dots\} \in [\omega]^\omega$, *the copy of $\mathcal{I}$ on X*:

  $$\mathcal{I}(X) = \{A \subseteq \omega : \{n \in \omega\colon x_n \in A\} \in \mathcal{I}\}.$$

  Analogously we can define the copy of a filter on an infinite set X.
- For an injective sequence $\bar{x} = \langle x_n : n \in \omega\rangle \in \omega^\omega$, *the copy of $\mathcal{I}$ on $\bar{x}$*:

  $$\mathcal{I}(\bar{x}) = \{A \subseteq \omega : \{n \in \omega : x_n \in A\} \in \mathcal{I}\}.$$

By the definitions both $\mathcal{I} \leq_{\mathrm{pm}} \mathcal{J}$ and $\mathcal{I} \leq_{\mathrm{s.i.}} \mathcal{J}$ implies $\mathcal{I} \leq_{1-1} \mathcal{J}$ which implies $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$. What about other implications?

Proposition 1.3 *The following hold for any ideal $\mathcal{I}$ containing all finite sets:*

(i) Fin $\leq_{\mathrm{pm}} \mathcal{J}$ *and* Fin $\leq_{\mathrm{s.i.}} \mathcal{J}$ *(so* Fin $\leq_{1-1} \mathcal{J}$ *as well).*

(ii) $\mathcal{I} \leq_{\mathrm{pm}}$ Fin *if and only if* $\mathcal{J} = $ Fin.

(iii) If $\sqsubseteq\ \in \{\leq_{\mathrm{s.i.}}, \leq_{1-1}, \leq_{\mathrm{KB}}, \leq_{\mathrm{K}}, \leq_{\mathrm{RB}}, \leq_{\mathrm{RK}}\}$ *then* $\mathcal{I} \sqsubseteq$ Fin *if and only if $\mathcal{I}$ is not tall.*

(iv) If $\mathcal{J}$ strictly extends Fin *then $\mathcal{I} \leq_{1-1} \mathcal{J}$ if and only if $\mathcal{I} \leq_{\mathrm{pm}} \mathcal{J}$.*

(v) If $\mathcal{J}$ strictly extends Fin *and $\mathcal{I} \leq_{\mathrm{s.i.}} \mathcal{J}$ then $\mathcal{I} \leq_{\mathrm{pm}} \mathcal{J}$.*

Proof. Of these (i) and (ii) are trivial. (iii) is a consequence of the easy fact that if $\mathcal{I}$ is not tall then $\mathcal{I} \leq_{\mathrm{RB}}$ Fin and if $\mathcal{I} \leq_{\mathrm{K}}$ Fin then $\mathcal{I}$ is not tall.
We will prove (iv) which implies (v). Assume f shows $\mathcal{I} \leq_{1-1} \mathcal{J}$. We can modify f on an infinite element A of $\mathcal{J}$ to be a permutation g such that $g \restriction (\omega \setminus A) \equiv f \restriction (\omega \setminus A)$ and $g[A] = f[A] \cup (\omega \setminus \mathrm{ran}(f))$. Then g shows $\mathcal{I} \leq_{\mathrm{pm}} \mathcal{J}$.
■

Now we present an example of two ideals $\mathcal{I}$, $\mathcal{J}$, such that $\mathcal{I} \leq_{\text{pm}} \mathcal{J}$ but $\mathcal{I} \not\leq_{\text{s.i.}} \mathcal{J}$. It follows that the $\leq_{\text{pm}}$ ordering is strictly stronger than s.i..

Example. Let (k_n) be a sequence of natural numbers such that $k_1 = 1$ and $k_{n+1} \geq n\, k_n$. Let f be the function defined by

$$f(k_n + l) = k_{n+1} - (l+1)$$

for $n \in \omega$ and $0 \leq l < k_{n+1}$. Loosely speaking f is the permutation inverting the order on intervals $[k_n, k_{n+1})$. Denote $\mathcal{I} = f[\mathcal{Z}]$. Clearly, $\mathcal{I} \leq_{\text{pm}} \mathcal{Z}$.
For a set $X \subseteq \omega$ let

$$d_*(X) = \liminf_{n \to \infty} \frac{|X \cap n|}{n}.$$

Of course $X \in \mathcal{Z}^*$ if and only if $d_*(X) = 1$.

Proposition 1.4 *If the ideal $\mathcal{I}$ is defined as above then $\mathcal{I} \not\leq_{\text{s.i.}} \mathcal{Z}$.*

Proof. Suppose *a contrario* that $P \subseteq \omega$ is a $\mathcal{I}$–intersection of f"$[\mathcal{Z}^*]$. Denote by $P_n = P \cap [k_n, k_{n+1})$.

