On measures on Polish spaces and on Boolean algebras

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Contents

1 Introduction

In this dissertation we explore properties of finite measures on Polish spaces and on a certain class of Boolean algebras. The thesis basically consists of two rather independent parts, of which the first is contained in Borodulin-Nadzieja, Plebanek [6] and the other in Borodulin-Nadzieja [7].

Although most of the results presented here concern measures, the methods used are more of set theoretic and topological provenience. It is essentially because the questions motivating our considerations come from these fields.

For example Chapter 2 inspired by problems posed by David Fremlin can be seen as an attempt to answer, how complicated the structure of Borel subsets of $[0,1]$ can be. Chapter 3 investigates the class of minimally generated Boolean algebras, a well–known tool for constructing peculiar small compact spaces. And again, by examining measures on this class of Boolean algebras we hope to understand minimal generation better, and thus to understand better small compact spaces.

Even problems from "pure" measure theory sometimes turn out to be undecidable within the standard set theoretic axioms. Perhaps this is the case of Problem (FN) formulated in Chapter 2, which has a positive answer under CH. One can have an impression that in other models of set theory the answer can be negative or, at least, the problem requires some subtle set theoretic techniques. Anyway, Chapter 2 proves that the set theoretic approach solve the problem partially.

The main results of the thesis are:

- Theorem 2.2.1, which points out a condition sufficient for the countable compactness of measure. It is used in following sections, but it might be of independent interest (it is cited by Fremlin in [17]);
- Theorem 2.4.1, stating that a measure defined on an uncountable product of Baire spaces is regularly monocompact if only it is inner regular with respect to zero sets;
- Theorem 2.5.2(a), which says that every measure on a sub–σ–algebra of Borel$[0,1]$ is an image of regularly monocompact measure and Theorem 2.5.2(b), which considerably generalizes one of Fremlin's theorems;

- the proof of Fremlin's Theorem 2.6.2, which says that every measure on a sub–σ–algebra of Borel$[0,1]$ is weakly–α–favourable; our proof has the advantage of showing directly the winning strategy;

- Theorem 3.2.7, which is a kind of dichotomy, connecting the existence of nonseparable measures with the existence of uniformly regular measures;

- Theorem 3.2.10, which says that measures on minimally generated Boolean algebras are "small". Combined with Theorem 3.2.17 it gives quite precise description of measures on this class of Boolean algebras;

- Theorem 4.1.2 and Theorem 4.1.5, demonstrating that even under Martin's Axiom some Efimov–like spaces can be constructed.

The thesis is divided into three chapters. Chapter 2 discusses problems related to countably compact measures. In Section 2.1 we introduce basic facts and definitions. Section 2.2 provides two useful facts, used in the following sections. In Sections 2.3, 2.4 and 2.5 we present several theorems partially answering an open problem posed by Fremlin. Section 2.6 explores the connections between countably compact measures and infinite games.

Chapter 3 is devoted to the study of minimally generated Boolean algebras, the notion introduced by Sabine Koppelberg. Section 3.1 provides mainly an overview of known results. Moreover, the reader will find here some lemmas useful for the following section. Section 3.2 contains an extensive analysis of measures on minimally generated Boolean algebras.

Chapter 4 is closely related to the previous one. We use minimally generated Boolean algebras to construct compact spaces with additional properties. In Section 4.1 we discuss the Efimov problem and in Section 4.2 a problem motivated by the theory of Banach spaces.

Each of chapters is preceded by a short introduction containing a more detailed description of its content. Also, we finish every chapter with a brief discussion of open problems and of the significance of the presented results.

We use the standard set theoretic, topological and measure theoretic notation. For any unexplained terminology the reader is referred to [34, 25, 5]. It is worth noticing that all measures here are finite even if it is not stated explicitly. When defined on a Boolean algebra a measure is assumed to be finitely additive, not necessarily σ–additive. Some authors prefer to use the name "charge" or "finitely additive measure" instead of "measure" in such case. We decided to call it “measure” since we consider such functions only on Boolean algebras, and thus we can “extend” each of them uniquely to the σ–additive measure on the Stone space (see Section 3.2). It is always clear from the context which type of measure is meant.

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2 On compactness of measures on Polish spaces

In this chapter, based on [6], we present some results concerning countably compact measures. Most of them were born in an attempt to answer the following question, posed by David Fremlin in [24, 26].

(FN) *Let μ be a measure defined on a σ–algebra $\Sigma \subseteq \mathrm{Borel}(X)$, where X is a Polish space. Is μ countably compact?*

Section 2.1 contains basic definitions and several historical remarks. We show here some lemmas useful for the following sections and, for completeness, two classical theorems: Ryll–Nardzewski's result establishing the connection between countably compact measures and perfect measures and Marczewski's example of a sub–σ–algebra of $\mathrm{Borel}[0,1]$ carrying a measure of Maharam type $\mathfrak{c}$.

In Section 2.2 we give two technical results helpful for constructing countably compact families. The following sections discuss properties of finite measures μ defined on $\Sigma \subseteq \mathrm{Borel}(X)$, where X is Polish.

So in Section 2.3 we prove that a measure μ like in Problem (FN) is countably compact under the additional assumption that μ is inner regular with respect to closed sets from Σ. David Fremlin remarked (in a private communication) that the above result in fact follows from the theorem due to Aldaz & Render [1]. Our proof of 2.3.2 has the advantage that it also gives a description of a countably compact family which approximates the measure in question. In the following sections, building on the same idea, we obtain a common generalization of the above result and Fremlin's theorem. Fremlin proved that every ω_1–generated sub–σ–algebra of $\mathrm{Borel}[0,1]$ is countably compact. We show that the assumption of ω_1–generation can be considerably weakened here (see Corollary 2.4.3, and Corollary 2.5.2).

In [22] Fremlin introduced the infinite game related to regularity properties of measures. It allowed to distinguish new interesting subclasses of perfect measures: weakly–α–favourable and α–favourable (see Section 2.6 for definitions and details). Fremlin proved that every measure as in Problem (FN) is weakly–α–favourable. It is not known if it is α–favourable. In Section 2.5 we show that every such measure is "an image" of some regularly monocompact measure; this result is based on a theorem from Section 2.4 on measures defined on uncountable products of Polish spaces. Regular monocompactness is a slightly weaker property than countable compactness but it is stronger than α–favourableness. Unfortunately, it is not clear if it is preserved by inverse–measure–preserving functions.

Finally, in Section 2.6 we give an alternate proof of the result mentioned above, stating that every μ as in question is weakly–α–favourable. Our proof is much simpler and, in a sense, more natural than the original one.

2.1 Preliminaries

In this section we consider only finite and σ–additive measures; concerning regularity properties of measures we follow the terminology of Fremlin [22] (note that some properties have different names in other sources!). If $\mathcal{K}$ is a family of sets, then we say that $\mathcal{K}$ is

> *countably compact* if every sequence $(A_n)_{n\in\omega}$ from $\mathcal{K}$ with the finite intersection property satisfies $\bigcap_{n\in\omega} A_n \neq \emptyset$;
>
> *monocompact* if $\bigcap_{n\in\omega} A_n \neq \emptyset$ whenever $(A_n)_{n\in\omega}$ is a decreasing sequence of nonempty elements from $\mathcal{K}$.

If (X, Σ, μ) is a measure space and $\mathcal{K} \subseteq \Sigma$, then μ is said to be *inner regular with respect to* $\mathcal{K}$ if

$$\mu(A) = \sup\{\mu(K)\colon K \subseteq A, K \in \mathcal{K}\},$$

for every $A \in \Sigma$ (sometimes we say in this case that $\mathcal{K}$ *approximates* μ *on* Σ). A measure μ is *countably compact* (*regularly monocompact*) if it is inner regular with respect to some family $\mathcal{K} \subseteq \Sigma$ which is countably compact (monocompact, respectively). Every countably compact measure is monocompact; [22] provides an example of a monocompact but not countably compact measure.

It is a nontrivial result due to Pachl [42] that a countably compact measure μ defined on some Σ remains countably compact when restricted to any sub–σ–algebra $\Sigma_0 \subseteq \Sigma$, see also Fremlin [21]. It is worth recalling that both proofs of Pachl's result use some external characterizations of countable compactness — it is not clear how to explicitly define a suitable countably compact family inside Σ_0.

If (X, Σ, μ) and $(Y, \mathcal{A}, \nu)$ are measure spaces and $f\colon X \to Y$ is a measurable function, then we say that f is *inverse–measure–preserving* if $\nu(A) = \mu(f^{-1}[A])$ for $A \in \mathcal{A}$. It can be derived from Pachl's results (see e.g. the lemma below) that if there is a such function and μ is countably compact, then so is ν.

Assume that (X, Σ, μ) is a measure space and consider a function $f \colon X \to Y$. Then we can consider a measure μ' defined on $\Sigma' = \{E \subseteq Y \colon f^{-1}[E] \in \Sigma\}$ such that $\mu'(E) = \mu(f^{-1}[E])$ for every $E \in \Sigma'$. We call μ' *the image* of μ and denote it by μf^{-1}.

Consider now a function $f \colon X \to Y$ and a measure space (Y, Σ, μ). The algebra Σ induces a σ–algebra $\Sigma' = \{f^{-1}[E] : E \in \Sigma\}$ on X, and we can define on Σ' a measure μ' by $\mu'(f^{-1}[E]) = \mu(E)$, which might be called *the preimage* of μ. It will be useful to note the following simple fact.

Lemma 2.1.1 *Let μ' be the preimage of μ (as described above).*

(a) If μ is inner regular with respect to some $\mathcal{K}$, then μ' is inner regular with respect to $\mathcal{C} = \{f^{-1}[K] : K \in \mathcal{K}\}$.

(b) If μ' is inner regular with respect to some $\mathcal{C} \subseteq \Sigma'$, then μ is inner regular with respect to $\mathcal{K} = \{E \in \Sigma \colon f^{-1}[E] \in \mathcal{C}\}$.

(c) The measure μ' is countably compact (respectively, regularly monocompact) if and only if μ is countably compact (respectively, regularly monocompact).

Proof. (a) For a given set $f^{-1}[E] \in \Sigma'$ and $\varepsilon > 0$ we can find $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu(E \setminus K) < \varepsilon$. Since $f^{-1}[E] \setminus f^{-1}[K] \subseteq f^{-1}[E \setminus K]$, we have

$$\varepsilon > \mu(E \setminus K) = \mu'(f^{-1}[E \setminus K]) \geq \mu'(f^{-1}[E] \setminus f^{-1}[K]).$$

(b) Let $E \in \Sigma$ and $\varepsilon > 0$. We can find a set $C \in \mathcal{C}$ such that $C \subseteq f^{-1}[E]$ and $\mu'(C) > \mu'(E) - \varepsilon$. Then the set $K \in \mathcal{K}$ such that $C = f^{-1}[K]$ is a subset of E and we have $\mu(K) = \mu'(C) > \mu'(f^{-1}[E]) - \varepsilon = \mu(E) - \varepsilon$. This shows that μ is inner regular with respect to $\mathcal{K}$.

(c) It is easy to check that $\mathcal{K}$ is countably compact or monocompact if and only if $\mathcal{C}$ has an analogous property (here, $\mathcal{K}$ and $\mathcal{C}$ are as in (b)). ∎

The class of countably compact measures was introduced by Marczewski [40] under the name *compact measures*. In the abstract setting (i.e. without referring to topology), such a notion singles out measures which are nice in the sense that they resemble the Lebesgue measure. It is well–known that, for a Polish space X, every finite measure on Borel(X) is inner regular with respect to compact sets, so is countably compact. On the other hand, the

Lebesgue measure restricted to a non–measurable subset V is usually not countably compact (e.g. if V is a Vitali set).

The research of Marczewski was continued by Ryll–Nardzewski, who introduced a slightly weaker notion of quasi–compact measures (see [51]). It turned out that quasi–compactness is equivalent to perfectness, a notion introduced few years earlier by Gnedenko and Kolmogorov.

There are many equivalent ways in which the definition of a *perfect measure* is formulated in the literature. We will say that a measure (X, Σ, μ) is perfect if for every measurable function $f\colon X \to [0,1]$ there is a Borel subset B of $f[X]$ such that $\mu f^{-1}(B) = 1$. In fact, we can see perfectness as a regularity property: if (X, Σ, μ) is perfect, then for every measurable function $f\colon X \to [0,1]$ the measure μf^{-1} is inner regular with respect to Borel subsets of $[0,1]$.

We present here one of the results of Ryll–Nardzewski from [51]. The following theorem implies that every countably compact measure is perfect and, moreover, indicates that perfectness is more a local property, while compactness refers to the whole σ–algebra. It is worth mentioning that the following proof is well–known although it is not the original one from [51]. We use here the powerful tool of *a characteristic function of a family of sets* (sometimes called a *Marczewski function*) introduced by Marczewski in [38].

Theorem 2.1.2 (Ryll–Nardzewski) *A measure μ on a σ-algebra Σ is perfect if and only if it is countably compact on every σ–generated $\Sigma_0 \subseteq \Sigma$.*

Proof. We can replace $[0,1]$ in the definition of perfect measure by any uncountable Polish space. Here, we will replace it by $\{0,1\}^\omega$.

Assume that a measure (X, Σ, μ) is perfect and Σ is σ–generated. We will show that μ is countably compact. Fix a sequence of generators $\{E_n \colon n \in \omega\}$ of Σ. Consider the following function $f\colon X \to \{0,1\}^\omega$:

$$f(x) = (\chi_{E_0}(x), \chi_{E_1}(x), \ldots).$$

Notice that $\Sigma = \{f^{-1}[B] \colon B \in \mathrm{Borel}\{0,1\}^\omega\}$. Because μ is perfect, there is $B \in \mathrm{Borel}\{0,1\}^\omega$ such that $B \subseteq f[X]$ and $\mu f^{-1}(B) = 1$. The measure μf^{-1} is countably compact on $\mathrm{Borel}\{0,1\}^\omega$ (as every Borel measure is countably compact). Combining these facts and Lemma 2.1.1, we conclude that μ is countably compact on Σ.

Assume now that a measure (X, Σ, μ) is countably compact on every σ–generated $\Sigma_0 \subseteq \Sigma$. To check that μ is perfect consider an arbitrary measurable function $f\colon X \to \{0,1\}^\omega$. Let $\Sigma_0 = \{f^{-1}[B] \colon B \in \mathrm{Borel}\{0,1\}^\omega\}$. It is easy to see that Σ_0 forms a σ–generated sub–σ–algebra of Σ. Therefore, μ is countably compact on Σ_0. Denote by $\mathcal{K}$ the countably compact class approximating μ on Σ_0.

We will show that there is a Borel set $B \subseteq f[X]$ such that $\mu f^{-1}(B) = 1$. For $n \in \omega$ denote $C_n = \{x \in \{0,1\}^\omega \colon x(n) = 1\}$. Fix $\varepsilon > 0$ and for $n \in \omega$ find $K_n^0, K_n^1 \in \mathcal{K}$ such that

$$K_n^0 \subseteq f^{-1}[C_n] \text{ and } \mu(f^{-1}[C_n] \setminus K_n^0) < \varepsilon/2^{n+1},$$

$$K_n^1 \subseteq f^{-1}[C_n^c] \text{ and } \mu(f^{-1}[C_n^c] \setminus K_n^1) < \varepsilon/2^{n+1}.$$

Let

$$K_\varepsilon = \bigcap_{n\in\omega} K_n^0 \cup K_n^1.$$

Consider the set $B_\varepsilon = f[K_\varepsilon]$. It is easy to check that $\mu f^{-1}(B_\varepsilon) \geq \mu(K_\varepsilon) \geq 1 - \varepsilon$. Of course $B_\varepsilon \subseteq f[X]$. We show that B_ε is closed. Let $y \in \{0,1\}^\omega$ be a condensation point of B_ε. There is $T \subseteq \omega$ such that

$$\{y\} = \bigcap_{n\in T} C_n \cap \bigcap_{n\in T^c} C_n^c.$$

Since y is a condensation point of B_ε, the family

$$\mathcal{K}_y = \{K_n^0 \colon n \in T\} \cup \{K_n^1 \colon n \in T^c\}$$

has the finite intersection property. As $\mathcal{K}$ is countably compact, there is $x \in \bigcap \mathcal{K}_y$. Obviously, $x \in K_\varepsilon$ and $f(x) = y$. Therefore, $y \in B_\varepsilon$ and, consequently, B_ε is closed.

Let $B = \bigcup_{n\in\omega} B_{1/n}$. It is Borel, $\mu f^{-1}(B) = 1$ and $B \subseteq f[X]$. Thus, μ is perfect.

■

Musiał [41] gave an example of a perfect measure which is not countably compact (the same result was announced earlier by Vinokurov). Under some mild set theoretic assumption there are even perfect measures which are not countably compact, and are of countable Maharam type (i.e. the underlying L_1 space is separable, see also Section 3.2), see Plebanek [46].

An extensive list of publications concerning countably compact and perfect measures can be found in Ramachandran's article [48] (see also [37]). This subject has been quite intensively studied up to the seventies but it has not been attracting so much attention over next years.

In 2000 Fremlin published the paper [22], where he explores several other subclasses of perfect measures. It was kind of turning point in the sense that some new areas for investigation have been opened. Fremlin's paper presents several subtle results on properties of measures related to infinite games. Most of them require advanced set theoretic techniques. We discuss some of his ideas in Section 2.6.

In [24, 26] Fremlin asked explicitly the following natural question.

(FN) *Let μ be a measure defined on a σ–algebra $\Sigma \subseteq \mathrm{Borel}[0,1]$. Is μ countably compact?*

As we mentioned above every Borel measure on a Polish space is countably compact. It is also of countable Maharam type (see Section 3.2 for more details on the notion of Maharam types). Measures defined on some $\Sigma \subseteq \mathrm{Borel}(X)$, however, can be more complicated as the following theorem shows. The proof presented below is not the original one by Marczewski (see [38]) but it is also well–known.

Theorem 2.1.3 (Marczewski) *The family of Borel subsets of $[0,1]$ contains a σ–independent family of size $\mathfrak{c}$. This family generates a σ–algebra Σ which carries a measure μ of Maharam type $\mathfrak{c}$.*

Proof. Denote $C = \{0,1\}^\omega$. We will look for the family with the desired properties in C^ω instead of $[0,1]$. We can do so without loss of generality since these two spaces are Borel isomorphic. Denote this isomorphism by $\phi\colon [0,1] \to C^\omega$. For a $c \in C$ let

$$Z_c = \{x \in C^\omega \colon \exists n\ x(n) = c\}.$$

Consider disjoint sequences $(s_n)_{n\in\omega}$, $(t_n)_{n\in\omega}$ of elements of C. Notice that if $x = (s_0, s_1, s_2, \ldots)$, then

$$x \in \bigcap_{n\in\omega} Z_{s_n} \cap \bigcap_{n\in\omega} Z^c_{t_n}.$$

Therefore, the family $\{Z_c \colon c \in C\}$ is σ–independent. Of course, it is of cardinality $\mathfrak{c}$. It consists of Borel sets since for every $c \in C$ the formula "$\exists n\ x(n) = c$" is Borel (Σ^0_1, in fact).

