Mathias Forcing with Filters

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Motivation: Given filter $\mathcal{F}$, add a pseudo-intersection of $\mathcal{F}$.
This also covers destroying MAD systems.
Be gentle.

Terminology

$\mathcal{F}$ is a filter means $\mathcal{F}$ is filter on ω and $\mathcal{F}$ extends the Fréchet filter.

For $f, g \in \omega^\omega$ is $f <^* g$ defined by $\{n \colon g(n) \leq f(n)\}$ is finite.

$\mathcal{P}(\omega) \simeq 2^\omega$
2^ω has the product topology.
Basic open sets $[q, n] = \{A \subset \omega \colon A \cap n = q\}$ for $n \in \omega, q \subset n$.

$[\omega]^\omega$ embeds in ω^ω via enumerating functions.
For $F, G \in [\omega]^\omega$ define $F <^* G$ by $e_F <^* e_G$.

$\mathcal{F} \subset \mathcal{P}(\omega)$ is unbounded, dominating, ...

Definition

Let $\mathcal{F}$ be a filter. $P \subset \omega$ is a *pseudo-intersection* of $\mathcal{F}$ iff $P \subset^* F$ for all $F \in \mathcal{F}$.

Remark

If P is a pseudo-intersection of $\mathcal{F}$ then P dominates $\mathcal{F}$.

Definition (Mathias forcing)

Let $\mathcal{F}$ be a filter on ω.
$\mathbb{M}_{\mathcal{F}} = \{\langle a, F\rangle \colon a \in [\omega]^{<\omega}, F \in \mathcal{F}\}$
$\langle a, F\rangle < \langle b, H\rangle$ iff $b \sqsubset a$, $F \subset H$, and $a \setminus b \subset H$.

Fact

$\mathbb{M}_{\mathcal{F}}$ adds a pseudo-intersection of $\mathcal{F}$.

$$P = \bigcup\{a \colon \langle a, F\rangle\} \in \mathcal{G}$$

Fact

$\mathbb{M}_{\mathcal{F}}$ is σ-centered.

Bounding-like properties

Definition

Let $\mathbb{P}$ be a forcing. We say that $\mathbb{P}$ (is):

ω^ω bounding	almost ω^ω bounding	does not add dominating reals
$\forall f \in \omega^\omega \cap V[G]$ $\exists g \in \omega^\omega \cap V$ $f <^* g$	$\forall \dot{f} \in V^{\mathbb{P}}\ \forall p \in \mathbb{P}$ $\exists g \in V\ \forall A \in [\omega]^\omega$ $\exists q < p$ $q \Vdash g \upharpoonright A \not<^* \dot{f} \upharpoonright A$	$\forall f \in \omega^\omega \cap V[G]$ $\exists g \in \omega^\omega \cap V$ $g \not<^* f$
preserves dominating sets as dominating	preserves unbounded sets as unbounded	preserves dominating sets as unbounded

Lemma

TFAE

1. $\mathbb{P}$ *is almost* ω^ω *bounding*
2. $\forall \dot{f} \in V^{\mathbb{P}}\, \forall p \in \mathbb{P}\, \exists g \in V\, \forall A \in [\omega]^\omega\, \exists q < p$
   $q \Vdash \{n \in A \colon \dot{f}(n) < g(n)\}$ *is infinite*
3. *preserves unbounded sets as unbounded*

$2 \Rightarrow 3.$

Let $\mathcal{A} \subset \omega^\omega$ be unbounded, $p \in \mathbb{P}$, $\dot{f} \in V^{\mathbb{P}}$.
There is g such that …
There is $a \in \mathcal{A}$ such that $A = \{n \in \omega \colon g(n) < a(n)\}$ is infinite.
There is $q < p$, $q \Vdash \{n \in A \colon \dot{f}(n) < g(n)\}$ is infinite,
hence $q \Vdash a \not<^* \dot{f}$.