CLAIM. The function defined by

$$g(n) = \frac{|P_n|}{|P \cap k_n|}$$

is unbounded.

First, notice that there is an $\varepsilon > 0$ such that for infinitely many n

$$\frac{|P_n|}{k_n} > \varepsilon.$$

Indeed, it is not hard to see that otherwise $I = f^{-1}[P] \in \mathcal{Z}$, and thus $f[\omega \setminus I] \in f$"$[\mathcal{F}^*]$ and $f[\omega \setminus I] \cap P = \emptyset$, so P cannot be an $\mathcal{I}$–intersection of f"$[\mathcal{Z}^*]$. Now, suppose for a contradiction that there is $C > 0$ such that $g(n) \leq C$ for every n. Therefore,

$$\varepsilon < \frac{|P_n|}{k_n} \leq \frac{C|P \cap k_n|}{k_n}$$

for infinitely many n. So, there is $\varepsilon' = \varepsilon / C$ such that for infinitely many n

$$\frac{|P \cap k_n|}{k_n} > \varepsilon'.$$

Therefore

$$\varepsilon' < \frac{|P \cap k_{n+1}|}{k_{n+1}} \leq \frac{|P_n| + k_n}{k_{n+1}} \leq \frac{C|P \cap k_n|}{k_{n+1}} + \frac{1}{n} \leq \frac{C\ k_n}{k_{n+1}} + \frac{1}{n} \leq \frac{C+1}{n},$$

for infinitely many n, a contradiction which proves the claim.

Now we are ready to construct a set $F \in \mathcal{Z}^*$ which will witness that P is not an $\mathcal{I}$–intersection of $f"[\mathcal{Z}^*]$. There is a sequence l_n such that $g(l_n) \geq n$. Without loss of generality we will assume that $g(n) \geq n$. Let F_n be the first $|P \cap k_n|$ many elements of P_n and define

$$F = \omega \setminus \bigcup_n f^{-1}[F_n].$$

We will show that $F \in \mathcal{Z}^*$ but $f[F] \notin \mathcal{Z}^*(P)$.

Notice that if $k_n \leq i < k_{n+1}$, then

$$\frac{|F \cap i|}{i} \geq \frac{k_n + (n-2)|P \cap k_n|}{k_n + n|P \cap k_n|} \geq 1 - \frac{2|P \cap k_n|}{k_n + nb_n} \geq 1 - \frac{2}{1+n}.$$

So, if $i > k_n$ then

$$\frac{|F \cap i|}{i} \geq 1 - \frac{2}{1+n}$$

and thus $F \in \mathcal{Z}^*$.

Now, denote $X = f[F]$. We will show that for every n there is $k_n \leq x < k_{n+1}$ such that $x = p_j$ for some j and

$$\frac{|\{i < j \colon p_i \in X\}|}{j} < \frac{1}{2}.$$

Indeed, let $n > 1$ and let x be the first element of $P_n \setminus F_n$. Since $x \in P$, there is $j \in \omega$ such that $x = p_j$. Then

$$\frac{|\{i < j \colon p_i \in X\}|}{j} \leq \frac{|P \cap k_n|}{|P \cap k_n| + |F_n|} \leq \frac{|P \cap k_n|}{2|P \cap k_n|} = \frac{1}{2},$$

so

$$\frac{|\{i < n \colon p_i \in X\}|}{n} < \frac{1}{2}$$

for infinitely many n. Hence, $X \notin \mathcal{Z}^*(P)$.

■

2 Characters of filters and orderings

The *character of a filter* $\mathcal{F}$, denoted by $\chi(\mathcal{F})$ is the minimal cardinality of a family generating $\mathcal{F}$. Similarly, the character of an ideal $\mathcal{I}$ is the character of its dual filter. The following theorem reveals some properties of the characters of non-feeble filters.

Theorem 2.1 (R. C. Solomon [7] and P. Simon [unpublished]) *If a filter has character less then $\mathfrak{b}$ then it is feeble but there is a non-feeble filter generated by $\mathfrak{b}$ sets.*

Now, we present a natural way of associating cardinal coefficients to partial orders on ideals. However, we will focus only on few of them.