Let Σ be the σ–algebra generated by the family $\{Z_c \colon c \in C\}$. There is a measure ν' such that $\nu'(Z_c) = 1/2$ for every $c \in C$. Indeed, consider a family of probability spaces $(C^\omega, \Sigma_c, \nu_c)$, where $\Sigma_c = \sigma(\{Z_c\})$ and $\nu_c(Z_c) = 1/2$.

Enumerate $C = \{c_\alpha \colon \alpha < \mathfrak{c}\}$. Let ν be the product measure on $\prod_{\alpha<\mathfrak{c}} C^\omega$, i.e. measure defined by

$$\nu(\prod_{\alpha<\beta} C^\omega \times A \times \prod_{\alpha>\beta} C^\omega) = \nu_{c_\beta}(A)$$

for every $\beta \in \mathfrak{c}$. Consider $\Delta = \{(c, c, \ldots) \colon c \in C^\omega\}$. Notice that Δ is homeomorphic to C^ω. Since $\{Z_c \colon c \in C\}$ is σ–independent, $\nu^*(\Delta) = 1$. Fix $\nu_0 = \nu|\Delta$.

For $s, t \in C$ such that $s \neq t$ we have

$$\nu_0(Z_s \triangle Z_t) = \nu_0(Z_s \cap Z_t^c) + \nu_0(Z_t \cap Z_s^c) = 1/2.$$

We see that $\{Z_c \colon c \in C\}$ cannot be approximated by less than $\mathfrak{c}$ many sets and, therefore, ν_0 is of Maharam type $\mathfrak{c}$. The measure $\mu = \nu_0\phi^{-1}$ fulfills the desired properties.

■

The above measure μ cannot be extended to Borel$[0,1]$ but, nevertheless, it is countably compact since the family $\{Z_c \colon c \in C\}$ is countably compact itself and it can be easily extended to a countably compact family approximating Σ with respect to μ.

If μ is a measure on $\Sigma \subseteq$ Borel$[0,1]$, then it is perfect, so countably compact whenever Σ is countably generated. In [24] Fremlin, based on his previous papers [19, 22] proved the following nontrivial generalization of this remark.

Theorem 2.1.4 (Fremlin) *If a σ–algebra $\Sigma \subseteq$ Borel(X), where X is a Polish space, is generated by ω_1 sets, then every finite measure on Σ is countably compact.*

It follows that under CH the problem FN has a positive solution. It is not known if FN can be resolved in ZFC. Let us remark that, under CH there is Σ built from Borel subsets of $[0,1]^2$ and a single non Borel set $\Delta \subseteq [0,1]^2$, which carries a perfect measure which is not countably compact, see Plebanek [46].

If (X, Σ, μ) is a measure space, then we denote by μ^* the corresponding outer measure. We repeatedly use the fact that μ^* is upward continuous, i.e. $\mu^*(\bigcup_n Z_n) = \lim \mu^*(Z_n)$ for an arbitrary sequence $Z_1 \subseteq Z_2 \subseteq \ldots \subseteq X$. It will be convenient to single out the following simple observation.

Lemma 2.1.5 *Let (X, Σ, μ) be a measure space and let $(Z_n)_n$ be an increasing sequence of arbitrary subsets of X with the union Z. For every $E \in \Sigma$ and $\varepsilon > 0$ there is a set $F \in \Sigma$ with $\mu(E \setminus F) < \varepsilon$, and a number $m \in \omega$ such that whenever $A \in \Sigma$, $A \subseteq F$, then $\mu^*(A \cap Z_m) = \mu^*(A \cap Z)$.*

Proof. Let $E \in \Sigma$ and $\varepsilon > 0$. Since outer measure is upward continuous we can find a number m such that $\mu^*(Z_m) > \mu^*(Z) - \varepsilon$. Let $F_1 \subseteq E$ be a measurable hull of the set $E \cap Z_m$ and F_2 be a measurable kernel of $E \cap Z^c$. Then for $F = F_1 \cup F_2$ we have $\mu(E \setminus F) < \varepsilon$, and $\mu^*(A \cap Z_m) = \mu(A \cap F_1) = \mu^*(A \cap Z)$ for every measurable $A \subseteq F$.

■

Given any measure space (X, Σ, μ), we say that a sequence $(E_n)_{n\in\omega}$ of measurable sets is *μ–centred* if $\mu(\bigcap_{k<n} E_k) > 0$ for every n.

2.2 Countably compact measures

We present in this section two auxiliary results on countably compact measures. A measure is countably compact if it is approximated by a countably compact family. We show that if we change this definition by replacing the countable compactness by slightly weaker conditions on approximating family, we will obtain the same class of measures. The first theorem is used directly in the proof of Theorem 2.3.1 below, while the second is related to game–theoretic properties of measures that are mentioned in Section 2.6.

Theorem 2.2.1 *Let (X, Σ, μ) be a measure space and suppose that $\mathcal{C} \subseteq \Sigma$ is a family such that the intersection of every μ–centred sequence $(F_n)_{n\in\omega}$ from $\mathcal{C}$ is not empty.*

If μ is inner regular with respect to $\mathcal{C}$, then μ is countably compact.

Proof. Let $\widehat{\Sigma}$ be the completion of Σ with respect to μ, and denote by $\mathfrak{A}$ the measure algebra of μ. For $A \in \Sigma$ we write $A^\bullet$ for the corresponding element of $\mathfrak{A}$. Let $\rho\colon \mathfrak{A} \to \widehat{\Sigma}$ be a lifting; i.e. ρ is a Boolean homomorphism such that $\rho(a)^\bullet = a$ for every $a \in \mathfrak{A}$ (see Fremlin's survey [18] for details).

We shall consider the family $\mathcal{C}'$ defined as follows

$$\mathcal{C}' = \left\{ \bigcap_{k\in\omega} F^k \;:\; F^k \in \mathcal{C},\; F^{k+1} \subseteq F^k \cap \rho(F^{k\bullet}) \text{ for every } k \right\}.$$

Let us check that μ is inner regular with respect to $\mathcal{C}'$. Take any set $F \in \mathcal{C}$ and $\varepsilon > 0$. We define a sequence of sets F^k from $\mathcal{C}$ in the following way. Put $F^1 = F$; if F^k is given, choose $F^{k+1} \in \mathcal{C}$ so that

$$F^{k+1} \subseteq F^k \cap \rho(F^{k\bullet}) \quad \text{and} \quad \mu((F^k \cap \rho(F^{k\bullet})) \setminus F^{k+1}) < \frac{\varepsilon}{2^k}.$$

Then the set $H = \bigcap_{k\in\omega} F^k$ is in $\mathcal{C}'$ and we have $\mu(F \setminus H) \leq \varepsilon$. As μ is inner regular with respect to $\mathcal{C}$, it is also inner regular with respect to $\mathcal{C}'$.

Now it remains to check that $\mathcal{C}'$ is countably compact. Consider any centred sequence $(C_n)_{n\in\omega}$ of sets from $\mathcal{C}'$. Every set C_n can be written as $C_n = \bigcap_{k\in\omega}^{\infty} F_n^k$, where the sets $F_n^k \in \mathcal{C}$ are as in the definition of $\mathcal{C}'$. Then

$$\bigcap_{n\in\omega} C_n = \bigcap_{n\geq 1} \bigcap_{k,m<n} F_m^k.$$

Observe that for every n

$$\bigcap_{k,m<n} \rho(F_m^{k\bullet}) \supseteq \bigcap_{k,m<n} F_m^k \cap \rho(F_m^{k\bullet}) \supseteq \bigcap_{k,m<n} F_m^{k+1} \supseteq \bigcap_{m<n} C_m \neq \emptyset.$$

Hence

$$\rho\left((\bigcap_{k,m<n} F_m^k)^\bullet \right) = \bigcap_{k,m<n} \rho(F_m^{k\bullet}) \neq \emptyset,$$

which means that $\mu(\bigcap_{k,m<n} F_m^k) > 0$. As the family of all F_m^k is μ–centred, by our assumption on $\mathcal{C}$ we get $\bigcap_{n\in\omega} C_n \neq \emptyset$. This completes the proof. ■

Proposition 2.2.2 *Let (X, Σ, μ) be any measure space and let $\Sigma^+ = \{A \in \Sigma : \mu(A) > 0\}$. Suppose that there is a function $\tau : \Sigma^+ \to \Sigma^+$ such that*

(i) $\tau(A) \subseteq A$ for every $A \in \Sigma^+$;

(ii) if $A_n \in \Sigma^+$ and the sequence $(\tau(A_n))_{n\in\omega}$ is μ–centred, then $\bigcap_{n\in\omega} A_n \neq \emptyset$.

Then the measure μ is countably compact.

Proof. For any $E \in \Sigma^+$ we let $\mathcal{T}(E)$ be the family of all finite unions of sets from $\{\tau(A) : A \in \Sigma^+, A \subseteq E\}$. Moreover we put

$$\mathcal{C} = \left\{ \bigcap_{k \in \omega} B^k : B^{k+1} \in \mathcal{T}(B^k) \text{ for every } k \right\}.$$

CLAIM 1. *μ is inner regular with respect to $\mathcal{C}$.*

Note first that $\mu(E) = \sup\{\mu(B) : B \in \mathcal{T}(E)\}$ for every $E \in \Sigma^+$. Indeed, by (i) E is a countable union, modulo a null set, of sets of the form $\tau(A)$, so $\mu(E)$ is approximated by $\mu(B)$ for $B \in \mathcal{T}(E)$. This implies in a standard way that μ is inner regular with respect to $\mathcal{C}$.

CLAIM 2. *If $B_n \in \mathcal{T}(E_n)$ and the sequence $(B_n)_{n\in\omega}$ is μ–centred, then $\bigcap_{n\in\omega} E_n \neq \emptyset$.*

This is so since if we write $B_n = \tau(A_{n,1}) \cup \tau(A_{n,2}) \cup \ldots \cup \tau(A_{n,k_n})$ for every n, then there is a function φ satisfying $\varphi(n) \leq k_n$ such that the sequence of sets $\tau(A_{n,\varphi(n)})$ is μ–centred, and the claim follows from (ii).

Now take a μ–centred sequence $(B_n)_{n\in\omega}$ from $\mathcal{C}$. Write $B_n = \bigcap_{k\in\omega} B_n^k$ as in the definition of $\mathcal{C}$. Then all the sets B_n^k, where $n \in \omega$, $k \geq 1$, are μ–centred, and by Claim 2 $\bigcap_{n\in\omega} B_n \neq \emptyset$. By Claim 1 and Lemma 2.2.1 μ is a countably compact measure.

■

2.3 Closed-regular measures

We denote by $\mathcal{N}$ the Baire space ω^ω. Recall that for every Polish space X and every $B \in \text{Borel}(X)$, B is analytic, i.e. is a continuous image of $\mathcal{N}$ (or is empty); see e.g. Kechris [32].

Theorem 2.3.1 *If Σ is any σ–algebra of subsets of $\mathcal{N}$ and a measure μ defined on Σ is inner regular with respect to closed subsets from Σ, then μ is countably compact.*

Proof. For any $n \in \omega$ and $\psi \in \omega^n$ define

$$V(\psi) = \{x \in \mathcal{N} : x(k) \leq \psi(k) \text{ for all } k < n\}.$$

Consider the family $\mathcal{C}$ of those closed sets F belonging to Σ for which there is a function $\phi : \omega \to \omega$ such that for every n

$$\mu^*(V(\phi|n) \cap F) = \mu(F).$$

We shall prove that $\mathcal{C}$ μ–approximates Σ and that every μ–centred sequence from $\mathcal{C}$ has a nonempty intersection; in view of Lemma 2.2.1 this will imply that μ is countably compact.

Take any $E \in \Sigma$ and $\varepsilon > 0$. We construct inductively a function $\phi \in \mathcal{N}$ such that for every n

$$\mu^*(V(\phi|n) \cap E) > \mu(E) - \frac{\varepsilon}{2}$$

If ϕ is defined on n, then from the fact that outer measure is upward continuous and that the sequence $V(\phi\frown m) \cap E$ converges to $V(\phi) \cap E$ as m goes to infinity we deduce that there exists m such that

$$\mu^*(V(\phi\frown m) \cap E) > \mu(E) - \frac{\varepsilon}{2},$$

and so we can set $\phi(n) = m$.

For every n we can choose a measurable hull $M_n \in \Sigma$ of $V(\phi|n) \cap E$, so that $E \supseteq M_1 \supseteq \ldots$ It follows that for $M = \bigcap_{n\in\omega} M_n$ we have $\mu(E \setminus M) \leq \varepsilon/2$. Now take any closed set $F \in \Sigma$ such that $F \subseteq M$ and $\mu(M \setminus F) < \varepsilon/2$. Then $\mu(E \setminus F) < \varepsilon$; for any n we have $F \subseteq M_n$, so $\mu(F) = \mu^*(F \cap V(\phi|n))$, which means that F is in our class $\mathcal{C}$.

Now consider any μ–centered sequence $(F_n)_{n\in\omega}$ from $\mathcal{C}$. Denote by ϕ a function $\omega \to \omega$ witnessing that $F_0 \in \mathcal{C}$. For every n, $\mu(\bigcap_{k\leq n} F_k) > 0$, so

$$\mu^*\Big(\bigcap_{k\leq n} F_k \cap V(\phi|n)\Big) > 0.$$

Thus we can choose $x_n \in \bigcap_{k\leq n} F_k$ such that $x_n(k) \leq \phi(k)$ for every $k < n$. It follows that the sequence x_n contains a subsequence converging to some $x \in \mathcal{N}$. Every F_k is closed and contains almost all x_n's, so $x \in F_k$ and therefore $\bigcap_{k\in\omega} F_k \neq \emptyset$.

■

Corollary 2.3.2 *If Σ is any σ-algebra of subsets of a Polish space X and the measure μ defined on Σ is inner regular with respect to closed subsets from Σ, then μ is countably compact.*

Proof. Take a continuous surjection $g : \mathcal{N} \to X$, and consider the σ–algebra $\Sigma' = \{g^{-1}[E] : E \in \Sigma\}$. It follows from Lemma 2.1.1 that the measure μ' on Σ' given by $\mu'(g^{-1}[E]) = \mu(E)$ is inner regular with respect to closed sets from Σ'. By the above theorem μ' is countably compact, and hence μ is countably compact by 2.1.1. ■

As we mentioned earlier, the above result in fact follows from the extension theorem due to Aldaz & Render [1]; see also Fremlin [25], 432D. Namely, if μ is a measure as in Corollary 2.3.2, then μ admits an extension to a Borel measure $\widehat{\mu}$ (which is countably compact), so in particular μ is countably compact as the restriction of $\widehat{\mu}$.

2.4 Measures on $\mathcal{N}^\kappa$

Let κ be any cardinal number. In the product space $\mathcal{N}^\kappa$ the family of all closed sets depending on countably many coordinates will be denoted by $Zero(\mathcal{N}^\kappa)$; such sets are often called zero sets. Recall that a set $A \subseteq \mathcal{N}^\kappa$ depends on coordinates in $I \subseteq \kappa$ if for every $x \in A$ and $y \in \mathcal{N}^\kappa$, $x(\alpha) = y(\alpha)$ for all $\alpha \in I$ implies $y \in A$. We shall write $A \sim I$ to indicate that A depends on coordinates in I. Recall that the σ–algebra $Baire(\mathcal{N}^\kappa)$ generated by $Zero(\mathcal{N}^\kappa)$, which is called the σ–algebra of Baire sets, is equal to the product of Borel σ–algebras on $\mathcal{N}$. Similar remarks apply to uncountable products of arbitrary Polish spaces; see Wheeler [55] for general background on measures on topological spaces, and Fremlin [23] for applications of sets depending on few coordinates to measure theory.

If μ is a measure on $Baire(\mathcal{N}^\kappa)$, then, using the fact that every measure on a Polish space is inner regular with respect to compact sets, one can check that μ is countably compact. The following theorem gives a partial generalization of this fact.

Theorem 2.4.1 *Let κ be any cardinal number and Σ any σ–algebra of subsets of $\mathcal{N}^\kappa$. If a measure μ defined on Σ is inner regular with respect to zero subsets from Σ, then μ is regularly monocompact.*

Proof. We shall identify the space $\mathcal{N}^\kappa$ with ω^κ and consider below partial functions from κ into ω. By saying that ϕ is a partial function on κ we mean that the domain of ϕ is a finite subset of κ and the values of ϕ are natural numbers. For every partial function ϕ on κ define

$$V(\phi) = \{x \in \omega^\kappa : \lambda \in Dom(\phi) \Longrightarrow x(\lambda) \le \phi(\lambda)\}.$$

Moreover, for any $\alpha < \kappa$ and $m \in \omega$ put

$$C_\alpha(m) = V(<\alpha, m>) = \{x \in \omega^\kappa : \ x(\alpha) \leq m\}.$$

For an arbitrary set $Y \subseteq \omega^\kappa$ and any $E \in \Sigma$, we introduce the following definitions.

(a) We call a partial function ϕ *Y–thick* if $\mu^*(Y \cap V(\phi)) = \mu^*(Y)$.

(b) We call a countable set $I \subseteq \kappa$ *good for E* if for every partial function ϕ on I and $\alpha \in I$, there is an extension of ϕ to an $E \cap V(\phi)$–thick partial function on $\mathrm{dom}(\phi) \cup \{\alpha\}$.

We shall consider the family $\mathcal{K}$ of sets F with the following properties:

(i) $F \in Zero(\omega^\kappa) \cap \Sigma$;

(ii) $\mu(F) > 0$;

(iii) there is a countable set $I \subseteq \kappa$ such that $F \sim I$ and I is good for F.

We first show that μ is inner regular with respect to $\mathcal{K}$ using the following claim.

CLAIM 1. *Let $E \in \Sigma$ depend on coordinates in a countable set $I \subseteq \kappa$. For every $\varepsilon > 0$ there is a set $F \in \Sigma \cap Zero(\omega^\kappa)$ with $F \subseteq E$, $\mu(E \setminus F) < \varepsilon$, such that for every function ϕ defined on a finite set $J \subseteq I$ and $\alpha \in I$,*

$$(*) \qquad \text{there is } m \text{ such that} \quad \mu^*(F \cap V(\phi) \cap C_\alpha(m)) = \mu^*(F \cap V(\phi)).$$

To prove this claim note that, for a fixed partial function ϕ on I and any $\alpha \in I$,

$$V(\phi) \cap C_\alpha(m) \nearrow V(\phi) \quad \text{as } m \to \infty,$$

so by Lemma 2.1.5 there is $F \subseteq E$ with $\mu(E \setminus F) < \varepsilon$ (which can be taken to be a zero set) such that (*) is satisfied. We have countably many pairs (ϕ, α) to consider, so repeating this argument we see that there is F such that (*) holds for every partial function on I and every $\alpha \in I$. This proves the claim.