$\neg 2 \Rightarrow \neg 3.$

$\exists \dot{f} \in V^{\mathbb{P}}\, \exists p \in \mathbb{P}\, \forall g \in V\, \exists A_g \in [\omega]^\omega\; p \Vdash g \restriction A_g <^* \dot{f} \restriction A_g.$
For $g \in \omega^\omega$ put $g'(n) = g(n)$ for $n \in A_g$ and $g'(n) = 0$ otherwise.
$\{g' \colon g \in \omega^\omega\}$ is unbounded.
$p \Vdash g' \leq^* \dot{f}$ for each $g \in \omega^\omega$.

Simple properties of Mathias forcing

Fact

$\mathbb{M}_{\mathcal{F}}$ is not ω^ω bounding.

Proof.

$\mathbb{M}_{\mathcal{F}}$ is σ-centered.

(Let $RO(\mathbb{P}) = \bigcup\{\mathcal{U}_n : n \in \omega\}, \mathcal{U}_n$ ultrafilter.
Define $\dot{f} \in \omega^\omega$ such that if $p \in \mathbb{P}$ decides $\dot{f}(n)$ then $p \notin \mathcal{U}_n$.)

Fact

If $\mathcal{F}$ unbounded then $\mathbb{M}_{\mathcal{F}}$ not almost ω^ω bounding.

Fact

If $\mathcal{F}$ dominating then $\mathbb{M}_{\mathcal{F}}$ adds a dominating real.

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If $\mathcal{F}$ dominating then $\mathbb{M}_{\mathcal{F}}$ adds a dominating real.

Fact

If $\mathcal{F}$ unbounded then $\mathbb{M}_{\mathcal{F}}$ not almost ω^ω bounding.

Theorem

$\mathbb{M}_{\mathcal{F}}$ does not add dominating reals iff $\mathcal{F}$ is Menger.

Theorem

$\mathbb{M}_{\mathcal{F}}$ is almost ω^ω bounding iff $\mathcal{F}$ is Hurewicz.

Let X be a topological space.

Definition

X is *Menger* if no continuous image of X in ω^ω is dominating.

Definition

X is *Hurewicz* if every continuous image of X in ω^ω is bounded.

$\mathcal{F}$ Hurewicz	$\mathcal{F}$ Menger
all images of $\mathcal{F}$ bounded	no images of $\mathcal{F}$ dominating
$\mathbb{M}_{\mathcal{F}}$ preserves unbounded sets as unbounded	$\mathbb{M}_{\mathcal{F}}$ preserves dominating sets as unbounded
$\mathbb{M}_{\mathcal{F}}$ almost ω^ω bounding	$\mathbb{M}_{\mathcal{F}}$ does not add dominating reals

Applications

Hurewicz and Menger classes are closed with respect to closed subsets, countable unions, products with compacts, continuous images, ...

Proposition

Let $\mathcal{F}$ be an analytic filter on ω. $\mathbb{M}_{\mathcal{F}}$ does not add a dominating real if and only if $\mathcal{F}$ is σ-compact.

Theorem

It is consistent with ZFC that every $\mathfrak{b}$-scale set is a γ-space.

Proposition

There exists a MAD family $\mathcal{A}$ on ω such that $\mathbb{M}_{\mathcal{F}(\mathcal{A})}$ adds a dominating real.

Proposition

If $\mathfrak{d} = \mathfrak{c}$, then there exists an infinite MAD family $\mathcal{A}$ such that $\mathbb{M}_{\mathcal{F}(\mathcal{A})}$ does not add a dominating real.

Cover of X means countable open cover of X.

Definition

$\mathcal{U}$ is a γ-*cover* of X if $\mathcal{U}$ is a cover of X and for every $x \in X$ the family $\{U \in \mathcal{U} \colon x \notin U\}$ is finite.

Definition

X is *Menger* if for every sequence $\{\mathcal{U}_n \colon n \in \omega\}$ of covers of X there is $\{\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega} \colon n \in \omega\}$ such that $\{\bigcup \mathcal{V}_n \colon n \in \omega\}$ is a cover of X.

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Example

$X \subset \omega^\omega$, X Hurewicz $\Rightarrow$ X bounded.

Proof.