Definition 2.2 *Let $\mathcal{F}$ be a family with SFIP (or simply a filter) and assume that $\mathcal{I}$ is an ideal. We say that $X = \{x_0 < x_1 < \ldots\} \in [\omega]^\omega$ is an $\mathcal{I}$-intersection of $\mathcal{F}$ if $\omega \setminus F \in \mathcal{I}(X)$ for each $F \in \mathcal{F}$; and an injective sequence $\bar{x} = \langle x_n : n \in \omega \rangle \in \omega^\omega$ is a weak–$\mathcal{I}$–intersection of $\mathcal{F}$ if $\omega \setminus F \in \mathcal{I}(\bar{x})$ for each $F \in \mathcal{F}$.*

In other words, a set X is an $\mathcal{I}$–intersection of $\mathcal{F}$ if every element of $\mathcal{F}$ is in the copy of $\mathcal{I}^*$ on X. Similarly, a set X is a weak–$\mathcal{I}$–intersection of $\mathcal{F}$ if we can reorder the elements of X in such a way that elements of $\mathcal{F}$ are in the copy of $\mathcal{I}^*$ on the rearranged X. It is trivial to check that $\mathcal{F}$ has an $\mathcal{I}$–intersection (a weak–$\mathcal{I}$–intersection) iff $\mathcal{F}^* \leq_{\mathrm{s.i.}} \mathcal{I}$ ($\mathcal{F}^* \leq_{1-1} \mathcal{I}$).
Notice that in the case of $\mathcal{I}$ generated by Fin and one infinite co–infinite set both of the above notions coincide with the notion of the pseudo-intersection of $\mathcal{F}$.

Definition 2.3 *Let $\sqsubseteq$ be a partial order on ideals. Then the $\sqsubseteq$-character of $\mathcal{I}$:*

$$\chi_{\sqsubseteq}(\mathcal{I}) = \min\{\chi(\mathcal{J}) : \mathcal{J} \not\sqsubseteq \mathcal{I}\}.$$

The $\sqsubseteq$-character number: $\mathfrak{p}(\sqsubseteq) = \sup\{\chi_{\sqsubseteq}(\mathcal{I}) : \mathcal{I}$ is an ideal on $\omega\}$.

Of course, these cardinals are not necessarily defined. We will abbreviate our notation. For example, we will write $\chi_{\mathrm{pm}}(\mathcal{I})$ instead of $\chi_{\leq_{\mathrm{pm}}}(\mathcal{I})$ and $\mathfrak{p}(\mathrm{pm})$ instead of $\mathfrak{p}(\leq_{\mathrm{pm}})$. Similarly we have $\chi_{1-1}(\mathcal{I})$ and $\mathfrak{p}(1-1)$, $\chi_{\mathrm{s.i.}}(\mathcal{I})$ and $\mathfrak{p}(\mathrm{s.i.})$, $\chi_{\mathrm{KB}}(\mathcal{I})$ and $\mathfrak{p}(\mathrm{KB})$, and so on.
Clearly, $\chi_{\mathrm{s.i.}}(\mathcal{I})$ ($\chi_{1-1}(\mathcal{I})$) is the smallest cardinality of a family $\mathcal{F}$ with SFIP but without a (weak–)$\mathcal{I}$–intersections. Furthermore $\chi_{\mathrm{pm}}(\mathrm{Fin}) = 1$, and if $\sqsubseteq \in \{\leq_{\mathrm{s.i.}}, \leq_{1-1}, \leq_{\mathrm{KB}}, \leq_{\mathrm{K}}, \leq_{\mathrm{RB}}, \leq_{\mathrm{RK}}\}$ then $\chi_{\sqsubseteq}(\mathrm{Fin}) = \mathfrak{p}$.
The cardinal $\mathfrak{p}(\mathrm{pm})$ is the smallest cardinal κ such that there is no ideal containing up to permutation all ideals generated by at most κ many elements.
The following facts are trivial consequences of the definitions:

Fact 2.4 *For ideals $\mathcal{I}$, $\mathcal{J}$ containing all finite sets the following facts hold*

(i) If $\mathcal{I} \sqsubseteq \mathcal{J}$ and $\chi_{\sqsubseteq}(\mathcal{J})$ is well-defined, then $\chi_{\sqsubseteq}(\mathcal{I})$ is also well-defined and $\chi_{\sqsubseteq}(\mathcal{I}) \leq \chi_{\sqsubseteq}(\mathcal{J})$.

(ii) If $\mathcal{I} \sqsubseteq_0 \mathcal{J}$ implies $\mathcal{I} \sqsubseteq_1 \mathcal{J}$ for each $\mathcal{I}$ and $\chi_{\sqsubseteq_1}(\mathcal{J})$ is well-defined, then $\chi_{\sqsubseteq_0}(\mathcal{J})$ is also well-defined and $\chi_{\sqsubseteq_0}(\mathcal{J}) \leq \chi_{\sqsubseteq_1}(\mathcal{J})$.

(iii) $\chi_{\mathrm{K}}(\mathcal{I})$ is well–defined for each ideal $\mathcal{I}$, since otherwise $\mathcal{I}$ would be a $\leq_K$–maximal ideal, which is impossible (since $\mathcal{I} \lneq_K \mathcal{I} \times \mathcal{I}$, where "$\times$" stands for the Fubini product, see [5]). So, all the cardinal invariants mentioned above are well–defined.