Let $A \in \Sigma$ and $\varepsilon > 0$ be given. We first find a measurable zero set F_0 and a countable $I_0 \subseteq \kappa$ such that $F_0 \sim I_0$, $F_0 \subseteq A$, and $\mu(A \setminus F_0) < \varepsilon/2$. We next apply the Claim to $E = F_0$, $I = I_0$ (and $\varepsilon/4$ in place of ε) to obtain a measurable zero set $F_1 \subseteq F_0$ and a countable $I_1 \supseteq I_0$ such that $F_1 \sim I_1$, $\mu(F_0 \setminus F_1) < \varepsilon/4$ and (*) holds for $F = F_1$ and any partial function ϕ on I_0 and $\alpha \in I_0$.

Continuing in the same manner we get a decreasing sequence of zero sets $F_n \in \Sigma$ and an increasing sequence I_n of countable sets such that $\mu(F_{n-1} \setminus F_n) < \varepsilon/2^{n+1}$, $F_n \sim I_n$, and (*) holds whenever ϕ is a partial function on I_{n-1} and $\alpha \in I_{n-1}$.

Finally we put $F = \bigcap_{n\in\omega} F_n$ and $I = \bigcup_{n\in\omega} I_n$. Then $\mu(A \setminus F) \leq \varepsilon$ and $F \sim I$. Moreover, I is good for F: If $J \subseteq I$ is finite, $\phi : J \to \omega$, $\alpha \in I$, then $J \cup \{\alpha\} \subseteq I_n$ for some n, so there is m such that

$$\mu^*(F_{n+1} \cap V(\phi) \cap C_\alpha(m)) = \mu^*(F_{n+1} \cap V(\phi))$$

and hence

$$\mu^*(F \cap V(\phi) \cap C_\alpha(m)) = \mu^*(F \cap V(\phi)).$$

In particular, we can extend any partial function ϕ to an $F \cap V(\phi)$–thick function by letting $\phi(\alpha) = m$. This shows that μ is regular with respect to $\mathcal{K}$.

Now it remains to verify that $\mathcal{K}$ is a monocompact class. Let $(F_n)_{n\in\omega}$ be a decreasing sequence of sets from $\mathcal{K}$. Then for every n there is a countable set $I_n \subseteq \kappa$ such that $F_n \sim I_n$ and I_n is good for F_n. Enumerate elements of $I = \bigcup_{n\in\omega} I_n$ as $I = \{\alpha_k : k \in \omega\}$ and write $T_k = \{\alpha_j : j < k\}$ for every k.

CLAIM 2. *There is a function $\tau : I \to \omega$ such that for every n and every k its restriction $\tau|(T_k \cap I_n)$ is F_n–thick.*

We define values of τ by induction. Suppose that τ is defined on T_k so that $\tau|(T_k \cap I_n)$ is F_n–thick for every n. There is a natural number p such that for every $n > p$ there is $j \leq p$ such that $T_{k+1} \cap I_n \subseteq T_{k+1} \cap I_j$.

For a given $j \leq p$ such that $\alpha_k \in I_j$ there is m_j such that the F_j–thick function $\tau|(T_k \cap I_j)$ can be extended to an F_j–thick function assuming the value m_j at α_k. We let $\tau(\alpha_k)$ be the maximum of these numbers m_j (where $j \leq p$).

In this way we have extended τ to T_{k+1} so that $\tau|(T_{k+1} \cap I_j)$ is F_j–thick for every $j \leq p$. For any $n > p$ we have $T_{k+1} \cap I_n \subseteq T_{k+1} \cap I_j$, where $j \leq p$. It follows that $\tau|(T_{k+1} \cap I_n)$ is F_j–thick (as the restriction of a thick function is thick). Therefore, $\tau|(T_{k+1} \cap I_n)$ is also F_n–thick (since $F_n \subseteq F_j$). This verifies the claim.

Using Claim 2 we can check that $\bigcap_{n\in\omega} F_n \neq \emptyset$. For every n the function $\tau|(T_n \cap I_n)$ is F_n–thick. Since $\mu(F_n) > 0$ there is $x_n \in F_n$ such that $x_n(\alpha) \leq \tau(\alpha)$ for $\alpha \in T_n \cap I_n$. We can moreover assume that

$$x_n(\alpha) = 0 \text{ for } \alpha \in (T_n \setminus I_n) \cup (\kappa \setminus I),$$

since F_n is determined by $I_n \subseteq I$. Now the sequence of x_n (dominated by τ) has a subsequence converging to some $x \in \omega^\kappa$. We have $x_n \in F_k$ for all $n \geq k$, so $x \in F_k$ (as F_k is closed). Finally, $x \in \bigcap_{n\in\omega} F_n$, and the proof is complete.

■

Let us remark that if we could refine this argument to prove that the measure in question is countably compact, then we would get the following result: *If a countable set I_j is good for F_j, $j = 1, 2$, then $I_1 \cup I_2$ is good for $F_1 \cap F_2$.* This can be done in case $\kappa = \omega_1$.

Theorem 2.4.2 *If Σ is any σ–algebra of subsets of $\mathcal{N}^{\omega_1}$, then every measure μ defined on Σ which is inner regular with respect to zero subsets from Σ is countably compact.*

Proof. We modify the argument from the previous proof as follows. Consider the family $\mathcal{K}$ of sets F with the following properties:

(i) $F \in Zero(\omega^\kappa) \cap \Sigma$;

(ii) $\mu(F) > 0$;

(iii) there is an *initial segment* I of ω_1 such that $F \sim I$ and I is good for F.

Since every initial segment of ω_1 is countable we can in a similar way verify that μ is again inner regular with respect to $\mathcal{K}$. The main difference is contained in the following claim.

CLAIM. *If $F, H \in \mathcal{K}$ and $\mu(F \cap H) > 0$, then $F \cap H \in \mathcal{K}$.*

Indeed, let I and J be good for F and H, respectively. We can assume that $I \subseteq J$, but in this case J is good for $F \cap H$, so $F \cap H \in \mathcal{K}$.

Now for any μ–centred sequence $(F_n)_{n\in\omega}$ of sets from $\mathcal{K}$ we have a decreasing sequence $H_n = F_1 \cap F_2 \cap \ldots \cap F_n \in \mathcal{K}$, so by the previous argument $\bigcap_{n\in\omega} H_n \neq \emptyset$, and we are done.

■

Corollary 2.4.3 *Let $X = \prod_{\alpha<\kappa} X_\alpha$, where every X_α is a Polish space. If Σ is a σ–algebra of subsets of X and μ is inner regular with respect to zero sets from Σ, then μ is regularly monocompact. If, moreover, $\kappa = \omega_1$, then μ is countably compact.*

Proof. For every α choose a continuous surjection $g_\alpha \colon \mathcal{N} \to X_\alpha$, and let

$$g = \prod_{\alpha<\kappa} g_\alpha : \mathcal{N}^\kappa \to X.$$

Then for every $Z \in Zero(X)$ we have $g^{-1}[Z] \in Zero(\mathcal{N}^\kappa)$, so we can argue as in Corollary 2.3.2. ■

2.5 Application to measures on Polish spaces

Our motivation for considering measures on uncountable products of Polish spaces came from the following result.

Lemma 2.5.1 *Let μ be a measure on a σ–algebra $\Sigma \subseteq \mathrm{Borel}(X)$, where X is a Polish space. Suppose that $\{B_\alpha : 1 \le \alpha < \kappa\}$ is a family of analytic subsets of X, and let $\mathcal{F}$ be a family of those sets $E \in \Sigma$ for which there is $\alpha < \kappa$ such that $E \subseteq B_\alpha$ is closed in B_α.*

If μ is inner regular with respect to $\mathcal{F}$, then there is a measure $\widehat{\mu}$ defined on some σ–algebra $\widehat{\Sigma}$ of subsets of $\mathcal{N}^\kappa$, which is inner regular with respect to $Zero(\mathcal{N}^\kappa) \cap \widehat{\Sigma}$, and an inverse–measure–preserving function $(\mathcal{N}^\kappa, \widehat{\Sigma}, \widehat{\mu}) \longrightarrow (X, \Sigma, \mu)$.

Proof. We can assume that $X = \mathcal{N}$. Every B_α is an analytic subset of $\mathcal{N}$, so there is a closed set $F_\alpha \subseteq \mathcal{N} \times \mathcal{N}$ such that $p[F_\alpha] = B_\alpha$, where $p : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ is the projection onto the first coordinate.

Let $\pi_\alpha : \mathcal{N}^\kappa \to \mathcal{N}$ be the projection onto the α's axis; we consider $\Delta \subseteq \mathcal{N}^\kappa$, where

$$\Delta = \{x \in \mathcal{N}^\kappa : \text{ for every } \alpha \ge 1, \text{ if } \pi_0(x) \in B_\alpha, \text{ then } (\pi_0(x), \pi_\alpha(x)) \in F_\alpha\}.$$

Let $g : \Delta \to \mathcal{N}$ be π_0 restricted to Δ. We endow Δ with the σ–algebra $\Sigma' = \{g^{-1}[E] \colon E \in \Sigma\}$ and the measure μ' on Σ' given by $\mu' g^{-1}(E) = \mu(E)$.

With every $E \in \mathcal{F}$ we can associate $Z(E) \in Zero(\mathcal{N}^\kappa)$ as follows. Choose $\alpha < \kappa$ such that $E \subseteq B_\alpha$ is closed; then $p^{-1}[E] \cap F_\alpha$ is a closed subset of $\mathcal{N} \times \mathcal{N}$. Now let

$$Z(E) = \{x \in \mathcal{N}^\kappa \colon (\pi_0(x), \pi_\alpha(x)) \in p^{-1}[E] \cap F_\alpha\}.$$

Note that

(i) $g^{-1}[E] = Z(E) \cap \Delta$ for $E \in \mathcal{F}$;

(ii) if $E_1, E_2 \in \mathcal{F}$ are disjoint, then $Z(E_1) \cap Z(E_2) = \emptyset$.

Let Σ'' be the σ–algebra of subsets of $\mathcal{N}^\kappa$ generated by the family

$$Z(\mathcal{F}) = \{Z(E) :\ E \in \mathcal{F}\},$$

and let $\mu''(C) = \mu'(C \cap \Delta)$ for $C \in \Sigma''$. Then for $E \in \mathcal{F}$ we have $\pi_0^{-1}[E] \supseteq Z(E)$ and

(iii) $\mu''(Z(E)) = \mu'(Z(E) \cap \Delta) = \mu'(g^{-1}[E]) = \mu(E)$.

Observe that, by (ii), (iii) and the $\mathcal{F}$–regularity of μ, for $E \in \mathcal{F}$ we have

$$\mu''(\mathcal{N}^\kappa \setminus Z(E)) = \sup\{\mu''(Z(F)) :\ F \in \mathcal{F}, Z(F) \cap Z(E) = \emptyset\}.$$

This implies that μ'' is inner regular with respect to the closure of the family $Z(\mathcal{F})$ with respect to finite unions and countable intersections. In particular, μ'' is regular with respect to zero sets lying inside Σ''.

We finally let $(\mathcal{N}^\kappa, \widehat{\Sigma}, \widehat{\mu})$ be the completion of $(\mathcal{N}^\kappa, \Sigma'', \mu'')$. Since μ'' is regularly monocompact by Theorem 2.4.1, so is the measure $\widehat{\mu}$.

By (iii) and the $\mathcal{F}$–regularity of μ, $\pi_0 : \mathcal{N}^\kappa \to \mathcal{N}$ is a measure–preserving function, and the proof is complete. ■

The above lemma, together with the result from Section 2.4 (and the fact that countable compactness is preserved by images) gives the following corollary.

Corollary 2.5.2 *Let μ be a measure on a σ–algebra $\Sigma \subseteq \mathrm{Borel}(X)$, where X is a Polish space.*

(a) There is a regularly monocompact measure space $(\widehat{X}, \widehat{\Sigma}, \widehat{\mu})$ and a inverse–measure–preserving function $(\widehat{X}, \widehat{\Sigma}, \widehat{\mu}) \longrightarrow (X, \Sigma, \mu)$.

(b) The measure μ is countably compact provided there is a family $\{B_\alpha : 1 \leq \alpha < \omega_1\}$ of analytic subsets of X such that μ is regular with respect to the family $\mathcal{F}$ of those $E \in \Sigma$ for which there is $\alpha < \omega_1$ such that $E \subseteq B_\alpha$ is closed in B_α.

Unfortunately, it is not known if regular monocompactness is preserved by inverse–measure–preserving mappings (see Fremlin [22]), so one cannot write in 2.5.2(a) that μ is simply regularly monocompact. Note that Theorem 2.1.4 follows from 2.5.2(b).

2.6 Measures and games

Let $\mathcal{A}$ be a family of sets. We define the Banach–Mazur game on $\mathcal{A}$ in the following way. The game $\Gamma(\mathcal{A})$ has two players I and II who choose sets $A_n, B_n \in \mathcal{A}$ respectively, so that $A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \ldots$. The player II wins if $\bigcap_{n\in\omega} A_n \neq \emptyset$.

Let (X, Σ, μ) be any measure space and write $\Sigma^+ = \{E \in \Sigma : \mu(E) > 0\}$. In [22] Fremlin considered Banach–Mazur games on Σ^+ for various measure spaces. Write $\Gamma(\mu)$ for $\Gamma(\Sigma^+)$.

Fremlin [22] calls the measure μ *weakly α–favourable* if Player II has a winning strategy in $\Gamma(\mu)$, and *α–favourable* if II has a winning tactic in this game, where tactic is a function $\tau : \Sigma^+ \to \Sigma^+$ such that II wins by playing $B_n = \tau(A_n)$ at each step. For such two classes of measure spaces we have the following implications:

regularly monocompact $\Longrightarrow$ α–favourable $\Longrightarrow$ weakly α–favourable $\Longrightarrow$ perfect.

For instance, if μ is inner regular with respect to a monocompact class $\mathcal{K}$, then II wins simply by choosing elements from $\mathcal{K} \cap \Sigma^+$. Fremlin [22] showed that the class of weakly α–favourable measures is properly contained in the class of perfect measures, and posed the question whether any of the first two implications can be reversed.

Perhaps the most attractive question here is the following: does the class of α–favourable measures differ from the class of weakly–α–favourable measures?

It is highly nonintuitive that there is a family $\mathcal{A}$ such that Player II has a winning strategy in $\Gamma(\mathcal{A})$ but has no winning tactic. There is, essentially, one example of such family (it is presented in [22]). The idea lying behind it can be used to produce an example of a topology (even completely regular, see [12]), which is weakly–α–favourable but not α–favourable. Unfortunately, it cannot be used to construct a measure space for a σ–finite measure.

Note that we could consider a less restrictive game $\Gamma'(\mu)$, in which the players form a sequence of sets which is μ–centred rather than decreasing. We can use Proposition 2.2.2 to show that, contrary to the case of $\Gamma(\mu)$, the existence of winning tactic for Player II in $\Gamma'(\mu)$ is a sufficient condition for countable compactness.

Fact 2.6.1 *Player II has a winning tactic in $\Gamma'(\mu)$ if and only if μ is countably compact.*

Proof. Of course if μ is countably compact, then Player II has a winning tactic in $\Gamma'(\mu)$. To see the opposite implication apply Proposition 2.2.2 to the winning tactic τ.

■

Fremlin showed in [22] that every weakly α–favourable measure defined on a σ–algebra Σ generated by ω_1 sets is countably compact, and in [24] he proved that μ is weakly α–favourable whenever μ is defined on some $\Sigma \subseteq \mathrm{Borel}(X)$, where X is a Polish space. We show below how one can apply some of the above ideas to prove the latter result; in fact our result is in a sense stronger. Indeed, in the case of $X = [0,1]$ we are able to explicitly construct a winning strategy for the second player. Fremlin's proof used involved techniques and it does not show, what a winning strategy for Player II looks like.

Theorem 2.6.2 (Fremlin) *If* $\Sigma \subseteq \mathrm{Borel}[0,1]$*, then every measure on* Σ *is weakly* α*–favourable.*

Proof. As in the proof of Theorem 2.4.1 we write

$$V(\psi) = \{x \in \mathcal{N} : \ x(k) \leq \psi(k) \text{ for all } k < n\},$$

for any $n \in \omega$ and $\psi \in \omega^n$. We shall work in the space $[0,1] \times \mathcal{N}$. Given $V(\psi)$ as above, we let $G(\psi) = [0,1] \times V(\psi)$. We denote by $\pi : [0,1] \times \mathcal{N} \to [0,1]$ the projection onto the first coordinate.

Every move A_n of the first player is a Borel set, so we can find a closed set $F_n \subseteq [0,1] \times \mathcal{N}$ such that $\pi[F_n] = A_n$. The second player defines inductively functions $\varphi_n : \omega \to \omega$ such that for every n the set

$$Y_n = \bigcap_{i \leq n} \pi[F_i \cap G(\varphi_i|n)]$$

satisfies $\mu^*(Y_n) > 0$, and for the n–th move chooses a set B_n which is a measurable hull of Y_n. Player I is obliged to choose $A_{n+1} \subseteq B_n$, so $\mu^*(\pi[F_{n+1}] \cap Y_n) = \mu(A_{n+1}) > 0$, and it is easily seen that one can define $\varphi_{n+1}|(n+1)$ and $\varphi_i(n)$ for $i \leq n$ in such a way that Y_{n+1} will be a set of positive outer measure.

By following this strategy Player II wins: For every n choose $t_n \in Y_n$. Then the sequence $t_n \in [0,1]$ has a subsequence converging to some t. Fix k. For every $n > k$ there is y_n such that $y_n \in V(\varphi_k|n)$ and $(t_n, y_n) \in F_k$. The sequence of y_n in turn has a subsequence that converges to some y. It follows that $(t,y) \in F_k$ since F_k is closed and $t = \pi(t,y) \in \pi[F_k] = A_k$. Finally, $t \in \bigcap_{k\in\omega} A_K$. This finishes the proof.

■

It is unclear if the above construction can be improved to obtain a winning tactic. Notice that if we can show that monocompactness is closed under inverse–measure–preserving functions, then Corollary 2.5.2 would imply the existence of winning tactic in the above case. However, it would not be clear, what this tactic looks like.

2.7 Remarks

The problem (FN) which motivated most of the considerations presented in Chapter 2 remains open: is it provable in ZFC that every measure defined on sub–σ–algebra of Borel$[0,1]$ is countably compact?

Monocompactness seems to be really close to countable compactness. Hence, one can see Corollary 2.5.2 as a strong premise that the answer to Problem FN is positive. On the other hand this impression may be illusory; e.g. Pachl's reasoning (mentioned in Section 2.1) proving that countable compactness of measure is closed under images cannot be generalized to monocompact measures. Anyway, Corollary 2.5.2 allows us to state the following conjecture.