Put $\mathcal{U}_n = \{U_n^i = \{f \in X \colon f(n) < i\} \colon i \in \omega\}$.
Pick $\mathcal{V}_n$ as in the definition.
Put $g(n) = \max\{i \colon U_n^i \in \mathcal{V}_n\}$, g dominates X.

For $a \subset \omega$ denote $\uparrow a = \{x \subset \omega \colon a \subset x\}$.

Fact

$\uparrow a$ is compact

a is finite $\Rightarrow \uparrow a$ is open

Definition

$\mathcal{U}$ is an $\uparrow$-*cover* of $X \subset 2^\omega$ if $\mathcal{U}$ is a cover of X consisting of sets of form $\uparrow a$, $a \in [\omega]^{<\omega}$.

Lemma

Let $\mathcal{F} \subset 2^\omega$ be a filter, $\mathcal{U}$ a cover of $\mathcal{F}$ (consisting of open subsets of 2^ω). There is an $\uparrow$-cover $\mathcal{O}$ of $\mathcal{F}$, such that $\mathcal{F} \subset \bigcup \mathcal{O} \subset \bigcup \mathcal{U}$.

Definition

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Lemma

Let $\mathcal{F} \subset 2^\omega$ be a filter, $\mathcal{U}$ a cover of $\mathcal{F}$ (consisting of open subsets of 2^ω). There is an $\uparrow$-cover $\mathcal{O}$ of F, such that $F \subset \bigcup \mathcal{O} \subset \bigcup \mathcal{U}$.

Corollary

For X filter, we can replace (γ-)cover by (γ-)$\uparrow$-cover in definitions of Menger and Hurewicz.

Corollary

Let $\mathcal{F}$ be a Menger (Hurewicz) filter. Then all finite powers of $\mathcal{F}$ are Menger (Hurewicz).

Lemma

Let $\mathcal{F} \subset 2^\omega$ be a filter, $\mathcal{U}$ a cover of $\mathcal{F}$ (consisting of open subsets of 2^ω). There is an $\uparrow$-cover $\mathcal{O}$ of $\mathcal{F}$, such that $\mathcal{F} \subset \bigcup \mathcal{O} \subset \bigcup \mathcal{U}$.

Proof.

Assume that $\mathcal{U}$ consists of basic open sets in 2^ω.
For each $F \in \mathcal{F}$ find a finite set $\mathcal{U}_F = \{[q_i, n_i] \colon i \in I_F\} \subset \mathcal{U}$, such that $\uparrow F \subset \bigcup \mathcal{U}_F$.
Put $n_F = \max\{n_i \colon i \in I_F\}$.

Claim

$\uparrow (F \cap n_F) \subset \bigcup \mathcal{U}_F$

If $A \in \uparrow (F \cap n_F)$, then $(A \cap n_F) \cup (\omega \setminus n_F) \in \uparrow F$,
there $i \in I_F$ such that $(A \cap n_F) \cup (\omega \setminus n_F) \in [q_i, n_i]$
hence $A \in [q_i, n_i]$.

Put $\mathcal{O} = \{\uparrow (F \cap n_F) \colon F \in \mathcal{F}\}$.

Definition

X is *Menger* if for every sequence $\{\mathcal{U}_n \colon n \in \omega\}$ of covers of X there is $\{\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega} \colon n \in \omega\}$ such that $\{\bigcup \mathcal{V}_n \colon n \in \omega\}$ is a cover of X.

Theorem

$\mathbb{M}_{\mathcal{F}}$ does not add dominating reals iff $\mathcal{F}$ is Menger.

Proof.

Theorem (Hrušák, Minami)

$\mathbb{M}_{\mathcal{F}}$ does not add dominating reals iff $\mathcal{F}^{<\omega}$ is a P^+-filter.

$\mathcal{F}^{<\omega}$ positive sets (i.e. $\mathcal{F}^{<\omega +}$) are exactly $\uparrow$-covers of $\mathcal{F}$.
$\mathcal{F}^{<\omega}$ is a P^+-filter iff $\mathcal{F}$ is Menger (with respect to $\uparrow$-covers).