Corollary 2.5 *If $\mathcal{I}$ strictly extends* Fin*, then $\mathfrak{p} \leq \chi_{\mathrm{s.i.}}(\mathcal{I}) \leq \chi_{\mathrm{pm}}(\mathcal{I}) = \chi_{1-1}(\mathcal{I})$ so $\mathfrak{p} \leq \mathfrak{p}(\mathrm{s.i.}) \leq \mathfrak{p}(\mathrm{pm}) = \mathfrak{p}(1-1)$. In a diagram for $\mathcal{I}$ strictly extending* Fin*:*

$$\begin{tikzcd}
\chi_{\mathrm{s.i.}}(\mathcal{I}) \arrow[r] & \chi_{\mathrm{pm}}(\mathcal{I}) = \chi_{1-1}(\mathcal{I}) \arrow[r] & \chi_{\mathrm{KB}}(\mathcal{I}) \arrow[r] & \chi_{\mathrm{K}}(\mathcal{I}) \\
\mathfrak{p} \arrow[u] & & \chi_{\mathrm{RB}}(\mathcal{I}) \arrow[u] \arrow[r] & \chi_{\mathrm{RK}}(\mathcal{I}) \arrow[u]
\end{tikzcd}$$

where one should read $a \to b$ as $a \leq b$.

In fact, Theorem 2.1 implies that for many ideals $\chi_{\mathrm{RB}}(\mathcal{I}) = \chi_{\mathrm{RK}}(\mathcal{I}) \leq \omega$. In case of this orderings the following definition makes more sense:

$$\chi_{\sqsupseteq}(\mathcal{I}) = \min\{\chi(\mathcal{J}) : \mathcal{J} \not\sqsupseteq \mathcal{I}\}.$$

E.g. Theorem 2.1 can be expressed as $\chi_{\geq_{\mathrm{RB}}}(\mathrm{Fin}) = \mathfrak{b}$. However, in what follows we will concentrate on $\chi_{s.i.}$ and χ_{pm}.
We have an easy upper bound for $\chi_{\mathrm{pm}}(\mathcal{I})$ for analytic $\mathcal{I}$.

Proposition 2.6 *If $\mathcal{I}$ is an analytic ideal, then $\chi_{\mathrm{pm}}(\mathcal{I}) \leq \mathfrak{b}$.*

We will see later that the assumption on definability of the ideal is necessary here.
Recall that for a tall ideal $\mathcal{I}$ the coefficient $\mathsf{add}^*(\mathcal{I})$ denotes the minimal cardinality of a family in $\mathcal{I}^*$ without a pseudo–intersection from $\mathcal{I}^*$. If $\mathfrak{p} < \mathsf{add}^*(\mathcal{I})$, then a filter witnessing $\mathfrak{p}$ cannot have a weak–$\mathcal{I}$–intersection. Therefore

Proposition 2.7 *If $\mathfrak{p} < \mathsf{add}^*(\mathcal{I})$ then $\chi_{\mathrm{pm}}(\mathcal{I}) = \mathfrak{p}$.*

It follows that under $\mathfrak{p} < \mathsf{add}^*(\mathcal{I})$ and $\mathfrak{p} < \mathfrak{b}$ there is a filter which is feeble but does not have a weak–$\mathcal{I}$–intersection.
We finish this section by an example of an ideal $\mathcal{I}$ such that $\mathcal{I} \leq_{\mathrm{pm}} \mathcal{Z}$ but $\mathcal{I} \not\leq_{s.i.} \mathcal{Z}$. The existence of this ideal implies that the property of possessing an $\mathcal{I}$–intersection is not closed under permutations and $\leq_{s.i.}$ is strictly stronger than $\leq_{\mathrm{pm}}$.

Problem 2.8 *Is it true that $\chi_{\mathrm{pm}}(\mathcal{I}) = \chi_{\mathrm{s.i.}}(\mathcal{I})$ for every ideal $\mathcal{I}$ strictly extending* Fin *or at least, does it hold for a large family of ideals? Is $\mathfrak{p}(\mathrm{pm}) = \mathfrak{p}(\mathrm{s.i.})$?*

3 Universal filters

It will be convenient to introduce the following notation.

Definition 3.1 *A filter $\mathcal{G}$ on ω is κ–universal if $\mathcal{F} \leq_{\mathrm{pm}} \mathcal{G}$ for every $\mathcal{F}$ of character κ.*

Clearly,

$$\chi_{pm}(\mathcal{I}) = \min\{\kappa\colon \mathcal{I}^* \text{ is not } \kappa-universal\}.$$

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