Conjecture 2.7.1 *Every measure defined on a sub–σ–algebra of* Borel$[0,1]$ *is regularly monocompact.*

Another interesting open problem in this subject was formulated in Section 2.6: is every weakly–α–favourable measure α–favourable? The analysis of the example from [22] mentioned in Section 2.6 leaded us to the following considerations.

Let $\mathcal{A}$ be a family of subsets of some set X and let $\mathcal{M}$ be an ideal on X. Define

$$\pi(\mathcal{A}) = \min\{|\mathcal{A}_0| \colon \mathcal{A}_0 \subseteq \mathcal{A},\ \forall A \in \mathcal{A}\ \exists A_0 \in \mathcal{A}_0\ A_0 \subseteq A\}$$

and

$$\mathrm{add}(\mathcal{M}) = \min\{|\mathcal{M}_0| \colon \mathcal{M}_0 \subseteq \mathcal{M},\ \bigcup \mathcal{M}_0 \notin \mathcal{M}\}.$$

Let $\mathcal{B} = \{U \triangle M \colon U \in \mathcal{U},\ M \in \mathcal{M}\}$. In [22] Fremlin proves that if $\mathcal{U}$ is the family of open subsets and $\mathcal{M}$ is the ideal of meager subsets of $[0,1]$, then $\mathcal{B}$ defined as above is weakly–α–favourable but not α–favourable. The only property of $\mathcal{U}$ and $\mathcal{M}$ used in the proof that $\mathcal{B}$ is not α–favourable is the inequality $\pi(\mathcal{U}) < \mathrm{add}(\mathcal{M})$. Thus, every weakly–$\alpha$–favourable $\mathcal{B}$ such that $\pi(\mathcal{U}) < \mathrm{add}(\mathcal{M})$ is not α–favourable. Unfortunately, such $\mathcal{B}$ cannot be a σ–algebra for a finite measure.

Problem 2.7.2 *Assume that $\mathcal{B}$ is a weakly–α–favourable family of subsets of some X and there is a family $\mathcal{U} \subseteq P(X)$ and an ideal $\mathcal{M}$ on X such that $\mathcal{B} = \{U \triangle M : U \in \mathcal{U},\ M \in \mathcal{M}\}$ and $\mathrm{add}(\mathcal{M}) \leq \pi(\mathcal{U})$. Is $\mathcal{B}$ α–favourable?*

A positive answer would imply that every weakly–α–favourable finite measure on $[0, 1]$ is α–favourable. Indeed, suppose that a measure μ is defined on a σ–algebra Σ. Denote by $\mathcal{M}$ the ideal of null sets. Let $\mathcal{B} = \Sigma \setminus \mathcal{M}$ and let $\mathcal{U}$ be such that $\mathcal{B} = \{U \triangle M : U \in \mathcal{U},\ M \in \mathcal{M}\}$. Then $\mathcal{U}$ is isomorphic to the measure algebra of μ (without the empty set). Hence, $\mathrm{add}(\mathcal{M}) \leq \pi(\mathcal{U})$ and, therefore, if μ is weakly–α–favourable, then it is α–favourable.

Perhaps one can solve Problem 2.7.2 using methods similar to those used in [28].

3 Measures on minimally generated Boolean algebras

In this chapter we study the notion of minimally generated Boolean algebras, mainly from the measure theoretic point of view. Its essential part is contained in [7].

Section 3.1 is devoted to the study of the Stone spaces of minimally generated algebras. We try to find their place among well–known classes of topological spaces. We have not been able to give a topological characterization of compact spaces whose algebras of clopen subsets are minimally generated. Most of the results contained in the section are direct applications of Koppelberg's theorems (repeated without proofs at the beginning of the section), so we decided to call such spaces *Koppelberg compacta*. Quite unexpectedly, it appeared that all monotonically normal spaces are Koppelberg compact. Also, we prove here a general fact about the existence of minimal extensions of certain types. We use it in following sections.

The essential part of Chapter 3 presents several results on measures on minimally generated algebras. It is done in Section 3.2. We show that all measures admitted by such algebras are separable (in fact, they fulfill a certain stronger regularity condition). It sheds some new light on similar results obtained for interval algebras and monotonically normal spaces (see [52, 11] respectively).

Also, in Section 3.2 we prove that a Boolean algebra carries either a nonseparable measure or a measure which is uniformly regular. It is shown that all measures on a free product $\mathfrak{A}\oplus\mathfrak{B}$ of Boolean algebras are weakly uniformly regular if only all measures on $\mathfrak{A}$ and $\mathfrak{B}$ are weakly uniformly regular. The section provides several results indicating that minimal generation cannot be characterized by measure theoretic conditions, at least not in any natural way. We point out that measures on retractive algebras can be nonseparable if CH is assumed. The retractive algebras are the only well–known subclass of small Boolean algebras which is not included in the class of minimally generated algebras. Using the above results we present some new examples of small (also, retractive) but not minimally generated Boolean algebras.

In Chapter 4 we show some applications of minimally generated Boolean algebras. We use there some of the results proved in this chapter.

3.1 Minimally generated Boolean algebras and their Stone spaces

In this section we mainly overview known results concerning minimal generation and translate them to the language of topology.

First, let us fix some notation concerning Boolean algebras. Throughout this section all "algebras" are Boolean algebras, even if it is not stated explicitly. We denote the Boolean operations like in algebras of sets ($\cup$, ^{c}, and so on). Given a Boolean algebra $\mathfrak{A}$ we denote by Stone($\mathfrak{A}$) its Stone space, i.e. the space of ultrafilters on $\mathfrak{A}$. A topological space is said to be *Boolean* if it is compact and zero–dimensional.

If $\mathcal{A}$ is a family of subsets of X, then $alg(\mathcal{A})$ is the subalgebra of $P(X)$ generated by $\mathcal{A}$. If $\mathfrak{A}$ is a Boolean algebra, then $\mathfrak{A}(B) = alg(\mathfrak{A} \cup \{B\})$. Recall that in $\mathfrak{A}(B)$ all elements are of the form $(B \cap A_1) \cup (B^c \cap A_2)$, where A_1, A_2 belong to $\mathfrak{A}$. By $Fin(X)$ we denote the family of finite subsets of X (write Fin if $X = \omega$) and by $Fin\text{–}Cofin(X)$ the algebra $alg(Fin(X))$.

Recall that $\mathfrak{A} \oplus \mathfrak{B}$ ($\mathfrak{A} \times \mathfrak{B}$) is a *free product* (*product*) of Boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$ if it is the algebra of clopen subsets of the product (disjoint union, respectively) of its Stone spaces.

For an algebra $\mathfrak{A}$ it is convenient to say that a sequence $(A_n)_{n\in\omega}$ in $\mathfrak{A}$ is *convergent to an ultrafilter* $p \in$ Stone($\mathfrak{A}$) if for every $U \in p$ we have $A_n \subseteq U$ for almost all n. We say that a sequence $(p_n)_{n\in\omega}$ in Stone($\mathfrak{A}$) is convergent to p if for every $U \in p$ we have $p_n \in U$ for almost all n.

We say that a Boolean algebra is *small* if it does not contain an uncountable independent sequence.

Now, we define the main notion of this section.

Definition 3.1.1 *We say that $\mathfrak{B}$ is a minimal extension of $\mathfrak{A}$ if $\mathfrak{A} \subseteq \mathfrak{B}$ and there is no algebra $\mathfrak{C}$ such that $\mathfrak{A} \subsetneq \mathfrak{C} \subsetneq \mathfrak{B}$.*
An algebra $\mathfrak{B}$ is minimally generated over $\mathfrak{A}$ if there is a continuous sequence of algebras $(\mathfrak{A}_\alpha)_{\alpha\leq\kappa}$, such that $\mathfrak{A}_0 = \mathfrak{A}$, $\mathfrak{A}_{\alpha+1}$ is a minimal extension of $\mathfrak{A}_\alpha$ for every $\alpha < \kappa$ and $\mathfrak{A}_\kappa = \mathfrak{B}$.
Finally, a Boolean algebra is minimally generated if it is minimally generated over $\{0, 1\}$.

Loosely speaking a Boolean algebra is minimally generated if it can be generated by small, indivisible steps

The notion of minimally generated Boolean algebra was introduced by Sabine Koppelberg in [33] although it was previously used implicitly by other authors.

The study originated in [33] was continued in [35], where some interesting counterexamples were indicated. In [36] one can find examples of forcing with minimally generated algebras. Several papers by Lutz Heindorf are closely related to the topic, see, e.g., [8]. This chapter presents a modest attempt to deepen the knowledge about this class of Boolean algebras.

The notion of minimal extension corresponds to the idea of a *simple extension* in the inverse limits setting. Indeed, many authors considering problems similar to those presented in this paper prefer to use the language of inverse limits (see e.g. [13, 15]).

Definition 3.1.2 *Let $(X_\alpha)_{\alpha\in\lambda}$ be an inverse limit and let $(f_{\alpha\beta})_{\alpha<\beta<\kappa}$ be the set of its bonding mappings. We say that $X_{\alpha+1}$ is a simple extension of X_α if there is exactly one point $x_\alpha \in X_\alpha$ such that $f^{-1}_{(\alpha)(\alpha+1)}(x)$ is a singleton for all $x \neq x_\alpha$ and consists of two points if $x = x_\alpha$.*

The connection can be explained by the following simple lemma. Indeed, if an algebra $\mathfrak{B}$ extends $\mathfrak{A}$ minimally, then all ultrafilters in $\mathfrak{A}$ but (possibly) one has unique extensions in $\mathfrak{B}$. It is stated (in a slightly different language) in [33], but we prove it here for the reader's convenience.

Lemma 3.1.3 *Let $\mathfrak{A} \subseteq \mathfrak{B}$. Then $\mathfrak{B}$ extends $\mathfrak{A}$ minimally if and only if the set*

$$\mathcal{U} = \{A \in \mathfrak{A} \colon \exists B \in \mathfrak{B}\ A \cap B \notin \mathfrak{A}\}$$

is an ultrafilter on $\mathfrak{A}$ and only this ultrafilter is split by $\mathfrak{B}$, i.e. only this ultrafilter can be extended to two different ultrafilters on $\mathfrak{B}$.

Proof. Let $\mathfrak{A} \subseteq \mathfrak{B}$. It is easy to check that if $A_0 \in \mathcal{U}$ and $A_0 \subseteq A_1$, then $A_1 \in \mathcal{U}$. If $B \in \mathfrak{B} \setminus \mathfrak{A}$, then for every $A \in \mathfrak{A}$ either $A \cap B \notin \mathfrak{A}$ or $A^c \cap B \notin \mathfrak{A}$. Therefore, if $\mathcal{U}$ is closed under finite intersections, then it is an ultrafilter.

Assume that $\mathfrak{B}$ extends $\mathfrak{A}$ minimally. Consider A_0, $A_1 \in \mathfrak{A}$ and B_0, $B_1 \in \mathfrak{B}$ such that $A_0 \cap B_0 \notin \mathfrak{A}$ and $A_1 \cap B_1 \notin \mathfrak{A}$. Suppose that $A_0 \cap A_1 \notin \mathcal{U}$. Then $C = A_0 \cap A_1 \cap B_0 \cap B_1 \in \mathfrak{A}$. Hence, $A_0 \cap B_0 \setminus C \notin \mathfrak{A}$ and $A_1 \cap B_1 \setminus C \notin \mathfrak{A}$ and $\mathfrak{A} \subsetneq \mathfrak{A}(A_0 \cap B_0 \setminus C) \subsetneq \mathfrak{A}(A_0 \cap B_0 \setminus C, A_1 \cap B_1 \setminus C) \subseteq \mathfrak{B}$, a contradiction. Thus, $\mathcal{U}$ is an ultrafilter.

Consider $p \in \mathrm{Stone}(\mathfrak{A})$ such that there is $A \in p \setminus \mathcal{U}$. Then $A \cap B \in \mathfrak{A}$ and $A \setminus B \in \mathfrak{A}$ for every $B \in \mathfrak{B} \setminus \mathfrak{A}$. Thus, either $A \cap B \in p$ and then we cannot extend p by B^c or $A \setminus B \in p$ but then we cannot extend p by B. Consequently, $\mathcal{U}$ is the only ultrafilter split by $\mathfrak{B}$.

It is easy to see that if $\mathfrak{B}$ is not a minimal extension of $\mathfrak{A}$, then there exist pairwise disjoint $B_0, B_1, B_2 \in \mathfrak{B} \setminus \mathfrak{A}$. Therefore, if $\mathcal{U}$ is an ultrafilter on $\mathfrak{A}$, then it can be extended to at least three ultrafilters on $\mathfrak{B}$.

■

This gives some idea how minimal extensions look like. The following remark is a simple consequence of the definition and of Lemma 3.1.3 but it simplifies many considerations included in the next sections.

Proposition 3.1.4 *Let $\mathfrak{B}$ be a minimal extension of $\mathfrak{A}$. The following facts hold:*

- *if $B \in \mathfrak{B} \setminus \mathfrak{A}$, then $\mathfrak{B} = \mathfrak{A}(B)$;*
- *if we consider disjoint elements A_0, A_1 of $\mathfrak{A}$ and any element B of $\mathfrak{B}$, then $A_0 \cap B \in \mathfrak{A}$ or $A_1 \cap B \in \mathfrak{A}$.*

Now we review some basic facts concerning minimally generated Boolean algebras. The proofs of Proposition 3.1.5 and of Theorem 3.1.6 can be found in [33].

Proposition 3.1.5 *The class of minimally generated algebras is closed under the following operations:*

(a) taking subalgebras;

(b) homomorphic images;

(c) finite products.

A Boolean algebra is called an *interval algebra* if it is generated by a subset linearly ordered under the Boolean partial order. Similarly, an algebra generated by a tree is called a *tree algebra*. Every tree algebra is embeddable into some interval algebra. A Boolean algebra $\mathfrak{A}$ is said to be *superatomic* if every nontrivial homomorphic image of $\mathfrak{A}$ has at least one atom. Recall also that a topological space is said to be *ordered* if its topology is generated by open intervals of some linear order (for Boolean spaces, if it is a Stone space of some interval algebra, equivalently). A topological space X is called *scattered* if for every closed subspace Y of X the isolated points of Y are dense in Y (i.e. if it is a Stone space of some superatomic algebra, in the case of Boolean spaces).

Theorem 3.1.6 (Koppelberg [33]) *The following classes are included in the class of minimally generated Boolean algebras:*

(a) subalgebras of interval algebras (and, thus, countable algebras, tree algebras);

(b) superatomic algebras.

If a Boolean algebra contains an uncountable independent set, then it cannot be minimally generated (see [33] or Theorem 3.2.10 in the next section). The algebra $\mathfrak{C}$ of clopen subsets of $[0,1) \times ([0,1) \cap \mathbb{Q})$, where $[0,1)$ is endowed with the Sorgenfrey line topology, is an example of a small algebra which is not minimally generated (see [35]). It also shows that a free product of minimally generated Boolean algebras does not need to be minimally generated.

We translate now Koppelberg's results to the language of topology. Most of the following reformulations are trivial. Say that a topological space is *Koppelberg compact* if it is Boolean and the algebra of its clopen subsets is minimally generated.

Proposition 3.1.7 *The class of Koppelberg compacta is closed under the following operations:*

(a) continuous images;

(b) taking closed subspaces;

(c) finite disjoint unions.

Proof. Clearly, (a) and (c) are direct consequences of Proposition 3.1.5. For Boolean algebras $\mathfrak{A}, \mathfrak{B}$ let $f\colon \mathrm{Stone}(\mathfrak{A}) \to \mathrm{Stone}(\mathfrak{B})$ be a continuous mapping. The set $\{f^{-1}(B)\colon B \in \mathfrak{B}\}$ forms a subalgebra of $\mathfrak{A}$, on the other hand it is isomorphic to $\mathfrak{B}$. We conclude that the minimal generation of $\mathfrak{A}$ implies the minimal generation of $\mathfrak{B}$, by (a) of Proposition 3.1.5. The proof of (b) is complete.

■

We translate in the same way Theorem 3.1.6. We first recall the notion of monotonically normal spaces which has been intensively studied in a number of papers over last years.

Definition 3.1.8 *A topological space X is monotonically normal if it is T_1 and for every open $U \subseteq X$ and $x \in U$ we can find an open subset $h(U, x)$ such that $x \in h(U, x) \subseteq U$ and*

- *$U \subseteq V$ implies $h(U, x) \subseteq h(V, x)$ for every $x \in U$;*
- *$h(x, X \setminus \{y\}) \cap h(y, X \setminus \{x\}) = \emptyset$ for $x \neq y$.*

Theorem 3.1.9 *A Boolean space K is Koppelberg compact if one of the following conditions is fulfilled:*

(a) K is metrizable;

(b) K is ordered;

(c) K is scattered;

(d) K is monotonically normal.

Proof. Of these (a), (b) and (c) are trivial since the ordered Boolean spaces coincide with the Stone spaces of interval algebras and the class of scattered Boolean spaces is exactly the class of Stone spaces of superatomic algebras. To prove (d) recall Rudin's theorem (see [50]) stating that every compact monotonically normal space is a continuous image of compact ordered space. By (a) of Proposition 3.1.7 we are done. ∎

The class of Koppelberg compact spaces is not included in any class mentioned in the above theorem, which is a trivial assertion in case of (a), (b) and (c). Also, monotone normality and minimal generation are not equivalent, even in the class of zero–dimensional spaces. Before exhibiting the example recall that by the result due to Heindorf (see [31]) every subalgebra of an interval algebra is generated by a pseudo–tree (a subfamily which is a pseudo–tree when considered as a partially ordered set under Boolean ordering).

The example is following. Consider an algebra $\mathfrak{A} = alg(Fin \cup \{A_\alpha \colon \alpha \in \mathfrak{c}\})$, where $(A_\alpha)_{\alpha\in\mathfrak{c}}$ is an almost disjoint family of subsets of ω. It is clear that $\mathfrak{A}$ is minimally generated and that we cannot generate $\mathfrak{A}$ by a pseudo–tree. Therefore, $\mathfrak{A}$ is not embeddable in an interval algebra and, by Rudin's result, Stone($\mathfrak{A}$) is not monotonically normal.

Anyway, the connection between the class of interval algebras, tree algebras and minimal generation is stronger than just the inclusion. The proof of following theorem can be found in [35].

Theorem 3.1.10 (Koppelberg) *If a Boolean algebra $\mathfrak{A}$ is minimally generated, then $\mathfrak{A}$ contains a dense tree subalgebra $\mathfrak{B}$ such that $\mathfrak{A}$ is minimally generated over $\mathfrak{B}$.*

The topological conclusion is as follows. Recall that two topological spaces are *co–absolute* if the algebras of their regular open sets are isomorphic.

Theorem 3.1.11 *Let K be Koppelberg compact. Then the following conditions are fulfilled for every closed subspace F of K:*

(a) F is co–absolute with an ordered space (i.e. its algebra of regular open sets is isomorphic to the algebra of regular open sets of some ordered space);

(b) F has a tree π–base.