Definition

X is *Hurewicz* if for every sequence $\{\mathcal{U}_n \colon n \in \omega\}$ of covers of X there is $\{\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega} \colon n \in \omega\}$ such that $\{\bigcup \mathcal{V}_n \colon n \in \omega\}$ is a γ-cover of X.

Theorem

$\mathbb{M}_{\mathcal{F}}$ is almost ω^ω bounding (preserves unbounded as unbounded) iff $\mathcal{F}$ is Hurewicz.

$\neg \Leftarrow \neg$.

Let $\mathcal{U}_n = \{\uparrow q_m(n) \colon m \in \omega\}$, where $q_m(n) \in [\omega]^{<\omega}$ be an $\uparrow$-cover witnessing non-Hurewicz of $\mathcal{F}$.
For $F \in \mathcal{F}$ define $x_F \in \omega^\omega$ by $x_F(n) = \min\{m \colon q_m(n) \subset F\}$.
$X = \{x_F \colon F \in \mathcal{F}\}$ is unbounded.
Let P be the generic pseudo-intersection added by $\mathbb{M}_{\mathcal{F}}$.
For every n there exists $g(n)$ such that $q_{g(n)}(n) \subset P \setminus n$ (by genericity).
Fix $F \in \mathcal{F}$. For any n such that $P \setminus n \subset F$ is

$$q_{g(n)}(n) \subset G \setminus n \subset F$$

i.e. $x_F(n) \leq g(n)$ and X is bounded by g.

Definition

X is *Hurewicz* if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of covers of X there is $\{\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega} : n \in \omega\}$ such that $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a γ-cover of X.

Theorem

$\mathbb{M}_{\mathcal{F}}$ is almost ω^ω bounding (preserves unbounded as unbounded) iff $\mathcal{F}$ is Hurewicz.

$\Leftarrow$.

Suppose there exists an unbounded $X \subset \omega^\omega$, $X \in V$, and a $\mathbb{M}_{\mathcal{F}}$-name $\dot{g}$ for a function dominating X.
For every $x \in X$ there is $n_x \in \omega$, $\langle s_x, F_x \rangle \in \mathbb{M}_{\mathcal{F}}$ such that

$$\langle s_x, F_x \rangle \Vdash x(n) < \dot{g}(n) \text{ for } n \geq n_x$$

We can assume $s_x = s^*$, $n_x = n^*$ for all $x \in X$.
Let $\mathcal{U}_m$ be set of those $s \in [\omega]^{<\omega}$, $s^* < s$ such that there is $F_s \in \mathcal{F}$,

$$\langle s^* \cup s, F_s \rangle \Vdash \dot{g}(m) = g_s(m).$$

$\mathcal{U}_m$ is an $\uparrow$-cover of $\mathcal{F}$.

Suppose there exists an unbounded $X \subset \omega^\omega$, $X \in V$, and a $\mathbb{M}_\mathcal{F}$-name $\dot{g}$ for a function dominating X.
For every $x \in X$ there is $n_x \in \omega$, $\langle s_x, F_x \rangle \in \mathbb{M}_\mathcal{F}$ such that

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$$\langle s^* \cup s, F_s \rangle \Vdash \dot{g}(m) = g_s(m).$$

$\mathcal{U}_m$ is an $\uparrow$-cover of $\mathcal{F}$.
There are $\mathcal{V}_m \in [U_m]^{<\omega}$, such that $\{\bigcup \mathcal{V}_m \colon m \in \omega\}$ is a γ-cover of $\mathcal{F}$.
Put $f(m) = \max\{g_s(m) \colon s \in \mathcal{V}_m\}$.
Fix $x \in X$.
There is $k_x \geq n^*$ such that for each $m \geq k_x$ there is $s_m \in \mathcal{V}_m$, $s_m \subset F_x$.
Now for $m \geq k_x$:

$$\langle s^*, F_x \rangle \Vdash x(m) < \dot{g}(m) \quad \text{and} \quad \langle s^* \cup s_m, F_{s_m} \rangle \Vdash \dot{g}(m) \leq f(m)$$

hence $x(m) < f(m)$, $\ldots X$ is bounded by f.