Proof. Both implications for $F = K$ are proved in [33]. By (b) of Proposition 3.1.7 we are done.

■

The class of spaces with tree π–bases is surprisingly wide. By the result due to Balcar, Pelant and Simon [4] (see Section 4.2 for more details) even $\beta\omega \setminus \omega$ has a tree π–base. This property is usually not inherited by all closed subspaces, though. It is the reason why we have formulated Theorem 3.1.11 in the above way. Nevertheless, it would be desirable to find some stronger conditions implied by minimal generation, in particular to have a topological characterization of the Koppelberg compacta. It could allow us to get rid of (artificial, in principle) assumption of zero–dimensionality in the definition without referring to the idea of inverse limits. We have not been able to exhibit any example of a space which is not Koppelberg compact such that every closed subspace and every continuous image of it has a tree π–base, but we believe the properties listed in Theorem 3.1.11 do not characterize the Koppelberg compacta.

It is worth here to recall the idea of *discretely generated* topological spaces (formulated by Dow, Tkachuk, Tkachenko and Wilson in [14]).

Definition 3.1.12 *A topological space X is called discretely generated if for every subset $A \subseteq X$ we have*

$$\mathrm{cl}(A) = \bigcup\{\mathrm{cl}(D) \colon D \subseteq A \text{ and } D \text{ is a discrete subspace of } X\}.$$

Problem 3.1.13 *Is every Koppelberg compactum discretely generated?*

One may ask when a given Boolean algebra $\mathfrak{A}$ has a proper minimal extension in a given algebra $\mathfrak{B} \supseteq \mathfrak{A}$. If $\mathfrak{B} = P(\mathrm{Stone}(\mathfrak{A}))$, then $\mathfrak{A}$ can be extended minimally by a point of its Stone space. On the other hand, in Section 5 we will consider only subalgebras of $P(\omega)$. In this case there do exist *maximal* minimally generated algebras, i.e. such subalgebras of $P(\omega)$ that no new subset of ω can extend them minimally. We present here a condition under which we can extend a Boolean algebra $\mathfrak{A}$ in $P(\mathrm{Stone}(\mathfrak{A}))$ in quite a natural way.

Lemma 3.1.14 *Let $(A_n)_{n\in\omega}$ be a disjoint sequence of clopen subsets of a compact space K converging to $p \in K$. Then we can extend $\mathfrak{A} = \mathrm{Clopen}(K)$ minimally by a set A of the form $A = \bigcup\{A_n : n \in T\}$, where T is an infinite co–infinite subset of ω. In particular, if $\mathfrak{B} \supseteq \mathfrak{A}$ is a σ–complete Boolean algebra, then we can extend $\mathfrak{A}$ minimally by an element of $\mathfrak{B}$.*

Proof. Let $Z = \bigcup_{n\in\omega} A_{2n}$. Of course, Z does not belong to $\mathfrak{A}$ as then either Z or Z^c would belong to p. $\mathfrak{A}(Z)$ splits the ultrafilter p but this is the only ultrafilter split by $\mathfrak{A}(Z)$.

Indeed, if $q \neq p$, then we have $B \in q$ such that $A_n \cap B = 0$ for almost all n. Let then

$$A = \bigcup\{A_n : A_n \cap B \neq \mathrm{O}\}.$$

Since $A \cap Z \in \mathfrak{A}$ either

- $A \cap Z \in q$ but then $(A \cap Z) \cap Z^c = 0$ so q can be extended only by Z or
- $(A \cap Z)^c \in q$. Thus, $B \cap (A \cap Z)^c \in q$ and $B \cap (A \cap Z)^c \cap Z = 0$ so we cannot extend q by Z.

■

Proposition 3.1.15 *If K is a compact space without isolated points and there is a G_δ point in K, then $\mathfrak{A} = \mathrm{Clopen}(K)$ can be extended minimally by an open F_σ subset of K.*

Proof. Assume p is a G_δ point in K. Enumerate by $(U_n)_{n\in\omega}$ a countable base of p. Let $A_0 = U_0 \setminus U_1$. For $n \in \omega$ let $A_{n+1} = \bigcup_{m\leq n} U_m \setminus U_{n+1}$. It is easy to check that $(A_n)_{n\in\omega}$ is a disjoint sequence converging to p. By Lemma 3.1.14 we are done.

■

It is easy to see that usually we can find many sequences witnessing that a Boolean algebra is minimally generated and these sequences can have different sizes. By the *length* of a minimally generated Boolean algebra $\mathfrak{A}$ we mean the least ordinal demonstrating the minimal generation of $\mathfrak{A}$.

3.2 Measures on minimally generated Boolean algebras

We recall several measure theoretic definitions. For a wider background the reader is referred to Fremlin's monograph [25].

By a measure on a Boolean algebra we mean a finitely additive function. We also occasionally mention Radon measures on topological spaces. If X is a topological space, then μ is a Radon measure on X if it is a σ–additive measure defined on the σ–algebra of Borel sets on X. We treat here only finite measures.

Let $\mathfrak{A}$ be a Boolean algebra and let K be its Stone space. Recall that every (finitely additive) measure on $\mathfrak{A}$ can be transferred to the algebra of clopen subsets of K and then extended to the unique Radon measure.

A measure μ on a Boolean algebra $\mathfrak{A}$ is *atomless* if for every $\varepsilon > 0$ there is a finite partition of 1 into elements of measure at most ε. In [10] such a measure is called "strongly continuous". Notice that there are different notions of atomlessness of measure, not necessarily equivalent to the above one.

We say that a measure μ on a topological space (a Boolean algebra) is *strictly positive* if $\mu(A) > 0$ for every nonempty open set (nonempty element of algebra) A. The following simple fact is proved in [35].

Fact 3.2.1 (Koppelberg) *If a Boolean algebra $\mathfrak{A}$ admits a strictly positive measure, then all trees in $\mathfrak{A}$ are countable.*

In Chapter 2 we mentioned the notion of Maharam type without defining it. We do it now.

Definition 3.2.2 *A measure μ on a Boolean algebra $\mathfrak{A}$ is said to be separable if there exists a countable $\mathcal{B} \subseteq \mathfrak{A}$ such that for every $A \in \mathfrak{A}$ and $\varepsilon > 0$ we have $B \in \mathcal{B}$ such that $\mu(A \triangle B) < \varepsilon$.*

A Radon measure satisfying the analogous condition is called a *measure of (Maharam) type* ω. The following two definitions are not so well–known as the above one.

Definition 3.2.3 *A measure* μ *on a Boolean algebra* $\mathfrak{A}$ *is uniformly regular if there is a countable set* $\mathcal{A} \subseteq \mathfrak{A}$ *such that* μ *is inner regular with respect to* $\mathcal{A}$ *(i.e., for every* $A \in \mathfrak{A}$ *and* $\varepsilon > 0$ *there is* $B \in \mathcal{A}$ *such that* $B \subseteq A$ *and* $\mu(A \setminus B) < \varepsilon$*). We say that* $\mathcal{A}$ *approximates* μ *from below.*

Sometimes such a measure is called "strongly countably determined", see [3] or [47] for further reading. The following simple modification of the above definition will be particularly useful.

Definition 3.2.4 *A measure* μ *on a Boolean algebra* $\mathfrak{A}$ *is weakly uniformly regular (w.u.r., for brevity) if there is a countable set* $\mathcal{A} \subseteq \mathfrak{A}$ *such that* μ *is inner regular with respect to the class* $\{A \setminus I \colon A \in \mathcal{A},\ \mu(I) = 0\}$*. We say that* $\mathcal{A}$ *weakly approximates* μ *from below.*

We can make this definition a little bit more understandable by switching to the topological point of view. A measure is weakly uniformly regular on $\mathrm{Clopen}(K)$, where K is a Boolean space, if the corresponding measure on K is uniformly regular on its support.

It is clear that the following implications hold:

$$\text{uniformly regular} \implies \text{weakly uniformly regular} \begin{cases} \implies \text{of Maharam type } \omega \\ \implies \text{has a separable support} \end{cases}$$

None of the above implications can be reversed. Consider the following examples:

(a) the usual 0–1 measure on the algebra $Fin\text{–}Cofin(\omega_1)$ is weakly uniformly regular but not uniformly regular;

(b) if $\mathfrak{A}$ is the algebra of Lebesgue measure on $[0, 1]$, then the standard measure on $\mathrm{Stone}(\mathfrak{A})$ is of Maharam type ω, its support is not separable, though, and thus it is not w.u.r.;

(c) the usual product measure on 2^{ω_1} has a separable support but is not of Maharam type ω (hence, is not w.u.r.).

We ought to remark here that example (b) exhibits one more property of uniform regularity. Notice that the Lebesgue measure on $[0,1]$ is uniformly regular but the measure from example (b) is not, although these measures has the same measure algebra. Hence, the uniform regularity of measure depends on its domain. This property plays no role in our considerations as we discuss here only measures on Boolean algebras and their Stone spaces.

Before we start an examination of measures on Koppelberg compacta, we prove a general theorem concerning the connections between uniformly regular measures and separable measures. Recall that if $\mathfrak{A}$ is contained in some larger algebra $\mathfrak{B}$, then every measure μ defined on $\mathfrak{A}$ can be extended to some measure ν defined on $\mathfrak{B}$. We say that $\mathfrak{A}$ is ν*–dense* in $\mathfrak{B}$ if

$$inf\{\nu(B \vartriangle A)\colon A \in \mathfrak{A}\} = 0$$

for every $B \in \mathfrak{B}$. We will need the following theorem due to Plachky (see [43]).

Theorem 3.2.5 (Plachky) *Let μ be a measure on a Boolean algebra $\mathfrak{B}$ containing an algebra $\mathfrak{A}$. The algebra $\mathfrak{A}$ is μ–dense in $\mathfrak{B}$ if and only if μ is an extreme point of the set*

$$\{\lambda\colon \lambda \textit{ is defined on } \mathfrak{B} \textit{ and } \lambda|\mathfrak{A} = \mu|\mathfrak{A}\}.$$

We use Plachky's criterion to prove the following result.

Lemma 3.2.6 *Let $\mathfrak{A}$ be a Boolean algebra carrying a measure μ. If $\mathfrak{A} \subseteq \mathfrak{B}$, then there is an extension of μ to a measure ν defined on $\mathfrak{B}$ such that $\mathfrak{A}$ is not ν–dense in $\mathfrak{B}$ if and only if there is $B \in \mathfrak{B}$ with the property $\mu_*(B) < \mu^*(B)$.*

Proof. Assume that $\mu_*(B_0) < \mu^*(B_0)$ for some $B_0 \in \mathfrak{B}$. It can be easily shown that the formulas

$$\mu'(B) = \mu^*(B \cap B_0) + \mu_*(B \setminus B_0),$$

$$\mu''(B) = \mu_*(B \cap B_0) + \mu^*(B \setminus B_0)$$

define extensions of μ to measures on the algebra $\mathfrak{A}(B_0)$. In turn, μ', μ'' can be extended to ν', ν'' on $\mathfrak{B}$. As $\nu' \neq \nu''$ it follows that $\nu = 1/2(\nu' + \nu'')$ is not an extreme extension, so by Plachky's criterion $\mathfrak{A}$ is not ν–dense in $\mathfrak{B}$.

The converse is obvious.

■

Theorem 3.2.7 *Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ carries either a uniformly regular measure or a measure which is not separable.*

Proof. Suppose that there is no uniformly regular measure on $\mathfrak{A}$. We construct a nonseparable measure ν defined on $\mathfrak{A}$. Namely, we construct a sequence of countable Boolean algebras $\{\mathfrak{B}_\alpha\colon \alpha < \omega_1\}$ and a sequence of measures $\{\mu_\alpha\colon \alpha < \omega_1\}$ such that for every $\alpha < \beta < \omega_1$ the following conditions are fulfilled:

- $\mathfrak{B}_\alpha$ carries μ_α;
- $\mathfrak{B}_\alpha \subseteq \mathfrak{B}_\beta \subseteq \mathfrak{A}$;
- μ_β extends μ_α;
- $\mathfrak{B}_\alpha$ is not μ_β–dense in $\mathfrak{B}_\beta$.

Assume that we have already constructed $\mathfrak{A}_\alpha$ and μ_α. We can extend μ_α to a measure τ on $\mathfrak{A}$. By our assumption, the measure τ is not uniformly regular so we can find an element A such that

$$\inf\{\tau(A \setminus U)\colon U \in \mathfrak{B}_\alpha,\, U \subseteq A\} > 0.$$

Set $\mathfrak{B}_{\alpha+1} = \mathfrak{B}_\alpha(A)$ and use Lemma 3.2.6 to find a measure $\mu_{\alpha+1}$ extending μ_α and such that $\mathfrak{B}_\alpha$ is not $\mu_{\alpha+1}$–dense in $\mathfrak{B}_{\alpha+1}$. At a limit step γ set $\mathfrak{B}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha$ and μ_γ to be the unique extension of all members of $\{\mu_\alpha\colon \alpha < \gamma\}$. Finally, set $\mathfrak{B} = \bigcup_{\alpha<\omega_1} \mathfrak{B}_\alpha$ and take the unique extension of all constructed μ_α's for μ. Every extension of μ to a measure ν on $\mathfrak{A}$ is not separable. ∎

We turn now to the proper topic of this section. First, we will see how a measure behaves when considered on a minimal extension of its domain.

Lemma 3.2.8 *Let μ be an atomless measure on a Boolean algebra $\mathfrak{A}$ and let $\mathfrak{B}$ be a minimal extension of $\mathfrak{A}$. Then for every $B \in \mathfrak{B}$ we have $\mu_*(B) = \mu^*(B)$.*

Proof. Consider $B \in \mathfrak{B}$ and $\varepsilon > 0$. We will show that $\mu^*(B) - \mu_*(B) < \varepsilon$. Assume that $(A_n)_{n<N}$ is a partition of $1_\mathfrak{A}$ witnessing that μ is atomless (for our ε). From Lemma 3.1.3 we deduce that there is only one $k < N$ such that $A_k \cap B \notin \mathfrak{A}$ (we exclude the trivial case of $B \in \mathfrak{A}$). Since

$$\sum_{k\neq n<N} \mu(A_n \cap B) = \mu(B \setminus A_k) \le \mu_*(B) \le \mu^*(B) \le \mu(B \setminus A_k) + \varepsilon,$$

we conclude that the desired inequality holds. As ε was arbitrary, $\mu_* = \mu^*$ on $\mathfrak{B}$.

$\blacksquare$

The above lemma expresses the fact that minimal extensions do not enrich atomless measures. This observation lies in the heart of the following facts.

Proposition 3.2.9 *If $\mathfrak{B}$ is minimally generated over $\mathfrak{A}$ and μ is a measure on $\mathfrak{B}$ such that $\mu|\mathfrak{A}$ is atomless and uniformly regular, then μ is uniformly regular.*

Proof. It is a direct consequence of Lemma 3.2.8.

$\blacksquare$

Theorem 3.1.9 allows us to see the following theorem as a generalization (of course only for the zero–dimensional case) of Theorem 9 of [11] (stating that every atomless measure on a monotonically normal space is of countable Maharam type) and of Theorem 3.2(i) of [52] (stating that every atomless measure on an ordered space is uniformly regular on its support).

Theorem 3.2.10 *Every measure μ on a minimally generated Boolean algebra $\mathfrak{A}$ is a sum of countably many weakly uniformly regular measures. Consequently, measures on minimally generated algebras are separable.*

Proof. Assume *a contrario* that there is a measure μ on $\mathfrak{A}$ which is not a sum of w.u.r. measures. Assume that the sequence $(\mathfrak{A}_\alpha)_{\alpha\leq\beta}$ witnesses that $\mathfrak{A}$ is minimally generated (where $\mathfrak{A}_\beta = \mathfrak{A}$) and let $\mu_\alpha = \mu|\mathfrak{A}_\alpha$ for every α. Denote

$$\kappa = \min\{\alpha\colon \mu_\alpha \text{ is not a sum of w.u.r. measures on } \mathfrak{A}_\alpha\}$$

and notice that $\mathrm{cf}(\kappa)$ is uncountable. Without loss of generality we can assume that μ_κ is atomless. If it is not then we can apply the Sobczyk–Hammer Decomposition Theorem (see Theorem 5.2.7 in [10]), i.e. split μ_κ into

$$\mu_\kappa = \nu_0 + \sum_{n\in\omega} a_n\nu_n,$$

where ν_0 is atomless and for $n \geq 1$ the measure ν_n is 0–1 valued. Of course each ν_n is weakly uniformly regular, so we can assume that $\mu_\kappa = \nu_0$. Denote now

$$\lambda = \min\{\alpha\colon \mu_\alpha \text{ is atomless}\}.$$

Of course $\lambda \leq \kappa$. Notice that $\mathrm{cf}(\lambda) = \aleph_0$. Indeed, if $\alpha(n)$ is the least ordinal such that there is a partition of 1 into sets from $\mathfrak{A}_{\alpha(n)}$ of μ–measure $< 1/n$, then μ on $\bigcup_{n\in\omega} \mathfrak{A}_{\alpha(n)}$ is atomless. Hence, $\lambda < \kappa$. But the measure μ_λ on $\mathfrak{A}_\lambda$ fulfils the conditions of Lemma 3.2.8 so for every $\alpha > \lambda$ the measure μ_α on $\mathfrak{A}_\alpha$ is a sum of w.u.r. measures, in particular so is μ on $\mathfrak{A}$, a contradiction.

Since every w.u.r. measure is separable and a sum of countably many separable measures is separable the second assertion follows.

■

The following corollary is proved directly in [33]. Recall that if we can map continuously a topological space K onto $\{0,1\}^{\omega_1}$, then there exists a measure of uncountable type on K (by Fremlin's theorem, under MA_{ω_1} the above conditions are in fact equivalent, see [20]). We should also remind here that a compact space K contains a copy of $\beta\omega$ if and only if it can be mapped continuously onto $\{0,1\}^{\mathfrak{c}}$. Now we can finally formulate the corollary.

Corollary 3.2.11 *If $\mathfrak{A}$ is a minimally generated Boolean algebra, then $\mathfrak{A}$ does not contain an uncountable independent sequence. Therefore,* Stone($\mathfrak{A}$) *cannot be mapped continuously onto $\{0,1\}^{\omega_1}$ and there is no copy of $\beta\omega$ in* Stone($\mathfrak{A}$).

It is worth to point out here one more remark. Some axioms (such as CH) imply the existence of examples of small Boolean algebras carrying non-separable measures. By Theorem 3.2.10 these examples turn out to be also examples of small but not minimally generated Boolean algebras.

The following fact can be easily deduced from the proof of Theorem 3.2.10.

Corollary 3.2.12 *Every atomless measure μ on a minimally generated Boolean algebra of length at most ω_1 is uniformly regular.*

We show that the above corollary cannot be strengthened in the obvious way.

Example 3.2.13 *There is a Boolean algebra of length at most $\omega_1 + \omega$ carrying an atomless measure which is not uniformly regular.*

Proof. Let $A(\omega_1)$ denote the Alexandrov compactification of ω_1 endowed with the discrete topology, i.e. the space $\omega_1 \cup \{\infty\}$ with the topology generated by $\{\alpha\}$ for $\alpha \in \omega_1$ and $\{\infty\} \cup (\omega_1 \setminus I)$ for finite sets I. Consider the algebra $\mathfrak{A} = \mathrm{Clopen}(A(\omega_1) \times C)$, where C is the Cantor set.

CLAIM 1. *The algebra $\mathfrak{A}$ is minimally generated.*

We can construct in a minimal way the algebra $\{0\} \times \mathrm{Clopen}(C)$ in the first ω steps. There are no obstacles (for the minimality of extensions) to repeat this construction for $\{1\} \times \mathrm{Clopen}(C)$ and proceed in this manner obtaining finally (in ω_1 steps) the algebra generated by sets of the form $\{\alpha\} \times K$, where $\alpha \in \omega_1$ and K is a clopen subset of C. Then we can add by minimal extensions all sets of the form $(\{\infty\} \cup \omega_1) \times K$, where K is a clopen subset of C. As a result, we obtain $\mathfrak{A}$.

Consider now the following measure μ on $\mathfrak{A}$:

$$\mu(A) = \lambda(A \cap (\{\infty\} \times C)),$$

where λ is the standard measure on C.

CLAIM 2. *The measure μ is atomless but not uniformly regular.*

Indeed, suppose that there is a countable family $\mathcal{A} \subseteq \mathfrak{A}$ approximating μ from below. For every $A \in \mathcal{A}$ of positive measure $\infty \in \pi(A)$, where $\pi \colon A(\omega_1) \times C \to A(\omega_1)$ is the projection to the first coordinate, so $\pi(A) = \omega_1 \setminus I_A$, where I_A is finite. Let

$$\alpha = sup \bigcup \{I_A \colon A \in \mathcal{A}\} + 1.$$

Let $B = (\{\infty\} \cup (\omega_1 \setminus \{\alpha\})) \times C$. It is easily seen that

- $B \in \mathfrak{A}$,
- $\mu(B) = 1$,
- there is no $A \in \mathcal{A}$ such that $\mu(A) > 0$ and $A \subseteq B$ (if $\mu(A) > 0$ and $A \in \mathcal{A}$ then by the definition of α we see that $\{\alpha\} \times C \subseteq A$).

■

From the above example we deduce that the length of a minimally generated algebra is not necessarily a cardinal number. The above algebra $\mathfrak{A}$ cannot be generated in ω_1 steps as then every atomless measure admitted by $\mathfrak{A}$ should be uniformly regular. Anyway, the following fact implies that the lengths of minimally generated algebras are limit ordinal numbers.

Proposition 3.2.14 *Let $\mathfrak{A}$ be a minimally generated subalgebra of a Boolean algebra $\mathfrak{C}$. Then the algebra $\mathfrak{A}(B)$ is minimally generated for every $B \in \mathfrak{C}$.*

Proof. Let $(A_\alpha)_{\alpha\in\kappa}$ be such that $\mathfrak{A}_{\alpha+1} = \mathfrak{A}_\alpha(A_\alpha)$ for every $\alpha < \lambda$, where $(\mathfrak{A}_\alpha)_{\alpha\in\lambda}$ is a sequence witnessing the minimal generation of $\mathfrak{A}$. We will construct a sequence of minimal extensions generating $\mathfrak{B} = \mathfrak{A}(B)$. Recall that an ordinal number λ is called *even* if it can be represented as $\lambda = \gamma + 2n$, where γ is a limit ordinal or 0 and $n \in \omega$. For such ordinals let $h(\gamma + 2n) = \gamma + n$.

Let $\mathfrak{B}_0 = \{0, 1, B, B^c\}$. Define

$$\mathfrak{B}_{\alpha+1} = \begin{cases} \mathfrak{B}_\alpha(B \cap A_{h(\alpha+2)}) & \text{if } \alpha \text{ is even;} \\ \mathfrak{B}_\alpha(A_{h(\alpha+1)}) & \text{else.} \end{cases}$$

At a limit step γ we set $\mathfrak{B}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{B}_\alpha$.

Our new sequence generates the desired algebra in a minimal way. Let ξ be even. Then $\mathfrak{B}_\xi$ is extended to $\mathfrak{B}_{\xi+1}$ by an element of the form $B \cap A$, where $A \in \mathfrak{A}$. We use Lemma 3.1.4. No element of $\mathfrak{A}\cap\mathfrak{B}_\xi$ can split $B\cap A$ into two elements not belonging to $\mathfrak{B}_\xi$. This holds because $\mathfrak{B}_{\xi+1}\cap\mathfrak{A}$ is minimally generated over $\mathfrak{B}_\xi \cap \mathfrak{A}$. Of course, B does not split $B \cap A$, too. It implies that no element of $\mathfrak{B}_\xi$ splits $B \cap A$ into two new elements, so our extension is minimal.

Similar arguments work for the case of odd ξ.

■

We will show now that the property of admitting only w.u.r. measures is closed under free products. By the result due to Sapounakis (see [52]) interval Boolean algebras admit only w.u.r. measures. It follows that Koppelberg's example $\mathfrak{C}$ mentioned on page 30 carries only w.u.r. measures (since it is a free product of interval algebras) but it is not minimally generated. Therefore, every measure on a minimally generated algebra is a countable sum of weakly uniformly regular measures but there is a Boolean algebra admitting only w.u.r. measures which is not minimally generated. Consequently, minimal generation cannot be characterized by any measure theoretic property mentioned in this section.

Theorem 3.2.15 *If every measure on a Boolean algebra $\mathfrak{A}$ is w.u.r. and every measure on $\mathfrak{B}$ is w.u.r., then every measure on $\mathfrak{A} \oplus \mathfrak{B}$ is w.u.r.*

Proof. For simplicity assume that the considered algebras are contained in $P(X)$ for some set X.

It is enough to show that we can weakly approximate from below all the rectangles since every member of $\mathfrak{A} \oplus \mathfrak{B}$ is a finite union of rectangles. Let

μ be a measure on $\mathfrak{A} \oplus \mathfrak{B}$. Define

$$\mu_1(A) = \mu(A \times X)$$

and for $A \in \mathfrak{A}$

$$\mu_A(B) = \mu(A \times B).$$

By the assumption the measure μ_1 is weakly uniformly regular so there is a countable set $\mathcal{A}$ weakly approximating μ_1 from below. For every $A \in \mathcal{A}$ the measure μ_A is also w.u.r. and has an approximating set $\mathcal{B}\ (A)$.

We will show that $\{A_0 \times B_0 \colon A_0 \in \mathcal{A}, B_0 \in \mathcal{B}\ (A_0)\}$ weakly approximates μ from below. Indeed, consider $A \in \mathfrak{A}, B \in \mathfrak{B}$ and $\varepsilon > 0$. Then by the definition we can find

- $A_0 \in \mathcal{A}$ such that $\mu_1(A \setminus A_0) < \frac{\varepsilon}{2}$ and $\exists F\ \mu_1(F) = 0, A_0 \setminus F \subseteq A$,
- $B_0 \in \mathcal{B}(A_0)$ such that $\mu_{A_0}(B \setminus B_0) < \varepsilon/2$ and $\exists G\ \mu_{A_0}(G) = 0, B_0 \setminus G \subseteq B$.

Now $\mu((A \times B) \setminus (A_0 \times B_0)) < \varepsilon$ since

$$(A \times B) \setminus (A_0 \times B_0) = A_0 \times (B \setminus B_0) \cup (A \setminus A_0) \times B$$

but

$$\mu(A_0 \times (B \setminus B_0)) = \mu_{A_0}(B \setminus B_0) < \varepsilon/2$$

and

$$\mu((A \setminus A_0) \times B) \leq \mu((A \setminus A_0) \times X) = \mu_1(A \setminus A_0) < \varepsilon/2.$$

It suffices to show that there exists an element H such that $\mu(H) = 0$ and $(A_0 \times B_0) \setminus H \subseteq (A \times B)$. Clearly, $H = (F \times X) \cup (A_0 \times G)$ is such an element.

■

We continue the measure theoretic examination of minimally generated Boolean algebras. The existence of uniformly regular measures on such algebras follows from Theorem 3.2.7 and Theorem 3.2.10. Anyway, such measures can be easily constructed directly using Theorem 3.1.10. Under certain conditions we can force these measures to have additional properties.

Theorem 3.2.16 *Let $\mathfrak{A}$ be an atomless minimally generated Boolean algebra. Then $\mathfrak{A}$ carries an atomless uniformly regular measure μ. Moreover, if any of the following conditions is fulfilled, then we can demand that μ is strictly positive as well:*

- *if $\mathfrak{A}$ carries a strictly positive measure;*
- *if $\mathfrak{A}$ is c.c.c. and the Suslin Conjecture is assumed;*
- *if $\mathfrak{A}$ is strongly c.c.c., i.e. it does not contain any uncountable set of pairwise incomparable elements.*

Proof. Let $T \subseteq \mathfrak{A}$ be a tree as in 3.1.10

We can easily find a countable dyadic tree $T_0 \subseteq T$. For an element $A \in T_0$ put $\mu(A) = 1/2^n$ if A belongs to the n–th level of T_0. In this way we obtain a measure defined on the algebra generated by T_0. It is atomless and uniformly regular, so by Lemma 3.2.8 its extension to ν defined on $\mathfrak{A}$ will be uniformly regular as well.

CLAIM. *If T can be assumed to be countable, then $\mathfrak{A}$ carries a strictly positive uniformly regular measure.*

Indeed, we can easily find a tree $T_0 \subseteq T$ isomorphic to $\omega^{<\omega}$ such that every level of T_0 forms a maximal antichain in $\mathfrak{A}$ and T_0 is dense in $\mathfrak{A}$. Define a strictly positive measure μ on T_0. By a similar argument as before the extension of μ to the measure ν on $\mathfrak{A}$ will be uniformly regular. Clearly, ν is strictly positive and the claim is proved.

To complete the proof we show that the assumptions listed above imply that T can be conceived as countable.

If $\mathfrak{A}$ carries a strictly positive measure then, according to Fact 3.2.1, every tree contained in $\mathfrak{A}$ is countable, and so is T.

If $\mathfrak{A}$ is c.c.c., then it does not contain neither an uncountable chain nor an uncountable antichain so every uncountable tree contained in $\mathfrak{A}$ is Suslin. Hence, the Suslin Conjecture implies that T is countable.

Finally, by the theorem of Baumgartner and Komjáth, if $\mathfrak{A}$ is strongly c.c.c., then it contains a countable dense subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$ (see [9] for the details). Therefore, the Stone space of $\mathfrak{A}$ is separable and thus it supports a strictly positive measure (for the proofs of the last implications we refer the reader to [53]).

■

It follows that in the class of Koppelberg compacta the property of having a strictly positive measure is equivalent to separability. If the Suslin Conjecture is assumed these properties are equivalent also to c.c.c. We can use these remarks to answer the question which seems to be natural in the context of Theorem 3.2.10.

Theorem 3.2.17 *There is a minimally generated Boolean algebra supporting a measure which is not w.u.r.*

Proof. Denote by $\mathfrak{B}$ the algebra of Lebesgue measure on $[0,1]$. Let $\mathfrak{A} \subseteq \mathfrak{B}$ be a minimally generated Boolean algebra such that for no $B \in \mathfrak{B} \setminus \mathfrak{A}$ the extension $\mathfrak{A}(B)$ is minimal over $\mathfrak{A}$. Notice that according to the proof of Theorem 3.1.15 and the completeness of $\mathfrak{B}$ no $p \in K = \text{Stone}(\mathfrak{A})$ is a G_δ point.

Since $\mathfrak{A}$ carries a strictly positive measure the space K is separable (by Theorem 3.2.16). Let $\{x_n \colon n \geq 1\}$ be dense in K. Consider the following measure:

$$\mu = \sum_{n \geq 1} \delta_{x_n}/2^n.$$

It is not w.u.r. Otherwise, it would be uniformly regular because μ is strictly positive. But δ_x is uniformly regular only if x is G_δ and there are no such points in K. Therefore, the measure δ_{x_1} is not uniformly regular and, accordingly, μ is not w.u.r.

∎

We finish this section with a short analysis of the behavior of measures on other well–known subclass of small Boolean algebras.

Definition 3.2.18 *A Boolean algebra $\mathfrak{A}$ is retractive if for every epimorphism $e \colon \mathfrak{A} \to \mathfrak{B}$ there is a monomorphism (lifting) $m \colon \mathfrak{B} \to \mathfrak{A}$ such that $e \circ m = id_{\mathfrak{B}}$.*

Notice that a Boolean algebra is retractive if and only if its Stone space K is co–retractive, i.e. every closed subspace of K is a retract of K. J. Donald Monk showed that no retractive Boolean algebra contains an uncountable independent sequence. It is also known that not every minimally generated algebra is retractive. In [35] Koppelberg gave an example of a retractive but not minimally generated Boolean algebra. However, the construction was carried out under CH. We present here an example of a retractive algebra which is not minimally generated and additionally carries a nonseparable measure. It requires the following assumption:

$$\text{cof}(\mathcal{N}) = min\{|\mathcal{A}| \colon \mathcal{A} \subseteq \mathcal{N} \ \forall N \in \mathcal{N} \ \exists A \in \mathcal{A} \ N \subseteq A\} = \omega_1,$$

where $\mathcal{N}$ denotes the ideal of Lebesgue measure zero sets. Of course CH implies $\text{cof}(\mathcal{N}) = \aleph_1$, on the other hand e.g. in the Sacks model $\mathfrak{c} = \aleph_2$ and, nevertheless, $\text{cof}(\mathcal{N}) = \aleph_1$.

In the following theorem we simply take advantage of the construction carried out by Plebanek in [44]. Recall that a Boolean space K is *Corson compact* if there exists a point–countable family $\mathcal{D}$ of clopen subsets of K such that $\mathcal{D}$ separates points of K. For our purposes it is important that separable Corson compacta are metrizable (see [2]).

Theorem 3.2.19 *Assume* $\mathrm{cof}(\mathcal{N}) = \aleph_1$. *Then there is a retractive Boolean algebra* $\mathfrak{A}$ *carrying a nonseparable measure and without a tree* π*–base.*

Proof. The equality $\mathrm{cof}(\mathcal{N}) = \aleph_1$ implies the existence of a Corson compact space K carrying a strictly positive nonseparable measure μ such that for every nowhere dense $F \subseteq K$ the set F is metrizable (see [44]).

To verify the retractiveness of Boolean algebra $\mathfrak{A} = \mathrm{Clopen}(K)$ one needs only to check if for every dense ideal $I \subseteq \mathfrak{A}$ the algebra $\mathfrak{A}/I$ is countable (see Theorem 4.3(c) in [49]). If an ideal I is dense, then $F = \mathrm{Stone}(\mathfrak{A}/I)$ is a closed nowhere dense subspace of K. Thus, it is metrizable. So $\mathfrak{A}/I$ is countable.

Assume now for a contradiction that $\mathfrak{A}$ has a tree π–base T. Since μ is strictly positive, by Fact 3.2.1, T has to be countable. Thus, K is separable and, since it is Corson compact, K is metrizable. It follows that every measure on K is of countable Maharam type, a contradiction.

■

On the other hand, as we have already mentioned, it is consistent to assume that small Boolean algebras carry only separable measures. Combining Fremlin's theorem mentioned on page 39 and the fact that retractive algebras are small we obtain the following.

Theorem 3.2.20 *If* MA_{ω_1} *holds, then retractive algebras admit only separable measures.*

It is not known if it is consistent to assume that every retractive Boolean algebra is minimally generated (or, at least, has a tree π–base).

3.3 Remarks

The study of measures on minimally generated Boolean algebras is the essential part of this section. The comparison of Theorem 3.2.10 and Theorem 3.2.17 indicates that our description of the behavior of measures on this class of Boolean algebras is quite precise. We have seen that the measure theoretic research can be used to prove some general theorems about minimally generated Boolean algebras.

In Section 3.1 we asked if we can find a reasonable topological characterization of Koppelberg compact spaces (without referring to inverse limits). On page 33 we asked several more detailed questions concerning this subject; see e.g. Problem 3.1.13.

4 Koppelberg compacta with additional properties

The aim of this chapter is to exhibit some applications of the notion of minimally generated Boolean algebras. Section 4.1 is based on one of the sections of [7], while the work presented in Section 4.2 is still in progress and, thus, the section is not particularly conclusive.

Section 4.1 deals with the connection between Koppelberg compacta and Efimov spaces, where by a Efimov space we mean a compact space that neither contains a nontrivial convergent sequence nor a copy of $\beta\omega$. It is not known if such spaces can be constructed in ZFC. However, many constructions of such spaces were carried out in several models of ZFC. Most of them (see [13, 15, 16]) use, explicitly or not, the notion of minimally generated Boolean algebra. Section 4.1 discusses this topic. We do not exhibit any new Efimov space, but we try to locate potential Efimov spaces within the class of Koppelberg compacta. We give here alternative and quite simple proof of Haydon's theorem stating that there is a compact but not sequentially compact space without a nonseparable measure. We finish with a construction of a *Efimov–like* space not involving minimally generated algebras.

In Section 4.2 we try to use the notion of Koppelberg compact space for a construction of a peculiar Banach space. The motivation for this section comes from the problem posed by Grzegorz Plebanek in [45]: is there a Banach space with the Mazur property but without the Gellfand–Philips property? We investigate a certain cardinal coefficient connected to the filter of density 1 sets. It is shown that an assumption on this coefficient implies a positive answer to Plebanek's question. A space of continuous functions on a certain Koppelberg compact space constitutes the desired example. Unfortunately, it is unclear whether this assumption is consistent with ZFC. We show that in Hechler's model there exists a Koppelberg compact space with a slightly weaker property.

4.1 Efimov spaces

We recall the longstanding Efimov problem.

Problem 4.1.1 *Is there an infinite compact space which neither contains a nontrivial convergent sequence nor a copy of $\beta\omega$?*

Such spaces (we call them *Efimov spaces*) can be constructed if certain set theoretic axioms are assumed. The question if one can construct a Efimov space in ZFC is still unanswered. For example, it is not known if Martin's Axiom implies the existence of Efimov spaces.

Consider a sequence $(r_n)_{n\in\omega}$ and a subsequence $(l_n)_{n\in\omega}$ in a topological space X. We say that $K \subseteq X$ separates $L = \{l_n : n \in \omega\}$ in $R = \{r_n : n \in \omega\}$ if $R \cap K = L$. To make a Boolean space Efimov we have to add many clopen sets to ensure that every sequence of distinct points has a subsequence separated by a clopen set. On the other hand, if our space is too rich, then it contains a sequence all of whose subsequences are separated and, thus, it would contain a copy of $\beta\omega$.

By Corollary 3.2.11 minimal generation gives us a tool for constructing compact zero–dimensional spaces without copies of $\beta\omega$. Fedorčuk's Efimov space (see [16]) has been constructed using simple extensions as well as the example presented by Dow in [13]. The first one requires CH, the latter a certain axiom connected to the notion of *splitting number*. For another construction (using $\diamondsuit$) see also [35].

We consider compactifications of ω. Notice at once that if there exists a Efimov space, then by taking the closure of countable discrete subspace we can obtain a compactification of ω which is Efimov.

We will employ the idea of pseudo–intersection number. Write $A \subseteq^* B$ if $A \setminus B$ is finite. We say that $P \subseteq X$ is a *pseudo–intersection* for a family $\mathcal{P} \subseteq P(X)$ provided for every $A \in \mathcal{P}$ we have $P \subseteq^* A$. A family $\mathcal{P}$ is said to have strong finite intersection property (*sfip* for brevity) if every finite subfamily has an infinite intersection. The definition of the *pseudo–intersection number* is as follows

$$\mathfrak{p} = min\{|\mathcal{P}| : \mathcal{P} \subseteq [\omega]^\omega \text{ has } \textit{sfip} \text{ but no } X \in [\omega]^\omega \text{ is a pseudo–intersection for } \mathcal{P}\}.$$

The assumption $\mathfrak{p} = \mathfrak{c}$ is equivalent to Martin's Axiom for σ–centered families (see, e.g., [27]).

For a topological space X and a cardinal α we say that $S \subseteq X$ is G_α if there is a family of open sets $\{U_\xi : \xi \in \alpha\}$ such that $S = \bigcap_{\xi\in\alpha} U_\xi$. It is convenient to say that S is $G_{<\alpha}$ if there is a $\beta < \alpha$ such that S is G_β.

Theorem 4.1.2 *There is a Koppelberg compactification K of ω without a convergent sequence of distinct $G_{<\mathfrak{p}}$ points. In particular, if MA is assumed, then K does not contain a convergent sequence of distinct $G_{<\mathfrak{c}}$ points.*

Proof. We will indicate a Koppelberg compactification of ω without a convergent subsequence of ω such that no point of its remainder is $G_{<\mathfrak{p}}$. We first show two claims.

CLAIM 1. *Let $\mathfrak{A}$ be a subalgebra of $P(\omega)$ containing the algebra Fin–$Cofin$. Then there is a nontrivial convergent subsequence of ω in $K=$ Stone$(\mathfrak{A})$ if and only if there is $p\in K$ with an infinite pseudo–intersection.*

Indeed, assume that a sequence $(n_k)_{k\in\omega}$ converges to p. Thus, for every $A\in p$ we have $N=\{n_0,n_1,...\}\subseteq^* A$ and, consequently, N is a pseudo–intersection of p. Conversely, an enumerated pseudo–intersection of p forms a subsequence of ω convergent to p.

CLAIM 2. *Let $\mathfrak{A}$ be an algebra minimally generated over Fin–$Cofin$ with an ultrafilter p with infinite pseudo–intersection P. Then $\mathfrak{A}(P)$ is a minimal extension of $\mathfrak{A}$.*

It is so because for every $A\in\mathfrak{A}$ either $A\cap P\in Fin$ or $P\subseteq^* A$ and, therefore, either $A\cap P\in\mathfrak{A}$ or $A^c\cap P\in\mathfrak{A}$. By Lemma 3.1.3 we are done.

Let $\mathfrak{A}\subseteq P(\omega)$ be a Boolean algebra minimally generated over Fin–$Cofin$ such that $\mathfrak{A}(A)$ is not a minimal extension of $\mathfrak{A}$ for any $A\in P(\omega)\setminus\mathfrak{A}$. By Claim 2 no $p\in K=$ Stone$(\mathfrak{A})$ has an infinite pseudo–intersection and by Claim 1 there is no convergent subsequence of ω in K. Since no $p\in K\setminus\omega$ is a $G_{<\mathfrak{p}}$ point and K is Koppelberg compact, we are done. ∎

As a corollary we get the following theorem proved by Haydon in [29].

Corollary 4.1.3 (Haydon) *There is a compact space which is not sequentially compact but which carries no measure of uncountable type.*

Proof. Let K be as in Theorem 4.1.2. Then the natural numbers form a sequence witnessing that K is not sequentially compact. By Theorem 3.2.10 every measure on K has a countable Maharam type. ∎

In fact, as can easily be seen in the proof of Theorem 4.1.2, every Boolean algebra $\mathfrak{A}$ minimally generated over Fin–$Cofin$ can be extended to $\mathfrak{B}\subseteq P(\omega)$ such that Stone$(\mathfrak{B})$ fulfills the conditions of Theorem 4.1.2 and Corollary 4.1.3. Thus, we can produce a lot of examples of such spaces.

Moreover, using Theorem 3.1.10 we can easily indicate tree algebras with the same property as in the above theorems. In fact, tree algebras can be unexpectedly rich. By the theorem already mentioned in Section 2 there is a tree algebra $\mathfrak{A}$ dense in $P(\omega)/Fin$, i.e. such that for every infinite $N \subseteq \omega$ there is an infinite set $M \subseteq^* N$ such that $M \in \mathfrak{A}$.

Theorem 4.1.2 can be counterpointed by the following theorem. Let us say that a compact space K is *Grothendieck* if $C(K)$ is Grothendieck, i.e. if every weak* convergent sequence in the space $C^*(K)$ weakly converges, which means that in a sense $C^*(K)$ does not contain nontrivial convergent sequences of measures and, thus, there is no nontrivial convergent sequences of points in K (as the convergence of $(x_n)_{n\in\omega}$ is equivalent to the convergence of $(\delta_{x_n})_{n\in\omega}$). So, the notion of a Grothendieck space is a strengthening of the property of not containing nontrivial convergent sequences.

Definition 4.1.4 *Let $\mathcal{F}$ be a family of subsets of a compact space K. We say that K contains a copy of $\beta\omega$ consisting of $\mathcal{F}$ sets if there is a disjoint sequence $(F_n)_{n\in\omega}$ of elements of $\mathcal{F}$ such that for every $T \subseteq \omega$ there is $A \in \mathrm{Clopen}(K)$ such that*

$$A \cap \bigcup_{n\in\omega} F_n = \bigcup_{n\in T} F_n.$$

Denote by (*) the following assumption:

$$2^\kappa \leq \mathfrak{c} \text{ if } \kappa < \mathfrak{c}.$$

Recall that (*) implies that $\mathfrak{c}$ is regular and that MA implies (*). We prove the following theorem.

Theorem 4.1.5 *There is a Grothendieck space not containing copies of $\beta\omega$ consisting of G_δ sets. Moreover, if (*) is assumed, then there is a Grothendieck space without copies of $\beta\omega$ consisting of $G_{<\mathfrak{c}}$ sets.*

Thus, although it is not known if one can construct a Efimov space under MA, some sorts of Efimov spaces can be, nevertheless, indicated: either if we admit the existence of a convergent sequence of $G_\mathfrak{c}$ points or if we admit $\beta\omega$ to be embeddable but only in such a way that natural numbers are mapped on $G_\mathfrak{c}$ sets.

In fact, our construction has a slightly stronger property. We say that a Boolean algebra $\mathfrak{A}$ has the *Subsequential Completeness Property* (*SCP*, for brevity) if for every disjoint sequence in $\mathfrak{A}$ there is an infinite co–infinite subset $T \subseteq \omega$ such that $(A_n)_{n\in T}$ has a least upper bound in $\mathfrak{A}$. A compact space K has SCP if Clopen(K) has SCP. Haydon showed that the spaces with SCP are Grothendieck (see [29]).

Definition 4.1.6 *Let $\mathfrak{A}$ be a Boolean algebra. Let $R = \{F_n : n \in \omega\}$ be a set of filters on $\mathfrak{A}$ and let $L = \{F_n : n \in T\}$ for some $T \subseteq \omega$. We say that $A \in \mathfrak{A}$ separates L in R if*

- *$A \in F_n$ for $n \in T$,*
- *$A^c \in F_n$ for $n \notin T$.*

The algebra $\mathfrak{A}$ separates L in R if there is $A \in \mathfrak{A}$ separating L in R.

Notice that a sequence $(F_n)_{n\in\omega}$ of closed sets in Stone$(\mathfrak{A})$ is a copy of $\beta\omega$ if and only if every subsequence of $(F_n)_{n\in\omega}$ is separated in $(F_n)_{n\in\omega}$ by $\mathfrak{A}$.

Thus, the assertion that K does not contain copies of $\beta\omega$ consisting of Clopen sets has a simple algebraic interpretation. It means that for every pairwise disjoint sequence $(A_n)_{n\in\omega}$ from $\mathfrak{A} = $ Clopen(K) the algebra $\mathfrak{A}$ contains a least upper bound of $(A_n)_{n\in T}$ for some infinite co–infinite $T \subseteq \omega$ but there is also $N \subseteq \omega$ such that $(A_n)_{n\in N}$ is non–separated in $(A_n)_{n\in\omega}$ by $\mathfrak{A}$.

The construction proceeds as follows, in the spirit of Haydon's construction from [29].

Consider a Boolean algebra $\mathfrak{A}$ and a sequence $(F_n)_{n\in\omega}$ of filters on $\mathfrak{A}$. We will say that a sequence $(p_n)_{n\in\omega}$ is *an extension of $(F_n)_{n\in\omega}$ in $\mathfrak{A}$* if p_n is an extension of F_n to an ultrafilter in $\mathfrak{A}$ for every $n \in \omega$. We will use the following trivial observation.

Fact 4.1.7 *Let R be a sequence of filters on a Boolean algebra $\mathfrak{A}$ with a subsequence L separated in R by $\mathfrak{A}$. If R' and L' are extensions of R and L in $\mathfrak{A}$ then L' is still separated in R' by $\mathfrak{A}$.*

Before we prove Theorem 4.1.5 we have to show the following lemma.

Lemma 4.1.8 *Let $\mathfrak{A} \subseteq P(X)$ be a Boolean algebra. Assume that $\{(L_\alpha, R_\alpha) : \alpha < \kappa < \mathfrak{c}\}$ is such that R_α is a nontrivial sequence in* Stone$(\mathfrak{A})$ *and L_α is its subsequence no separated in R_α by $\mathfrak{A}$ for every $\alpha < \kappa$. Let $(A_n)_{n\in\omega}$ be a disjoint sequence in $\mathfrak{A}$. Then there is an infinite, co–infinite $\sigma \subseteq \omega$ and a collection $\{(L'_\alpha, R'_\alpha) : \alpha < \kappa < \mathfrak{c}\}$ such that for every $\alpha < \kappa$ and $n \in \omega$ we have: $R'_\alpha(n)$ is an extension of $R_\alpha(n)$ to an ultrafilter in $\mathfrak{A}(\bigcup_{n\in\sigma} A_n)$, L'_α is the corresponding subsequence of R'_α and $\mathfrak{A}(\bigcup_{n\in\sigma} A_n)$ does not separate L'_α in R'_α.*

Proof. For $\sigma \subseteq \omega$ denote

$$A_\sigma = \bigcup_{n \in \sigma} A_n.$$

Consider the algebras $\mathfrak{A}(A_\sigma)$ for $\sigma \subseteq \omega$. Fix $\alpha < \kappa$ and $n \in \omega$. We define $R^\sigma_\alpha(n)$ in the following way. If A_σ does not split the ultrafilter $\mathcal{F} = R_\alpha(n)$, then $R^\sigma_\alpha(n)$ is the unique extension of $\mathcal{F}$ in $\mathfrak{A}(A_\sigma)$. If A_σ splits $\mathcal{F}$, then let $R^\sigma_\alpha(n)$ be defined as the extension of $\mathcal{F}$ by A^c_σ. $L^\sigma_\alpha(n) = R^\sigma_\alpha(m)$ if $L_\alpha(n) = R_\alpha(m)$.

Consider an almost disjoint family Σ of infinite subsets of ω of cardinality $\mathfrak{c}$. We show that there is $\sigma \in \Sigma$ such that no L^σ_α is separated in R^σ_α by $\mathfrak{A}(A_\sigma)$. Suppose otherwise; then by a cardinality argument, there are $\alpha < \kappa$, $\sigma, \tau \in \Sigma$ and $U_1, U_2 \in \mathfrak{A}$ such that $\sigma \neq \tau$ and

$$Z_\sigma = (A_\sigma \cap U_1) \cup (A^c_\sigma \cap U_2) \text{ separates } L^\sigma_\alpha \text{ in } R^\sigma_\alpha$$

and

$$Z_\tau = (A_\tau \cap U_1) \cup (A^c_\tau \cap U_2) \text{ separates } L^\tau_\alpha \text{ in } R^\tau_\alpha.$$

Set

$$A = (A_{\sigma\cap\tau} \cap U_1) \cup (A^c_{\sigma\cap\tau} \cap U_2),$$

and notice that $A \in \mathfrak{A}$ (as $\sigma \cap \tau$ is finite). It suffices to show that the set A separates L_α in R_α.

Consider $\mathcal{F} = L_\alpha(n)$ for some $\alpha < \kappa$ and $n \in \omega$. We show that $A \in \mathcal{F}$. Denote $\mathcal{F}^\sigma = L^\sigma_\alpha(n)$ and $\mathcal{F}^\tau = L^\tau_\alpha(n)$. Obviously, $Z_\sigma \in \mathcal{F}^\sigma$ and $Z_\tau \in \mathcal{F}^\tau$. It means that either $Z^1_\sigma = (A_\sigma \cap U_1) \in \mathcal{F}^\sigma$ or $Z^2_\sigma = (A^c_\sigma \cap U_2) \in \mathcal{F}^\sigma$ and either $Z^1_\tau = (A_\tau \cap U_1) \in \mathcal{F}^\tau$ or $Z^2_\tau = (A^c_\tau \cap U_2) \in \mathcal{F}^\tau$. To show that $A \in \mathcal{F}$ we have to consider three cases. We will repeatedly use basic properties of ultrafilters.

1. If $Z^1_\sigma \in \mathcal{F}^\sigma$ and $Z^2_\tau \in \mathcal{F}^\tau$ or $Z^2_\sigma \in \mathcal{F}^\sigma$ and $Z^1_\tau \in \mathcal{F}^\tau$, then both U_1, U_2 belong to $\mathcal{F}$ and, since either $A_{\sigma\cap\tau} \in \mathcal{F}$ or $A^c_{\sigma\cap\tau} \in \mathcal{F}$, $A \in \mathcal{F}$.

2. If $Z^2_\sigma \in \mathcal{F}^\sigma$ and $Z^2_\tau \in \mathcal{F}^\tau$, then the set $A_{\sigma\cap\tau}$ cannot belong to $\mathcal{F}$ (because then $\emptyset \in \mathcal{F}^\sigma$), so $A^c_{\sigma\cap\tau} \in \mathcal{F}$ but $U_2 \in \mathcal{F}$ and, therefore, $A \in \mathcal{F}$.

3. Assume that $Z^1_\sigma \in \mathcal{F}^\sigma$ and $Z^1_\tau \in \mathcal{F}^\tau$. Notice first that in this case $\mathcal{F}^\sigma, \mathcal{F}^\tau$ have to be unique extensions of $\mathcal{F}$ (by A_σ, A_τ respectively). Therefore, $\mathcal{F}$ has a unique extension in $\mathfrak{A}(A_\sigma, A_\tau)$ and $Z^1_\sigma \cap Z^1_\tau = A_{\sigma\cap\tau} \cap U_1$ belongs to this extension. But $A_{\sigma\cap\tau} \cap U_1 \in \mathfrak{A}$ and, again, $A \in \mathcal{F}$.

Similar methods are used to prove that for every element $\mathcal{F}$ of R_α not belonging to L_α we have $A \notin \mathcal{F}$. Hence, A separates L_α in R_α, a contradiction.

It follows that there is $\sigma \in \Sigma$ such that $\mathfrak{A}(A_\sigma)$ does not separate $L'_\alpha = L^\sigma_\alpha$ in $R'_\alpha = R^\sigma_\alpha$ for $\alpha < \kappa$; σ is infinite and co–infinite.

■

Proof. (of Theorem 4.1.5) Let $\rho\colon \mathfrak{c} \to \mathfrak{c} \times \mathfrak{c}$ be a surjection such that if $\rho(\alpha) = (\gamma, \beta)$, then $\gamma \leq \alpha$ and $\rho(0) = (0, 0)$. We construct an increasing sequence of Boolean algebras $(\mathfrak{A}_\alpha)_{\alpha\in\mathfrak{c}}$ each of size less than $\mathfrak{c}$. For every $\xi < \mathfrak{c}$ fix

- an enumeration $\{\mathcal{A}^\xi_\beta \colon \beta < \mathfrak{c}\}$ of disjoint sequences in $\mathfrak{A}_\xi$,
- an enumeration $\{S^\xi_\beta \colon \beta < \mathfrak{c}\}$ of disjoint sequences of G_δ sets in $\mathrm{Stone}(\mathfrak{A}_\xi)$.

Let $\mathfrak{A}_0 = \mathrm{Clopen}(2^\omega)$. Fix $R^0_0(0)$ to be an extension of S^0_0 in $\mathfrak{A}_0$ and $L^0_0(0)$ to be some non–separated subsequence.

Assume that $\mathfrak{A}_\alpha$ is constructed and we have a family $\{(L^\xi_\delta(\alpha), R^\xi_\delta(\alpha))\colon (\xi, \delta) = \rho(\eta), \eta < \alpha\}$ of sequences of ultrafilters and their non–separated subsequences. Let $\rho(\alpha) = (\gamma, \beta)$. Define $R^\gamma_\beta(\alpha)$ to be an extension of S^γ_β in $\mathfrak{A}_\alpha$. Fix a subsequence $L^\gamma_\beta(\alpha)$ non–separated by $\mathfrak{A}_\alpha$ (such a subsequence exists since $|\mathfrak{A}_\alpha| < \mathfrak{c}$). Apply Lemma 4.1.8 to the sequence $\mathcal{A}^\gamma_\beta$ and to $\{(L^\xi_\delta(\alpha), R^\xi_\delta(\alpha))\colon (\xi, \delta) = \rho(\eta), \eta \leq \alpha\}$ to produce $\mathfrak{A}_{\alpha+1}$. Let $L^\xi_\delta(\alpha + 1) = L'_{\rho^{-1}(\xi,\delta)}$ and $R^\xi_\delta(\alpha + 1) = R'_{\rho^{-1}(\xi,\delta)}$ for every pair (ξ, δ) such that there is $\eta \leq \alpha$ and $\rho(\eta) = (\xi, \delta)$.

On a limit step α take $\mathfrak{A}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta$. Set $R^\xi_\delta(\alpha)$ to be the unique extension of all $R^\xi_\delta(\beta)$ and $L^\xi_\delta(\alpha)$ to be the unique extension of all $L^\xi_\delta(\beta)$ for $\beta < \alpha$ and pair (ξ, δ) such that there is $\eta < \alpha$ and $\rho(\eta) = (\xi, \delta)$. It is easy to see that in this way the limit steps preserve the property that L^ξ_δ is non–separated in R^ξ_δ.

Finally, let $\mathfrak{A} = \bigcup_{\alpha<\mathfrak{c}} \mathfrak{A}_\alpha$ and $K = \mathrm{Stone}(\mathfrak{A})$. We demonstrate that K satisfies all the required conditions.

Indeed, it is easy to see that K has SCP (and, therefore, is Grothendieck). If $\mathcal{A} = \{A_n \colon n \in \omega\}$ is a disjoint sequence in $\mathfrak{A}$, then there is $\alpha < \mathfrak{c}$ such that $A_n \in \mathfrak{A}_\alpha$ for every n. It is then enumerated as $\mathcal{A}^\alpha_\beta$ for some β and, thus, $\bigcup_{n\in N} A_n$ is added at step $\rho^{-1}(\alpha, \beta)$, for some infinite N.

Similarly, consider a disjoint sequence $(F_n)_{n\in\omega}$ of closed G_δ sets together with fixed countable bases. Since the cofinality of $\mathfrak{c}$ is uncountable all elements of these bases appear in $\mathfrak{A}_\alpha$ for some $\alpha < \mathfrak{c}$. The sequence $(F_n| \, \mathfrak{A}_\alpha)_{n\in\omega}$ is labeled as R^α_β for some β. From that point using Fact 4.1.7 we bother to keep L^α_β not separated in R^α_β. Therefore, $(F_n)_{n\in\omega}$ is not a copy of $\beta\omega$.

If we assume (*), then for every $\xi < \mathfrak{c}$ the set of disjoint sequences of closed sets in $\mathfrak{A}_\xi$ is of cardinality $\mathfrak{c}$. Therefore, for every $\xi < \mathfrak{c}$ we can think about $\{S^\xi_\beta\colon \beta < \mathfrak{c}\}$ as being an enumeration of disjoint sequences of closed sets in $\mathfrak{A}_\xi$. Since (*) also implies that $\mathfrak{c}$ is regular, the above proof shows that K does not contain copies of $\beta\omega$ consisting of $G_{<\mathfrak{c}}$ sets.

■

By Argyros's theorem (see [30]), every Boolean algebra with SCP contains an independent sequence of size ω_1, so K from the above theorem is not Koppelberg compact and, what is more important, under CH $\beta\omega$ is embeddable in K. Therefore, one cannot hope that the above example will turn out to be a Efimov space in ZFC.

4.2 Condensed and feeble filters

In the previous section we have seen that one can quite easily construct a Koppelberg compactifications of ω without convergent subsequences of ω. Now we will try to construct such spaces with some additional properties. This section is an interim report on a work in progress and, thus, it is more laconic than the previous ones.

In [45] Grzegorz Plebanek asked the following question:

(GM) *Is there a Banach space satisfying the Mazur property but without the Gellfand–Philips property?*

Defining the above properties exceeds the scope of this section. The reader is referred to [45] for the definitions and background. We will consider a statement in the language of Boolean algebras, which implies a positive answer to the above question.

Recall that *the asymptotic density* of a set $A \subseteq \omega$, denoted here by $d(A)$, is defined as

$$d(A) = \lim_{n\to\infty} \frac{|A \cap n|}{n},$$

provided this limit exists. For an infinite $B = \{b_0 < b_1 < b_2 < \dots\} \subseteq \omega$ let $d_B(A) = d(\{n\colon b_n \in A\})$ if this limit exists. We say that $A \subseteq \omega$ is *a condenser* for a filter $\mathcal{F}$ on ω if $d_A(F) = 1$ for every $F \in \mathcal{F}$. Notice that the idea of condenser is in a sense analogous to the idea of pseudo–intersection. However, it is unclear if the property of having a condenser is preserved by bijective images. This motivated us to introduce the following, more general (or perhaps equivalent), notion: a filter $\mathcal{F}$ is *condensed* if there is a bijection $f: \omega \to \omega$ such that $f[F]$ is of density 1 for every $F \in \mathcal{F}$.

Problem 4.2.1 *Is there a minimally generated Boolean subalgebra $\mathfrak{A}$ of $P(\omega)$ such that no $x \in K = \mathrm{Stone}(\mathfrak{A})$ has an infinite pseudo–intersection but all $x \in K \setminus \omega$ are condensed (as filters on ω)?*

If K is as above, then it has the following properties:

(a) no $x \in K \setminus \omega$ is a limit of a sequence of natural numbers in K;

(b) for every $x \in K$ there is a sequence $(n_i)_{i\in\omega}$ of natural numbers such that
$$\delta_x = \lim_{k\to\infty} \frac{1}{k} \sum_{1\leq i\leq k} \delta_{n_i},$$
where "lim" is meant in the sense of *weak** topology.

There is no nontrivial convergent subsequence of ω in K (see the previous section) and thus (a) is satisfied. To see (b) notice that for every $A \in x$ we have
$$\lim_{k\to\infty} \frac{1}{k} \sum_{1\leq i\leq k} \delta_{f^{-1}(i)}(A) = 1,$$
where f witnesses that x is condensed. It follows directly from the equality $\delta_{f^{-1}(i)}(A) = \delta_i(f[A])$ and the definition of density function. The sequence $(f^{-1}(i))_{i\in\omega}$ satisfies the claimed property.

The properties listed above imply that if K is as in Problem 4.2.1, then $C(K)$ is as in Problem (GM). We will not prove it here. However, it is worth mentioning that both these properties refer to the space of the finite measures on K with the *weak** topology. Property (a) ensures that $C(K)$ does not have the Gellfand–Philips property, while (b) and Theorem 3.2.10 imply that each measure on K can be "approximated" by the measures of the form δ_n, for $n \in \omega$. This is a sufficient condition for the Mazur property.

Problem 4.2.1 has a negative answer, e.g. if CH is assumed. Indeed, then K has to be of cardinality $2^{\mathfrak{c}}$ if all convergent subsequences of ω have to be killed. But there are only $\mathfrak{c}$ many bijections on ω, so only $\mathfrak{c}$ many elements of K can be condensed.

We investigate if it is consistent to assume that there exists K as in Problem 4.2.1. Consider the following cardinal coefficient, analogous to the pseudo–intersection number:

$$\mathfrak{k} = \min\{|\mathcal{A}|: \mathcal{A} \text{ generates a non–condensed filter on } \omega\}.$$

We try to place $\mathfrak{k}$ among well–known cardinal coefficients. We need a couple of definitions.

Definition 4.2.2 *A filter $\mathcal{F} \subseteq P(\omega)$ is feeble provided there is a finite–to–one function $f\colon \omega \to \omega$ such that $f[F]$ is co–finite for every $F \in \mathcal{F}$.*

Fact 4.2.3 *For a filter $\mathcal{F} \subseteq P(\omega)$ the following implications hold:*

(a) if $\mathcal{F}$ has an infinite pseudo–intersection, then it is condensed;

(b) if $\mathcal{F}$ is condensed, then it is feeble.

Proof. Assume that $\mathcal{F} \subseteq P(\omega)$ is a filter and P is a pseudo–intersection of $\mathcal{F}$. Every bijection $f\colon \omega \to \omega$ such that $f[P]$ is of density 1 witnesses that $\mathcal{F}$ is condensed.

To prove the second implication notice that the function $f(n) = [\log_2(n)]$ demonstrates the feebleness of the filter of density 1 sets. Indeed, if $A \subseteq \omega$ is not co–finite, then $f^{-1}[A]$ cannot have density greater than 1/2. If a filter $\mathcal{F}$ is condensed, then there is a bijection $g\colon \omega \to \omega$ such that $g[F]$ is of density 1. Hence, $f \circ g$ proves that $\mathcal{F}$ is feeble. ∎

Recall that *the bounding number* $\mathfrak{b}$ is the smallest cardinality of any unbounded family in ω^ω (with respect to $\leq^*$). Every filter generated by less than $\mathfrak{b}$ sets is feeble and, by a result due to Simon, there is a non–feeble filter generated by $\mathfrak{b}$ many sets (see [5]).

Fact 4.2.4 *The following inequalities hold:*

(a) $\mathfrak{p} \leq \mathfrak{k}$,

(b) $\mathfrak{k} \leq \mathfrak{b}$,

(c) if $2^\alpha > \mathfrak{c}$, then $\mathfrak{k} \leq \alpha$.

Proof. Of these (a) and (b) are simple consequences of Fact 4.2.3 and of Simon's result. To check the last inequality consider a subalgebra $\mathfrak{A}$ of $P(\omega)$ generated by an independent sequence of size α (we skip the trivial case $\alpha > \mathfrak{c}$). Then $|K| = 2^\alpha > \mathfrak{c}$. Since there are only $\mathfrak{c}$ many bijections on ω, there is a non–condensed $x \in K = \mathrm{Stone}(\mathfrak{A})$. But every $x \in K = \mathrm{Stone}(\mathfrak{A})$ is generated by α many sets. Hence, $\mathfrak{k} \leq \alpha$.

$\blacksquare$

It should be pointed out that Fact 4.2.4(c) implies that it is consistent to assume that $\mathfrak{k} < \mathfrak{b}$ (so, that there is a feeble but non–condensed filter).

Problem 4.2.5 *$Con(\mathfrak{p} < \mathfrak{k})$?*

We show that a certain assumption on the coefficient $\mathfrak{k}$ implies a positive answer to Problem 4.2.1.

Consider a $\subseteq^*$–tree $\mathcal{T} = \{T_\sigma \colon \sigma \in \kappa^{<\lambda}\}$ on ω, i.e. a subfamily of $P(\omega)$ with the properties:

- $T_\emptyset = \omega$;
- $T_{\sigma^\wedge\alpha} \subseteq^* T_\sigma$ for every $\sigma \in \kappa^{<\lambda}$ and $\alpha \in \kappa$;
- $T_\sigma \cap T_\tau =^* \emptyset$ if $|\sigma| = |\tau|$ and $\sigma \neq \tau$.

Let $\mathfrak{T}$ be a subalgebra of $P(\omega)$ generated by $\mathcal{T}$. Assume $x \in K = \mathrm{Stone}(\mathfrak{T})$. If there is $\sigma \in \kappa^\lambda$ such that x is generated by the family $\{T_{\sigma|\gamma} \colon \gamma < \lambda\}$, then we say that x is *a branch*. Otherwise, we say that x is *a knot*. Every knot is generated by a family of the form

$$\{T_{\sigma|\gamma} \colon \gamma \leq |\sigma|\} \cup \{T^c_{\sigma^\wedge\alpha} \colon \alpha \in \kappa\},$$

where $\sigma \in \kappa^{<\lambda}$ (we assume here that κ is infinite). It is often convenient to parametrize a $\subseteq^*$–tree by a proper subset of $\kappa^{<\lambda}$. In this case knots in K can have other forms. For example, if $\mathcal{T}$ is parametrized by the sequences of lengths being successor ordinals, then knots are generated by

$$\{T_{\sigma|\gamma} \colon \gamma \leq |\sigma|\} \cup \{T^c_{\sigma^\wedge\alpha} \colon \alpha \in \kappa\},$$

if $|\sigma|$ is a successor ordinal and

$$\{T_{\sigma|\gamma} \colon \gamma < |\sigma|\} \cup \{T^c_{\sigma^\wedge\alpha} \colon \alpha \in \kappa\},$$

if $|\sigma|$ is limit.

In Chapter 3 we mentioned the Balcar– Simon–Pelant theorem. We recall it once again in its full strength. Here, $\mathfrak{h}$ stands for *the distributivity number* (see [5] for the definition and basic properties). Recall that $\mathcal{A} \subseteq P(\omega)$ is *a m.a.d. family* if it is maximal pairwise almost disjoint. We find it convenient to say that for a filter $\mathcal{F} \subseteq P(\omega)$ a family $\mathcal{A} \subseteq P(\omega)$ is *a m.a.d. family below* $\mathcal{F}$, if it is a maximal family such that $\mathcal{A}$ is pairwise almost disjoint and consists of pseudo–intersections of $\mathcal{F}$.

Theorem 4.2.6 (Balcar, Simon, Pelant [4]) *There is a family of infinite sets $\mathcal{S} \subseteq P(\omega)$ such that*

- *$\mathcal{S}$ is a $\subseteq^*$–tree of height $\mathfrak{h}$,*
- *each level of $\mathcal{S}$, except of the root (which is ω), is a m.a.d. family,*
- *every infinite $A \subseteq \omega$ has a subset in $\mathcal{S}$.*

For our purposes it is not important that $\mathcal{S}$ is dense in $[\omega]^\omega$, but rather that it is splitting, i.e. for every infinite $A \subseteq \omega$ there is $T \in \mathcal{S}$ such that both $A \cap T$ and $A \setminus T$ are infinite. This property ensures that no ultrafilter in the algebra generated by $\mathcal{S}$ has an infinite pseudo–intersection.

A $\subseteq^*$–tree satisfying the above properties is often called a *base matrix tree*. We can assume that its each node has $\mathfrak{c}$ immediate successors. Thus, it can be indexed by the set $\Lambda = \{\sigma \in \mathfrak{c}^{<\mathfrak{h}} \colon |\sigma| \text{ is a successor ordinal or } |\sigma| = 0\}$.

Lemma 4.2.7 *Let $\mathcal{A}$ be a m.a.d. family below a filter $\mathcal{F}$ and let $f \colon \omega \to \omega$ be a bijection. Then there is a refinement $\mathcal{B}$ of $\mathcal{A}$ (i.e. for every $B \in \mathcal{B}$ there is $A \in \mathcal{A}$ such that $B \subseteq^* A$) such that $\mathcal{B}$ is a m.a.d. family below $\mathcal{F}$ and $f[B]$ has density 0 for every $B \in \mathcal{B}$.*

Proof. Let $\mathcal{B}$ be a maximal family such that

- $\mathcal{B}$ is pairwise almost disjoint,
- $\mathcal{B}$ is a refinement of $\mathcal{A}$,
- if $B \in \mathcal{B}$, then $f[B]$ has density 0.

The family $\mathcal{B}$ is a m.a.d. family below $\mathcal{F}$. Indeed, assume that there is an infinite $N \notin \mathcal{B}$ such that $N \cap B$ is finite for every $B \in \mathcal{B}$. Clearly, $N \cap A$ is infinite for some $A \in \mathcal{A}$. Hence, every infinite $M \subseteq A \cap N$ such that $f[M]$ is of density 0 contradicts the maximality assumption.

■

Theorem 4.2.8 *Assume $\aleph_1 = \mathfrak{h} < \mathfrak{b}$. Then there is a Koppelberg compactification K of ω such that no subsequence of ω is convergent and every $x \in K \setminus \omega$ is feeble (as a filter on ω).*

Proof. Let $\mathcal{S} = \{S_\sigma \colon \sigma \in \Lambda\}$ be a splitting $\subseteq^*$–tree of height $\mathfrak{h}$. We will construct new $\subseteq^*$–splitting tree $\mathcal{T} = \{T_\sigma \colon \sigma \in \Lambda\}$ such that $\mathcal{T}$ is a refinement of $\mathcal{S}$ and for every $\sigma \in \mathfrak{c}^{<\mathfrak{h}}$ there is a bijection $f_\sigma \colon \omega \to \omega$ such that

(a) $f_\sigma[T_\tau]$ has density 1 for every $\tau \in \Lambda$ such that σ extends τ;

(b) $f_\sigma[T_{\sigma^\wedge\alpha}]$ has density 0 for every $\alpha \in \mathfrak{c}$.

We define the levels of $\mathcal{T}$ inductively modifying the levels of $\mathcal{S}$ as follows.

Let $T_\emptyset = S_\emptyset = \omega$. Let $\xi < \mathfrak{h}$. Assume that we have defined T_σ for every $|\sigma| \leq \xi$ ($|\sigma| < \xi$ for a limit ξ) and consider σ of length ξ. Put $\mathcal{P} = \{T_\tau \colon \tau \in \Lambda$ and σ extends $\tau\}$. Let f_σ be such that $f_\sigma[T]$ has density 1 for every $P \in \mathcal{P}$. Such function exists since $\mathcal{P}$ has an infinite pseudo–intersection. Let $\mathcal{A}$ be a maximal infinite family such that

- $\mathcal{A}$ is pairwise almost disjoint,
- $\mathcal{A}$ refines the $(\xi + 1)$–st level of $\mathcal{S}$,
- if $A \in \mathcal{A}$, then A is a pseudo–intersection of $\mathcal{P}$.

Use Lemma 4.2.7 to find a refinement $\mathcal{B}$ of $\mathcal{A}$ below $\mathcal{P}$ such that $f_\sigma[B]$ has density 0 for every $B \in \mathcal{B}$. Enumerate $\mathcal{B} = \{\mathcal{T}_{\sigma^\wedge\alpha} \colon \alpha \in \mathfrak{c}\}$ and notice that for every $\sigma \in \mathfrak{c}^\xi$ we can repeat the above procedure independently.

The $\subseteq^*$–tree $\mathcal{T}$ constructed in this way is splitting since for every successor $\xi < \mathfrak{h}$ its ξ–th level refines the ξ–th level of $\mathcal{S}$. Therefore, no ultrafilter on $\mathfrak{T} = alg(\mathcal{T})$ has an infinite pseudo–intersection. The proof is completed by showing that $K = \mathrm{Stone}(\mathfrak{T})$ satisfies the other desired properties.

The algebra $\mathfrak{T}$ is minimally generated since it is a $\subseteq^*$–tree; so K is Koppelberg compact.

Consider $x \in K$. If x is a knot, then it is generated by families $\{T^c_{\sigma^\wedge\alpha} \colon \alpha \in \mathfrak{c}\}$ and $\{T_\tau \colon \tau \in \Lambda$ and σ extends $\tau\}$ for some $\sigma \in \mathfrak{c}^{<\mathfrak{h}}$. Therefore, the bijection f_σ proves that x is condensed and, hence, feeble.

Assume now that x is a branch. Then it is generated by $\mathfrak{h}$ many sets. By our assumption $\mathfrak{h} < \mathfrak{b}$, so x is, again, feeble.

■

The following theorem can be proved in exactly the same way, since its assumption implies that each branch in a base matrix tree is condensed.

Theorem 4.2.9 *Assume $\aleph_1 = \mathfrak{h} < \mathfrak{k}$. Then there is a Koppelberg compactification K of ω such that no subsequence of ω is convergent and every $x \in K \setminus \omega$ is condensed.*

Of course, the conclusion of Theorem 4.2.9 is stronger than that of 4.2.8, but it is not known if its assumption does not contradict ZFC (see Problem 4.2.5), while $\aleph_1 = \mathfrak{h} < \mathfrak{b}$ occurs in the standard Hechler model. Note that a positive answer to Problem 4.2.1 would be consistent also if we manage to prove that in standard Hechler's or Dordal's model every ω_1–tower is condensed.

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